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STABILITY AND MEMORY EFFECTS IN A HOMOGENIZED MODEL GOVERNING THE ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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We present a macroscopic model of electrical conduction in biological tissues. This model is derived via a homogenization limit by a microscopic formulation based on Maxwell's equations, taking into account the periodic geometry of the microstructure. We also study the asymptotic behavior of the model for large times. Our results imply that periodic boundary data lead to an asymptotically periodic solution. The model is relevant to applications like electric impedance tomography.

1. Introduction

In this paper we deal with a model of electrical conduction in composite media and, specifically, conduction in biological tissues. The classical governing equation is

$$-\operatorname{div}(\kappa \nabla u_t + \sigma \nabla u) = 0\,, \tag{1-1}$$

which is derived from the Maxwell equations in the quasistationary approximation (see for example, [Novožilov and Yappa 1978]). Here, u is the electrical potential and κ, σ are the permittivity and the conductivity of the material, respectively. The geometry of the composite media we have in mind is a periodic array of the unit cell depicted in Figure 1. More precisely, we look at a phase E_1^η which models the cell cytosol, coated by a shell Γ^η which models the cell membrane, included in a phase E_2^η which models the extracellular fluid [Foster and Schwan 1989]. In particular, the permittivity κ in E_1^η and E_2^η is lower, and the conductivity σ is higher, than in Γ^η. The diameter of the cell is of the order of tens of micrometers, while the width of the membrane is of the order of ten nanometers. This suggests that the thin shell Γ^η could be preferably modeled as a two dimensional interface Γ, in order to get a simpler model and, possibly, a better understanding of the effect of the geometric features of the microscopic structure. This simpler model can be obtained from Equation (1-1) via a concentration-of-capacity procedure [Amar et al. 2006], leading to Problem (2-1)–(2-6), below. In particular, Equation (2-3) takes into account the conductive/capacitive behavior of the concentrated membrane. As shown in (2-3), the electric potential jumps across the interface Γ, and its jump satisfies a dynamical condition (roughly speaking, in the form of a hyperbolic differential equation on the interface itself).

Our model is designed to investigate the response of biological tissues to the injection of electrical currents in the radio frequency range, that is, the Maxwell–Wagner interfacial polarization effect [Foster and Schwan 1989; Bisegna et al. 2001], at higher frequencies than those considered in [Amar et al. 2003; 2004b; 2005; 2006; 2008]. This effect is relevant to clinical applications like electric impedance tomography and body composition [De Lorenzo et al. 1997; Bronzino 1999].

Keywords: asymptotic decay, stability, homogenization, memory effects, electrical conduction, biological tissues.

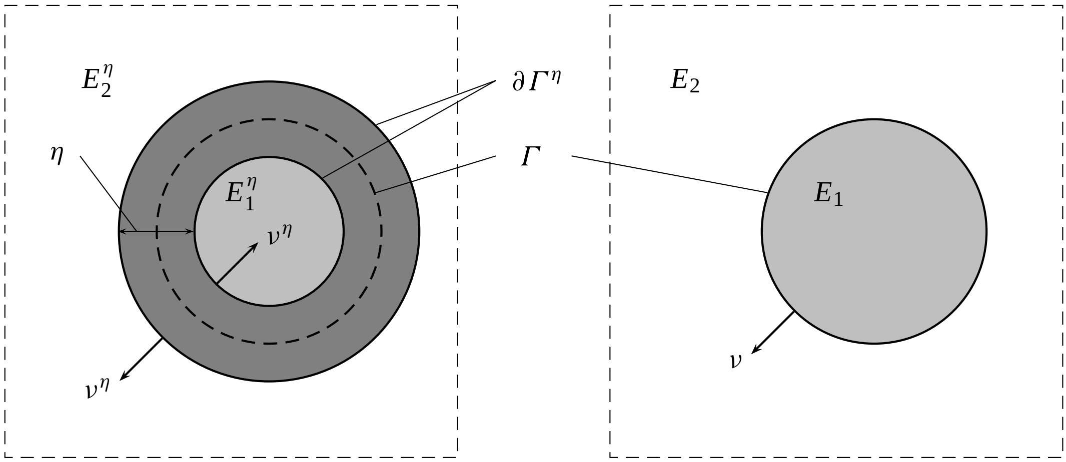

Figure 1. The periodic cell Y. Left: before concentration; Γ^η is the dark gray region, and $E^\eta = E_1^\eta \cup E_2^\eta$ is the union of the light gray and white regions. Right: after concentration; Γ^η shrinks to Γ as $\eta \to 0$.

Problem (2-1)–(2-6) contains a small parameter ε, coinciding with the period of the microstructure. The typical structure of the periodic array we have in mind is given in Figure 2. Some applications deal with measurements of the electric potential at the macroscopic (body) scale: this suggests that it would be advantageous to investigate the homogenization limit of Problem (2-1)–(2-6) when we let $\varepsilon \to 0$. Extensive surveys on this topic are, for example, in [Bensoussan et al. 1978; Sánchez-Palencia

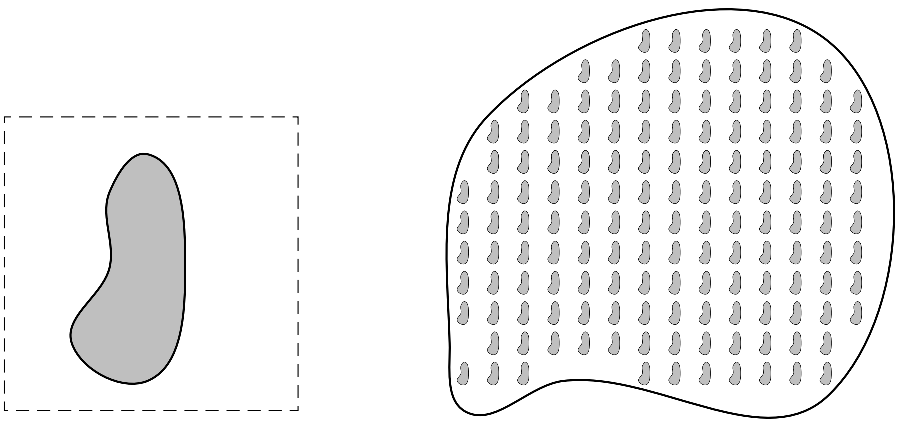

Figure 2. Left: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\mathbb{R}^2$. Here E_1 is the light gray region and Γ is its boundary. The remaining part of Y (the white region) is E_2. Right: the corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here Ω_1^ε is the light gray region and Γ^ε is its boundary. The remaining part of Ω (the white region) is Ω_2^ε.

1980; Lions 1981; Attouch 1984; Bakhvalov and Panasenko 1989; Oleinik et al. 1992; Sánchez-Hubert and Sánchez-Palencia 1992; Jikov et al. 1994; Braides and Defranceschi 1998; Cioranescu and Donato 1999]. It turns out that the partial differential equation obtained in the limit is nonstandard; see (3-39) below. Indeed, it is an equation exhibiting memory effects, that is, it contains explicitly the history of the unknown and hence is markedly different from the Laplace equation presently used as a standard in the bioelectrical impedance literature [Bronzino 1999].

Our model can be compared to some papers where homogenization theory is applied to linear stationary elliptic problems involving imperfect interfaces arising in fields like elasticity [Lene and Leguillon 1981] or heat conduction [Lipton 1998]. See also [Donato et al. 2007; Sánchez-Palencia 1980], where hyperbolic problems with interfaces are considered in the framework of elastodynamics and electrodynamics.

In view of the applications, it is also of interest to study the time evolution of the homogenized potential (see Section 2). In particular, it is of interest to show that time-harmonic boundary data elicits a time-harmonic solution for large times. In this regard, following the same reasoning as that presented in [Amar et al. 2008], it is enough to prove that the solution u_0 of (3-39) exponentially decays to zero as time increases, provided that a zero Dirichlet boundary condition is assigned (see Theorem 2.1 and Corollary 2.2).

From a mathematical point of view, the asymptotic behavior of evolutive equations with memory is a classical problem [Fichera 1979; Slemrod 1981; Fabrizio and Morro 1988; Lazzari and Vuk 1992], currently drawing much interest in the literature [Lazzari and Nibbi 2002; Giorgi et al. 2001; 2005; Medjden and Tatar 2005; Appleby et al. 2006]. We note that the exponential decay of the memory kernel, in general, does not imply the existence of bounded solutions, as shown by a counterexample presented in Section 5 (see also, [Fichera 1979; Fabrizio and Morro 1988]).

We finally note that our methods could be easily applied to study the homogenization problem and the time-asymptotic behavior of Kelvin–Voigt viscoelastic composites with coated inclusions.

2. Position of the problem and main results

We look at the homogenization limit ($\varepsilon \to 0$) of the following problem for $u_\varepsilon(x,t)$:

$$-\operatorname{div}(\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)=0 \qquad \text{in } (\Omega_1^\varepsilon\cup\Omega_2^\varepsilon)\times(0,+\infty); \tag{2-1}$$

$$[(\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)\cdot\nu]=0 \qquad \text{on } \Gamma^\varepsilon\times(0,+\infty); \tag{2-2}$$

$$(\alpha/\varepsilon)\partial[u_\varepsilon]/\partial t+(\beta/\varepsilon)[u_\varepsilon]=((\kappa\nabla u_{\varepsilon t}+\sigma\nabla u_\varepsilon)\cdot\nu)^{(2)} \qquad \text{on } \Gamma^\varepsilon\times(0,+\infty); \tag{2-3}$$

$$u_\varepsilon(x,t)=0 \qquad \text{on } \partial\Omega\times(0,+\infty); \tag{2-4}$$

$$\nabla u_\varepsilon(x,0)=\boldsymbol{G}_\varepsilon(x) \qquad \text{in } \Omega_1^\varepsilon\cup\Omega_2^\varepsilon; \tag{2-5}$$

$$[u_\varepsilon](x,0)=S_\varepsilon(x) \qquad \text{on } \Gamma^\varepsilon. \tag{2-6}$$

The operators div and ∇ act with respect to the space variable x; $\Omega=\Omega_1^\varepsilon\cup\Omega_2^\varepsilon\cup\Gamma^\varepsilon$, where Ω_1^ε and Ω_2^ε are two disjoint open subsets of Ω, and $\Gamma^\varepsilon=\partial\Omega_1^\varepsilon\cap\Omega=\partial\Omega_2^\varepsilon\cap\Omega$; ν is the normal unit vector pointing into Ω_2^ε; the typical geometry we have in mind is depicted in Figure 2. We refer to Section 2 for a precise definition of the structure of Ω_1^ε, Ω_2^ε, Γ^ε.

Moreover, we assume that

$$\begin{aligned} \alpha > 0\,; \qquad \beta \geq 0\,; \qquad & \kappa = \kappa_1 > 0\,,\ \sigma = \sigma_1 > 0 \quad \text{in } \Omega_1^\varepsilon; \\ & \kappa = \kappa_2 > 0\,,\ \sigma = \sigma_2 > 0 \quad \text{in } \Omega_2^\varepsilon; \end{aligned} \tag{2-7}$$

where κ_1, κ_2, σ_1, σ_2, α, and β are constants. From a physical point of view, Γ^ε represents the cell membranes having capacitance α/ε and conductance β/ε per unit area, whereas Ω_1^ε (respectively, Ω_2^ε) is the intracellular (respectively, extracellular) space, having permittivity κ_1 (respectively, κ_2) and conductivity σ_1 (respectively, σ_2).

Since u_ε is not, in general, continuous across Γ^ε, we have set

$$u_\varepsilon^{(2)} := \text{ trace of } u_{\varepsilon|\Omega_2^\varepsilon} \text{ on } \Gamma^\varepsilon\,, \quad u_\varepsilon^{(1)} := \text{ trace of } u_{\varepsilon|\Omega_1^\varepsilon} \text{ on } \Gamma^\varepsilon\,, \quad \text{and} \quad [u_\varepsilon] := u_\varepsilon^{(2)} - u_\varepsilon^{(1)}\,.$$

A similar convention is employed for the current flux density across the membrane $(\kappa\nabla u_{\varepsilon t} + \sigma\nabla u_\varepsilon)\cdot\nu$.

We assume that the restrictions of $\boldsymbol{G}_\varepsilon$ to Ω_1^ε and Ω_2^ε are gradients of scalar fields, and that $\boldsymbol{G}_\varepsilon$ strongly converges in L^2. Moreover, we assume that $S_\varepsilon \in H^1(\Omega)$, and that $S_\varepsilon/\varepsilon$ strongly converges in L^2. These assumptions are introduced in order to rule out the appearance of an initial layer (see [Amar et al. 2009]). Further assumptions on $\boldsymbol{G}_\varepsilon$ and S_ε are introduced in the next paragraph.

Geometry. Following [Amar et al. 2004b], we introduce a periodic open subset E of $\mathbb{R}^N$, so that $E+z = E$ for all $z \in \boldsymbol{Z}^N$. For all $\varepsilon > 0$ we define $\Omega_1^\varepsilon = \Omega \cap \varepsilon E$, $\Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}$, $\Gamma^\varepsilon = \Omega \cap \partial(\varepsilon E)$. We assume that Ω, E have a regular boundary, say of class C^∞ for the sake of simplicity. We also employ the notation $Y = (0,1)^N$, $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. We stipulate that E_1 is a connected smooth subset of Y such that $\text{dist}(\overline{E}_1, \partial Y) > 0$. Some generalizations may be possible, but we do not dwell on this point here. Finally, we assume that $\text{dist}(\Gamma^\varepsilon, \partial\Omega) > \gamma\varepsilon$ for some constant $\gamma > 0$ independent of ε, by dropping the inclusions contained in the cells $\varepsilon(Y+z)$, $z \in \boldsymbol{Z}^N$ which intersect $\partial\Omega$ (see Figure 2). For later usage, we introduce the set

$$\boldsymbol{Z}_\varepsilon^N := \{z \in \boldsymbol{Z}^N : \varepsilon(Y+z) \subseteq \Omega\}\,. \tag{2-8}$$

Energy estimate. Multiply (2-1) by u_ε and integrate by parts. Using (2-2)–(2-6), we arrive, for all $t > 0$, to the energy estimate

$$\begin{aligned} \int_\Omega \frac{\kappa}{2}|\nabla u_\varepsilon(x,t)|^2\,\mathrm{d}x + \int_0^t\!\!\int_\Omega \sigma|\nabla u_\varepsilon(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon(x,t)]^2\,\mathrm{d}\sigma \\ + \frac{\beta}{\varepsilon}\int_0^t\!\!\int_{\Gamma^\varepsilon}[u_\varepsilon(x,\tau)]^2\,\mathrm{d}\sigma\,\mathrm{d}\tau = \int_\Omega \frac{\kappa}{2}|\boldsymbol{G}_\varepsilon(x)|^2\,\mathrm{d}x + \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon} S_\varepsilon^2(x)\,\mathrm{d}\sigma\,. \end{aligned} \tag{2-9}$$

We assume that

$$\int_\Omega \frac{\kappa}{2}|\boldsymbol{G}_\varepsilon(x)|^2\,\mathrm{d}x + \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon} S_\varepsilon^2(x)\,\mathrm{d}\sigma < \gamma\,, \tag{2-10}$$

for a constant γ independent of ε. In fact (2-9), coupled with the Poincaré's inequality (Lemma 4.1), is a main tool in the rigorous proof of convergence of u_ε to its limit. In particular, up to a subsequence, u_ε converges weakly in $L^2(\Omega \times (0,\overline{T}))$ as $\varepsilon \to 0$ to a limit u_0, for every $\overline{T} > 0$. The equation satisfied by u_0 will be formally derived via a homogenization procedure in Section 3.

Exponential decay.

Theorem 2.1. *Let Ω_1^ε, Ω_2^ε, Γ^ε be as before. Assume that (2-7) holds, and that the initial data $\boldsymbol{G}_\varepsilon$ are gradients of scalar fields and together with S_ε satisfy (2-10). Let u_ε be the solution of (2-1)–(2-6). Then*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C(\varepsilon + \mathrm{e}^{-\lambda t}) \quad \text{a.e. in } (0,+\infty), \tag{2-11}$$

where C and λ are positive constants independent of ε. Moreover, if $\beta > 0$, or else if S_ε has null mean average over each connected component of Γ^ε, it follows that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C\,\mathrm{e}^{-\lambda t} \quad \text{a.e. in } (0,+\infty). \tag{2-12}$$

This result easily yields the following exponential time-decay estimate for the limit u_0 under homogeneous Dirichlet boundary data:

Corollary 2.2. *Under the assumptions of Theorem 2.1, if $u_\varepsilon \to u_0$ weakly in $L^2(\Omega\times(0,\overline{T}))$ for every $\overline{T} > 0$, then*

$$\|u_0(\cdot,t)\|_{L^2(\Omega)} \le C\,\mathrm{e}^{-\lambda t} \quad \text{a.e. in } (0,+\infty). \tag{2-13}$$

3. Formal homogenization

To establish the notation, we summarize here some well known asymptotic expansions needed in the two-scale method (see, for example, [Bensoussan et al. 1978], [Sánchez-Palencia 1980]). Introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$, assuming

$$u_\varepsilon = u_\varepsilon(x,y,t) = u_0(x,y,t) + \varepsilon u_1(x,y,t) + \varepsilon^2 u_2(x,y,t) + \dots . \tag{3-1}$$

Note that u_0, u_1, u_2 are periodic in y, and u_1, u_2 are assumed to have zero integral average over Y. Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon}\operatorname{div}_y + \operatorname{div}_x\,, \qquad \nabla = \frac{1}{\varepsilon}\nabla_y + \nabla_x\,, \tag{3-2}$$

we compute, for example,

$$\nabla u_\varepsilon = \frac{1}{\varepsilon}\nabla_y u_0 + \bigl(\nabla_x u_0 + \nabla_y u_1\bigr) + \varepsilon\bigl(\nabla_y u_2 + \nabla_x u_1\bigr) + \dots . \tag{3-3}$$

We also stipulate

$$\boldsymbol{G}_\varepsilon = \boldsymbol{G}_\varepsilon(x,y) = \boldsymbol{G}_0(x,y) + \varepsilon \boldsymbol{G}_1(x,y) + \varepsilon^2 \boldsymbol{G}_2(x,y) + \dots , \tag{3-4}$$

$$S_\varepsilon = S_\varepsilon(x,y) = S_0(x,y) + \varepsilon S_1(x,y) + \varepsilon^2 S_2(x,y) + \dots , \tag{3-5}$$

where the restrictions of $\boldsymbol{G}_0(x,\cdot)$, $\boldsymbol{G}_1(x,\cdot)$, ... to E_1 and E_2 are the gradients of scalar fields. According to Equation (2-10), recalling that $|\Gamma^\varepsilon|_{N-1} \sim 1/\varepsilon$, we assume $S_0 \equiv 0$ in (3-5). Moreover, according to the assumption on the strong convergence of $\boldsymbol{G}_\varepsilon$ and $S_\varepsilon/\varepsilon$, the functions $\boldsymbol{G}_0(x,y)$ and $S_1(x,y)$ do not depend on y, that is $\boldsymbol{G}_0(x,y) = \boldsymbol{G}_0(x)$ and $S_1(x,y) = S_1(x)$.

For the sake of brevity, we introduce the operator

$$\mathcal{D} := \kappa\frac{\partial}{\partial t} + \sigma\,. \tag{3-6}$$

Applying (3-2)–(3-3) to Problem (2-1)–(2-6), one readily sees, by matching corresponding powers of ε, that u_0 solves,

$$-\mathcal{D}\Delta_y u_0 = 0 \qquad \text{in } (E_1 \cup E_2)\times(0,+\infty); \tag{3-7}$$

$$[\mathcal{D}\nabla_y u_0 \cdot \nu] = 0 \qquad \text{on } \Gamma \times (0,+\infty); \tag{3-8}$$

$$\alpha \frac{\partial [u_0]}{\partial t} + \beta [u_0] = (\mathcal{D}\nabla_y u_0 \cdot \nu)^{(2)} \qquad \text{on } \Gamma \times (0,+\infty); \tag{3-9}$$

$$\nabla_y u_{0|t=0} = 0 \qquad \text{on } E_1 \cup E_2; \tag{3-10}$$

$$[u_0]_{|t=0} = 0 \qquad \text{on } \Gamma. \tag{3-11}$$

Reasoning as in Section 2, we obtain an energy estimate for (3-7)–(3-11), which implies that $[u_0] = 0$ for all times, and

$$u_0 = u_0(x,t)\,.$$

Next, we find for u_1 that

$$-\mathcal{D}\Delta_y u_1 = 0 \qquad \text{in } (E_1 \cup E_2)\times(0,+\infty); \tag{3-12}$$

$$[\mathcal{D}(\nabla_y u_1 + \nabla_x u_0)\cdot \nu] = 0 \qquad \text{on } \Gamma \times (0,+\infty); \tag{3-13}$$

$$\alpha \frac{\partial [u_1]}{\partial t} + \beta [u_1] = (\mathcal{D}(\nabla_y u_1 + \nabla_x u_0)\cdot \nu)^{(2)} \qquad \text{on } \Gamma \times (0,+\infty); \tag{3-14}$$

$$\nabla_y u_{1|t=0} + \nabla_x u_{0|t=0} = \boldsymbol{G}_0 \qquad \text{on } E_1 \cup E_2; \tag{3-15}$$

$$[u_1]_{|t=0} = S_1 \qquad \text{on } \Gamma. \tag{3-16}$$

Since both u_0 and $\boldsymbol{G}_0$ do not depend on y, Equation (3-15) implies $\nabla_y u_{1|t=0} = 0$ on $E_1 \cup E_2$.

In order to represent u_1 in a suitable way, let $\boldsymbol{g} \in L^2(E_1 \cup E_2)$ and $s \in L^2(\Gamma)$ be assigned such that the restrictions of $\boldsymbol{g}$ to E_1 and E_2 are gradients of scalar fields, and consider the problem

$$-\mathcal{D}\Delta_y v = 0\,, \qquad \text{in } (E_1 \cup E_2)\times(0,+\infty); \tag{3-17}$$

$$[\mathcal{D}\nabla_y v \cdot \nu] = 0\,, \qquad \text{on } \Gamma \times (0,+\infty); \tag{3-18}$$

$$\alpha \frac{\partial [v]}{\partial t} + \beta [v] = (\mathcal{D}\nabla_y v \cdot \nu)^{(2)} \qquad \text{on } \Gamma \times (0,+\infty). \tag{3-19}$$

$$\nabla_y v_{|t=0} = \boldsymbol{g} \qquad \text{on } E_1 \cup E_2; \tag{3-20}$$

$$[v]_{|t=0} = s \qquad \text{on } \Gamma. \tag{3-21}$$

where v is a periodic function in Y, such that $\int_Y v(y,t)\,\mathrm{d}y = 0$. Define the transform $\mathcal{T}$ by

$$\mathcal{T}(\boldsymbol{g}, s)(y,t) = v(y,t)\,, \qquad y \in Y\,, t > 0\,.$$

Then, introduce the cell functions $\boldsymbol{\chi}^0 : Y \to \mathbb{R}^N$ and $\boldsymbol{\chi}^1 : Y \times (0,+\infty) \to \mathbb{R}^N$, whose components χ_h^0 and $\chi_h^1(\cdot,t)$, $h = 1, \dots, N$, are required to be periodic functions with vanishing integral averages over

Y for $t \geq 0$. The function χ_h^0 of the components of $\boldsymbol{\chi}^0$ satisfies

$$-\kappa \Delta_y \chi_h^0 = 0 \qquad \text{in } E_1 \cup E_2; \tag{3-22}$$

$$[\kappa(\nabla_y \chi_h^0 - \boldsymbol{e}_h) \cdot \nu] = 0 \qquad \text{on } \Gamma; \tag{3-23}$$

$$\alpha[\chi_h^0] = (\kappa(\nabla_y \chi_h^0 - \boldsymbol{e}_h) \cdot \nu)^{(2)} \qquad \text{on } \Gamma. \tag{3-24}$$

The initial value $\chi_h^1(\cdot\,,0)$ of the components of $\boldsymbol{\chi}^1$ satisfies

$$-\kappa \Delta_y \chi_h^1(\cdot\,,0) - \sigma \Delta_y \chi_h^0 = 0\,, \qquad \text{in } E_1 \cup E_2; \tag{3-25}$$

$$[(\kappa \nabla_y \chi_h^1(\cdot\,,0) + \sigma(\nabla_y \chi_h^0 - \boldsymbol{e}_h)) \cdot \nu] = 0 \qquad \text{on } \Gamma; \tag{3-26}$$

$$((\kappa \nabla_y \chi_h^1(\cdot\,,0) + \sigma(\nabla_y \chi_h^0 - \boldsymbol{e}_h)) \cdot \nu)^{(2)} = \alpha[\chi_h^1(\cdot\,,0)] + \beta[\chi_h^0] \qquad \text{on } \Gamma. \tag{3-27}$$

Finally, χ_h^1 is defined for $t > 0$ by

$$\chi_h^1 = \mathcal{T}\left(\nabla_y \chi_h^1(\cdot\,,0), [\chi_h^1(\cdot\,,0)]\right). \tag{3-28}$$

Straightforward calculations show that u_1 may be written in the form

$$u_1(x, y, t) = -\boldsymbol{\chi}^0(y) \cdot \nabla_x u_0(x,t) - \int_0^t \boldsymbol{\chi}^1(y, t-\tau) \cdot \nabla_x u_0(x,\tau)\, \mathrm{d}\tau \\ + \mathcal{T}\big(\nabla_y(\boldsymbol{\chi}^0 \cdot \boldsymbol{G}_0(x)), S_1(x) + [\boldsymbol{\chi}^0] \cdot \boldsymbol{G}_0(x)\big)(y,t)\,, \tag{3-29}$$

so that

$$\mathcal{D}u_1(x, y, t) = -\kappa \boldsymbol{\chi}^0(y) \cdot \nabla_x u_{0t}(x,t) - (\kappa \boldsymbol{\chi}^1(y,0) + \sigma \boldsymbol{\chi}^0(y)) \cdot \nabla_x u_0(x,t) \\ - \int_0^t (\mathcal{D}\boldsymbol{\chi}^1)(y, t-\tau) \cdot \nabla_x u_0(x,\tau)\, \mathrm{d}\tau \\ + \mathcal{D}\mathcal{T}\big(\nabla_y(\boldsymbol{\chi}^0 \cdot \boldsymbol{G}_0(x)), S_1(x) + [\boldsymbol{\chi}^0] \cdot \boldsymbol{G}_0(x)\big)(y,t)\,. \tag{3-30}$$

Next we find for u_2 that

$$-\mathcal{D}\left(\Delta_y u_2 + 2\frac{\partial^2 u_1}{\partial x_j \partial y_j} + \Delta_x u_0\right) = 0\,, \qquad \text{in } (E_1 \cup E_2) \times (0, +\infty); \tag{3-31}$$

$$[\mathcal{D}(\nabla_y u_2 + \nabla_x u_1) \cdot \nu] = 0 \qquad \text{on } \Gamma \times (0, +\infty); \tag{3-32}$$

$$(\mathcal{D}(\nabla_y u_2 + \nabla_x u_1) \cdot \nu)^{(2)} = \alpha \frac{\partial [u_2]}{\partial t} + \beta[u_2] \qquad \text{on } \Gamma \times (0, +\infty). \tag{3-33}$$

$$\nabla_y u_{2|t=0} + \nabla_x u_{1|t=0} = \boldsymbol{G}_1 \qquad \text{on } E_1 \cup E_2; \tag{3-34}$$

$$[u_2]_{|t=0} = S_2 \qquad \text{on } \Gamma. \tag{3-35}$$

Let us find the solvability conditions for this problem. Integrating by parts the partial differential equations (3-31) solved by u_2, both in E_1 and in E_2, adding the two contributions, and using (3-32), we get

$$\left[\int_{E_1}+\int_{E_2}\right]\mathfrak{D}\left\{\Delta_x u_0(x,t)+2\frac{\partial^2 u_1}{\partial x_j\partial y_j}\right\}\mathrm{d}y=-\int_{\Gamma}[\mathfrak{D}\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma\,. \tag{3-36}$$

Thus, we obtain

$$\left(\kappa_0\frac{\partial}{\partial t}+\sigma_0\right)\Delta_x u_0=2\int_{\Gamma}[\mathfrak{D}\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma-\int_{\Gamma}[\mathfrak{D}\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma=\int_{\Gamma}[\mathfrak{D}\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma\,, \tag{3-37}$$

where

$$\kappa_0=\kappa_1|E_1|+\kappa_2|E_2|\,;\quad \sigma_0=\sigma_1|E_1|+\sigma_2|E_2|\,. \tag{3-38}$$

Then, we substitute the representation (3-29) into Equation (3-37) and, after simple algebra, obtain the homogenized equation for u_0 in $\Omega\times(0,+\infty)$ as

$$-\operatorname{div}\left(K\nabla_x u_{0t}+A\nabla_x u_0+\int_0^t B(t-\tau)\nabla_x u_0(\cdot\,,\tau)\,\mathrm{d}\tau-\mathcal{F}\right)=0\,, \tag{3-39}$$

where the matrices K, A, $B(t)$, and the vector $\mathcal{F}(x,t)$ are defined as follows:

$$K=\kappa_0 I+\int_{\Gamma}\nu\otimes[\kappa\boldsymbol{\chi}^0(y)]\,\mathrm{d}\sigma\,, \tag{3-40}$$

$$A=\sigma_0 I+\int_{\Gamma}\nu\otimes[\kappa\boldsymbol{\chi}^1(y,0)+\sigma\boldsymbol{\chi}^0(y)]\,\mathrm{d}\sigma\,, \tag{3-41}$$

$$B(t)=\int_{\Gamma}\nu\otimes[(\mathfrak{D}\boldsymbol{\chi}^1)(y,t)]\,\mathrm{d}\sigma\,, \tag{3-42}$$

$$\mathcal{F}(x,t)=\int_{\Gamma}[\mathfrak{D}\mathcal{T}\big(\nabla_y(\boldsymbol{\chi}^0\cdot\boldsymbol{G}_0(x)),S_1(x)+[\boldsymbol{\chi}^0]\cdot\boldsymbol{G}_0(x)\big)(y,t)]\nu\,\mathrm{d}\sigma\,. \tag{3-43}$$

Equation (3-39) is complemented with the initial condition

$$\nabla_x u_{0|t=0}=\boldsymbol{G}_0\,,\qquad \text{on }\Omega. \tag{3-44}$$

Finally, integrating Equation (3-39) over time, changing the order in the double integral that results, and using (3-44), we obtain also the following formulation

$$-\operatorname{div}\left(K\nabla_x u_0+\int_0^t\left(A+\int_0^{t-s}B(\tau)\,\mathrm{d}\tau\right)\nabla_x u_0(\cdot\,,s)\,\mathrm{d}s-K\boldsymbol{G}_0-\int_0^t\mathcal{F}(\cdot\,,\tau)\,\mathrm{d}\tau\right)=0\,, \tag{3-45}$$

which shows that the homogenized equation has exactly the form of an equation with memory of the type derived in [Amar et al. 2003; 2004b] and studied in [Amar et al. 2004a].

4. Time-exponential asymptotic decay: proof of Theorem 2.1

The case $\beta>0$ is quite simple. We introduce the space

$$H^1_\varepsilon(\Omega):=\{v\in L^2(\Omega)\ :\ v_{|\Omega^\varepsilon_i}\in H^1(\Omega^\varepsilon_i),\ i=1,2;\ v=0\text{ on }\partial\Omega\}\,. \tag{4-1}$$

It turns out that, for all $v \in H^1_\varepsilon(\Omega)$,

$$\int_\Omega \sigma |\nabla v|^2 \,\mathrm{d}x + \frac{\beta}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \geq \lambda \left(\int_\Omega \frac{\kappa}{2} |\nabla v|^2 \,\mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \right), \tag{4-2}$$

for $\lambda = \min\{2\sigma_1/\kappa_1, 2\sigma_2/\kappa_2, 2\beta/\alpha\}$. Taking $v = u_\varepsilon(\cdot\,, t)$ in the previous estimate and using equations (2-9), (2-10), and the differential version of Gronwall's Lemma, we obtain

$$\int_\Omega \frac{\kappa}{2} |\nabla u_\varepsilon(\cdot\,, t)|^2 \,\mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(\cdot\,, t)]^2 \,\mathrm{d}\sigma \leq \gamma \,\mathrm{e}^{-\lambda t}, \quad \text{a.e. in } (0, +\infty), \tag{4-3}$$

and (2-12) follows from Poincaré's inequality (Lemma 4.1).

Now we consider the case $\beta = 0$. We introduce the space $\widetilde{H}^{1/2}(\Gamma^\varepsilon) \subset H^{1/2}(\Gamma^\varepsilon)$ of the functions which have a null average over each connected component of Γ^ε, that is, on $\varepsilon(\Gamma + z)$, for each z belonging to the set $\boldsymbol{Z}^N_\varepsilon$ defined in (2-8). We decompose the initial datum $S_\varepsilon(x)$ in (2-6) as $S_\varepsilon(x) = \overline{S}_\varepsilon(x) + \widetilde{S}_\varepsilon(x)$, where

$$\begin{aligned} &\overline{S}_\varepsilon(x) = \fint_{\varepsilon(\Gamma+z)} S_\varepsilon \,\mathrm{d}\sigma =: C_{\varepsilon z} \qquad \text{on each } \varepsilon(\Gamma + z),\ z \in \boldsymbol{Z}^N_\varepsilon\,; \\ &\widetilde{S}_\varepsilon(x) \in \widetilde{H}^{1/2}(\Gamma^\varepsilon)\,, \end{aligned} \tag{4-4}$$

and the initial datum $\boldsymbol{G}_\varepsilon(x)$ in (2-5) as $\boldsymbol{G}_\varepsilon(x) = \overline{\boldsymbol{G}}_\varepsilon(x) + \widetilde{\boldsymbol{G}}_\varepsilon(x)$, where $\overline{\boldsymbol{G}}_\varepsilon(x) = 0$ and $\widetilde{\boldsymbol{G}}_\varepsilon(x) = \boldsymbol{G}_\varepsilon(x)$. Accordingly, the solution u_ε to Problem (2-1)–(2-6) is decomposed as $\overline{u}_\varepsilon + \tilde{u}_\varepsilon$. Clearly,

$$\overline{u}_\varepsilon(x, t) = \begin{cases} 0 & \text{for } (x, t) \in \Omega^\varepsilon_2 \times (0, +\infty)\,, \\ -C_{\varepsilon z} & \text{for } (x, t) \in (\varepsilon(E_1 + z)) \times (0, +\infty),\ z \in \boldsymbol{Z}^N_\varepsilon\,. \end{cases} \tag{4-5}$$

Using the previous equation, we compute

$$\int_\Omega |\overline{u}_\varepsilon|^2 \,\mathrm{d}x = \sum_{z \in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(E_1+z)} |\overline{u}_\varepsilon|^2 \,\mathrm{d}x = \varepsilon^N |E_1| \sum_{z \in \boldsymbol{Z}^N_\varepsilon} \left| \fint_{\varepsilon(\Gamma+z)} S_\varepsilon \,\mathrm{d}\sigma \right|^2. \tag{4-6}$$

On the other hand, by Hölder's inequality, we estimate

$$\sum_{z \in \boldsymbol{Z}^N_\varepsilon} \left| \fint_{\varepsilon(\Gamma+z)} S_\varepsilon \,\mathrm{d}\sigma \right|^2 \leq \frac{\gamma}{\varepsilon^{N-1}} \int_{\Gamma^\varepsilon} S^2_\varepsilon \,\mathrm{d}\sigma\,. \tag{4-7}$$

Hence, as a consequence of (2-10), it follows that

$$\|\overline{u}_\varepsilon(\cdot\,, t)\|_{L^2(\Omega)} \leq C\varepsilon\,, \tag{4-8}$$

where C is a constant independent of ε.

In order to obtain an estimate for $\tilde{u}_\varepsilon$, we introduce the space

$$\widetilde{H}^1_\varepsilon(\Omega) := \{v \in H^1_\varepsilon(\Omega) \ : [v] \in \widetilde{H}^{1/2}(\Gamma^\varepsilon)\}\,, \tag{4-9}$$

and, using Lemma 4.2 and Remark 4.3 below, we compute, for every $v \in \widetilde{H}^1_\varepsilon(\Omega)$,

$$\int_\Omega \sigma|\nabla v|^2 \, \mathrm{d}x \geq \sum_{z\in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(Y+z)} \sigma|\nabla v|^2 \, \mathrm{d}x \geq \frac{\alpha\widetilde{\lambda}}{\varepsilon} \sum_{z\in \boldsymbol{Z}^N_\varepsilon} \int_{\varepsilon(\Gamma+z)} [v]^2 \, \mathrm{d}\sigma = \frac{\alpha\widetilde{\lambda}}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \, \mathrm{d}\sigma \,, \tag{4-10}$$

where $\widetilde{\lambda}$ is defined in (4-15) and is independent of ε. Hence,

$$\int_\Omega \sigma|\nabla v|^2 \, \mathrm{d}x \geq \lambda \left(\int_\Omega \frac{\kappa}{2}|\nabla v|^2 \, \mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \, \mathrm{d}\sigma \right), \tag{4-11}$$

for $\lambda = (\max\{\kappa_1/(2\sigma_1), \kappa_2/(2\sigma_2)\} + 1/(2\widetilde{\lambda}))^{-1}$.

On the other hand, reasoning as in Section 2 and using (4-4) and (2-10), we get that $\tilde{u}_\varepsilon$ satisfies the energy estimate

$$\int_\Omega \frac{\kappa}{2}|\nabla \tilde{u}_\varepsilon(x,t)|^2 \, \mathrm{d}x + \int_0^t \int_\Omega \sigma|\nabla \tilde{u}_\varepsilon(x,\tau)|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\tilde{u}_\varepsilon(x,t)]^2 \, \mathrm{d}\sigma < \gamma \,. \tag{4-12}$$

Hence, by using (4-11) written for $\tilde{u}_\varepsilon(\cdot\,,t)$ and the differential version of Gronwall's Lemma, we obtain

$$\int_\Omega \frac{\kappa}{2}|\nabla \tilde{u}_\varepsilon(\cdot\,,t)|^2 \, \mathrm{d}x + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\tilde{u}_\varepsilon(\cdot\,,t)]^2 \, \mathrm{d}\sigma \leq \gamma \, \mathrm{e}^{-\lambda t} \,, \quad \text{a.e. in } (0,+\infty), \tag{4-13}$$

and (2-11) follows from Poincaré's inequality (Lemma 4.1) and (4-8).

Lemma 4.1 (Poincaré's inequality [Hummel 2000; Amar et al. 2004b]). *Let v belong to the space $H^1_\varepsilon(\Omega)$ introduced in Equation (4-1). Then,*

$$\int_\Omega v^2 \, \mathrm{d}x \leq C \left\{ \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \varepsilon^{-1} \int_{\Gamma^\varepsilon} [v]^2 \, \mathrm{d}\sigma \right\}. \tag{4-14}$$

Here C depends only on Ω and E.

Lemma 4.2 [Amar et al. 2008]. *Set $\widetilde{H}^1(Y) := \{v \in L^2(Y) \,:\, v_{|E_i} \in H^1(E_i),\ i = 1,2,\ [v] \in \widetilde{H}^{1/2}(\Gamma)\}$, where $\widetilde{H}^{1/2}(\Gamma)$ is comprised of the functions of $H^{1/2}(\Gamma)$ with null integral average. Then,*

$$\tilde{\lambda} := \min_{v\in \widetilde{H}^1(Y),\ [v]\not\equiv 0} \frac{\displaystyle\int_Y \sigma|\nabla v|^2 \, \mathrm{d}y}{\alpha \displaystyle\int_\Gamma [v]^2 \, \mathrm{d}\sigma} > 0 \,. \tag{4-15}$$

Remark 4.3 [Amar et al. 2008]. The change of variables $y = x/\varepsilon$ applied to Equation (4-15) yields

$$\min_{\substack{v\in \widetilde{H}^1(\varepsilon Y)\\ [v]\not\equiv 0}} \frac{\displaystyle\int_{\varepsilon Y} \sigma|\nabla v|^2 \, \mathrm{d}x}{\displaystyle\frac{\alpha}{\varepsilon} \int_{\varepsilon\Gamma} [v]^2 \, \mathrm{d}\sigma} = \tilde{\lambda} > 0 \,, \tag{4-16}$$

where $\widetilde{H}^1(\varepsilon Y) := \{v \in L^2(\varepsilon Y) \,:\, v_{|\varepsilon E_i} \in H^1(\varepsilon E_i),\ i = 1,2,\ [v] \in \widetilde{H}^{1/2}(\varepsilon\Gamma)\}$, $\widetilde{H}^{1/2}(\varepsilon\Gamma)$ is comprised of the functions of $H^{1/2}(\varepsilon\Gamma)$ with null integral average, and $\tilde{\lambda}$ is the positive constant introduced in Lemma 4.2.

5. A counterexample

As pointed out in the Introduction, the structure of (3-39) is not enough to imply that the solution exponentially decays to zero, nor does it imply the solution's boundedness, even if an exponentially decaying memory kernel and source are considered. Indeed, let $\Omega = (-1, 1)$, $\mu > 0$, $a > 0$, $b \in \mathbb{R}$, and $f(x), h(x)$ be smooth functions. Consider the problem

$$\begin{cases} -\left(u_{0xt} + a u_{0x} + b \displaystyle\int_0^t \mathrm{e}^{-\mu(t-\tau)} u_{0x}(x,\tau)\,\mathrm{d}\tau + f(x)\mathrm{e}^{-\mu t} \right)_x = 0\,, \\ u_0(\pm 1, 0) = 0\,, \\ u_{0x}(x,0) = h(x)\,. \end{cases} \tag{5-1}$$

Multiplying the previous equation by $\mathrm{e}^{\mu t}$, we obtain

$$u_{0xxt}\mathrm{e}^{\mu t} + a u_{0xx}\mathrm{e}^{\mu t} + b \int_0^t \mathrm{e}^{\mu\tau} u_{0xx}(x,\tau)\,\mathrm{d}\tau = f'(x)\,. \tag{5-2}$$

Setting $v(x,t) = u_{0xx}\mathrm{e}^{\mu t}$ and differentiating with respect to t, Equation (5-2) can be rewritten as

$$v_{tt} + (a-\mu)v_t + bv = 0\,,$$

which must be complemented with the initial conditions

$$\begin{cases} v(x,0) = h'(x)\,, \\ v_t(x,0) = f'(x) + (\mu - a)h'(x)\,. \end{cases}$$

This last equation has an explicit solution (if $(\mu-a)^2 - 4b > 0$) of the form,

$$v(x,t) = C_1(x)\exp\left(\frac{\mu - a + \sqrt{(\mu-a)^2 - 4b}}{2}t\right) + C_2(x)\exp\left(\frac{\mu - a - \sqrt{(\mu-a)^2 - 4b}}{2}t\right),$$

where $C_1(x)$ and $C_2(x)$ are easily determined by using the initial conditions, thus implying that

$$u_{0xx}(x,t) = C_1(x)\exp\left(\frac{-\mu - a + \sqrt{(\mu-a)^2 - 4b}}{2}t\right) + C_2(x)\exp\left(\frac{-\mu - a - \sqrt{(\mu-a)^2 - 4b}}{2}t\right).$$

Hence, u_0 can be obtained by integrating twice with respect to x and using the previous mentioned boundary conditions.

Note that in general, if b is negative and $-b > \mu a$, the first exponential tends to infinity as $t \to +\infty$. With the exception of particular choices of the initial data, C_1 is different from zero, and hence solutions to Problem (5-1) do not, in general, decay exponentially in time.

References

[Amar et al. 2003] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "Homogenization limit for electrical conduction in biological tissues in the radio-frequency range", *Comptes Rendus Mécanique* **331**:7 (2003), 503–508.

[Amar et al. 2004a] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "An elliptic equation with history", *C. R. Acad. Sci. Paris Sér. I Math.* **338**:8 (2004), 595–598.

[Amar et al. 2004b] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "Evolution and memory effects in the homogenization limit for electrical conduction in biological tissues", *Math. Models Methods Appl. Sci.* **14**:9 (2004), 1261–1295.

[Amar et al. 2005] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics", *Nonlinear Anal. Real World Appl.* **6**:2 (2005), 367–380.

[Amar et al. 2006] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "On a hierarchy of models for electrical conduction in biological tissues", *Math. Methods Appl. Sci.* **29**:7 (2006), 767–787.

[Amar et al. 2008] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "Exponential asymptotic stability for an elliptic equation with memory arising in electrical conduction", Preprint, Università di Roma "La Sapienza", 2008.

[Amar et al. 2009] M. Amar, D. Andreucci, P. Bisegna, and R. Gianni, "Homogenization limit for electrical conduction in biological tissues in the high radiofrequency range", Preprint, Università di Roma "La Sapienza", 2009.

[Appleby et al. 2006] J. A. D. Appleby, M. Fabrizio, B. Lazzari, and D. W. Reynolds, "On exponential asymptotic stability in linear viscoelasticity", *Math. Models Methods Appl. Sci.* **16**:10 (2006), 1677–1694.

[Attouch 1984] H. Attouch, *Variational convergence for functionals and operators*, Pitman, London, 1984.

[Bakhvalov and Panasenko 1989] N. Bakhvalov and G. Panasenko, *Homogenization: averaging processes in periodic media: mathematical problems in the mechanics of composite materials*, Mathematics and its Applications **36**, Kluwer, Dordrecht, 1989.

[Bensoussan et al. 1978] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North Holland, Amsterdam, 1978.

[Bisegna et al. 2001] P. Bisegna, G. Caruso, and F. Lebon, "Bioelectrical impedance analysis: a matter of homogenization of composites with imperfect interfaces", in *Proceedings of the 15th AIMETA Congress of Theoretical and Applied Mechanics* (Taormina), edited by G. Augusti et al., 2001.

[Braides and Defranceschi 1998] A. Braides and A. Defranceschi, *Homogenization of multiple integrals*, Oxford University Press, New York, 1998.

[Bronzino 1999] J. D. Bronzino, *The biomedical engineering handbook*, 2nd ed., CRC Press, Boca Raton, FL, 1999.

[Cioranescu and Donato 1999] D. Cioranescu and P. Donato, *An introduction to homogenization*, Oxford University Press, New York, 1999.

[De Lorenzo et al. 1997] A. De Lorenzo, A. Andreoli, J. Matthie, and P. Withers, "Predicting body cell mass with bioimpedence by using theoretical methods: a technological review", *J. Appl. Physiol.* **82**:5 (1997), 1542–1558.

[Donato et al. 2007] P. Donato, L. Faella, and S. Monsurrò, "Homogenization of the wave equation in composites with imperfect interface: a memory effect", *J. Math. Pures Appl.* **87**:2 (2007), 119–143.

[Fabrizio and Morro 1988] M. Fabrizio and A. Morro, "Viscoelastic relaxation functions compatible with thermodynamics", *J. Elasticity* **19**:1 (1988), 63–75.

[Fichera 1979] G. Fichera, "Avere una memoria tenace crea gravi problemi", *Arch. Ration. Mech. An.* **70**:2 (1979), 101–112.

[Foster and Schwan 1989] K. R. Foster and H. P. Schwan, "Dielectric properties of tissues and biological materials: a critical review", *Crit. Rev. Biomed. Eng.* **17** (1989), 25–104.

[Giorgi et al. 2001] C. Giorgi, M. G. Naso, and V. Pata, "Exponential stability in linear heat conduction with memory: a semigroup approach", *Commun. Appl. Anal.* **5** (2001), 121–133.

[Giorgi et al. 2005] C. Giorgi, M. G. Naso, and V. Pata, "Energy decay of electromagnetic systems with memory", *Math. Models Methods Appl. Sci.* **15**:10 (2005), 1489–1502.

[Hummel 2000] H.-K. Hummel, "Homogenization for heat transfer in polycrystals with interfacial resistances", *Appl. Anal.* **75**:3–4 (2000), 403–424.

[Jikov et al. 1994] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer, Berlin, 1994.

[Lazzari and Nibbi 2002] B. Lazzari and R. Nibbi, "Sufficient conditions for the exponential stability in linear conductors with memory", *Int. J. Eng. Sci.* **40**:10 (2002), 1131–1149.

[Lazzari and Vuk 1992] B. Lazzari and E. Vuk, "Constitutive equations and quasi-static problem in linear thermoviscoelaticity", *Int. J. Eng. Sci.* **30**:4 (1992), 533–544.

[Lene and Leguillon 1981] F. Lene and D. Leguillon, "Étude de l'influence d'un glissement entre les constituants d'un matériau composite sur ses coefficients de comportement effectifs", *J. Méc.* **20** (1981), 509–536.

[Lions 1981] J. L. Lions, *Some methods in the mathematical analysis of systems and their controls*, Science Press, Beijing, 1981.

[Lipton 1998] R. Lipton, "Heat conduction in fine scale mixtures with interfacial contact resistance", *SIAM J. Appl. Math.* **58**:1 (1998), 55–72.

[Medjden and Tatar 2005] M. Medjden and N.-E. Tatar, "Asymptotic behavior for a viscoelastic problem with not necessarily decreasing kernel", *Appl. Math. Comput.* **167**:2 (2005), 1221–1235.

[Novožilov and Yappa 1978] Y. J. Novožilov and Y. A. Yappa, Электродинамика, Nauka, Moscow, 1978. Translated as *Electrodynamics*, Mir, Moscow, 1981.

[Oleinik et al. 1992] O. A. Oleinik, A. S. Shamaev, and G. A. Yosifian, *Mathematical problems in elasticity and homogenization*, North-Holland, Amsterdam, 1992.

[Sánchez-Hubert and Sánchez-Palencia 1992] J. Sánchez-Hubert and E. Sánchez-Palencia, *Introduction aux méthodes asymptotiques et à l'homogénéisation*, Masson, Paris, 1992.

[Sánchez-Palencia 1980] E. Sánchez-Palencia, *Non-homogeneous media and vibration theory*, Lecture Notes in Physics **127**, Springer, Berlin, 1980.

[Slemrod 1981] M. Slemrod, "Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional non-linear thermoelasticity", *Arch. Ration. Mech. An.* **76**:2 (1981), 97–133.

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