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## ELASTIC SOLUTIONS FOR AN INCLINED TRANSVERSELY ISOTROPIC MATERIAL DUE TO THREE-DIMENSIONAL POINT LOADS

JYH-JONG LIAO, TIN-BIN HU AND CHENG-DER WANG

We present the elastic solutions for displacements and stresses due to three-dimensional point loads in a transversely isotropic material (rock), for which the transversely isotropic full planes are inclined with respect to the horizontal loading surface. The closed-form solutions are derived by applying an efficient method, the double Fourier transform, to obtain the integral expressions for displacements and stresses. Subsequently, the double inverse Fourier transform and residue calculus are utilized to integrate the contours. Utilizing the double Fourier transform in a Cartesian coordinate system is a new approach to solving the displacement and stress components that result from three-dimensional point loads applied to an inclined transversely isotropic medium. In addition, it is the first presentation of the exact closed-form characteristic roots for this special material anisotropy. The proposed solutions demonstrate that the displacements and stresses are profoundly influenced by the rotation of the transversely isotropic planes ( $\phi$ ), the type and degree of material anisotropy ( $E/E'$ ,  $\nu/\nu'$ ,  $G/G'$ ), the geometric position ( $r$ ,  $\phi$ ,  $\zeta$ ), and the type of three-dimensional loading ( $P_x$ ,  $P_y$ ,  $P_z$ ). The present solutions are identical to previously published solutions if the planes of transverse isotropy are parallel to the horizontal loading surface. A parametric study is conducted to elucidate the influence of the aforementioned factors on the displacements and stresses. The computed results reveal that the induced displacements and stresses in the inclined isotropic/transversely isotropic rocks by a vertical point load are quite different from the displacements that result from previous solutions in which  $\phi = 0$ . The numerical results presented here are interesting for their ability to describe the physical features of inclined transversely isotropic rocks. Hence, the dip at an angle of inclination should be considered in computing the displacements and stresses in a transversely isotropic material due to applied loads.

### Introduction

Generally, estimates for the magnitudes of displacements and stresses in a solid are made using solutions that model a material as a homogeneous, linearly elastic, and isotropic medium. However, among geomaterials, these models are unable to describe the properties of some natural soils which have deposited by means of sedimentation over a long period of time, for example flocculated clays, varved silts or sands, or rock masses cut by discontinuities such as cleavages, foliations, stratifications, schistosity, or joints. The deformability properties of these inhomogeneous materials should result in structural anisotropy. From the standpoint of practical considerations in engineering, anisotropic bodies are often modeled as orthotropic or transversely isotropic materials. In particular, when discontinuities of rock masses dip at an angle from the horizontal surface, their effects on displacement and stress could be essential to

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*Keywords:* displacements, stresses, inclined transversely isotropic material, double Fourier transform, residue calculus, material anisotropy.

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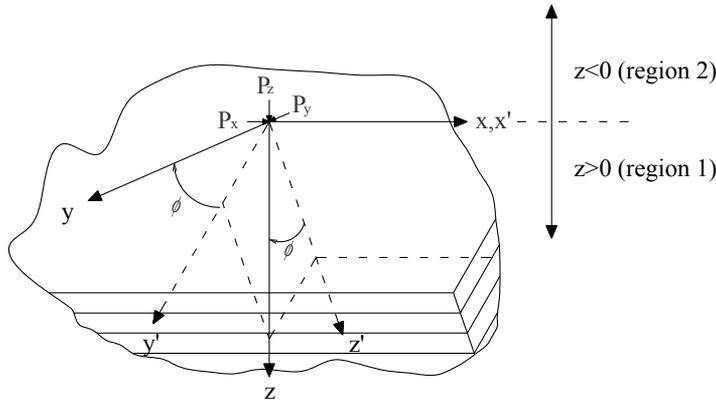
the design of structures. Hence, we study here the elastic loading problem for an inclined transversely isotropic material.

Elastic solutions to the problem of a point load acting on the interior of a full space are called the fundamental solutions or the elastic Green's function solutions [Tarn and Wang 1987]. These solutions in exact closed-form have always played an important role in applied mechanics and in particular numerical formulations of boundary element methods [Liew et al. 2001]. In [Liao and Wang 1998] we detailed the existing solutions for transversely isotropic full/half-spaces in which the planes of transverse isotropy are assumed to be parallel to the horizontal ground surface, subjected to three-dimensional point loads. However, recent work that was not cited in that paper should be mentioned here: [Ding et al. 1997; Hanson 1999; Pan and Tonon 2000; Pan and Yuan 2000a; Pan and Yuan 2000b; Tonon et al. 2001; Pan 2002; Fabrikant 2004; Ding et al. 2006]. Hu et al. [2007] presented the analytical solutions for displacements caused by three-dimensional point loads ( $P_x, P_y, P_z$ ) in a transversely isotropic full space, where the transversely isotropic planes are inclined with respect to the horizontal loading surface. In their derivation, the triple Fourier transform was employed to yield integral expressions of Green's displacement. Subsequently, the triple inverse Fourier transform and residue calculus were performed to integrate the contours. It is known that the stress components due to three-dimensional point loads on an inclined infinite space can be obtained by employing the coordinate transformation formulae with respect to the applied loads. Nevertheless, the displacement components cannot be solved by the same approach. Hence, their solutions for the displacements in an inclined space are new and cannot be obtained directly from the aforementioned point load solutions (see [Liao and Wang 1998], for example) with a linear rotation of the Cartesian coordinate system. Nevertheless, the derivations using the triple Fourier transform were not very efficient, and the presentations were rather lengthy. Yet, to the best of our knowledge, no solutions for displacements and stresses resulting from three-dimensional point loads acting on an inclined transversely isotropic material have been performed by using the double Fourier transform in a Cartesian coordinate system. The advantage of deriving stress components by applying the double Fourier transform is that the proposed solutions for the inclined transversely isotropic infinite space are parts of the point force problem in an inclined transversely isotropic half-space, achieved by satisfying the surface boundary conditions and thus utilizing the principle of superposition. Therefore, the yielded solutions are valuable and would be straightforward to extend to the half-space/bimaterial/layered material problems.

In the solutions presented here, it is found that both the displacement and stress solutions are governed by: (1) the rotation of the transversely isotropic planes ( $\phi$ ), (2) the type and degree of material anisotropy ( $E/E', \nu/\nu', G/G'$ ), (3) the geometric position ( $r, \varphi, \zeta$ ), and (4) the type of three-dimensional loading ( $P_x, P_y, P_z$ ). Two examples are given to illustrate the generated solutions and clarify how the rotation of the transversely isotropic planes ( $\phi$ ), the geometric position ( $\varphi$ ), and the type and degree of rock anisotropy would affect the displacements and stresses in inclined isotropic/transversely isotropic rocks subjected to a vertical point load ( $P_z$ ).

### 1. Displacements and stresses in an inclined transversely isotropic full space due to three-dimensional point loads

We start with the generalized Hooke's law for a transversely isotropic material in a Cartesian coordinate system ( $x', y', z'$ ), where  $z'$  is the rotation axis associated with elastic symmetry, and the  $x'$  and  $y'$  axes



**Figure 1.**  $(P_x, P_y, P_z)$  acting in an inclined transversely isotropic full space.

are in the plane of transverse isotropy (see [Figure 1](#)). We have

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \sigma_{z'z'} \\ \tau_{y'z'} \\ \tau_{z'x'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \\ \gamma_{y'z'} \\ \gamma_{z'x'} \\ \gamma_{x'y'} \end{bmatrix},$$

where  $\sigma_{x'x'}$ ,  $\sigma_{y'y'}$ ,  $\sigma_{z'z'}$  are the normal stresses,  $\varepsilon_{x'x'}$ ,  $\varepsilon_{y'y'}$ ,  $\varepsilon_{z'z'}$  the normal strains,  $\tau_{y'z'}$ ,  $\tau_{z'x'}$ ,  $\tau_{x'y'}$  the shear stresses,  $\gamma_{y'z'}$ ,  $\gamma_{z'x'}$ ,  $\gamma_{x'y'}$  the shear strains, and  $C_{11}$ ,  $C_{12}$ ,  $C_{13}$ ,  $C_{33}$ ,  $C_{44}$ ,  $C_{66}$  the elastic moduli.

Because  $C_{12} = C_{11} - 2C_{66}$ , only  $C_{11}$ ,  $C_{13}$ ,  $C_{33}$ ,  $C_{44}$ ,  $C_{66}$  are independent for a transversely isotropic material, so it is convenient to work in terms of the constants  $c_1, \dots, c_5$  defined in terms of the  $C_{ij}$  by

$$\begin{aligned} c_1 = C_{11} &= \frac{E(1 - (E/E')\nu^2)}{(1 + \nu)(1 - \nu - (2E/E')\nu^2)}, & c_2 = C_{33} &= \frac{E'(1 - \nu)}{1 - \nu - (2E/E')\nu^2}, & c_5 = C_{44} &= G', \\ c_3 = C_{13} + C_{44} &= \frac{Ev'}{1 - \nu - (2E/E')\nu^2} + C_{44}, & c_4 = C_{66} &= \frac{C_{11} - C_{12}}{2} = \frac{E}{2(1 + \nu)}. \end{aligned} \tag{1}$$

[Equation \(1\)](#) also displays the relation between  $c_1, \dots, c_5$  and the traditional constants  $E$  and  $E'$  (Young's moduli in the plane of transverse isotropy and in the normal direction to it),  $\nu$  and  $\nu'$  (Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively), and  $G'$  (the shear modulus in planes normal to the plane of transverse isotropy). In this notation we have

$$\begin{bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \sigma_{z'z'} \\ \tau_{y'z'} \\ \tau_{z'x'} \\ \tau_{x'y'} \end{bmatrix} = \begin{bmatrix} c_1 & c_1 - 2c_4 & c_3 - c_5 & 0 & 0 & 0 \\ c_1 - 2c_4 & c_1 & c_3 - c_5 & 0 & 0 & 0 \\ c_3 - c_5 & c_3 - c_5 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_4 \end{bmatrix} \begin{bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{z'z'} \\ \gamma_{y'z'} \\ \gamma_{z'x'} \\ \gamma_{x'y'} \end{bmatrix} \tag{2}$$

If a new coordinate system  $x, y, z$  is obtained from the original system  $x', y', z'$  by rotation through an angle  $\phi$  about an axis parallel to the strike direction, then  $x = x'$ . (See again Figure 1.) The direction cosines  $l_{ij}$  ( $i, j = 1, 2, 3$ ) of the transformation are given by

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}, \quad (3)$$

and in the new coordinate system the matrix of elastic moduli is

$$[q_{ij}]^T [C_{ij}] [q_{ij}] =: [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}, \quad (4)$$

where

$$[q_{ij}] = \begin{bmatrix} l_{11}^2 & l_{12}^2 & l_{13}^2 & l_{12}l_{13} & l_{13}l_{11} & l_{12}l_{11} \\ l_{21}^2 & l_{22}^2 & l_{23}^2 & l_{23}l_{22} & l_{23}l_{21} & l_{22}l_{21} \\ l_{31}^2 & l_{32}^2 & l_{33}^2 & l_{33}l_{32} & l_{33}l_{31} & l_{32}l_{31} \\ 2l_{31}l_{21} & 2l_{32}l_{22} & 2l_{33}l_{23} & l_{33}l_{22}+l_{32}l_{23} & l_{33}l_{21}+l_{31}l_{23} & l_{31}l_{22}+l_{32}l_{21} \\ 2l_{31}l_{11} & 2l_{32}l_{12} & 2l_{33}l_{13} & l_{33}l_{12}+l_{32}l_{13} & l_{33}l_{11}+l_{31}l_{13} & l_{31}l_{12}+l_{32}l_{11} \\ 2l_{21}l_{11} & 2l_{22}l_{12} & 2l_{23}l_{13} & l_{13}l_{22}+l_{12}l_{23} & l_{13}l_{21}+l_{11}l_{23} & l_{11}l_{22}+l_{12}l_{21} \end{bmatrix}, \quad (5)$$

$T$  denotes transposition, and the elastic constants  $a_{ij}$  have the following explicit expressions (those not listed vanish):

$$a_{11} = c_1,$$

$$a_{12} = a_{21} = (c_1 - 2c_4) \cos^2 \phi + (c_3 - c_5) \sin^2 \phi,$$

$$a_{13} = a_{31} = (c_3 - c_5) \cos^2 \phi + (c_1 - 2c_4) \sin^2 \phi,$$

$$a_{14} = a_{41} = (c_1 - c_3 - 2c_4 + c_5) \cos \phi \sin \phi,$$

$$a_{22} = c_1 \cos^4 \phi + 2c_3 + 2c_5 \cos^2 \phi \sin^2 \phi + c_2 \sin^4 \phi,$$

$$a_{23} = a_{32} = \frac{1}{8}(c_1 + c_2 + 6c_3 - 10c_5 - (c_1 + c_2 - 2c_3 - 2c_5) \cos 4\phi),$$

$$a_{24} = a_{42} = \frac{1}{4}(c_1 - c_2 + (c_1 + c_2 - 2c_3 - 2c_5) \cos 2\phi) \sin 2\phi,$$

$$a_{33} = c_2 \cos^4 \phi + 2c_3 + 2c_5 \cos^2 \phi \sin^2 \phi + c_1 \sin^4 \phi,$$

$$a_{34} = a_{43} = -\frac{1}{4}(-c_1 + c_2 + (c_1 + c_2 - 2c_3 - 2c_5) \cos 2\phi) \sin 2\phi,$$

$$a_{44} = \frac{1}{8}(c_1 + c_2 - 2c_3 + 6c_5 - (c_1 + c_2 - 2c_3 - 2c_5) \cos 4\phi),$$

$$a_{55} = c_5 \cos^2 \phi + c_4 \sin^2 \phi,$$

$$a_{56} = a_{65} = (c_4 - c_5) \cos \phi \sin \phi,$$

$$a_{66} = c_4 \cos^2 \phi + c_5 \sin^2 \phi.$$

Next we use the strain-displacement relation under the small strain condition in Cartesian coordinates:

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} -\partial u_x / \partial x \\ -\partial u_y / \partial y \\ -\partial u_z / \partial z \\ -\partial u_y / \partial z - \partial u_z / \partial y \\ -\partial u_x / \partial z - \partial u_z / \partial x \\ -\partial u_y / \partial x - \partial u_x / \partial y \end{bmatrix}, \quad (6)$$

where  $u_x$ ,  $u_y$ , and  $u_z$  are the components of the displacement.

Finally, the equilibrium equation is

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}, \quad (7)$$

where  $(F_x, F_y, F_z)$  are the three-dimensional point loads. Hence, the generalized Hooke's law expressed in the  $xyz$ -coordinates,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}, \quad (8)$$

gives rise to the system of equations

$$\sigma_{xx} = a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{yz} = -a_{11}\frac{\partial u_x}{\partial x} - a_{12}\frac{\partial u_y}{\partial y} - a_{13}\frac{\partial u_z}{\partial z} - a_{14}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad (9a)$$

$$\sigma_{yy} = a_{12}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{yz} = -a_{12}\frac{\partial u_x}{\partial x} - a_{22}\frac{\partial u_y}{\partial y} - a_{23}\frac{\partial u_z}{\partial z} - a_{24}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad (9b)$$

$$\sigma_{zz} = a_{13}\varepsilon_{xx} + a_{23}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{yz} = -a_{13}\frac{\partial u_x}{\partial x} - a_{23}\frac{\partial u_y}{\partial y} - a_{33}\frac{\partial u_z}{\partial z} - a_{34}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad (9c)$$

$$\tau_{yz} = a_{14}\varepsilon_{xx} + a_{24}\varepsilon_{yy} + a_{34}\varepsilon_{zz} + a_{44}\gamma_{yz} = -a_{14}\frac{\partial u_x}{\partial x} - a_{24}\frac{\partial u_y}{\partial y} - a_{34}\frac{\partial u_z}{\partial z} - a_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad (9d)$$

$$\tau_{zx} = a_{55}\gamma_{zx} + a_{56}\gamma_{xy} = -a_{55}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - a_{56}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad (9e)$$

$$\tau_{xy} = a_{56}\gamma_{zx} + a_{66}\gamma_{xy} = -a_{56}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) - a_{66}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right). \quad (9f)$$

Substituting  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ,  $\tau_{yz}$ ,  $\tau_{zx}$ ,  $\tau_{xy}$  from (9) into (7) enables the equations to be regrouped as Navier–Cauchy equations for a transversely isotropic material as

$$\begin{aligned}
 a_{11} \frac{\partial^2 u_x}{\partial x^2} + a_{66} \frac{\partial^2 u_x}{\partial y^2} + a_{55} \frac{\partial^2 u_x}{\partial z^2} + 2a_{56} \frac{\partial^2 u_x}{\partial y \partial z} + (a_{12} + a_{66}) \frac{\partial^2 u_y}{\partial x \partial y} + (a_{14} + a_{56}) \frac{\partial^2 u_y}{\partial x \partial z} \\
 + (a_{14} + a_{56}) \frac{\partial^2 u_z}{\partial x \partial y} + (a_{13} + a_{55}) \frac{\partial^2 u_z}{\partial x \partial z} + F_x = 0, \quad (10a)
 \end{aligned}$$

$$\begin{aligned}
 (a_{12} + a_{66}) \frac{\partial^2 u_x}{\partial x \partial y} + (a_{14} + a_{56}) \frac{\partial^2 u_x}{\partial x \partial z} + a_{66} \frac{\partial^2 u_y}{\partial x^2} + a_{22} \frac{\partial^2 u_y}{\partial y^2} + a_{44} \frac{\partial^2 u_y}{\partial z^2} + 2a_{24} \frac{\partial^2 u_y}{\partial y \partial z} \\
 + a_{56} \frac{\partial^2 u_z}{\partial x^2} + a_{24} \frac{\partial^2 u_z}{\partial y^2} + a_{34} \frac{\partial^2 u_z}{\partial z^2} + (a_{23} + a_{44}) \frac{\partial^2 u_z}{\partial y \partial z} + F_y = 0, \quad (10b)
 \end{aligned}$$

$$\begin{aligned}
 (a_{14} + a_{56}) \frac{\partial^2 u_x}{\partial x \partial y} + (a_{13} + a_{55}) \frac{\partial^2 u_x}{\partial x \partial z} + a_{56} \frac{\partial^2 u_y}{\partial x^2} + a_{24} \frac{\partial^2 u_y}{\partial y^2} + a_{34} \frac{\partial^2 u_y}{\partial z^2} + (a_{23} + a_{44}) \frac{\partial^2 u_y}{\partial y \partial z} \\
 + a_{55} \frac{\partial^2 u_z}{\partial x^2} + a_{44} \frac{\partial^2 u_z}{\partial y^2} + a_{33} \frac{\partial^2 u_z}{\partial z^2} + 2a_{34} \frac{\partial^2 u_z}{\partial y \partial z} + F_z = 0. \quad (10c)
 \end{aligned}$$

The point loads ( $F_x, F_y, F_z$ ) applied at the origin of the coordinate system of a full space can be described in terms of body forces ( $P_x, P_y, P_z$ ) as

$$F_x = P_x \delta(x) \delta(y) \delta(z), \quad F_y = P_y \delta(x) \delta(y) \delta(z), \quad F_z = P_z \delta(x) \delta(y) \delta(z), \quad (11)$$

where  $\delta$  is the Dirac delta function.

We will now use the double Fourier transform of the displacement components to solve the governing equations (10a)–(10c). We introduce the transform variables  $\alpha, \beta$  corresponding to  $x, y$ , respectively, and consider the double Fourier transform of  $u_i$  ( $i = x, y, z$ ),

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_i(x, y, z) e^{-i(\alpha x + \beta y)} dx dy = \bar{u}_i(\alpha, \beta, z).$$

As is well-known, we have

$$\begin{aligned}
 \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u_i(x, y, z)}{\partial x} e^{-i(\alpha x + \beta y)} dx dy &= i\alpha \bar{u}_i(\alpha, \beta, z), \\
 \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial u_i(x, y, z)}{\partial y} e^{-i(\alpha x + \beta y)} dx dy &= i\beta \bar{u}_i(\alpha, \beta, z), \\
 \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 u_i(x, y, z)}{\partial x^2} e^{-i(\alpha x + \beta y)} dx dy &= -\alpha^2 \bar{u}_i(\alpha, \beta, z), \quad \dots
 \end{aligned}$$

Since  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) dx dy = 1$ , the double Fourier transforms of (11) reduce to

$$\bar{F}_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_x \delta(x) \delta(y) \delta(z) e^{-i(\alpha x + \beta y)} dx dy = \frac{P_x}{2\pi} \delta(z), \quad (12a)$$

$$\bar{F}_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_y \delta(x) \delta(y) \delta(z) e^{-i(\alpha x + \beta y)} dx dy = \frac{P_y}{2\pi} \delta(z), \quad (12b)$$

$$\bar{F}_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_z \delta(x) \delta(y) \delta(z) e^{-i(\alpha x + \beta y)} dx dy = \frac{P_z}{2\pi} \delta(z). \quad (12c)$$

When  $z \neq 0$ , this allows us to rewrite the Navier–Cauchy equations (10a)–(10c) as a system of linear ordinary differential equations:

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (13)$$

where

$$d_{11} = a_{11}\alpha^2 + a_{66}\beta^2 - a_{55} \frac{d^2}{dz^2} - 2ia_{56}\beta \frac{d}{dz}, \quad (14a)$$

$$d_{12} = d_{21} = (a_{12} + a_{66})\alpha\beta - i(a_{14} + a_{56})\alpha \frac{d}{dz}, \quad (14b)$$

$$d_{13} = d_{31} = (a_{14} + a_{56})\alpha\beta - i(a_{13} + a_{55})\alpha \frac{d}{dz}, \quad (14c)$$

$$d_{22} = a_{66}\alpha^2 + a_{22}\beta^2 - a_{44} \frac{d^2}{dz^2} - 2ia_{24}\beta \frac{d}{dz}, \quad (14d)$$

$$d_{23} = d_{32} = a_{56}\alpha^2 + a_{24}\beta^2 - a_{34} \frac{d^2}{dz^2} - i(a_{23} + a_{44})\beta \frac{d}{dz}, \quad (14e)$$

$$d_{33} = a_{55}\alpha^2 + a_{44}\beta^2 - a_{33} \frac{d^2}{dz^2} - 2ia_{34}\beta \frac{d}{dz}. \quad (14f)$$

From the elementary theory of linear ordinary differential equations we know that we need to solve the characteristic equation

$$\det [d_{ij}(\alpha, \beta, u)] = 0, \quad (15)$$

where the entries  $d_{ij}(\alpha, \beta, u)$  are obtained from (14) by substituting the unknown  $u$  (representing the eigenvalue) for  $d/dz$ :

$$d_{11}(\alpha, \beta, u) = a_{11}\alpha^2 + a_{66}\beta^2 + a_{55}(iu)^2 - 2a_{56}\beta(iu), \quad (16a)$$

$$d_{12}(\alpha, \beta, u) = d_{21}(\alpha, \beta, u) = (a_{12} + a_{66})\alpha\beta - (a_{14} + a_{56})\alpha(iu), \quad (16b)$$

$$d_{13}(\alpha, \beta, u) = d_{31}(\alpha, \beta, u) = (a_{14} + a_{56})\alpha\beta - (a_{13} + a_{55})\alpha(iu), \quad (16c)$$

$$d_{22}(\alpha, \beta, u) = a_{66}\alpha^2 + a_{22}\beta^2 + a_{44}(iu)^2 - 2a_{24}\beta(iu), \quad (16d)$$

$$d_{23}(\alpha, \beta, u) = d_{32}(\alpha, \beta, u) = a_{56}\alpha^2 + a_{24}\beta^2 + a_{34}(iu)^2 - (a_{23} + a_{44})\beta(iu), \quad (16e)$$

$$d_{33}(\alpha, \beta, u) = a_{55}\alpha^2 + a_{44}\beta^2 + a_{33}(iu)^2 - 2a_{34}\beta(iu). \quad (16f)$$

An algebraic manipulation (details and the physical basis of which are given in the [Appendix](#)) shows that, if we set

$$A_1 = \frac{c_4}{c_5}, \quad A_2 = \frac{1}{2} \left[ \frac{c_5^2 + c_1c_2 - c_3^2}{c_2c_5} + \left\{ \left( \frac{c_5^2 + c_1c_2 - c_3^2}{c_2c_5} \right)^2 - 4 \frac{c_1}{c_2} \right\}^{1/2} \right], \quad (17)$$

$$A_3 = \frac{1}{2} \left[ \frac{c_5^2 + c_1c_2 - c_3^2}{c_2c_5} - \left\{ \left( \frac{c_5^2 + c_1c_2 - c_3^2}{c_2c_5} \right)^2 - 4 \frac{c_1}{c_2} \right\}^{1/2} \right],$$

the characteristic equation takes on the form

$$c_2 c_5^2 \prod_{j=1}^3 \left( A_j (-u^2 + \alpha^2 + \beta^2 - (iu \cos \phi + \beta \sin \phi)^2) + (iu \cos \phi + \beta \sin \phi)^2 \right) = 0, \quad (18)$$

and its six eigenroots  $u_1, \dots, u_6$  can be expressed as

$$\left. \begin{aligned} u_j &= \frac{-i\beta \sin \phi \cos \phi (-1 + A_j) - \sqrt{A_j(\beta^2 + \alpha^2(\cos^2 \phi + A_j \sin^2 \phi))}}{\cos^2 \phi + A_j \sin^2 \phi} \\ u_{3+j} &= \frac{-i\beta \sin \phi \cos \phi (-1 + A_j) + \sqrt{A_j(\beta^2 + \alpha^2(\cos^2 \phi + A_j \sin^2 \phi))}}{\cos^2 \phi + A_j \sin^2 \phi} \end{aligned} \right\} (i = 1, 2, 3). \quad (19)$$

(In particular, the real parts of  $u_1, u_2, u_3$  are negative and those of  $u_4, u_5, u_6$  are positive.)

Equations (19) represent the first instance in which exact closed-form eigenroots for the inclined transversely isotropic media are proposed. To derive explicitly the solutions to (10a)–(10c), define three displacement functions as follows:

for  $z > 0$  (region 1 in Figure 1),

$$\bar{u}_{x1}(\alpha, \beta, z) = A_{x1}^1 e^{u_1 z} + A_{x1}^2 e^{u_2 z} + A_{x1}^3 e^{u_3 z} + A_{x1}^4 e^{u_4 z} + A_{x1}^5 e^{u_5 z} + A_{x1}^6 e^{u_6 z}, \quad (20a)$$

$$\bar{u}_{y1}(\alpha, \beta, z) = A_{y1}^1 e^{u_1 z} + A_{y1}^2 e^{u_2 z} + A_{y1}^3 e^{u_3 z} + A_{y1}^4 e^{u_4 z} + A_{y1}^5 e^{u_5 z} + A_{y1}^6 e^{u_6 z}, \quad (20b)$$

$$\bar{u}_{z1}(\alpha, \beta, z) = A_{z1}^1 e^{u_1 z} + A_{z1}^2 e^{u_2 z} + A_{z1}^3 e^{u_3 z} + A_{z1}^4 e^{u_4 z} + A_{z1}^5 e^{u_5 z} + A_{z1}^6 e^{u_6 z}, \quad (20c)$$

and for  $z < 0$  (region 2 in Figure 1),

$$\bar{u}_{x2}(\alpha, \beta, z) = A_{x2}^1 e^{u_1 z} + A_{x2}^2 e^{u_2 z} + A_{x2}^3 e^{u_3 z} + A_{x2}^4 e^{u_4 z} + A_{x2}^5 e^{u_5 z} + A_{x2}^6 e^{u_6 z}, \quad (21a)$$

$$\bar{u}_{y2}(\alpha, \beta, z) = A_{y2}^1 e^{u_1 z} + A_{y2}^2 e^{u_2 z} + A_{y2}^3 e^{u_3 z} + A_{y2}^4 e^{u_4 z} + A_{y2}^5 e^{u_5 z} + A_{y2}^6 e^{u_6 z}, \quad (21b)$$

$$\bar{u}_{z2}(\alpha, \beta, z) = A_{z2}^1 e^{u_1 z} + A_{z2}^2 e^{u_2 z} + A_{z2}^3 e^{u_3 z} + A_{z2}^4 e^{u_4 z} + A_{z2}^5 e^{u_5 z} + A_{z2}^6 e^{u_6 z}. \quad (21c)$$

In (20), the undetermined coefficients  $A_{x1}^j, A_{y1}^j$ , and  $A_{z1}^j$  ( $j = 1, \dots, 6$ ) can be obtained by assuming the displacements in region 1,  $u_{x1}, u_{y1}$ , and  $u_{z1}$  must be finite when  $z$  approaches  $\infty$ . Hence,  $A_{x1}^4 = A_{x1}^5 = A_{x1}^6 = 0, A_{y1}^4 = A_{y1}^5 = A_{y1}^6 = 0$ , and  $A_{z1}^4 = A_{z1}^5 = A_{z1}^6 = 0$ . Similarly, in region 2,  $u_{x2}, u_{y2}$ , and  $u_{z2}$  also must be finite when  $z$  approaches  $-\infty$ . Therefore,  $A_{x2}^1 = A_{x2}^2 = A_{x2}^3 = 0, A_{y2}^1 = A_{y2}^2 = A_{y2}^3 = 0$ , and  $A_{z2}^1 = A_{z2}^2 = A_{z2}^3 = 0$ .

Now, let

$$\begin{aligned} \frac{A_{x1}^j}{D_{11}(\alpha, \beta, u_j)} &= \frac{A_{y1}^j}{D_{21}(\alpha, \beta, u_j)} = \frac{A_{z1}^j}{D_{31}(\alpha, \beta, u_j)} = C_d^j \quad (j = 1, 2, 3), \\ \frac{A_{x2}^j}{D_{11}(\alpha, \beta, u_j)} &= \frac{A_{y2}^j}{D_{21}(\alpha, \beta, u_j)} = \frac{A_{z2}^j}{D_{31}(\alpha, \beta, u_j)} = C_u^j \quad (j = 4, 5, 6), \end{aligned} \quad (22)$$

where  $D_{11}$ ,  $D_{21}$ , and  $D_{31}$  are second-order determinants<sup>1</sup> involving the functions  $d_{ij}$  ( $i, j = 1, 2, 3$ ) of (16):

$$D_{11}(\alpha, \beta, u_j) = \det \begin{bmatrix} d_{22}(\alpha, \beta, u_j) & d_{23}(\alpha, \beta, u_j) \\ d_{32}(\alpha, \beta, u_j) & d_{33}(\alpha, \beta, u_j) \end{bmatrix}, \quad (23a)$$

$$D_{21}(\alpha, \beta, u_j) = -\det \begin{bmatrix} d_{12}(\alpha, \beta, u_j) & d_{13}(\alpha, \beta, u_j) \\ d_{32}(\alpha, \beta, u_j) & d_{33}(\alpha, \beta, u_j) \end{bmatrix}, \quad (23b)$$

$$D_{31}(\alpha, \beta, u_j) = \det \begin{bmatrix} d_{12}(\alpha, \beta, u_j) & d_{13}(\alpha, \beta, u_j) \\ d_{22}(\alpha, \beta, u_j) & d_{23}(\alpha, \beta, u_j) \end{bmatrix}. \quad (23c)$$

The stress components of (9) are transformed by the double Fourier transform as

$$\bar{\sigma}_{xx} = -i\alpha a_{11}\bar{u}_x - i\beta a_{12}\bar{u}_y - a_{13}\frac{\partial\bar{u}_z}{\partial z} - a_{14}\left(\frac{\partial\bar{u}_y}{\partial z} + i\beta\bar{u}_z\right), \quad (24a)$$

$$\bar{\sigma}_{yy} = -i\alpha a_{12}\bar{u}_x - i\beta a_{22}\bar{u}_y - a_{23}\frac{\partial\bar{u}_z}{\partial z} - a_{24}\left(\frac{\partial\bar{u}_y}{\partial z} + i\beta\bar{u}_z\right), \quad (24b)$$

$$\bar{\sigma}_{zz} = -i\alpha a_{13}\bar{u}_x - i\beta a_{23}\bar{u}_y - a_{33}\frac{\partial\bar{u}_z}{\partial z} - a_{34}\left(\frac{\partial\bar{u}_y}{\partial z} + i\beta\bar{u}_z\right), \quad (24c)$$

$$\bar{\tau}_{yz} = -i\alpha a_{14}\bar{u}_x - i\beta a_{24}\bar{u}_y - a_{34}\frac{\partial\bar{u}_z}{\partial z} - a_{44}\left(\frac{\partial\bar{u}_y}{\partial z} + i\beta\bar{u}_z\right), \quad (24d)$$

$$\bar{\tau}_{zx} = -a_{55}\left(\frac{\partial\bar{u}_x}{\partial z} + i\alpha\bar{u}_z\right) - ia_{56}(\beta\bar{u}_x + \alpha\bar{u}_y), \quad (24e)$$

$$\bar{\tau}_{xy} = -a_{56}\left(\frac{\partial\bar{u}_x}{\partial z} + i\alpha\bar{u}_z\right) - ia_{66}(\beta\bar{u}_x + \alpha\bar{u}_y). \quad (24f)$$

We next consider the plane  $z = 0$  separating regions 1 and 2 of Figure 1, and write the pertinent continuity and discontinuity conditions along this plane:

$$\tau_{zx1}(x, y, 0) - \tau_{zx2}(x, y, 0) = P_x\delta(x)\delta(y), \quad u_{x1}(x, y, 0) = u_{x2}(x, y, 0), \quad (25a)$$

$$\tau_{zy1}(x, y, 0) - \tau_{zy2}(x, y, 0) = P_y\delta(x)\delta(y), \quad u_{y1}(x, y, 0) = u_{y2}(x, y, 0), \quad (25b)$$

$$\sigma_{zz1}(x, y, 0) - \sigma_{zz2}(x, y, 0) = P_z\delta(x)\delta(y), \quad u_{z1}(x, y, 0) = u_{z2}(x, y, 0), \quad (25c)$$

where  $-\infty < x < \infty$ ,  $-\infty < y < \infty$ . The subscripts 1 and 2 denote the limits as  $z \rightarrow 0^+$  and  $z \rightarrow 0^-$ .

Taking the double Fourier transform of (25), we obtain

$$\bar{\tau}_{zx1}(\alpha, \beta, 0) - \bar{\tau}_{zx2}(\alpha, \beta, 0) = P_x/2\pi, \quad \bar{u}_{x1}(\alpha, \beta, 0) = \bar{u}_{x2}(\alpha, \beta, 0), \quad (26a)$$

$$\bar{\tau}_{zy1}(\alpha, \beta, 0) - \bar{\tau}_{zy2}(\alpha, \beta, 0) = P_y/2\pi, \quad \bar{u}_{y1}(\alpha, \beta, 0) = \bar{u}_{y2}(\alpha, \beta, 0), \quad (26b)$$

$$\bar{\sigma}_{zz1}(\alpha, \beta, 0) - \bar{\sigma}_{zz2}(\alpha, \beta, 0) = P_z/2\pi, \quad \bar{u}_{z1}(\alpha, \beta, 0) = \bar{u}_{z2}(\alpha, \beta, 0). \quad (26c)$$

We further introduce the convention

$$D_{i1}^j := D_{i1}(\alpha, \beta, u_j). \quad (27)$$

<sup>1</sup>There are similar determinants  $D_{ij}$  for all  $i, j = 1, 2, 3$ , but we don't need the other ones here.

Then we can rewrite the linear system (26) in terms of unknowns  $C_d^1, C_d^2, C_d^3, C_u^4, C_u^5, C_u^6$ :

$$[f_{ij}] \begin{bmatrix} C_d^1 \\ C_d^2 \\ C_d^3 \\ C_u^4 \\ C_u^5 \\ C_u^6 \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{28}$$

where

$$\begin{aligned} f_{1j} &= -i(a_{56}(\beta D_{11}^j + \alpha D_{21}^j) + a_{55}(\alpha D_{31}^j - i D_{11}^j u_j)) & (j = 1, 2, 3), \\ f_{1j} &= i(a_{56}(\beta D_{11}^j + \alpha D_{21}^j) + a_{55}(\alpha D_{31}^j - i D_{11}^j u_j)) & (j = 4, 5, 6), \\ f_{2j} &= -i(\alpha a_{14} D_{11}^j + \beta a_{24} D_{21}^j + \beta a_{44} D_{31}^j) - (a_{44} D_{21}^j + a_{34} D_{31}^j) u_j & (j = 1, 2, 3), \\ f_{2j} &= i(\alpha a_{14} D_{11}^j + \beta a_{24} D_{21}^j + \beta a_{44} D_{31}^j) + (a_{44} D_{21}^j + a_{34} D_{31}^j) u_j & (j = 4, 5, 6), \\ f_{3j} &= -i(\alpha a_{13} D_{11}^j + \beta a_{23} D_{21}^j + \beta a_{34} D_{31}^j) - (a_{34} D_{21}^j + a_{33} D_{31}^j) u_j & (j = 1, 2, 3), \\ f_{3j} &= i(\alpha a_{13} D_{11}^j + \beta a_{23} D_{21}^j + \beta a_{34} D_{31}^j) + (a_{34} D_{21}^j + a_{33} D_{31}^j) u_j & (j = 4, 5, 6), \\ f_{4j} &= D_{11}^j, \quad f_{5j} = D_{21}^j, \quad f_{6j} = D_{31}^j & (j = 1, 2, 3), \\ f_{4j} &= -D_{11}^j, \quad f_{5j} = -D_{21}^j, \quad f_{6j} = -D_{31}^j & (j = 4, 5, 6). \end{aligned}$$

Now  $u_x, u_y,$  and  $u_z$  can be obtained by taking the double inverse Fourier transform of (20) and (21):

For  $z > 0$  (region 1),

$$u_{x1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j D_{11}(\alpha, \beta, u_j) e^{i(\alpha x + \beta y) + u_j z} d\alpha d\beta, \tag{29a}$$

$$u_{y1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j D_{21}(\alpha, \beta, u_j) e^{i(\alpha x + \beta y) + u_j z} d\alpha d\beta, \tag{29b}$$

$$u_{z1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j D_{31}(\alpha, \beta, u_j) e^{i(\alpha x + \beta y) + u_j z} d\alpha d\beta. \tag{29c}$$

For  $z < 0$  (region 2),

$$u_{x2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j D_{11}(\alpha, \beta, u_j) e^{i(\alpha x + \beta y) + u_j z} d\alpha d\beta, \tag{30a}$$

$$u_{y2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j D_{21}(\alpha, \beta, u_j) e^{i(\alpha x + \beta y) + u_j z} d\alpha d\beta, \tag{30b}$$

$$u_{z2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j D_{31}(\alpha, \beta, u_j) e^{i(\alpha x + \beta y) + u_j z} d\alpha d\beta. \tag{30c}$$

The desired  $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{yz}, \tau_{zx},$  and  $\tau_{xy}$  can also be obtain by the double inverse Fourier transform:

for  $z > 0$  (region 1),

$$\sigma_{xx1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j \bar{\sigma}_{xx}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (31a)$$

$$\sigma_{yy1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j \bar{\sigma}_{yy}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (31b)$$

$$\sigma_{zz1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j \bar{\sigma}_{zz}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (31c)$$

$$\tau_{yz1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j \bar{\tau}_{yz}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (31d)$$

$$\tau_{zx1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j \bar{\tau}_{zx}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (31e)$$

$$\tau_{xy1}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^3 C_d^j \bar{\tau}_{xy}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (31f)$$

for  $z < 0$  (region 2),

$$\sigma_{xx2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j \bar{\sigma}_{xx}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (32a)$$

$$\sigma_{yy2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j \bar{\sigma}_{yy}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (32b)$$

$$\sigma_{zz2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j \bar{\sigma}_{zz}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (32c)$$

$$\tau_{yz2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j \bar{\tau}_{yz}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (32d)$$

$$\tau_{zx2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j \bar{\tau}_{zx}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (32e)$$

$$\tau_{xy2}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=4}^6 C_u^j \bar{\tau}_{xy}^j e^{i(ax+\beta y)+u_j z} d\alpha d\beta, \quad (32f)$$

where

$$\bar{\sigma}_{xx}^j = -i(\alpha a_{11} D_{11}^j + \beta a_{12} D_{21}^j + \beta a_{14} D_{31}^j - i(a_{14} D_{21}^j + a_{13} D_{31}^j)u_j), \quad (33a)$$

$$\bar{\sigma}_{yy}^j = -i(\alpha a_{12} D_{11}^j + \beta a_{22} D_{21}^j + \beta a_{24} D_{31}^j - i(a_{24} D_{21}^j + a_{23} D_{31}^j)u_j), \quad (33b)$$

$$\bar{\sigma}_{zz}^j = -i(\alpha a_{13} D_{11}^j + \beta a_{23} D_{21}^j + \beta a_{34} D_{31}^j - i(a_{34} D_{21}^j + a_{33} D_{31}^j)u_j), \quad (33c)$$

$$\bar{\tau}_{yz}^j = -i(\alpha a_{14} D_{11}^j + \beta a_{24} D_{21}^j + \beta a_{44} D_{31}^j - i(a_{44} D_{21}^j + a_{34} D_{31}^j)u_j), \quad (33d)$$

$$\bar{\tau}_{zx}^j = -i(a_{56}(\beta D_{11}^j + \alpha D_{21}^j) + a_{55}(\alpha D_{31}^j - i D_{11}^j u_j)), \quad (33e)$$

$$\bar{\tau}_{xy}^j = -i(a_{66}(\beta D_{11}^j + \alpha D_{21}^j) + a_{56}(\alpha D_{31}^j - i D_{11}^j u_j)). \quad (33f)$$

In (33),  $j = 1, 2, 3$  for  $z > 0$  (region 1), and  $j = 4, 5, 6$  for  $z < 0$  (region 2).

Now introduce polar coordinates  $(k, \theta_x)$  in the  $\alpha\beta$ -plane, so

$$\alpha = k \cos \theta_x \quad \text{and} \quad \beta = k \sin \theta_x.$$

Substituting into (19), we obtain for  $j = 1, 2, 3$

$$u_j = k \frac{-i \sin \theta_x \sin \phi \cos \phi (-1 + A_j) - \sqrt{A_j (\sin^2 \theta_x + \cos^2 \theta_x (\cos^2 \phi + A_j \sin^2 \phi))}}{\cos^2 \phi + A_j \sin^2 \phi}, \quad (34)$$

$$u_{j+3} = k \frac{-i \sin \theta_x \sin \phi \cos \phi (-1 + A_j) + \sqrt{A_j (\sin^2 \theta_x + \cos^2 \theta_x (\cos^2 \phi + A_j \sin^2 \phi))}}{\cos^2 \phi + A_j \sin^2 \phi},$$

where  $0 < k < \infty$  and  $0 < \theta_x < 2\pi$ .

We can then rewrite (27) and (22) in terms of  $k$  and  $\theta_x$ :

$$D_{11}^j(k, \theta_x) = k^4 D_{11}^j(\theta_x), \quad (35a)$$

$$D_{21}^j(k, \theta_x) = k^4 D_{21}^j(\theta_x), \quad (35b)$$

$$D_{31}^j(k, \theta_x) = k^4 D_{31}^j(\theta_x), \quad (35c)$$

$$C_d^j(k, \theta_x) = k^{-5} C_d^j(\theta_x), \quad (35d)$$

$$C_u^j(k, \theta_x) = k^{-5} C_u^j(\theta_x). \quad (35e)$$

Equations (33) can also be rewritten as

$$\bar{\sigma}_{xx}^j(k, \theta_x) = k^5 \bar{\sigma}_{xx}^j(\theta_x), \quad (36a)$$

$$\bar{\sigma}_{yy}^j(k, \theta_x) = k^5 \bar{\sigma}_{yy}^j(\theta_x), \quad (36b)$$

$$\bar{\sigma}_{zz}^j(k, \theta_x) = k^5 \bar{\sigma}_{zz}^j(\theta_x), \quad (36c)$$

$$\bar{\tau}_{yz}^j(k, \theta_x) = k^5 \bar{\tau}_{yz}^j(\theta_x), \quad (36d)$$

$$\bar{\tau}_{zx}^j(k, \theta_x) = k^5 \bar{\tau}_{zx}^j(\theta_x), \quad (36e)$$

$$\bar{\tau}_{xy}^j(k, \theta_x) = k^5 \bar{\tau}_{xy}^j(\theta_x). \quad (36f)$$

The exponential terms in (29)–(32) become

$$i(\alpha x + \beta y) + u_j z = k \psi_j(\theta_x), \quad (37)$$

while the element  $d\alpha d\beta$  equals

$$d\alpha d\beta = k dk d\theta_x. \quad (38)$$

Using (34), (35) and Equation (37), we can rewrite (29) and (30) as follows:

For  $z > 0$  (region 1),

$$u_{x1}(x, y, z) = -\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) D_{11}^j(\theta_x)}{\psi_j(\theta_x)} d\theta_x, \quad (39a)$$

$$u_{y1}(x, y, z) = -\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) D_{21}^j(\theta_x)}{\psi_j(\theta_x)} d\theta_x, \quad (39b)$$

$$u_{z1}(x, y, z) = -\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) D_{31}^j(\theta_x)}{\psi_j(\theta_x)} d\theta_x. \quad (39c)$$

For  $z < 0$  (region 2),

$$u_{x2}(x, y, z) = -\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=4}^6 \frac{C_u^j(\theta_x) D_{11}^j(\theta_x)}{\psi_j(\theta_x)} d\theta_x, \quad (40a)$$

$$u_{y2}(x, y, z) = -\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=4}^6 \frac{C_u^j(\theta_x) D_{21}^j(\theta_x)}{\psi_j(\theta_x)} d\theta_x, \quad (40b)$$

$$u_{z2}(x, y, z) = -\frac{1}{2\pi} \int_0^{2\pi} \sum_{j=4}^6 \frac{C_u^j(\theta_x) D_{31}^j(\theta_x)}{\psi_j(\theta_x)} d\theta_x. \quad (40c)$$

Likewise, using (34)–(37), we rewrite (32) for  $z > 0$  as

$$\sigma_{xx1}(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) \bar{\sigma}_{xx}^j(\theta_x)}{(\psi_j(\theta_x))^2} d\theta_x, \quad (41a)$$

$$\sigma_{yy1}(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) \bar{\sigma}_{yy}^j(\theta_x)}{(\psi_j(\theta_x))^2} d\theta_x, \quad (41b)$$

$$\sigma_{zz1}(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) \bar{\sigma}_{zz}^j(\theta_x)}{(\psi_j(\theta_x))^2} d\theta_x, \quad (41c)$$

$$\tau_{yz1}(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) \bar{\tau}_{yz}^j(\theta_x)}{(\psi_j(\theta_x))^2} d\theta_x, \quad (41d)$$

$$\tau_{zx1}(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) \bar{\tau}_{zx}^j(\theta_x)}{(\psi_j(\theta_x))^2} d\theta_x, \quad (41e)$$

$$\tau_{yz1}(x, y, z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^3 \frac{C_d^j(\theta_x) \bar{\tau}_{yz}^j(\theta_x)}{(\psi_j(\theta_x))^2} d\theta_x, \tag{41f}$$

and similarly for  $z < 0$ , with 1 replaced by 2 in the subscripts on the left-hand side and the sum ranging from 4 to 6.

We now introduce  $\omega = e^{i\theta_x}$  (whence in particular  $d\omega = i\omega d\theta_x$ ), and

$$\psi_7(\omega) = \psi_1(\omega)\psi_4(\omega) = \frac{1}{4\omega^2}(\eta_1^1\omega^4 + \eta_2^1\omega^2 + \eta_3^1) = \frac{\eta_1^1}{4\omega^2}(\omega^2 - \alpha_1^2)(\omega^2 - \beta_1^2), \tag{42a}$$

$$\psi_8(\omega) = \psi_2(\omega)\psi_5(\omega) = \frac{1}{4\omega^2}(\eta_1^2\omega^4 + \eta_2^2\omega^2 + \eta_3^2) = \frac{\eta_1^2}{4\omega^2}(\omega^2 - \alpha_2^2)(\omega^2 - \beta_2^2), \tag{42b}$$

$$\psi_9(\omega) = \psi_3(\omega)\psi_6(\omega) = \frac{1}{4\omega^2}(\eta_1^3\omega^4 + \eta_2^3\omega^2 + \eta_3^3) = \frac{\eta_1^3}{4\omega^2}(\omega^2 - \alpha_3^2)(\omega^2 - \beta_3^2), \tag{42c}$$

where, for  $j = 1, 2, 3$ ,

$$\eta_1^j = \frac{((ix + y) \cos \phi + z \sin \phi)^2 - A_j \sin \phi (2(ix + y)z \cos \phi + ((x - iy)^2 + z^2) \sin \phi)}{\cos^2 \phi + A_j \sin^2 \phi}, \tag{43a}$$

$$\eta_2^j = -2 \left( x^2 + y^2 - z^2 - \frac{z(y \sin 2\phi (-1 + A_j) - z(1 + 2A_j))}{\cos^2 \phi + A_j \sin^2 \phi} \right), \tag{43b}$$

$$\eta_3^j = \frac{((-ix + y) \cos \phi + z \sin \phi)^2 - A_j \sin \phi (2(-ix + y)z \cos \phi + ((x + iy)^2 + z^2) \sin \phi)}{\cos^2 \phi + A_j \sin^2 \phi}, \tag{43c}$$

$$\alpha_j^2 = \frac{-\eta_2^j + \sqrt{\eta_2^j - 4\eta_1^j \eta_3^j}}{2\eta_1^j}, \quad \beta_j^2 = \frac{-\eta_2^j - \sqrt{\eta_2^j - 4\eta_1^j \eta_3^j}}{2\eta_1^j}. \tag{43d}$$

In this notation, Equations (39), valid for  $z > 0$ , become

$$u_{x1}(x, y, z) = -\frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) D_{11}^j(\omega) \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} d\omega, \tag{44a}$$

$$u_{y1}(x, y, z) = -\frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) D_{21}^j(\omega) \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} d\omega, \tag{44b}$$

$$u_{z1}(x, y, z) = -\frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) D_{31}^j(\omega) \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} d\omega, \tag{44c}$$

while (40), valid for  $z < 0$ , takes on an analogous form, with 1 replaced by 2 on the left-hand sides and the roles of  $j$  and  $j + 3$  on the right-hand sides reversed.

Next, Equations (41), valid for  $z > 0$ , become

$$\sigma_{xx1}(x, y, z) = \frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) \bar{\sigma}_{xx}^j(\omega) \left( \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} \right)^2 d\omega, \quad (45a)$$

$$\sigma_{yy1}(x, y, z) = \frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) \bar{\sigma}_{yy}^j(\omega) \left( \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} \right)^2 d\omega, \quad (45b)$$

$$\sigma_{zz1}(x, y, z) = \frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) \bar{\sigma}_{zz}^j(\omega) \left( \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} \right)^2 d\omega, \quad (45c)$$

$$\tau_{yz1}(x, y, z) = \frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) \bar{\tau}_{yz}^j(\omega) \left( \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} \right)^2 d\omega, \quad (45d)$$

$$\tau_{zx1}(x, y, z) = \frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) \bar{\tau}_{zx}^j(\omega) \left( \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} \right)^2 d\omega, \quad (45e)$$

$$\tau_{xy1}(x, y, z) = \frac{1}{2\pi} \oint_C \frac{1}{i\omega} \sum_{j=1}^3 C_d^j(\omega) \bar{\tau}_{xy}^j(\omega) \left( \frac{\psi_{j+3}(\omega)}{\psi_{j+6}(\omega)} \right)^2 d\omega. \quad (45f)$$

The counterparts for  $z < 0$  are, as before, obtained by replacing 1 with 2 on the left-hand sides and interchanging  $j$  and  $j + 3$  on the right-hand sides.

In (42), Cauchy's theory of residues can be used to integrate the contours. If we set

$$\psi_j = \frac{\Phi_j}{\omega} \quad (j = 1, 2, 3), \quad (46)$$

and substitute (42) into (44), the result is a new expression for  $u_x$ ,  $u_y$ , and  $u_z$  valid for  $z > 0$  (region 1):

$$u_{x1}(x, y, z) = -4 \sum_{j=1}^3 \left( C_d^j(\alpha_j) D_{11}^j(\alpha_j) \frac{\Phi_{j+3}(\alpha_j)}{\eta_1^j(2\alpha_j)(\alpha_j^2 - \beta_j^2)} + C_d^j(-\alpha_j) D_{11}^j(-\alpha_j) \frac{\Phi_{j+3}(-\alpha_j)}{\eta_1^j(-2\alpha_j)(\alpha_j^2 - \beta_j^2)} \right), \quad (47a)$$

$$u_{y1}(x, y, z) = -4 \sum_{j=1}^3 \left( C_d^j(\alpha_j) D_{21}^j(\alpha_j) \frac{\Phi_{j+3}(\alpha_j)}{\eta_1^j(2\alpha_j)(\alpha_j^2 - \beta_j^2)} + C_d^j(-\alpha_j) D_{21}^j(-\alpha_j) \frac{\Phi_{j+3}(-\alpha_j)}{\eta_1^j(-2\alpha_j)(\alpha_j^2 - \beta_j^2)} \right), \quad (47b)$$

$$u_{z1}(x, y, z) = -4 \sum_{j=1}^3 \left( C_d^j(\alpha_j) D_{31}^j(\alpha_j) \frac{\Phi_{j+3}(\alpha_j)}{\eta_1^j(2\alpha_j)(\alpha_j^2 - \beta_j^2)} + C_d^j(-\alpha_j) D_{31}^j(-\alpha_j) \frac{\Phi_{j+3}(-\alpha_j)}{\eta_1^j(-2\alpha_j)(\alpha_j^2 - \beta_j^2)} \right), \quad (47c)$$

Since  $\Phi_4(-\alpha_1) = \Phi_5(-\alpha_2) = \Phi_6(-\alpha_3) = 0$ , this reduces to (still for  $z > 0$ )

$$u_{x1}(x, y, z) = -4 \sum_{j=1}^3 C_d^j(\alpha_j) D_{11}^j(\alpha_j) \frac{\Phi_{j+3}(\alpha_j)}{\eta_1^j(2\alpha_j)(\alpha_j^2 - \beta_j^2)}, \tag{48a}$$

$$u_{y1}(x, y, z) = -4 \sum_{j=1}^3 C_d^j(\alpha_j) D_{21}^j(\alpha_j) \frac{\Phi_{j+3}(\alpha_j)}{\eta_1^j(2\alpha_j)(\alpha_j^2 - \beta_j^2)}, \tag{48b}$$

$$u_{z1}(x, y, z) = -4 \sum_{j=1}^3 C_d^j(\alpha_j) D_{31}^j(\alpha_j) \frac{\Phi_{j+3}(\alpha_j)}{\eta_1^j(2\alpha_j)(\alpha_j^2 - \beta_j^2)}, \tag{48c}$$

The analogous formulas for  $z < 0$ , taking into account that in this case  $\Phi_1(\alpha_1) = \Phi_2(\alpha_2) = \Phi_3(\alpha_3) = 0$ , read as follows:

$$u_{x2}(x, y, z) = -4 \sum_{j=1}^3 C_u^{j+3}(-\alpha_j) D_{11}^{j+3}(-\alpha_j) \frac{\Phi_j(-\alpha_j)}{\eta_1^j(-2\alpha_j)(\alpha_j^2 - \beta_j^2)}, \tag{49a}$$

$$u_{y2}(x, y, z) = -4 \sum_{j=1}^3 C_u^{j+3}(-\alpha_j) D_{21}^{j+3}(-\alpha_j) \frac{\Phi_j(-\alpha_j)}{\eta_1^j(-2\alpha_j)(\alpha_j^2 - \beta_j^2)}, \tag{49b}$$

$$u_{z2}(x, y, z) = -4 \sum_{j=1}^3 C_u^{j+3}(-\alpha_j) D_{31}^{j+3}(-\alpha_j) \frac{\Phi_j(-\alpha_j)}{\eta_1^j(-2\alpha_j)(\alpha_j^2 - \beta_j^2)}, \tag{49c}$$

Using the same approach, with the substitution of (42) into (45), one obtains for  $z > 0$

$$\sigma_{xx1}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_d^j(\omega) \bar{\sigma}_{xx}^j(\omega) \left( \frac{\Phi_{j+3}(\omega)}{\eta_1^j(\omega + \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{50a}$$

$$\sigma_{yy1}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_d^j(\omega) \bar{\sigma}_{yy}^j(\omega) \left( \frac{\Phi_{j+3}(\omega)}{\eta_1^j(\omega + \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{50b}$$

$$\sigma_{zz1}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_d^j(\omega) \bar{\sigma}_{zz}^j(\omega) \left( \frac{\Phi_{j+3}(\omega)}{\eta_1^j(\omega + \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{50c}$$

$$\tau_{yz1}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_d^j(\omega) \bar{\tau}_{yz}^j(\omega) \left( \frac{\Phi_{j+3}(\omega)}{\eta_1^j(\omega + \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{50d}$$

$$\tau_{zx1}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_d^j(\omega) \bar{\tau}_{zx}^j(\omega) \left( \frac{\Phi_{j+3}(\omega)}{\eta_1^j(\omega + \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{50e}$$

$$\tau_{xy1}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_d^j(\omega) \bar{\tau}_{xy}^j(\omega) \left( \frac{\Phi_{j+3}(\omega)}{\eta_1^j(\omega + \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{50f}$$

and similarly for  $z < 0$

$$\sigma_{xx2}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_u^{j+3}(\omega) \bar{\sigma}_{xx}^{j+3}(\omega) \left( \frac{\Phi_j(\omega)}{\eta_1^j(\omega - \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{51a}$$

$$\sigma_{yy2}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_u^{j+3}(\omega) \bar{\sigma}_{yy}^{j+3}(\omega) \left( \frac{\Phi_j(\omega)}{\eta_1^j(\omega - \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{51b}$$

$$\sigma_{zz2}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_u^{j+3}(\omega) \bar{\sigma}_{zz}^{j+3}(\omega) \left( \frac{\Phi_j(\omega)}{\eta_1^j(\omega - \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{51c}$$

$$\tau_{yz2}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_u^{j+3}(\omega) \bar{\tau}_{yz}^{j+3}(\omega) \left( \frac{\Phi_j(\omega)}{\eta_1^j(\omega - \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{51d}$$

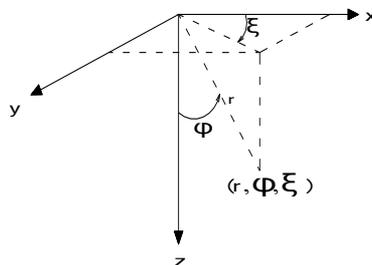
$$\tau_{zx2}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_u^{j+3}(\omega) \bar{\tau}_{zx}^{j+3}(\omega) \left( \frac{\Phi_j(\omega)}{\eta_1^j(\omega - \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{51e}$$

$$\tau_{xy2}(x, y, z) = \sum_{j=1}^3 \frac{d}{d\omega} \left\{ \omega C_u^{j+3}(\omega) \bar{\tau}_{xy}^{j+3}(\omega) \left( \frac{\Phi_j(\omega)}{\eta_1^j(\omega - \alpha_j)(\omega^2 - \beta_j^2)} \right)^2 \right\}_{\omega \rightarrow \alpha_j}, \tag{51f}$$

**2. Illustrative examples**

The closed-form solutions given above demonstrate that several factors can affect the displacements and stresses in an inclined transversely isotropic material. They include: (1) the rotation of the transversely isotropic planes ( $\phi$ ), (2) the type and degree of material anisotropy ( $E/E'$ ,  $\nu/\nu'$ ,  $G/G'$ ), (3) the geometric position ( $r, \phi, \zeta$ ), as seen in [Figure 2](#), and (4) the type of three-dimensional loading ( $P_x, P_y, P_z$ ). Based on Equations (48) and (49) for region 1 ( $z > 0$ ) and their counterparts for region 2, a Mathematica program was written to clarify the effect of these factors on the induced displacements and stresses. In this program, the displacements and stresses at any point in the full space can be calculated.

A parametric study is conducted to illustrate the generated analytical solutions and investigate the influence of the rotation of transversely isotropic planes, the geometric position, and the degree and type of rock anisotropy on the displacements and stresses. Two examples will be discussed: the first example presents the effect of  $\phi$  on the displacements and stresses of a material subjected to a vertical point load  $P_z$  at  $x = y = z = 1$  (as shown in [Figure 3](#) for displacements, and [Figure 4](#) for stresses); the second



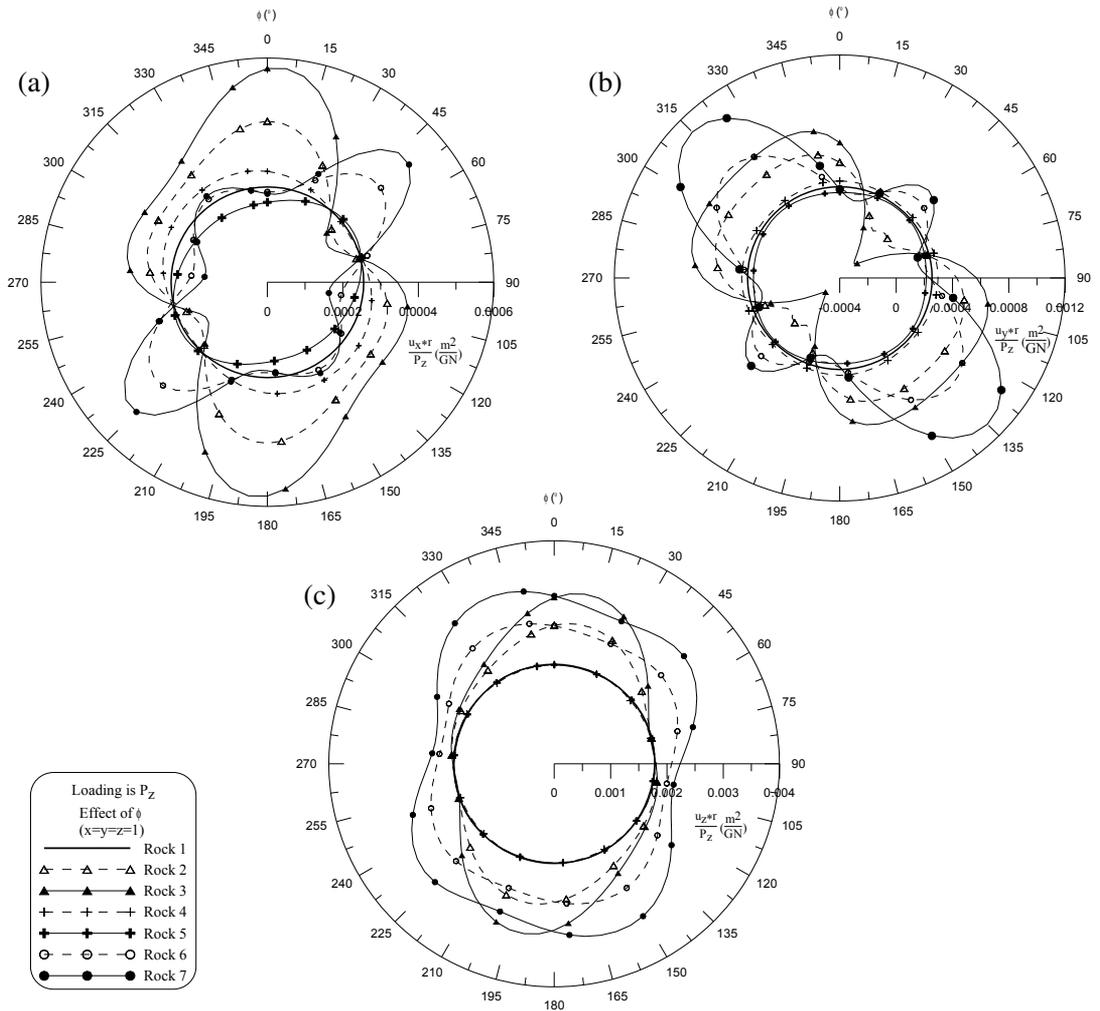
**Figure 2.** Spherical coordinate system ( $r, \phi, \zeta$ ).

	$E/E'$	$\nu/\nu'$	$G/G'$
Rock 1. Isotropic	1.0	1.0	1.0
Rock 2. Transversely isotropic	2.0	1.0	1.0
Rock 3. Transversely isotropic	3.0	1.0	1.0
Rock 4. Transversely isotropic	1.0	0.75	1.0
Rock 5. Transversely isotropic	1.0	1.5	1.0
Rock 6. Transversely isotropic	1.0	1.0	2.0
Rock 7. Transversely isotropic	1.0	1.0	3.0

**Table 1.** Elastic properties of the hypothetical rocks used in the examples. In each case,  $E = 50$  GPa and  $\nu = 0.25$ .

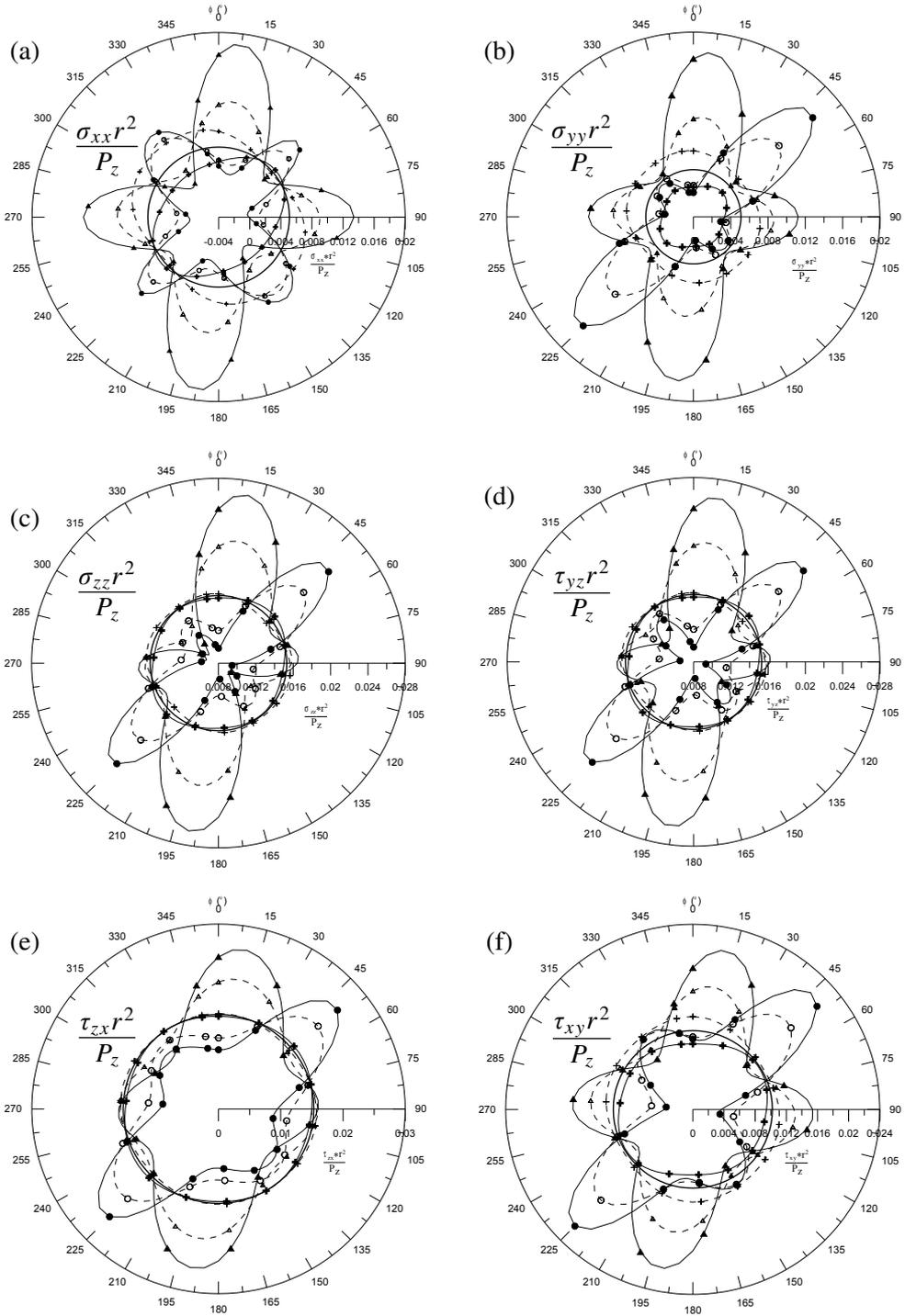
example exhibits the effect of  $\phi$  on the stresses due to  $P_z$  at  $\phi = 90^\circ$  and  $\zeta = 45^\circ$  (as depicted in Figure 5). Seven types of isotropic and transversely isotropic rocks are considered in our model of the foundation materials. For typical ranges of transversely isotropic rocks, Gerrard [1977] and Amadei et al. [1987] suggested that the ratios  $E/E'$  and  $G/G'$  range from 1.0 to 3.0, and that  $\nu/\nu'$  vary between 0.75 and 1.5. Hence, the degree of rock anisotropy, specified by the ratios  $E/E'$ ,  $\nu/\nu'$ , and  $G/G'$  is accounted for in the investigation of the anisotropy effect on displacements and stresses. Table 1 lists the rock type and elastic properties for the hypothetical rocks. The values chosen for  $E$  and  $\nu$  are 50 GPa and 0.3.

Figure 3 shows the normalized displacements  $u_x r/P_z$ ,  $u_y r/P_z$  and  $u_z r/P_z$  versus rotation of the transversely isotropic planes ( $\phi$ ), due to a vertical point load ( $P_z$ ), at  $x = y = z = 1$ , for the constituted isotropic/transversely isotropic rocks (rock 1/rocks 2–7, Table 1). Figure 3(a) depicts the normalized displacement  $u_x$  of the rocks, induced by  $P_z$ . It is observed that any value in each curve is symmetric with respect to the origin of the coordinates, and the ratios  $E/E'$  (rocks 2 and 3),  $\nu/\nu'$  (rocks 4 and 5), and  $G/G'$  (rocks 6 and 7) all strongly influence this displacement. This figure also shows that the magnitude of the normalized induced displacement ( $0.00026 \text{ m}^2/\text{GN}$ ) for rock 1 is independent of the change in  $\phi$ . However, for rocks 2 and 3, the displacement is maximal at about  $\phi = 0^\circ\text{--}180^\circ$ , and is minimal at approximately  $\phi = 60^\circ\text{--}240^\circ$ . As for rocks 6 and 7, the displacement is maximal at around  $\phi = 50^\circ\text{--}230^\circ$ , and is minimal at about  $\phi = 100^\circ\text{--}280^\circ$ . Figure 3(b) presents the normalized displacement  $u_y$  of the rocks, due to  $P_z$ . This figure clearly reveals that the displacement induced in transversely isotropic rocks is deeply affected by the ratios  $E/E'$  (rocks 2 and 3) and  $G/G'$  (rocks 6 and 7), but is only slightly influenced by  $\nu/\nu'$  (rocks 4 and 5). Notably, the normalized displacement ( $0.00026 \text{ m}^2/\text{GN}$ ) of the isotropic rock (rock 1) is also independent of  $\phi$ . Nevertheless, it is found that the values of induced displacement for rocks 2 and 3 would be partially within the range of  $-0.0004$  to 0, meaning there could be an opposite-direction displacement occurring in these media. Figure 3(c) displays the normalized displacement  $u_z$  of the rocks, subjected to  $P_z$ . Clearly, the ratios  $E/E'$  (rocks 2 and 3) and  $G/G'$  (rocks 6 and 7) profoundly impact the induced displacement, but the effect of  $\nu/\nu'$  (rocks 4 and 5) is small. The magnitude of the normalized induced displacement for rock 1 is always  $0.00179 \text{ m}^2/\text{GN}$ ; however, for rocks 2, 3, 6, and 7, the values of  $u_z$  are nearly greater than those of rock 1. The calculated results for the displacement fields are all in good agreement with Wang and Liao's solutions [Wang and Liao 1999] if the full space is homogeneous, linearly elastic, and the planes of transverse isotropy are parallel to the horizontal axes.



**Figure 3.** At the position  $x = y = z = 1$ , the effect of  $\phi$  on the normalized displacement (a)  $u_x r / P_z$ , (b)  $u_y r / P_z$ , (c)  $u_z r / P_z$ .

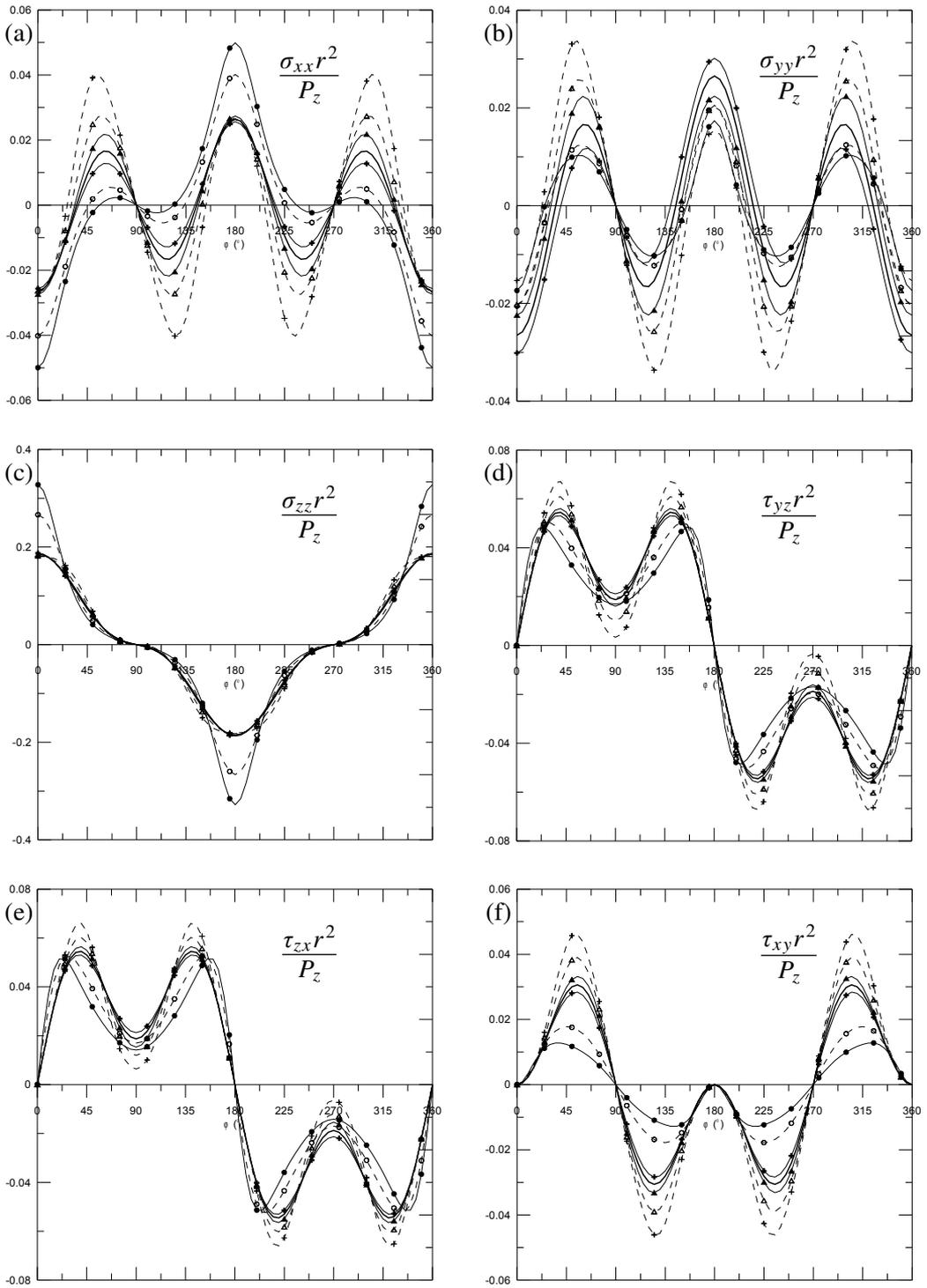
Figure 4 plots the nondimensional normal stresses  $\sigma_{xx} r^2 / P_z$ ,  $\sigma_{yy} r^2 / P_z$ ,  $\sigma_{zz} r^2 / P_z$  and the nondimensional shear stresses  $\tau_{yz} r^2 / P_z$ ,  $\tau_{zx} r^2 / P_z$ ,  $\tau_{xy} r^2 / P_z$ , versus the rotation of the transversely isotropic planes ( $\phi$ ), subjected to a vertical point load ( $P_z$ ), at  $x = y = z = 1$ , for the isotropic (rock 1) and transversely isotropic rocks (rocks 2–7). Figure 4(a) illustrates the effect of  $\phi$  on  $\sigma_{xx} r^2 / P_z$ , for rocks 1–7. This figure shows that the induced stress for the isotropic rock (rock 1) has the same value (0.005105) that is again independent of  $\phi$ . However, it is found that the values of induced stress for rocks 1–7 vary between  $-0.004$  and  $0.02$ , namely, that there is an obvious tensile stress occurring in rock 7. In addition, any value in each curve is symmetric with respect to the origin of the coordinates. Hence, from this figure, it is apparently revealed that the induced stress is greatly influenced by the rotation of the transversely isotropic planes ( $\phi$ ), and the type and degree of rock anisotropy ( $E/E'$ ,  $\nu/\nu'$ ,  $G/G'$ ). Figure 4(b) presents



**Figure 4.** At the position  $x = y = z = 1$ , the effect of  $\phi$  on the nondimensional normal and shear stresses. For the code indicating the type of rock, see [Figure 3](#).

the effect of  $\phi$  on  $\sigma_{yy}r^2/P_z$ , for rocks 1–7. Notably, the value in the curves is also symmetric with respect to the origin of the coordinates, and the ratios  $E/E'$  (rocks 2 and 3),  $\nu/\nu'$  (rocks 4 and 5), and  $G/G'$  (rocks 6 and 7) do also have a considerable influence on the stress. This graph shows that the magnitude of the nondimensional normal stress ( $\sigma_{yy}r^2/P_z$ ) for rock 1 (0.005105) is also independent of  $\phi$ , and the value of the nondimensional stress is within 0.06. In particular, the computed results for rocks 4 and 5 are, respectively, greater than or less than those of rock 1. **Figure 4(c)** depicts the effect of  $\phi$  on  $\sigma_{zz}r^2/P_z$ , for rocks 1–7. This stress depends heavily on the ratios  $E/E'$  (rocks 2 and 3) and  $G/G'$  (rocks 6 and 7). Nevertheless, the effect of the ratios  $\nu/\nu'$  (rocks 4 and 5) on the stress is slight. The maximum value of the nondimensional stress approaches 0.026. **Figure 4(d)** plots the effect of  $\phi$  on  $\tau_{yz}r^2/P_z$ , for rocks 1–7. Evidently, the ratios  $E/E'$  (rocks 2 and 3) and  $G/G'$  (rocks 6 and 7) could intensely affect the induced stress. However, the effect of the ratios  $\nu/\nu'$  (rocks 4 and 5) on the induced stress is still small. The trend of these stress curves in this figure is similar to that in **Figure 4(c)**. **Figure 4(e)** displays the effect of  $\phi$  on  $\tau_{zx}r^2/P_z$ , for rocks 1–7. The maximum value of the nondimensional stress is about 0.026. **Figure 4(f)** shows the effect of  $\phi$  on  $\tau_{xy}r^2/P_z$ , for rocks 1–7. The effect of the ratios  $\nu/\nu'$  for rocks 4 and 5 in this figure is more explicit than the effect on other shear stresses (**Figures 4(d) and 4(e)**). Especially, the calculated results of rocks 4 and 5 are, respectively, greater than or less than those of rock 1. The maximum value of the nondimensional stress is within the range of 0.024. The computed results for the stress fields are identical to those estimated from Wang and Liao's solutions [1999], in which the planes of transverse isotropic full space are parallel to the horizontal loading surface.

**Figure 5** plots the nondimensional normal stresses ( $\sigma_{xx}r^2/P_z$ ,  $\sigma_{yy}r^2/P_z$ ,  $\sigma_{zz}r^2/P_z$ ) and the nondimensional shear stresses ( $\tau_{yz}r^2/P_z$ ,  $\tau_{zx}r^2/P_z$ ,  $\tau_{xy}r^2/P_z$ ) versus the geometric position  $\phi$  (from  $0^\circ$  to  $360^\circ$ ), due to a vertical point load ( $P_z$ ), at the rotation of the transversely isotropic planes  $\phi = 90^\circ$  and the geometric position  $\zeta = 45^\circ$ , for the constituted isotropic/transversely isotropic rocks (rock 1/rocks 2–7). **Figure 5(a)** clarifies the effect of  $\phi$  on  $\sigma_{xx}r^2/P_z$ , for rocks 1–7. It is observed that the magnitudes of the estimated stresses are symmetric with respect to  $\phi = 180^\circ$ . The upper/lower part of this figure denotes the compressive/tensile stress occurring in the rock media. The maximum values of tensile/compressive stress appeared at  $\phi = 0^\circ/180^\circ$  in rock 7. In addition, the induced stresses are found to be influenced by the ratios  $E/E'$  (rocks 2 and 3),  $\nu/\nu'$  (rocks 4 and 5),  $G/G'$  (rocks 6 and 7), and they are all zero at  $\phi = 90^\circ$  and  $270^\circ$ . **Figure 5(b)** demonstrates the effect of  $\phi$  on  $\sigma_{yy}r^2/P_z$ , for rocks 1–7. Results reveal that the magnitudes of the computed stresses are also symmetric with respect to  $\phi = 180^\circ$ , and the tensile and compressive stresses would be expected to occur in all media. However, the maximum values of tensile/compressive stress appeared at approximately  $\phi = 125^\circ$  and  $235^\circ/55^\circ$  and  $305^\circ$  in rock 4. This means that at a given position ( $\phi = 90^\circ$  and  $\zeta = 45^\circ$ ), the decrease in the ratio  $\nu/\nu'$  from 1.0 (rock 1) to 0.75 (rock 4) could remarkably affect the stress ( $\sigma_{yy}$ ). **Figure 5(c)** shows the induced nondimensional normal stress  $\sigma_{zz}r^2/P_z$  for rocks 1–7. The distributions and magnitudes of the calculated stresses are quite different from those of **Figures 5(a) and 5(b)**. The tensile/compressive stress can be found within  $\phi = 0^\circ\text{--}90^\circ$  and  $270^\circ\text{--}360^\circ/90^\circ\text{--}270^\circ$ . Moreover, the stress ( $\sigma_{zz}$ ) is apparently impacted by the ratios  $G/G'$  (rocks 6 and 7). Nevertheless, the stress is affected to only a small degree by the ratios  $E/E'$  (rocks 2 and 3) and  $\nu/\nu'$  (rocks 4 and 5). The induced nondimensional shear stress  $\tau_{yz}r^2/P_z$  for rocks 1–7 is depicted in **Figure 5(d)**. It is noted that the positive/negative values of  $\tau_{yz}$  are symmetric with respect to  $\phi = 180^\circ$ . Additionally, the computed stresses are all zero at  $\phi = 0^\circ$ ,  $180^\circ$ , and  $360^\circ$ . The results of rocks 2, 4, 6, 7 are rather distinct from those of rocks 1, 3, 5. Similarly, the trends can be discovered in



**Figure 5.** At the position  $\phi = 90^\circ$ ,  $\zeta = 45^\circ$ , the effect of  $\varphi$  on the nondimensional normal and shear stresses. For the code indicating the type of rock, see [Figure 3](#).

Figure 5(e) for  $\tau_{zx}r^2/P_z$ . Eventually, the induced nondimensional shear stress  $\tau_{xy}r^2/P_z$  for rocks 1–7 is displayed in Figure 5(f). The calculated positive/negative values of  $\tau_{xy}$  are symmetric with  $\varphi = 90^\circ$  and  $270^\circ$ . The zero values for  $\tau_{xy}$  are found at  $\varphi = 0^\circ, 90^\circ, 180^\circ, 270^\circ$ , and  $360^\circ$ . Furthermore, the influences of the type and degree of rock anisotropy in this figure are more explicit than those in Figure 5(d,e). That means again that at  $\phi = 90^\circ$  and  $\zeta = 45^\circ$ , the normal and shear stresses due to a vertical point load are strongly impacted by the geometric position ( $\varphi$ ) and rock anisotropy ( $E/E', \nu/\nu', G/G'$ ).

The examples are presented to illustrate the derived solutions and demonstrate how the rotation of transversely isotropic planes ( $\phi$ ), the geometric position ( $r, \varphi, \zeta$ ), and the degree and type of material anisotropy ( $E/E', \nu/\nu', G/G'$ ) would influence the normalized displacements and nondimensional normal and shear stresses. Results reveal that the displacements and stresses in the inclined isotropic or transversely isotropic rocks (rock 1/rocks 2–7) due to a vertical point load are quite different from the displacements and stresses calculated by assuming the transversely isotropic planes are parallel to the horizontal surface. Hence, it is imperative to consider the dip at an angle of inclination when calculating the induced displacements and stresses in a transversely isotropic material by applied loads.

### Conclusions

Analytical solutions are presented for displacements and stresses in a transversely isotropic material (in which the transversely isotropic planes are oriented with respect to the horizontal axes) subjected to three-dimensional point loads. It is known that the stress components due to three-dimensional point loads in an inclined infinite space can be obtained by employing the coordinate transformation formulae with respect to the applied loads. Nevertheless, the displacement components cannot be solved by the same approach. Hence, an efficient method that employs the double Fourier transform in a Cartesian coordinate system is proposed to yield the fundamental solutions for displacements and stresses. First, the characteristic roots for this special material anisotropy are addressed in this article. Next, the double inverse Fourier transform and residue calculus are applied to integrate the contours. The generated solutions for displacements are the same as those of [Hu et al. 2007], which were arrived at by using the triple Fourier transform for an inclined transversely isotropic full space. In addition, they are identical with those of [Wang and Liao 1999] if the planes of transverse isotropy are parallel to the horizontal loading surface. The proposed solutions clarify that the displacements and stresses are governed by

- (1) the rotation of the transversely isotropic planes ( $\phi$ ),
- (2) the type and degree of material anisotropy ( $E/E', \nu/\nu', G/G'$ ),
- (3) the geometric position ( $r, \varphi, \zeta$ ), and
- (4) the type of three-dimensional loading ( $P_x, P_y, P_z$ ).

The present elastic fundamental solutions could not only realistically imitate the actual stratum of loading circumstances in many fields of practical engineering, but also provide the foundations of the boundary integral equation method (BIEM) or the boundary element method (BEM) for solving numerous boundary value problems. Moreover, the addressed solutions can be extended to generate the displacements and stresses due to three-dimensional point loads in the inclined transversely isotropic half-spaces/bimaterials/layered materials. These solutions could be very valuable to solid mechanics, and they will be presented in the near future.

### Appendix

In this appendix we give a parallel, more leisurely derivation of the characteristic equation (18). Recall that the  $x'$  and  $y'$  axes are in the plane of transversely isotropy, and that the generalized Hooke's law for a transversely isotropic material is expressed by Equation (2).

Let  $u_{x'}, u_{y'}, u_{z'}$  be the displacements of a point on the axes of a Cartesian coordinate system. The strain-displacement relationship for the small strain condition is given by (6), with  $x, y, z$  replaced by  $x', y', z'$ .

The equation of force equilibrium is

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'z'} \end{bmatrix} \begin{bmatrix} \partial/\partial x' \\ \partial/\partial y' \\ \partial/\partial z' \end{bmatrix} - \begin{bmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{bmatrix} = \rho \begin{bmatrix} \partial^2 u_{x'}/\partial t^2 \\ \partial^2 u_{y'}/\partial t^2 \\ \partial^2 u_{z'}/\partial t^2 \end{bmatrix}.$$

If we set  $(F_{x'}, F_{y'}, F_{z'}) = (0, 0, 0)$ , this becomes

$$\rho \frac{\partial^2 u_{x'}}{\partial t^2} = c_1 \frac{\partial^2 u_{x'}}{\partial x'^2} + c_4 \frac{\partial^2 u_{x'}}{\partial y'^2} + c_5 \frac{\partial^2 u_{x'}}{\partial z'^2} + (c_1 - c_4) \frac{\partial^2 u_{y'}}{\partial x' \partial y'} + c_3 \frac{\partial^2 u_{z'}}{\partial x' \partial z'}, \tag{52a}$$

$$\rho \frac{\partial^2 u_{y'}}{\partial t^2} = (c_1 - c_4) \frac{\partial^2 u_{x'}}{\partial x' \partial y'} + c_4 \frac{\partial^2 u_{y'}}{\partial x'^2} + c_1 \frac{\partial^2 u_{y'}}{\partial y'^2} + c_5 \frac{\partial^2 u_{y'}}{\partial z'^2} + c_3 \frac{\partial^2 u_{z'}}{\partial y' \partial z'}, \tag{52b}$$

$$\rho \frac{\partial^2 u_{z'}}{\partial t^2} = c_3 \frac{\partial^2 u_{x'}}{\partial x' \partial z'} + c_3 \frac{\partial^2 u_{y'}}{\partial y' \partial z'} + c_5 \frac{\partial^2 u_{z'}}{\partial x'^2} + c_5 \frac{\partial^2 u_{z'}}{\partial y'^2} + c_2 \frac{\partial^2 u_{z'}}{\partial z'^2}. \tag{52c}$$

For the elastic dynamic problem, an arbitrary time-harmonic body force in the  $x', y'$ , and  $z'$  directions with angular frequency  $\omega$  can be written as

$$u_{x'}(x', y', z', t) = u_{x'}^*(x', y', z') \exp(-i\omega t), \tag{53a}$$

$$u_{y'}(x', y', z', t) = u_{y'}^*(x', y', z') \exp(-i\omega t), \tag{53b}$$

$$u_{z'}(x', y', z', t) = u_{z'}^*(x', y', z') \exp(-i\omega t), \tag{53c}$$

where  $u_{x'}^*, u_{y'}^*$ , and  $u_{z'}^*$  represent the complex amplitude of the body force.

Taking the triple Fourier transform of (53) we obtain

$$\bar{u}_{x'}(\alpha, \beta, \gamma) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{x'}(x', y', z') e^{-i(\alpha x' + \beta y' + \gamma z')} dx' dy' dz',$$

$$\bar{u}_{y'}(\alpha, \beta, \gamma) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{y'}(x', y', z') e^{-i(\alpha x' + \beta y' + \gamma z')} dx' dy' dz',$$

$$\bar{u}_{z'}(\alpha, \beta, \gamma) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_{z'}(x', y', z') e^{-i(\alpha x' + \beta y' + \gamma z')} dx' dy' dz'.$$

Substituting this and (53) into (52), we have the triple Fourier-type integrals

$$\begin{aligned}\rho\omega^2\bar{u}_{x'}^* &= c_1\alpha^2\bar{u}_{x'}^* + c_4\beta^2\bar{u}_{x'}^* + c_5\gamma^2\bar{u}_{x'}^* + (c_1 - c_4)\alpha\beta\bar{u}_{y'}^* + c_3\alpha\gamma\bar{u}_{z'}^*, \\ \rho\omega^2\bar{u}_{y'}^* &= (c_1 - c_4)\alpha\beta\bar{u}_{x'}^* + c_4\alpha^2\bar{u}_{y'}^* + c_1\beta^2\bar{u}_{y'}^* + c_5\gamma^2\bar{u}_{y'}^* + c_3\beta\gamma\bar{u}_{z'}^*, \\ \rho\omega^2\bar{u}_{z'}^* &= c_3\alpha\gamma\bar{u}_{x'}^* + c_3\beta\gamma\bar{u}_{y'}^* + c_5\alpha^2\bar{u}_{z'}^* + c_5\beta^2\bar{u}_{z'}^* + c_2\gamma^2\bar{u}_{z'}^*.\end{aligned}$$

Rearranging, we obtain

$$\begin{bmatrix} d_{11} - \rho\omega^2 & d_{12} & d_{13} \\ d_{21} & d_{22} - \rho\omega^2 & d_{23} \\ d_{31} & d_{32} & d_{33} - \rho\omega^2 \end{bmatrix} \begin{bmatrix} \bar{u}_{x'}^* \\ \bar{u}_{y'}^* \\ \bar{u}_{z'}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (54)$$

where

$$[d_{ij}] = \begin{bmatrix} c_1\alpha^2 + c_4\beta^2 + c_5\gamma^2 & (c_1 - c_4)\alpha\beta & c_3\alpha\gamma \\ (c_1 - c_4)\alpha\beta & c_4\alpha^2 + c_1\beta^2 + c_5\gamma^2 & c_3\beta\gamma \\ c_3\alpha\gamma & c_3\beta\gamma & c_5\alpha^2 + c_5\beta^2 + c_2\gamma^2 \end{bmatrix}$$

Expanding the determinant of the matrix in (54) and substituting the expressions of the  $d_{ij}$  shows that the eigenvalues of the matrix  $[d_{ij}]$  are

$$\rho\omega^2 = \begin{cases} c_5\gamma^2 + c_4(\alpha^2 + \beta^2), \\ \frac{1}{2}(c_1(\alpha^2 + \beta^2) + c_2\gamma^2 + c_5(\alpha^2 + \beta^2 + \gamma^2) + \Delta), \\ \frac{1}{2}(c_1(\alpha^2 + \beta^2) + c_2\gamma^2 + c_5(\alpha^2 + \beta^2 + \gamma^2) - \Delta), \end{cases}$$

where

$$\Delta = \sqrt{((c_1 - c_5)(\alpha^2 + \beta^2) - (c_2 - c_5)\gamma^2)^2 + 4c_3^2(\alpha^2 + \beta^2)\gamma^2}.$$

If we introduce spherical coordinates  $(k, \theta_{x'}, \theta_{z'})$  in  $\alpha\beta\gamma$ -space (so  $\alpha = k \sin \theta_{z'} \cos \theta_{x'}$ ,  $\beta = k \sin \theta_{z'} \sin \theta_{x'}$ ,  $\gamma = k \cos \theta_{z'}$ ), we see that  $\Delta$  can be expressed as

$$\Delta = \sqrt{((c_1 - c_5) \sin^2 \theta_{z'} - (c_2 - c_5) \cos^2 \theta_{z'})^2 + 4c_3^2 \sin^2 \theta_{z'} \cos^2 \theta_{z'}}.$$

Introducing the quantity  $V^2 = \omega^2/k^2$ , the square of the body-wave velocity, leads to a familiar appearance for the eigenvalues:

$$\begin{aligned}V_{\text{SH},\theta,z'} &= \sqrt{\frac{c_5 \cos^2 \theta_{z'} + c_4 \sin^2 \theta_{z'}}{\rho}}, \\ V_{\text{P},\theta,z'} &= \sqrt{\frac{c_1 \sin^2 \theta_{z'} + c_2 \cos^2 \theta_{z'} + c_5 + \Delta}{2\rho}}, \\ V_{\text{SV},\theta,z'} &= \sqrt{\frac{c_1 \sin^2 \theta_{z'} + c_2 \cos^2 \theta_{z'} + c_5 - \Delta}{2\rho}}.\end{aligned}$$

Note that  $\theta_{z'}$  can be interpreted as the angle between the direction of wave travel and the  $z'$ -axis. The determinant of  $[d_{ij}]$  is written as

$$\begin{aligned} D &= \det [d_{ij}] = \rho^3 k^6 (V_{SH,\theta,z'} V_{P,\theta,z'} V_{SV,\theta,z'})^2 \\ &= (c_5 \gamma^2 + c_4 (\alpha^2 + \beta^2)) [(c_1 (\alpha^2 + \beta^2) + c_5 \gamma^2) (c_5 (\alpha^2 + \beta^2) + c_2 \gamma^2) - c_3^2 (\alpha^2 + \beta^2) \gamma^2] \\ &= c_2 c_5^2 \prod_{i=1}^3 (A_i (\alpha^2 + \beta^2) + \gamma^2) = c_2 c_5^2 k^6 \prod_{i=1}^3 (A_i \sin^2 \theta_{z'} + \cos^2 \theta_{z'}), \end{aligned} \tag{55}$$

where  $A_1, A_2, A_3$  are defined in (17).

As depicted in Figure 1, a new coordinate system  $x, y, z$  is obtained from the original system  $x', y', z'$  by rotating through an angle  $\phi$  about the  $x = x'$  axis. Then the value of  $D$  in (55) becomes

$$D = \rho^3 k^6 (V_{SH,\theta,t} V_{P,\theta,t} V_{SV,\theta,t})^2 = c_2 c_5^2 k^6 \prod_{i=1}^3 (A_i \sin^2 \theta_t + \cos^2 \theta_t), \tag{56}$$

where  $\theta_t$  is the angle between the vector  $(\alpha, \beta, \gamma)$  and the  $z'$  axis, which can be expressed in terms of  $\alpha, \beta, \gamma$ , and  $\phi$  as

$$\cos \theta_t = \frac{\gamma \cos \phi - \beta \sin \phi}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}, \tag{57}$$

that is,

$$\sin^2 \theta_t = \frac{\alpha^2 + \beta^2 + \gamma^2 - (\gamma \cos \phi - \beta \sin \phi)^2}{\alpha^2 + \beta^2 + \gamma^2}.$$

Hence, (56) can be rearranged as

$$\begin{aligned} D &= c_2 c_5^2 k^6 \prod_{i=1}^3 (A_i \sin^2 \theta_t + \cos^2 \theta_t) \\ &= \frac{c_2 c_5^2 k^6}{(\alpha^2 + \beta^2 + \gamma^2)^3} \prod_{i=1}^3 \left( A_i (\alpha^2 + \beta^2 + \gamma^2 - (\gamma \cos \phi - \beta \sin \phi)^2) + (\gamma \cos \phi - \beta \sin \phi)^2 \right) \\ &= c_2 c_5^2 \prod_{i=1}^3 \left( A_i (\alpha^2 + \beta^2 + \gamma^2 - (\gamma \cos \phi - \beta \sin \phi)^2) + (\gamma \cos \phi - \beta \sin \phi)^2 \right). \end{aligned}$$

If we further set  $i\gamma = u$ , this becomes

$$\begin{aligned} D &= \det [d_{ij}] \\ &= c_2 c_5^2 \prod_{i=1}^3 \left( A_i (-u^2 + \alpha^2 + \beta^2 - (iu \cos \phi + \beta \sin \phi)^2) + (iu \cos \phi + \beta \sin \phi)^2 \right). \end{aligned}$$

The six eigenroots can be generated by setting  $D = 0$  in this equation. They are given in (19).

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