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CRACK FRONT POSITION AND CRACK BACK POSITION TECHNIQUES FOR EVALUATING THE *T*-STRESS AT CRACK TIP USING FUNCTIONS OF A COMPLEX VARIABLE

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In this paper, the crack front position and the crack back position techniques for evaluating the T-stress using complex variables are suggested. In the crack front technique, an expression for stress components in the crack front position is expressed through a complex variable. The limit value of the expression from the crack front position will give the T-stress. In the crack back technique, the other expression of stress components in the crack back position is expressed through the complex variable. The limit value of the expression from the crack back position will give the T-stress. The suggested techniques are used to evaluate T-stress in the arc crack and the curved crack problems. It is found from a detailed derivation that both techniques give the same result in the crack problem. Numerical examples are carried out for two problems: an elliptic crack with a central crack and a curved crack with parabolic configuration.

1. Introduction

The T-stress term at the vicinity of a crack tip was introduced in earlier years [Williams 1957; Rice 1974]. The T-stress term may affect the plastic zone ahead of crack tip [Larsson and Carlsson 1973; Betegon and Hancock 1991]. In addition, the T-stress has significant influence on the directional stability for the crack growth path [Rice 1974; Melin 2002]. A maximum tensile stress criterion for the onset of crack growth was suggested, which considers the role of the stress intensity factors and T-stress [Smith et al. 2006]. The T-stress before and after crack kinking in two-dimensional elastic solids was studied [Li and Xu 2007]. Contributions from the T-stress before crack kinking to the T-stress and stress intensity factors of the kinked crack are clearly described.

A variety of methods were used to evaluate the T-stress. The Eshelby technique was used [Kfouri 1998]. The T-stress evaluation is completed by using the weight function method [Sham 1989; 1991; Chen 1997]. The boundary collocation method and the weight function were developed to evaluate the T-stress [Fett 1997; 1998a; 1998b; 2001; Fett and Rizzi 2005].

The finite element method was used to evaluate the T-stress in crack problems [Ayatollahi et al. 1998; Chen et al. 2001]. A hybrid finite element at the vicinity of the crack tip was suggested [Tong et al. 1973; Cheung and Chen 1991; Karihaloo and Xiao 2001; Xiao and Karihaloo 2002; Xiao et al. 2004]. The formulation of a hybrid finite element depends on the Williams expansion. The element is embedded in the usual finite elements. Once the problem is solved, the higher-order terms as well as the T-stress in the expansion are obtainable. Using the HCE (hybrid crack element), the higher-order terms in the stress distribution of a three-point bend beam are evaluated [Karihaloo and Xiao 2001]. The problem for an edge crack in a finite plate with wedge force on the crack face was studied [Xiao and Karihaloo 2002].

Keywords: T-stress, crack, crack front position technique, crack back position technique, arc crack, numerical solution.

The problem was reduced to a problem of a traction-free edge crack with loading on the outer boundary. The usage of the Williams expansion and the boundary collocation method gave the final solution.

A stress difference method was developed to evaluate the T-stress in the crack problem [Yang and Ravi-Chandar 1999]. It was proved that a limit of the difference between two normal stress components ahead of a crack tip would give the T-stress.

Using the dislocation distribution method and the singular integral equation, several T-stress problems were solved [Broberg 2005]. Those problems include the problems of: (i) two collinear cracks, (ii) an edge crack, and (iii) cracks emanating from a circular hole. A Fredholm integral equation was used to evaluate the T-stress in the multiple crack problems [Chen 1994]. The solved problems were limited to the line crack case. In addition, a compendium of the T-stress solutions in the crack problems was carried out [Sherry et al. 1995].

From the methodology for evaluating the T-stress, researchers suggested two techniques for obtaining the T-stress in the line crack case. In the stress difference method, the T-stress is obtained from the stress difference in crack front position [Yang and Ravi-Chandar 1999]. In the mode I fracture case, the T-stress evaluation was related to a stress evaluation in the crack back position [Ayatollahi et al. 1998]. The mentioned derivations were related to the real analysis only. It is seen that those methods are not easy to use in some complicated cases, for example, for evaluating the T-stress in the curved crack problem.

It is known that in most cases the complex potentials in the plane elasticity crack problem can be formulated successfully [Savruk 1981; Chen and Lin 2006]. In addition, the stress components can be expressed by the complex potentials explicitly. Therefore, it is a particular advantage to use a complex variable for evaluating the SIF (stress intensity factor) as well as the T-stress. In this paper, the crack front position and crack back position techniques for evaluating T-stress using the complex variable are suggested. In the crack front technique, an expression for stress components in the crack front position is expressed through the complex variable. The limit value of the expression from the crack front position will give the T-stress. In addition, in the crack back technique, the other expression for stress components in the crack back position is expressed through the complex variable. The limit value of the expression from the crack back position will give the T-stress. It is found from a detailed derivation that both techniques give the same result in the crack problem.

2. Basic equations in the crack front position and the crack back position techniques

In the crack front position technique, the T-stress is evaluated in the front position of the crack. In addition, in the crack back position technique, the T-stress is evaluated in the back position of the crack. The two techniques with usage of a complex variable are introduced below.

2.1. ***The stress expansions in the vicinity of crack tip.*** The stress distribution near a crack tip under the traction-free crack face was investigated early on by Williams [1957]. A little modification for the Williams expansion is suggested below. It is assumed that the crack face has the following loadings (Figure 1)

$$\sigma_y^+ = \sigma_y^- = p_c, \qquad \sigma_{xy}^+ = \sigma_{xy}^- = q_c. \tag{1}$$

From the Williams expansion, the stresses at the crack tip area can be expressed as

$$\begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix} f_{11}(\theta) & f_{12}(\theta) \\ f_{12}(\theta) & f_{22}(\theta) \end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix} g_{11}(\theta) & g_{12}(\theta) \\ g_{12}(\theta) & g_{22}(\theta) \end{bmatrix} + \begin{bmatrix} T & q_c \\ q_c & p_c \end{bmatrix}, \tag{2}$$

where the first two terms in the expansion form are singular at the crack tip, K_I, K_{II} denote the mode I and mode II stress intensity factors respectively, and the functions $f_{ij}(\theta)$, $g_{ij}(\theta)$ represent the angular distributions of stresses at crack tip. In Equation (2), the third term is finite and bounded. The term T is denoted as the T-stress and can be regarded as the stress acting parallel to the crack flanks. In Equation (1) the term $O(r^{1/2})$ has been neglected for clarity. In addition, the angular distribution can be expressed as [Williams 1957]

$$\begin{bmatrix} f_{11} \\ f_{12} \\ f_{22} \end{bmatrix} = \cos(\theta/2) \begin{bmatrix} 1-\sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) \\ 1+\sin(\theta/2)\sin(3\theta/2) \end{bmatrix}, \qquad \begin{bmatrix} g_{11} \\ g_{12} \\ g_{22} \end{bmatrix} = \begin{bmatrix} -\sin(\theta/2)[2+\cos(\theta/2)\cos(3\theta/2)] \\ \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{bmatrix}. \tag{3}$$

Clearly, substituting $\theta = \pm\pi$ into Equation (2) will yield the stresses $\sigma_y = p_c$ and $\sigma_{xy} = q_c$, which are applied on the crack face.

Note that Equation (2) represents a pattern of stress distribution at the vicinity of crack tip. It is easily seen that the stress field defined by Figure 1 was solely determined by two factors: (1) the tractions $\sigma_y^+ = \sigma_y^- = p_c$, $\sigma_{xy}^+ = \sigma_{xy}^- = q_c$ applied on the crack face, (2) the tractions applied along the outer boundary CDEFGH in Figure 1. Therefore, the tractions $\sigma_y^+ = \sigma_y^- = p_c$, $\sigma_{xy}^+ = \sigma_{xy}^- = q_c$ applied on the crack face cannot alone determine the K_I, K_{II} and T values. Alternatively, there is no definite relation between (i) the tractions $\sigma_y^+ = \sigma_y^- = p_c$, $\sigma_{xy}^+ = \sigma_{xy}^- = q_c$ applied on the crack face and (ii) the stress field and the T-stress at the crack tip. In this study, it is assumed that p_c and q_c are given beforehand. In this case, the K_I, K_{II} and T values will be determined by the tractions applied on the boundary CDEFGH.

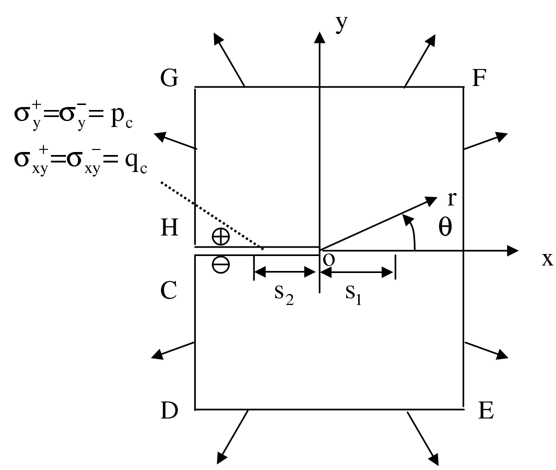

Figure 1. A finite cracked plate with loadings on the crack face.

Let $\theta = 0$ and $r = s_1$ in Figure 1, from Equations (2), (3), we have

$$\sigma_x = \frac{K_I}{\sqrt{2\pi s_1}} + T + O(s_1^{1/2}), \qquad \sigma_y = \frac{K_I}{\sqrt{2\pi s_1}} + p_c + O(s_1^{1/2}). \tag{4}$$

Therefore, from Equation (4), the following equation is obtained

$$T = \lim_{s_1 \to 0} (\sigma_x - \sigma_y) + p_c \qquad \text{or} \qquad T = -\lim_{s_1 \to 0} (\sigma_y - \sigma_x) + p_c \tag{5}$$

This equation will be used for the crack front position technique. In fact, this equation was suggested in [Yang and Ravi-Chandar 1999].

The necessity for introducing Equations (1), (2), and (5) can be seen from an example described in Section 5. In the example, the T-stress in the curve crack problem is evaluated. In the problem, the stress field of the original problem must be decomposed into the uniform stress field and the perturbation stress field. In the perturbation stress field, the crack face is applied by the normal and shear tractions. Clearly, for investigating the T-stress in the perturbation field (in Figure 2b), the usage of Equations (1), (2), and (5) is necessary simply because the normal and shear tractions are applied on the crack face. When some solutions to the traction-free condition on the crack face are available, we simply let $p_c = 0$ and $q_c = 0$ in the relevant equations.

Alternatively, let $\theta = \pi$ and $r = s_2$ in Figure 1, and from Equations (2), (3) we have

$$\sigma_x^+ = -\frac{2K_{II}}{\sqrt{2\pi s_2}} + T + O(s_2^{1/2}), \qquad \sigma_y^+ = p_c + O(s_2^{1/2}). \tag{6}$$

In addition, let $\theta = -\pi$ and $r = s_2$, and we have

$$\sigma_x^- = \frac{2K_{II}}{\sqrt{2\pi s_2}} + T + O(s_2^{1/2}), \qquad \sigma_y^- = p_c + O(s_2^{1/2}). \tag{7}$$

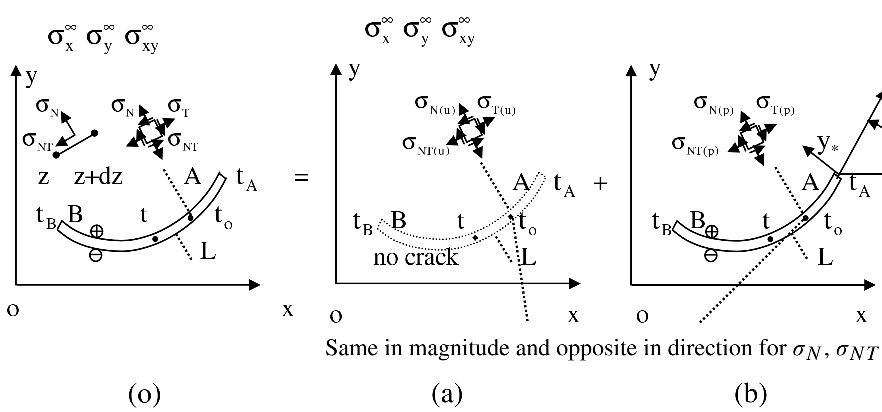

Figure 2. Superposition method: (o) the original field, a curved crack in an infinite plate with the remote loading $\sigma_y^\infty = p_2$, or $\sigma_x^\infty = p_1$, or $\sigma_{xy}^\infty = q$; (a) the uniform field, a perfect plate with remote loading $\sigma_y^\infty = p_2$, or $\sigma_x^\infty = p_1$, or $\sigma_{xy}^\infty = q$, the subscript u denoting the uniform field; (b) the perturbation field, a curved crack with loading on the crack face, the subscript p denoting the perturbation field.

In Equations (6) and (7), σ_x^+, $\sigma_y^+(\sigma_x^-, \sigma_y^-)$ denote the stress component along the upper (lower) side, respectively (Figure 1). Therefore, from Equations (6) and (7), the following equation is obtained:

$$T = \lim_{s_2\to 0}\{[(\sigma_x^+ + \sigma_y^+) + (\sigma_x^- + \sigma_y^-)]/2\} - p_c \qquad \text{or} \qquad T = \lim_{s_2\to 0}\{(\sigma_x^+ + \sigma_x^-)/2\}. \tag{8}$$

Equation (8) can be used for the general case, which is first suggested by us in this paper.

In the case of a mode I fracture, or $K_{II} = 0$, the above-mentioned equations can be reduced to

$$T = \lim_{s_2\to 0}\{\sigma_x^+ + \sigma_y^+\} - p_c = \lim_{s_2\to 0}\{\sigma_x^- + \sigma_y^-\} - p_c \qquad \text{or} \qquad T = \lim_{s_2\to 0}\sigma_x^+ = \lim_{s_2\to 0}\sigma_x^-. \tag{9}$$

This equation will be used in the crack back position technique. In fact, Equation (9) was suggested in [Ayatollahi et al. 1998].

2.2. *T-stress expressions in crack front position and crack back position techniques using a complex variable.* The following analysis depends on the complex variable function method in plane elasticity [Muskhelishvili 1953]. In this method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y), and the displacements (u, v) are expressed in terms of complex potentials $\phi(z)$, $\psi(z)$, $\Phi(z) = \phi'(z)$, and $\Psi(z) = \psi'(z)$ such that

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\Phi(z), \tag{10}$$

$$\sigma_y - i\sigma_{xy} = 2\,\mathrm{Re}\,\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)}, \tag{11}$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2(\bar{z}\Phi'(z) + \Psi(z)), \tag{12}$$

or

$$\sigma_y - \sigma_x - 2i\sigma_{xy} = 2(z\overline{\Phi'(z)} + \overline{\Psi(z)}), \tag{13}$$

$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}, \tag{14}$$

$$2G(u + iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \tag{15}$$

where $z = x + iy$ denotes a complex variable, G is the shear modulus of elasticity, $\kappa = (3 - \nu)/(1 + \nu)$ is for the plane stress problems, $\kappa = 3 - 4\nu$ is for the plane strain problems, and ν is the Poisson's ratio. In the present study, the plane strain condition is assumed thoroughly.

It is assumed that a concrete crack problem, for example, the problem shown by Figure 1 has been solved. Alternatively, the relevant complex potentials are obtained in advance. Therefore, from Equations (5) and (12), we have

$$\begin{aligned} T &= -\,\mathrm{Re}(\sigma_y - \sigma_x + 2i\sigma_{xy})|_{z=s_1, s_1\to 0} + p_c \\ &= -2\,\mathrm{Re}(\bar{z}\Phi'(z) + \Psi(z))|_{z=s_1, s_1\to 0} + p_c, \end{aligned} \tag{16}$$

(see Figure 1). This is the formula for evaluating the T-stress in the crack front position technique.

In addition, from Equations (8) and (10) we have (Figure 1)

$$\begin{aligned} T &= \lim_{s_2\to 0}\{[(\sigma_x^+ + \sigma_y^+) + (\sigma_x^- + \sigma_y^-)]/2\} - p_c \\ &= 2\,\mathrm{Re}(\Phi^+(z) + \Phi^-(z))|_{z=-s_2, s_2\to 0} - p_c. \end{aligned} \tag{17}$$

This is the formula for evaluating the T-stress in the crack back position technique.

3. Closed form solution for the *T*-stress in an arc crack using the crack front position and crack back position techniques

A closed form solution for the *T*-stress in the arc crack by using the crack front position and crack back position techniques is introduced below. The configuration of the arc crack is shown in Figure 3. The arc crack has a spanning angle 2α with the remote loading σ_x^∞, σ_y^∞, σ_{xy}^∞. In addition to two complex potentials $\Phi(z)$ and $\Psi(z)$, the following complex potential $\Omega(z)$ is introduced:

$$\Omega(z) = \bar{\Phi}\Big(\frac{1}{z}\Big) - \frac{1}{z}\bar{\Phi}'\Big(\frac{1}{z}\Big) - \frac{1}{z^2}\bar{\Psi}\Big(\frac{1}{z}\Big). \tag{18}$$

Here and after, for example, the following definition is used [Muskhelishvili 1953]

$$\bar{\Phi}\Big(\frac{1}{z}\Big) = \overline{\Phi\Big(\frac{1}{\bar{z}}\Big)}. \tag{19}$$

After some manipulation, the stress components in (r, θ) coordinates can be expressed as [Muskhelishvili 1953]

$$\sigma_r + \sigma_\theta = 4\,\mathrm{Re}\,\Phi(z), \tag{20}$$

$$\sigma_r - \sigma_\theta - 2i\sigma_{r\theta} = -2\Phi(z) + 2\bar{\Omega}\Big(\frac{1}{z}\Big) + 2z\Big(z - \frac{1}{\bar{z}}.\Big)\Psi(z). \tag{21}$$

For the arc crack problem under the remote loading σ_x^∞, σ_y^∞, σ_{xy}^∞, there is a solution as follows [Muskhelishvili 1953]:

$$\Phi(z) = F_1(z) + F_2(z), \qquad \Omega(z) = F_1(z) + F_2(z), \tag{22}$$

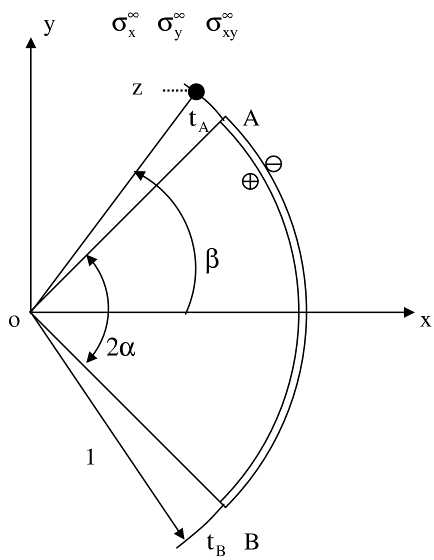

Figure 3. An arc crack.

where

$$F_1(z) = \frac{1}{2X(z)}\Big(C_o z + C_1 + \frac{D_1}{z} + \frac{D_2}{z^2}\Big), \qquad F_2(z) = \frac{D_o}{2} + \frac{\bar{\Gamma}_1}{2z^2}, \tag{23}$$

$$\begin{aligned} C_o &= \frac{1}{2}(\Gamma_1 - \bar{\Gamma}_1)\sin^2(\alpha/2) + \frac{4\Gamma + (\Gamma_1 + \bar{\Gamma}_1)\sin^2(\alpha/2)\cos^2(\alpha/2)}{2(1+\sin^2(\alpha/2))}, \\ C_1 &= -C_o\cos\alpha, \\ D_o &= 2\Gamma - C_o, \\ D_1 &= -\bar{\Gamma}_1\cos\alpha, \\ D_2 &= \bar{\Gamma}_1, \end{aligned} \tag{24}$$

$$\Gamma = \frac{\sigma_x^\infty + \sigma_y^\infty}{4}, \qquad \Gamma_1 = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + i\sigma_{xy}^\infty, \tag{25}$$

$$X(z) = \sqrt{(z-\exp(-i\alpha))(z-\exp(i\alpha))} \quad \text{(taking the branch } \lim_{z\to\infty} X(z)/z = 1). \tag{26}$$

The T-stress at the crack tip A is evaluated by using the crack front position technique. Similar to Equation (16), the T-stress at the crack tip A can be evaluated by (Figure 3)

$$T_A = -\operatorname{Re}\{\sigma_r - \sigma_\theta - 2i\sigma_{r\theta}\}|_{z=\exp(i\beta),\beta>\alpha,\beta\to\alpha} + p_c, \tag{27}$$

where the point $z = \exp(i\beta)$ with $\beta > \alpha$ is actually located in front of the crack tip A (Figure 3).

Considering (i) the traction-free crack face, where $p_c = 0$, (ii) $z - 1/\bar{z} = 0$ for $z = \exp(i\beta)$, and (iii) substituting Equation (21) into (27), the above-mentioned equation can be reduced to

$$T_A = 2\operatorname{Re}\Big(\Phi(z) - \bar{\Omega}\Big(\frac{1}{z}\Big)\Big)_{z=\exp(i\beta),\beta>\alpha,\beta\to\alpha}. \tag{28}$$

Substituting Equations (22), (23), and (24) into (28) yields

$$\begin{aligned} T_A = 2\operatorname{Re}(D_o + \bar{\Gamma}_1\exp(-2i\alpha)) = \sigma_x^\infty\Big(-\cos 2\alpha + \frac{(1-\cos\alpha)(3+\cos\alpha)}{2(3-\cos\alpha)}\Big) \\ + \sigma_y^\infty\Big(\cos 2\alpha + \frac{(1-\cos\alpha)^2}{2(3-\cos\alpha)}\Big) - 2\sigma_{xy}^\infty\sin 2\alpha, \end{aligned} \tag{29}$$

which was obtained previously by using a different method [Chen 2000].

Since $\sigma_x + \sigma_y (= \sigma_r + \sigma_\theta)$ is invariant, similar to Equation (17), the T-stress in the crack back position technique can be defined as

$$T_A = \{[(\sigma_r^+ + \sigma_\theta^+) + (\sigma_r^- + \sigma_\theta^-)]/2\}|_{t=\exp(i\beta),\beta<\alpha,\beta\to\alpha} - p_c. \tag{30}$$

Considering the traction-free crack face, where $p_c = 0$, and substituting Equations (20) and (22) into (30), the above-mentioned equation can be reduced to

$$T_A = 2\operatorname{Re}(\Phi^+(t) + \Phi^-(t))|_{t=\exp(i\beta),\beta<\alpha,\beta\to\alpha}. \tag{31}$$

In the crack back position, or for $t = \exp(i\beta)$, where $\beta < \alpha$, we have $X^+(t) = -X^-(t)$, where $X^+(t)$, $X^-(t)$ denotes the value of $X(z)$ at $z = t^+$, the positive side, and $z = t^-$, the negative side, respectively. Considering this point and substituting (22) into (31) yields the same result as shown by Equation (29):

$$T_A = 2\,\mathrm{Re}(D_o + \bar{\Gamma}_1 \exp(-2i\alpha)). \tag{29}$$

4. Evaluation of the *T*-stress for an elliptic plate with a central crack and normal loading on the contour

In the following analysis, we can let $\omega(z) = z\bar{\phi}'(z) + \bar{\psi}(z)$, $\Omega(z) = \omega'(z)$. From Equations (10)–(15), the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y), and the displacements (u, v) are expressed in terms of the complex potentials $\phi(z)$ and $\omega(z)$ in the following form:

$$\begin{aligned} \sigma_x + \sigma_y &= 4\,\mathrm{Re}\,\Phi(z), \\ \sigma_x - \sigma_y + 2i\sigma_{xy} &= 2\overline{\Phi(z)} - 2(z-\bar{z})\overline{\Phi'(z)} - 2\Omega(\bar{z}), \end{aligned} \tag{32}$$

$$f = -Y + iX = \phi(z) + (z-\bar{z})\overline{\phi'(z)} + \omega(\bar{z}), \tag{33}$$

$$2G(u+iv) = \kappa\phi(z) - (z-\bar{z})\overline{\phi'(z)} - \omega(\bar{z}). \tag{34}$$

For an elliptic plate with a crack under the condition of symmetric loading (Figure 4), the complex potentials can be expressed in the form [Chen 1983]:

$$\phi(z) = \phi_1(z) + \phi_2(z), \tag{35}$$

$$\omega(z) = \omega_1(z) + \omega_2(z), \tag{36}$$

where

$$\phi_1(z) = \omega_1(z) = \sum_{k=1}^{M} a_k X(z) z^{2k-2}, \tag{37}$$

$$\phi_2(z) = -\omega_2(z) = \sum_{k=1}^{M} b_k z^{2k-1}, \tag{38}$$

$$X(z) = \sqrt{z^2 - a^2} \quad \text{(taking the branch } \lim_{z\to\infty} X(z)/z = 1). \tag{39}$$

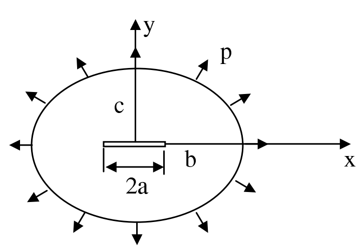

Figure 4. An elliptic plate with a central crack and normal loading p.

In Equations (37) and (38), a_k and b_k ($k = 1, 2, \ldots, M$) are real undetermined coefficients. The complex potentials shown by Equations (37) and (38) satisfy the traction-free condition [Chen 1983]. Therefore, only the condition on the elliptic boundary needs to be satisfied for the complex potentials. The eigenexpansion variational method is used to evaluate the coefficients a_k and b_k ($k = 1, 2, \ldots, M$) [Chen 1983]. In Equations (37) and (38), we can denote

$$\Phi_1(z) = \phi_1'(z) = \Omega_1(z) = \omega_1'(z), \tag{40}$$

$$\Phi_2(z) = \phi_2'(z) = -\Omega_2(z) = -\omega_2'(z). \tag{41}$$

Note that, for $z = s_1$, $s_1 > a$ and $s_1 \to a$, we have $\overline{\Phi_1(z)} - \Omega_1(\bar{z}) = 0$ and $\overline{\Phi_2(z)} - \Omega_2(\bar{z}) = 2\Phi_2(z)$. If the crack front position method is used, from Equations (5) and (32), and $p_c = 0$, we have

$$T = \lim_{z=s_1,\, s_1>a,\, s_1\to a} (\sigma_x - \sigma_y) = 4 \lim_{z=s_1,\, s_1>a,\, s_1\to a} \Phi_2(z) = 4\sum_{k=1}^{M} b_k(2k-1)a^{2k-2}. \tag{42}$$

If the crack back position method is used, from Equations (8) and (32), and $p_c = 0$, we have

$$T = \lim_{z=s_2,\, s_2<a,\, s_2\to a} \{[(\sigma_x^+ + \sigma_y^+) + (\sigma_x^- + \sigma_y^-)]/2\} = \lim_{z=s_2,\, s_2<a,\, s_2\to a} 2\,\mathrm{Re}\{\Phi^+(z) + \Phi^-(z)\}. \tag{43}$$

For $z = s_2$, $s_2 < a$ and $s_2 \to a$, both $\Phi_1^+(z)$ and $\Phi_1^-(z)$ take pure imaginary values and $\Phi_2^+(z) = \Phi_2^-(z)$ takes a real value. Therefore, from Equations (43), the same result is obtainable

$$T = 4\sum_{k=1}^{M} b_k(2k-1)a^{2k-2}. \tag{44}$$

For the normal loading p (Figure 4), we choose $M = 15$ in Equations (37) and (38) in the eigenexpansion variational method [Chen 1983]. Finally, the computed results for the T-stress is expressed as

$$T = H(c/b, a/b)p, \tag{45}$$

which are tabulated in Table 1. From Table 1 we see that, for a circular plate ($c/b = 1$) with a central crack, we have $T = -0.0199P$ for $a/b = 0.1$, which is a rather small value. However, we have $T = -3.8334p$ for $a/b = 0.9$, which is a rather large value. It is also seen from tabulated results that the deviation of the computed results for the $c/b = 1$ case from previously obtained results [Fett 2001] is rather small.

5. Evaluation for the T-stress in a curved crack using the crack front position and crack back position techniques

The T-stress in the curved crack problem can also be evaluated with the usage of two techniques and the relevant complex potentials. In addition, one numerical example is presented below.

For evaluating the T-stress at the crack tip, it is suitable to use the superposition method. The original problem is shown in Figure 2o. Without losing generality, it is assumed that the remote loading is $\sigma_y^\infty = p_2$, or $\sigma_x^\infty = p_1$, or $\sigma_{xy}^\infty = q$. The original field can be considered as a superposition of a uniform field and a perturbation field, which are shown by Figure 2a–b, respectively. The T-stress at the crack

$a/b=$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$c/b=$									
0.5	−0.0529	−0.1701	−0.2713	−0.3012	−0.2418	−0.0982	0.0904	0.1489	−0.9572
1.0	−0.0199	−0.0778	−0.1705	−0.2960	−0.4603	−0.6870	−1.0447	−1.7523	−3.8334
1.0†	−0.0216	−0.0806	−0.1712	−0.2937	−0.4568	−0.6868	−1.0515	−1.7589	−3.8466
1.5	−0.0114	−0.0464	−0.1079	−0.2023	−0.3439	−0.5636	−0.9370	−1.6859	−3.5710
2.0	−0.0090	−0.0373	−0.0887	−0.1713	−0.3012	−0.5118	−0.8803	−1.6178	−3.2109

Table 1. Nondimensional T-stresses $H(c/b, a/b)$ for an elliptic plate with a central crack and the normal loading p on contour (see Figure 4 and Equation (45)). † From an equation in [Fett 2001].

tip A is denoted by T_A, which is composed of two portions and can be expressed as

$$T_A = T_{A(u)} + T_{A(p)}, \tag{46}$$

where $T_{A(u)}$ and $T_{A(p)}$ are derived from the uniform field and the perturbation field, respectively.

The uniform field is defined for an infinite perfect plate with the remote loading $\sigma_y^\infty = p_2$, $\sigma_x^\infty = p_1$, or $\sigma_{xy}^\infty = q$ (Figure 2a). This stress field is easy to evaluate. The stress components along the prospective site of the crack are denoted by $\sigma_{N(u)}$, $\sigma_{T(u)}$, $\sigma_{NT(u)}$, where the subscript (u) denotes the stress components defined in the uniform field. Clearly, for the right crack tip A, we have the following T-stress contribution (Figure 2a):

$$T_{A(u)} = \sigma_{T(u),\ \text{at point}\ t_A} = \sigma_{T(u)}(t_A). \tag{47}$$

In the notation for $\sigma_{T(u)}(t_A)$, the subscript T denotes the stress in the T-direction, (u) represents the stress from the uniform field, and t_A denotes the location of a point for finding the T-stress.

In the following, the perturbation field for the curved crack is studied (Figure 2b). It was proved that the complex potentials for this field could be expressed as [Savruk 1981; Chen and Lin 2006]

$$\phi'(z) = \Phi(z) = \frac{1}{2\pi}\int_L \frac{g'(t)dt}{t-z}, \qquad \phi''(z) = \Phi'(z) = \frac{1}{2\pi}\int_L \frac{g'(t)dt}{(t-z)^2}, \tag{48}$$

$$\psi'(z) = \Psi(z) = \frac{1}{2\pi}\int_L \frac{\overline{g'(t)}d\bar{t}}{t-z} - \frac{1}{2\pi}\int_L \frac{\bar{t}g'(t)dt}{(t-z)^2}, \tag{49}$$

where $g'(t)$ denotes the dislocation distribution along the curved crack and is defined by

$$g'(t) = \frac{dg(t)}{dt}, \quad (t \in L,\ L - \text{the curved crack}). \tag{50}$$

In Equation (50), $g(t)$ is the COD (crack opening displacement) function defined by

$$g(t) = -\frac{2Gi}{\kappa+1}\{(u(t)+iv(t))^+ - (u(t)+iv(t))^-\}, \quad (t \in L,\ L - \text{the curved crack}), \tag{51}$$

where $(u(t)+iv(t))^+((u(t)+iv(t))^-)$ denotes the displacement in the upper side (lower side) of the curved crack, respectively.

For the curved crack problem, a singular integral equation was suggested previously [Savruk 1981; Chen and Lin 2006]:

$$\frac{1}{\pi}\int_L \frac{g'(t)dt}{t-t_o} + M(t_o) = \sigma_{N(p)}(t_o) + i\sigma_{NT(p)}(t_o), \quad (t_o \in L), \tag{52}$$

where

$$M(t_o) = \frac{1}{2\pi}\int_L K_1(t, t_o) g'(t)dt + \frac{1}{2\pi}\int_L K_2(t, t_o)\overline{g'(t)}d\bar{t}, \tag{53}$$

L denotes the curved crack configuration, and

$$K_1(t, t_o) = \frac{d}{dt_o}\left\{\ln\frac{t-t_o}{\bar{t}-\bar{t}_o}\right\} = -\frac{1}{t-t_o} + \frac{1}{\bar{t}-\bar{t}_o}\frac{d\bar{t}_o}{dt_o}, \tag{54}$$

$$K_2(t, t_o) = -\frac{d}{dt_o}\left\{\frac{t-t_o}{\bar{t}-\bar{t}_o}\right\} = \frac{1}{\bar{t}-\bar{t}_o} - \frac{t-t_o}{(\bar{t}-\bar{t}_o)^2}\frac{d\bar{t}_o}{dt_o}. \tag{55}$$

For the perturbation field, the applied tractions on the crack face must be opposite to those from the uniform field (Figure 2a–b). Clearly, the right hand term in Equation (52) is defined by

$$\sigma_{N(p)}(t_o) + i\sigma_{NT(p)}(t_o) = -(\sigma_{N(u)}(t_o) + i\sigma_{NT(u)}(t_o)),\ (t_o \in L). \tag{56}$$

In addition, the dislocation distribution $g'(t)$ should satisfy the following single-valued condition of displacements [Savruk 1981; Chen and Lin 2006],

$$\int_L g'(t)dt = 0. \tag{57}$$

Substituting Equations (48) and (49) into (12) yields

$$\sigma_y - \sigma_x - 2i\sigma_{xy} = \frac{1}{\pi}\int_L \frac{g'(t)dt}{\bar{t}-\bar{z}} - \frac{1}{\pi}\int_L \frac{(t-z)\overline{g'(t)}d\bar{t}}{(\bar{t}-\bar{z})^2}. \tag{58}$$

In the vicinity of the right crack tip A, we can assume the coordinates Ax_*y_* (Figure 2b). In these coordinates, we have

$$(\sigma_y-\sigma_x-2i\sigma_{xy})_* = (\sigma_y-\sigma_x-2i\sigma_{xy})\exp(-2i\delta) = \left(\frac{1}{\pi}\int_L \frac{g'(t)dt}{\bar{t}-\bar{z}} - \frac{1}{\pi}\int_L \frac{(t-z)\overline{g'(t)}d\bar{t}}{(\bar{t}-\bar{z})^2}\right)\frac{d\bar{t}_A}{dt_A}, \tag{59}$$

with $d\bar{t}_A/dt_A = \exp(-2i\delta)$. It is convenient to introduce the following equality

$$\mathrm{Re}\left(-\frac{1}{\pi}\int_L \frac{g'(t)dt}{t-z} + \frac{1}{\pi}\int_L \frac{\overline{g'(t)}\,d\bar{t}}{\bar{t}-\bar{z}}\right) = 0. \tag{60}$$

Therefore, from Equations (59) and (60) we have

$$(\sigma_y - \sigma_x)_* = \mathrm{Re}\left\{\left(\frac{1}{\pi}\int_L \frac{g'(t)dt}{\bar{t}-\bar{z}} - \frac{1}{\pi}\int_L \frac{(t-z)\overline{g'(t)}d\bar{t}}{(\bar{t}-\bar{z})^2}\right)\frac{d\bar{t}_A}{dt_A}\right\} + \mathrm{Re}\left(-\frac{1}{\pi}\int_L \frac{g'(t)dt}{t-z} + \frac{1}{\pi}\int_L \frac{\overline{g'(t)}d\bar{t}}{\bar{t}-\bar{z}}\right). \tag{61}$$

Similar to Equation (16), the T-stress in the perturbation field can be defined by

$$T_{A(p)} = -(\sigma_y - \sigma_x)_*|_{z\to t_A} + p_c, \tag{62}$$

where t_A denotes the complex value for the crack tip A, and $z \to t_A$ represents a limit from the crack front position. In this case, we have

$$p_c = \sigma_{N(p)}(t_A) = -\sigma_{N(u)}(t_A). \tag{63}$$

Substituting Equations (61) and (63) into (62) yields

$$T_{A(p)} = -\sigma_{N(u)}(t_A) - 2\,\mathrm{Re}\, M(t_A), \tag{64}$$

where the integral $M(t_A) = M(t_A)|_{t_o=t_A}$ has been defined by Equation (53). Finally, from Equations (46), (47), and (64), we have

$$T_A = T_{A(u)} + T_{A(p)} = \sigma_{T(u)}(t_A) - \sigma_{N(u)}(t_A) - 2\,\mathrm{Re}\, M(t_A). \tag{65}$$

A particular case is introduced below. It is assumed that there is a line crack in an infinite plate. In this case, from Equations (53), (54), and (55) we find $M(t_o) = 0$ and $M(t_A) = 0$, and Equation (65) can be reduced to

$$T_A = \sigma_{T(u)}(t_A) - \sigma_{N(u)}(t_A). \tag{66}$$

Therefore, the term $-2\,\mathrm{Re}(M(t_A))$ in Equation (65) represents the influence caused by curvature to the T-stress. For a straight-line crack, this term $-2\,\mathrm{Re}(M(t_A))$ is generally equal to zero.

On the other hand, the crack back technique is introduced below. From Equations (17) and (48), we have

$$T_{A(p)} = 2\,\mathrm{Re}\big(\Phi^+(t_o) + \Phi^-(t_o)\big)|_{t_o\to t_A} - \sigma_{N(p)}(t_A) = \mathrm{Re}\Big(\frac{2}{\pi}\int_L \frac{g'(t)dt}{t-t_o}\Big)_{t_o\to t_A} - \sigma_{N(p)}(t_A). \tag{67}$$

In Equation (67), the Plemelj formula is used for obtaining $\Phi^+(t_o)$ and $\Phi^-(t_o)$ [Muskhelishvili 1953].

Substituting Equation (52) into (67), the same result as shown by Equation (64) will be found. It is seen that the two techniques give the same result.

Similarly, at the left crack tip B we have

$$T_B = T_{B(u)} + T_{B(p)} = \sigma_{T(u)}(t_B) - \sigma_{N(u)}(t_B) - 2\,\mathrm{Re}\big(M(t_B)\big). \tag{68}$$

Once the solution for the function $g'(t)$ is obtained, the SIFs (stress intensity factors) at the right crack tip A and the left crack tip B can be evaluated by [Savruk 1981; Chen and Lin 2006]

$$\begin{aligned}(K_1 - iK_2)_A &= -\sqrt{2\pi}\lim_{t\to t_A}\sqrt{|t-t_A|}\,g'(t),\\ (K_1 - iK_2)_B &= \sqrt{2\pi}\lim_{t\to t_B}\sqrt{|t-t_B|}\,g'(t).\end{aligned} \tag{69}$$

To obtain the final solution, successive steps for evaluating the T-stresses and SIFs in the numerical solution are summarized as follows.

(i) The first step is to obtain the dislocation function $g'(t)$ from the integral equation pair composed of Equations (52) and (57).

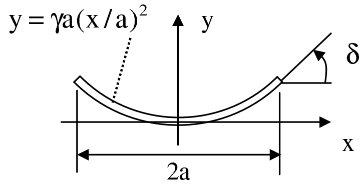

Figure 5. A parabolic crack.

(ii) The second step is to obtain two values $M(t_A) = M(t_o)|_{t_o=t_A}$ and $M(t_B) = M(t_o)|_{t_o=t_B}$, where $M(t_o)$ was defined by Equation (53).

(iii) The third step is to evaluate the T-stresses and SIFs at the crack tips by using Equations (65), (68), and (69).

One numerical example is carried out: a parabolic curved crack is defined by (see Figure 5)

$$y = \gamma a(x/a)^2. \tag{70}$$

In computation, the curve length coordinates method is used to solve the singular integral equation [Chen 2004]. The computed T-stresses is expressed as

$$T = G_1(\gamma)p \quad (\text{for } \sigma_y^\infty = p \text{ case}), \tag{71}$$

$$T = G_2(\gamma)p \quad (\text{for case } \sigma_x^\infty = \sigma_y^\infty = p). \tag{72}$$

The computed results for $\gamma = 0.1, 0.2, \ldots, 1.0$ are listed in Table 2. It is known that for a line crack in horizontal position under loading $\sigma_y^\infty = p$ we have $T = -p$. In addition, for a line crack in vertical position under loading $\sigma_y^\infty = p$ we have $T = p$. The tabulated results for $G_1(\gamma)$ (for $\sigma_y^\infty = p$) reflect the following property. If the tangential angle δ at the crack tip is changed gradually, the relevant T-stress is also changed simultaneously, for example from $T = -0.9051p$ (for $\delta = \arctan 0.2$) to from $T = 0.6527p$ (for $\delta = \arctan 2.0$).

6. Conclusions

It is known that the T-stress is a particular term for a stress component parallel to the crack face in the vicinity of the crack tip. However, depending on the position where the stress component is evaluated, the situations for this particular term are quite different. We assume that the fracture is mode I and the crack face is traction-free. In this case, the T-stress is a term embedded in the singular value of

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$G_1(\gamma)$	−0.9051	−0.6674	−0.3793	−0.1062	0.1185	0.2813	0.4276	0.5183	0.5936	0.6527
$G_2(\gamma)$	0.0184	0.0603	0.1015	0.1299	0.1409	0.1326	0.1301	0.1107	0.0946	0.0791

Table 2. Nondimensional T-stresses $G_1(\gamma)$ (for $\sigma_y^\infty = p$ case) and $G_2(\gamma)$ (for $\sigma_x^\infty = \sigma_y^\infty = p$ case) for a parabolic crack (see Figure 5 and Equations (71), (72)).

the stress component if the stress distribution at the crack front position is considered. However, if the stress distribution at the crack back position is considered under the condition $K_{II} = 0$, the T-stress is a regular term in the stress component. All of those situations cause people to investigate the crack front position and crack back position techniques for evaluating the T-stress. It is known that in the crack front technique, one needs to evaluate a limit taking the form of $\lim_{r\to 0}(\sigma_x - \sigma_y)$. However, in this case, this limit generally takes the type $\infty - \infty$. This is an inconvenient point in computation.

In this study, all derivations including the T-stress are related to complex potentials. In this case, one can use the two techniques to evaluate T-stress in more complicated crack problems, for example, in the curved crack problem. In addition, the obtained T-stress expression, or the equation shown by Equation (65) for the curved crack, are a regular integral plus some terms.

References

[Ayatollahi et al. 1998] M. R. Ayatollahi, M. J. Pavier, and D. J. Smith, "Determination of T-stress from finite element analysis for mode I and mixed mode I/II loading", *Int. J. Fract.* **91**:3 (1998), 283–298.

[Betegon and Hancock 1991] C. Betegon and J. W. Hancock, "Two parameter characterization of elastic-plastic crack-tip fields", *J. Appl. Mech.* (*ASME*) **58**:1 (1991), 104–110.

[Broberg 2005] K. B. Broberg, "A note on T-stress determination using dislocation arrays", *Int. J. Fract.* **131**:1 (2005), 1–14.

[Chen 1983] Y. Z. Chen, "An investigation of the stress intensity factor for a finite internally cracked plate by using variational method", *Eng. Fract. Mech.* **17**:5 (1983), 387–394.

[Chen 1994] Y. Z. Chen, "T-stress in multiple crack problem for an infinite plate", *Eng. Fract. Mech.* **48**:5 (1994), 641–647.

[Chen 1997] Y. Z. Chen, "Novel weight function approach for evaluating T-stress in plane elasticity crack problem", *Int. J. Fract.* **85**:4 (1997), L35–L40.

[Chen 2000] Y. Z. Chen, "Closed form solutions of T-stress in plane elasticity crack problems", *Int. J. Solids Struct.* **37**:11 (2000), 1629–1637.

[Chen 2004] Y. Z. Chen, "Solution of integral equation in curve crack problem by using curve length coordinate", *Eng. Anal. Bound. Elem.* **28**:8 (2004), 989–994.

[Chen and Lin 2006] Y. Z. Chen and X. Y. Lin, "Complex potentials and integral equations for curved crack and curved rigid line problems in plane elasticity", *Acta Mech.* **182**:3–4 (2006), 211–230.

[Chen et al. 2001] C. S. Chen, R. Krause, R. G. Pettit, L. Banks-Sills, and A. R. Ingraffea, "Numerical assessment of T-stress computation using a p-version finite element method", *Int. J. Fract.* **107**:2 (2001), 177–199.

[Cheung and Chen 1991] Y. K. Cheung and Y. Z. Chen, "Hybrid finite element formulation in plane elasticity by using complex variable function", pp. 1149–1155 in *Computational mechanics: proceedings of the Asian Pacific Conference on Computational Mechanics* (Hong Kong, 1991), edited by Y. K. Cheung et al., Balkema, Brookfield, VT, 1991.

[Fett 1997] T. Fett, "A Green's function for T-stresses in an edge-cracked rectangular plate", *Eng. Fract. Mech.* **57**:4 (1997), 365–373.

[Fett 1998a] T. Fett, "Stress intensity factors and T-stress in edge-cracked rectangular plates under mixed boundary condition", *Eng. Fract. Mech.* **60**:5–6 (1998), 625–630.

[Fett 1998b] T. Fett, "T-stresses in rectangular plates and circular disks", *Eng. Fract. Mech.* **60**:5–6 (1998), 631–652.

[Fett 2001] T. Fett, "Stress intensity factors and T-stress for internally cracked circular disks under various boundary conditions", *Eng. Fract. Mech.* **68**:9 (2001), 1119–1136.

[Fett and Rizzi 2005] T. Fett and G. Rizzi, "Weight functions for stress intensity factors and T-stress for oblique cracks in a half-space", *Int. J. Fract.* **132**:1 (2005), L9–L16.

[Karihaloo and Xiao 2001] B. L. Karihaloo and Q. Z. Xiao, "Higher order terms of the crack tip asymptotic field for a notched three-point bend beam", *Int. J. Fract.* **112**:2 (2001), 111–128.

[Kfouri 1998] A. P. Kfouri, "Some evaluations of the elastic T-term using Eshelby's method", *Int. J. Fract.* **30**:4 (1998), 301–315.

[Larsson and Carlsson 1973] S. G. Larsson and A. J. Carlsson, "Influence of non-singular stress terms and specimen geometry on small-scale yielding at crack tips in elastic-plastic materials", *J. Mech. Phys. Solids* **21**:4 (1973), 263–277.

[Li and Xu 2007] X. F. Li and L. R. Xu, "T-stresses across static crack kinking", *J. Appl. Mech.* (*ASME*) **74**:2 (2007), 181–190.

[Melin 2002] S. Melin, "The influence of the T-stress on the directional stability of cracks", *Int. J. Fract.* **114**:3 (2002), 259–265.

[Muskhelishvili 1953] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity: fundamental equations, plane theory of elasticity, torsion, and bending*, Noordhoff, Groningen, 1953.

[Rice 1974] J. R. Rice, "Limitations to the small scale yielding approximation of elastic-plastic crack-tip fields", *J. Mech. Phys. Solids* **22**:1 (1974), 17–26.

[Savruk 1981] M. P. Savruk, Двухмерные задачи упругости для тел с трещинами, Naukova Dumka, Kiev, 1981.

[Sham 1989] T. L. Sham, "The theory of higher order weight functions for linear elastic plane problems", *Int. J. Solids Struct.* **25**:4 (1989), 357–380.

[Sham 1991] T. L. Sham, "The determination of the elastic T-term using higher order weight functions", *Int. J. Fract.* **48**:2 (1991), 81–102.

[Sherry et al. 1995] A. H. Sherry, C. C. France, and M. R. Goldthorpe, "Compendium of T-stress solutions for two and three dimensional cracked geometries", *Fatigue Fract. Eng. Mater. Struct.* **18**:1 (1995), 141–155.

[Smith et al. 2006] D. J. Smith, M. R. Ayatollahi, and M. J. Pavier, "On the consequences of T-stress in elastic brittle fracture", *Proc. R. Soc. Lond. A* **462**:2072 (2006), 2415–2437.

[Tong et al. 1973] P. Tong, T. H. H. Pian, and S. J. Lasry, "A hybrid-element approach to crack problems in plane elasticity", *Int. J. Numer. Methods Eng.* **7**:3 (1973), 297–308.

[Williams 1957] M. L. Williams, "On the stress distribution at the base of a stationary crack", *J. Appl. Mech.* (*ASME*) **24** (1957), 109–114.

[Xiao and Karihaloo 2002] Q. Z. Xiao and B. L. Karihaloo, "Approximate Green's functions for singular and higher order terms of an edge crack in a finite plate", *Eng. Fract. Mech.* **69**:8 (2002), 959–981.

[Xiao et al. 2004] Q. Z. Xiao, B. L. Karihaloo, and X. Y. Liu, "Direct determination of SIF and higher order terms of mixed mode cracks by a hybrid crack element", *Int. J. Fract.* **125**:3–4 (2004), 207–225.

[Yang and Ravi-Chandar 1999] B. Yang and K. Ravi-Chandar, "Evaluation of elastic T-stress by the stress difference method", *Eng. Fract. Mech.* **64**:5 (1999), 589–605.

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