STABILITY AND MEMORY EFFECTS IN A HOMOGENIZED MODEL GOVERNING THE ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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We present a macroscopic model of electrical conduction in biological tissues. This model is derived via a homogenization limit by a microscopic formulation based on Maxwell’s equations, taking into account the periodic geometry of the microstructure. We also study the asymptotic behavior of the model for large times. Our results imply that periodic boundary data lead to an asymptotically periodic solution. The model is relevant to applications like electric impedance tomography.

1. Introduction

In this paper we deal with a model of electrical conduction in composite media and, specifically, conduction in biological tissues. The classical governing equation is

\[- \text{div}(\kappa \nabla u_t + \sigma \nabla u) = 0, \quad (1-1)\]

which is derived from the Maxwell equations in the quasistationary approximation (see for example, [Novožilov and Yappa 1978]). Here, \(u\) is the electrical potential and \(\kappa, \sigma\) are the permittivity and the conductivity of the material, respectively. The geometry of the composite media we have in mind is a periodic array of the unit cell depicted in Figure 1. More precisely, we look at a phase \(E_1^n\) which models the cell cytosol, coated by a shell \(\Gamma^n\) which models the cell membrane, included in a phase \(E_2^n\) which models the extracellular fluid [Foster and Schwan 1989]. In particular, the permittivity \(\kappa\) in \(E_1^n\) and \(E_2^n\) is lower, and the conductivity \(\sigma\) is higher, than in \(\Gamma^n\). The diameter of the cell is of the order of tens of micrometers, while the width of the membrane is of the order of ten nanometers. This suggests that the thin shell \(\Gamma^n\) could be preferably modeled as a two dimensional interface \(\Gamma\), in order to get a simpler model and, possibly, a better understanding of the effect of the geometric features of the microscopic structure. This simpler model can be obtained from Equation (1-1) via a concentration-of-capacity procedure [Amar et al. 2006], leading to Problem (2-1)–(2-6), below. In particular, Equation (2-3) takes into account the conductive/capacitive behavior of the concentrated membrane. As shown in (2-3), the electric potential jumps across the interface \(\Gamma\), and its jump satisfies a dynamical condition (roughly speaking, in the form of a hyperbolic differential equation on the interface itself).

Our model is designed to investigate the response of biological tissues to the injection of electrical currents in the radio frequency range, that is, the Maxwell–Wagner interfacial polarization effect [Foster and Schwan 1989; Bisegna et al. 2001], at higher frequencies than those considered in [Amar et al. 2003; 2004b; 2005; 2006; 2008]. This effect is relevant to clinical applications like electric impedance tomography and body composition [De Lorenzo et al. 1997; Bronzino 1999].

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Figure 1. The periodic cell $Y$. Left: before concentration; $\Gamma^\eta$ is the dark gray region, and $E^\eta = E_1^\eta \cup E_2^\eta$ is the union of the light gray and white regions. Right: after concentration; $\Gamma^\eta$ shrinks to $\Gamma$ as $\eta \to 0$.

Problem (2-1)–(2-6) contains a small parameter $\varepsilon$, coinciding with the period of the microstructure. The typical structure of the periodic array we have in mind is given in Figure 2. Some applications deal with measurements of the electric potential at the macroscopic (body) scale: this suggests that it would be advantageous to investigate the homogenization limit of Problem (2-1)–(2-6) when we let $\varepsilon \to 0$. Extensive surveys on this topic are, for example, in [Bensoussan et al. 1978; Sánchez-Palencia

Figure 2. Left: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\mathbb{R}^2$. Here $E_1$ is the light gray region and $\Gamma$ is its boundary. The remaining part of $Y$ (the white region) is $E_2$. Right: the corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here $\Omega_1^\varepsilon$ is the light gray region and $\Gamma^\varepsilon$ is its boundary. The remaining part of $\hat{\Omega}$ (the white region) is $\Omega_2^\varepsilon$. 
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It turns out that the partial differential equation obtained in the limit is nonstandard; see (3-39) below. Indeed, it is an equation exhibiting memory effects, that is, it contains explicitly the history of the unknown and hence is markedly different from the Laplace equation presently used as a standard in the bioelectrical impedance literature [Bronzino 1999].

Our model can be compared to some papers where homogenization theory is applied to linear stationary elliptic problems involving imperfect interfaces arising in fields like elasticity [Lene and Leguillon 1981] or heat conduction [Lipton 1998]. See also [Donato et al. 2007; Sánchez-Palencia 1980], where hyperbolic problems with interfaces are considered in the framework of elastodynamics and electrodynamics.

In view of the applications, it is also of interest to study the time evolution of the homogenized potential (see Section 2). In particular, it is of interest to show that time-harmonic boundary data elicits a time-harmonic solution for large times. In this regard, following the same reasoning as that presented in [Amar et al. 2008], it is enough to prove that the solution \( u_0 \) of (3-39) exponentially decays to zero as time increases, provided that a zero Dirichlet boundary condition is assigned (see Theorem 2.1 and Corollary 2.2).

From a mathematical point of view, the asymptotic behavior of evolutive equations with memory is a classical problem [Fichera 1979; Slemrod 1981; Fabrizio and Morro 1988; Lazzari and Vuk 1992], currently drawing much interest in the literature [Lazzari and Nibbi 2002; Giorgi et al. 2001; 2005; Medjden and Tatar 2005; Appleby et al. 2006]. We note that the exponential decay of the memory kernel, in general, does not imply the existence of bounded solutions, as shown by a counterexample presented in Section 5 (see also, [Fichera 1979; Fabrizio and Morro 1988]).

We finally note that our methods could be easily applied to study the homogenization problem and the time-asymptotic behavior of Kelvin–Voigt viscoelastic composites with coated inclusions.

2. Position of the problem and main results

We look at the homogenization limit \( (\varepsilon \to 0) \) of the following problem for \( u_\varepsilon(x, t) \):

\[
\begin{align*}
- \text{div}(\kappa \nabla u_\varepsilon + \sigma \nabla u_\varepsilon) &= 0 \quad &\text{in} \ (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times (0, +\infty); \\
[(\kappa \nabla u_\varepsilon + \sigma \nabla u_\varepsilon) \cdot n] &= 0 \quad &\text{on} \ \Gamma^\varepsilon \times (0, +\infty); \\
(\alpha/\varepsilon)\partial[u_\varepsilon]/\partial t + (\beta/\varepsilon)[u_\varepsilon] &= ((\kappa \nabla u_\varepsilon + \sigma \nabla u_\varepsilon) \cdot n)^{(2)} \quad &\text{on} \ \Gamma^\varepsilon \times (0, +\infty); \\
u_\varepsilon(x, t) &= 0 \quad &\text{on} \ \partial \Omega \times (0, +\infty); \\
\nabla u_\varepsilon(x, 0) &= G_\varepsilon(x) \quad &\text{in} \ \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \\
u_\varepsilon(x, 0) &= S_\varepsilon(x) \quad &\text{on} \ \Gamma^\varepsilon.
\end{align*}
\]

The operators \( \text{div} \) and \( \nabla \) act with respect to the space variable \( x \); \( \Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon \), where \( \Omega_1^\varepsilon \) and \( \Omega_2^\varepsilon \) are two disjoint open subsets of \( \Omega \), and \( \Gamma^\varepsilon = \partial \Omega_1^\varepsilon \cap \Omega = \partial \Omega_2^\varepsilon \cap \Omega \); \( n \) is the normal unit vector pointing into \( \Omega_2^\varepsilon \); the typical geometry we have in mind is depicted in Figure 2. We refer to Section 2 for a precise definition of the structure of \( \Omega_1^\varepsilon, \Omega_2^\varepsilon, \Gamma^\varepsilon \).
Moreover, we assume that

\[ \alpha > 0; \quad \beta \geq 0; \quad \kappa = \kappa_1 > 0, \quad \sigma = \sigma_1 > 0 \quad \text{in } \Omega_1^\varepsilon; \]
\[ \kappa = \kappa_2 > 0, \quad \sigma = \sigma_2 > 0 \quad \text{in } \Omega_2^\varepsilon, \]

(2-7)

where \( \kappa_1, \kappa_2, \sigma_1, \sigma_2, \alpha, \) and \( \beta \) are constants. From a physical point of view, \( \Gamma^\varepsilon \) represents the cell membranes having capacitance \( \alpha / \varepsilon \) and conductance \( \beta / \varepsilon \) per unit area, whereas \( \Omega_1^\varepsilon \) (respectively, \( \Omega_2^\varepsilon \)) is the intracellular (respectively, extracellular) space, having permittivity \( \kappa_1 \) (respectively, \( \kappa_2 \)) and conductivity \( \sigma_1 \) (respectively, \( \sigma_2 \)).

Since \( u_\varepsilon \) is not, in general, continuous across \( \Gamma^\varepsilon \), we have set

\[ u_\varepsilon^{(2)} := \text{trace of } u_\varepsilon|_{\Omega_2^\varepsilon} \text{ on } \Gamma^\varepsilon, \quad u_\varepsilon^{(1)} := \text{trace of } u_\varepsilon|_{\Omega_1^\varepsilon} \text{ on } \Gamma^\varepsilon, \quad \text{and} \quad [u_\varepsilon] := u_\varepsilon^{(2)} - u_\varepsilon^{(1)}. \]

A similar convention is employed for the current flux density across the membrane \((\kappa \nabla u_\varepsilon + \sigma \nabla u_\varepsilon) \cdot v\).

We assume that the restrictions of \( G_\varepsilon \) to \( \Omega_1^\varepsilon \) and \( \Omega_2^\varepsilon \) are gradients of scalar fields, and that \( G_\varepsilon \) strongly converges in \( L^2 \). Moreover, we assume that \( S_\varepsilon \in H^1(\Omega) \), and that \( S_\varepsilon / \varepsilon \) strongly converges in \( L^2 \). These assumptions are introduced in order to rule out the appearance of an initial layer (see [Amar et al. 2009]). Further assumptions on \( G_\varepsilon \) and \( S_\varepsilon \) are introduced in the next paragraph.

**Geometry.** Following [Amar et al. 2004b], we introduce a periodic open subset \( E \) of \( \mathbb{R}^N \), so that \( E + z = E \) for all \( z \in \mathbb{Z}^N \). For all \( \varepsilon > 0 \) we define \( \Omega_1^\varepsilon = \Omega \cap \varepsilon E, \Omega_2^\varepsilon = \Omega \setminus \varepsilon \bar{E}, \Gamma^\varepsilon = \partial \Omega \setminus \partial \varepsilon E \). We assume that \( \Omega, E \) have a regular boundary, say of class \( C^\infty \) for the sake of simplicity. We also employ the notation \( Y = (0, 1)^N, E_1 = E \cap Y, E_2 = Y \setminus \bar{E}, \Gamma = \partial E \cap \bar{Y}. \) We stipulate that \( E_1 \) is a connected smooth subset of \( Y \) such that \( \text{dist}(E_1, \partial Y) > 0 \). Some generalizations may be possible, but we do not dwell on this point here. Finally, we assume that \( \text{dist}(\Gamma^\varepsilon, \partial \Omega) > \gamma \varepsilon \) for some constant \( \gamma > 0 \) independent of \( \varepsilon \), by dropping the inclusions contained in the cells \( \varepsilon (Y + z), z \in \mathbb{Z}^N \) which intersect \( \partial \Omega \) (see Figure 2).

For later usage, we introduce the set

\[ \mathbb{Z}_\varepsilon^N := \{ z \in \mathbb{Z}^N : \varepsilon (Y + z) \subseteq \Omega \}. \]

(2-8)

**Energy estimate.** Multiply (2-1) by \( u_\varepsilon \) and integrate by parts. Using (2-2)–(2-6), we arrive, for all \( t > 0, \) to the energy estimate

\[
\int_{\Omega} \frac{\kappa}{2} |\nabla u_\varepsilon(x, t)|^2 \, dx + \int_0^t \int_{\Omega} \sigma |\nabla u_\varepsilon(x, \tau)|^2 \, dx \, d\tau + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(x, \tau)]^2 \, d\sigma \\
+ \frac{\beta}{\varepsilon} \int_0^t \int_{\Gamma^\varepsilon} [u_\varepsilon(x, \tau)]^2 \, d\sigma \, d\tau = \int_{\Omega} \frac{\kappa}{2} |G_\varepsilon(x)|^2 \, dx + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, d\sigma. \tag{2-9}
\]

We assume that

\[
\int_{\Omega} \frac{\kappa}{2} |G_\varepsilon(x)|^2 \, dx + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, d\sigma < \gamma, \tag{2-10}
\]

for a constant \( \gamma \) independent of \( \varepsilon \). In fact (2-9), coupled with the Poincaré’s inequality (Lemma 4.1), is a main tool in the rigorous proof of convergence of \( u_\varepsilon \) to its limit. In particular, up to a subsequence, \( u_\varepsilon \) converges weakly in \( L^2(\Omega \times (0, T)) \) as \( \varepsilon \to 0 \) to a limit \( u_0 \), for every \( T > 0 \). The equation satisfied by \( u_0 \) will be formally derived via a homogenization procedure in Section 3.
Exponential decay.

**Theorem 2.1.** Let $\Omega^e_1, \Omega^e_2, \Gamma^e$ be as before. Assume that (2-7) holds, and that the initial data $G_e$ are gradients of scalar fields and together with $S_e$ satisfy (2-10). Let $u_e$ be the solution of (2-1)–(2-6). Then

$$
\|u_e(\cdot, t)\|_{L^2(\Omega)} \leq C(\varepsilon + e^{-\lambda t}) \quad \text{a.e. in } (0, +\infty),
$$

(2-11)

where $C$ and $\lambda$ are positive constants independent of $\varepsilon$. Moreover, if $\beta > 0$, or else if $S_e$ has null mean average over each connected component of $\Gamma^e$, it follows that

$$
\|u_e(\cdot, t)\|_{L^2(\Omega)} \leq Ce^{-\lambda t} \quad \text{a.e. in } (0, +\infty).
$$

(2-12)

This result easily yields the following exponential time-decay estimate for the limit $u_0$ under homogeneous Dirichlet boundary data:

**Corollary 2.2.** Under the assumptions of Theorem 2.1, if $u_e \to u_0$ weakly in $L^2(\Omega \times (0, T))$ for every $T > 0$, then

$$
\|u_0(\cdot, t)\|_{L^2(\Omega)} \leq C e^{-\lambda t} \quad \text{a.e. in } (0, +\infty).
$$

(2-13)

3. Formal homogenization

To establish the notation, we summarize here some well known asymptotic expansions needed in the two-scale method (see, for example, [Bensoussan et al. 1978], [Sánchez-Palencia 1980]). Introduce the microscopic variables $y \in Y, y = x/\varepsilon$, assuming

$$
u_e = u_e(x, y, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \ldots.
$$

(3-1)

Note that $u_0, u_1, u_2$ are periodic in $y$, and $u_1, u_2$ are assumed to have zero integral average over $Y$. Recalling that

$$
\text{div} = \frac{1}{\varepsilon} \text{div}_y + \text{div}_x, \quad \nabla = \frac{1}{\varepsilon} \nabla_y + \nabla_x
$$

(3-2)

we compute, for example,

$$
\nabla u_e = \frac{1}{\varepsilon} \nabla_y u_0 + (\nabla_x u_0 + \nabla_y u_1) + \varepsilon (\nabla_y u_2 + \nabla_x u_1) + \ldots.
$$

(3-3)

We also stipulate

$$
G_e = G_0(x, y) = G_0(x, y) + \varepsilon G_1(x, y) + \varepsilon^2 G_2(x, y) + \ldots,
$$

(3-4)

$$
S_e = S_0(x, y) = S_0(x, y) + \varepsilon S_1(x, y) + \varepsilon^2 S_2(x, y) + \ldots,
$$

(3-5)

where the restrictions of $G_0(x, \cdot), G_1(x, \cdot), \ldots$ to $E_1$ and $E_2$ are the gradients of scalar fields. According to Equation (2-10), recalling that $|\Gamma^e|_{N-1} \sim 1/\varepsilon$, we assume $S_0 \equiv 0$ in (3-5). Moreover, according to the assumption on the strong convergence of $G_e$ and $S_e/\varepsilon$, the functions $G_0(x, y)$ and $S_1(x, y)$ do not depend on $y$, that is $G_0(x, y) = G_0(x)$ and $S_1(x, y) = S_1(x)$.

For the sake of brevity, we introduce the operator

$$
\mathcal{D} := \kappa \frac{\partial}{\partial t} + \sigma.
$$

(3-6)
Applying (3-2)–(3-3) to Problem (2-1)–(2-6), one readily sees, by matching corresponding powers of \( \varepsilon \), that \( u_0 \) solves,

\[
-\Delta_y u_0 = 0 \quad \text{in} \ (E_1 \cup E_2) \times (0, +\infty); \tag{3-7}
\]

\[
[\Delta_y u_0 \cdot v] = 0 \quad \text{on} \ \Gamma \times (0, +\infty); \tag{3-8}
\]

\[
a \frac{\partial[u_0]}{\partial t} + \beta[u_0] = (\Delta_y u_0 \cdot v)^{(2)} \quad \text{on} \ \Gamma \times (0, +\infty); \tag{3-9}
\]

\[
\nabla_y u_0|_{t=0} = 0 \quad \text{on} \ E_1 \cup E_2; \tag{3-10}
\]

\[
[u_0]|_{t=0} = 0 \quad \text{on} \ \Gamma. \tag{3-11}
\]

Reasoning as in Section 2, we obtain an energy estimate for (3-7)–(3-11), which implies that \([u_0] = 0\) for all times, and

\[
u_0 = u_0(x, t).\]

Next, we find for \( u_1 \) that

\[
-\Delta_y u_1 = 0 \quad \text{in} \ (E_1 \cup E_2) \times (0, +\infty); \tag{3-12}
\]

\[
[\Delta_y (\nabla_y u_1 + \nabla_x u_0) \cdot v] = 0 \quad \text{on} \ \Gamma \times (0, +\infty); \tag{3-13}
\]

\[
a \frac{\partial[u_1]}{\partial t} + \beta[u_1] = (\Delta_y (\nabla_y u_1 + \nabla_x u_0) \cdot v)^{(2)} \quad \text{on} \ \Gamma \times (0, +\infty); \tag{3-14}
\]

\[
\nabla_y u_1|_{t=0} + \nabla_x u_0|_{t=0} = G_0 \quad \text{on} \ E_1 \cup E_2; \tag{3-15}
\]

\[
[u_1]|_{t=0} = S_1 \quad \text{on} \ \Gamma. \tag{3-16}
\]

Since both \( u_0 \) and \( G_0 \) do not depend on \( y \), Equation (3-15) implies \( \nabla_y u_1|_{t=0} = 0 \) on \( E_1 \cup E_2 \).

In order to represent \( u_1 \) in a suitable way, let \( g \in L^2(E_1 \cup E_2) \) and \( s \in L^2(\Gamma) \) be assigned such that the restrictions of \( g \) to \( E_1 \) and \( E_2 \) are gradients of scalar fields, and consider the problem

\[
-\Delta_y v = 0 \quad \text{in} \ (E_1 \cup E_2) \times (0, +\infty); \tag{3-17}
\]

\[
[\Delta_y v \cdot v] = 0 \quad \text{on} \ \Gamma \times (0, +\infty); \tag{3-18}
\]

\[
a \frac{\partial[v]}{\partial t} + \beta[v] = (\Delta_y v \cdot v)^{(2)} \quad \text{on} \ \Gamma \times (0, +\infty). \tag{3-19}
\]

\[
\nabla_y v|_{t=0} = g \quad \text{on} \ E_1 \cup E_2; \tag{3-20}
\]

\[
[v]|_{t=0} = s \quad \text{on} \ \Gamma. \tag{3-21}
\]

where \( v \) is a periodic function in \( Y \), such that \( \int_Y v(y, t) \, dy = 0 \). Define the transform \( \mathcal{T} \) by

\[
\mathcal{T}(g, s)(y, t) = v(y, t), \quad y \in Y, t > 0.
\]

Then, introduce the cell functions \( \chi^0 : Y \to \mathbb{R}^N \) and \( \chi^1 : Y \times (0, +\infty) \to \mathbb{R}^N \), whose components \( \chi^0_h \) and \( \chi^1_h (\cdot, t), h = 1, \ldots, N \), are required to be periodic functions with vanishing integral averages over
Finally, \( \chi_h^1 \) is defined for \( t > 0 \) by
\[
\chi_h^1 = T \left( \nabla_y \chi_h^1(\cdot, 0), [\chi_h^1(\cdot, 0)] \right).
\]

Straightforward calculations show that \( u_1 \) may be written in the form
\[
u_1(x, y, t) = -\chi^0(y) \cdot \nabla_x u_0(x, t) \int_0^t \chi^1(y, t - \tau) \cdot \nabla_x u_0(x, \tau) \, d\tau
+ \nabla_y (\chi^0 \cdot \mathbf{G}_0(x)), S_1(x) + [\chi^0] \cdot \mathbf{G}_0(x))(y, t). \tag{3-29}
\]
so that
\[
\begin{align*}
\mathbb{D} u_1(x, y, t) &= -\kappa \chi^0(y) \cdot \nabla_x u_0(x, t) - (\kappa \chi^1(y, 0) + \sigma \chi^0(y)) \cdot \nabla_x u_0(x, t) \\
&\quad - \int_0^t (\mathbb{D} \chi^1)(y, t - \tau) \cdot \nabla_x u_0(x, \tau) \, d\tau \\
&\quad + \mathbb{D} \nabla_y (\chi^0 \cdot \mathbf{G}_0(x)), S_1(x) + [\chi^0] \cdot \mathbf{G}_0(x))(y, t). \tag{3-30}
\end{align*}
\]

Next we find for \( u_2 \) that
\[
-\mathbb{D} \left( \Delta_y u_2 + 2 \frac{\partial^2 u_2}{\partial x_j \partial y_j} + \Delta_x u_0 \right) = 0, \quad \text{in } (E_1 \cup E_2) \times (0, +\infty); \tag{3-31}
\]
\[
\mathbb{D} \nabla_y u_2 + \nabla_x u_1) \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty); \tag{3-32}
\]
\[
(\mathbb{D} (\nabla_y u_2 + \nabla_x u_1) \cdot \nu)^{(2)} = \alpha \frac{\partial [u_2]}{\partial t} + \beta [u_2] \quad \text{on } \Gamma \times (0, +\infty). \tag{3-33}
\]
\[
\nabla_y u_2|_{t=0} + \nabla_x u_1|_{t=0} = \mathbf{G}_1 \quad \text{on } E_1 \cup E_2; \tag{3-34}
\]
\[
[u_2]|_{t=0} = S_2 \quad \text{on } \Gamma. \tag{3-35}
\]

Let us find the solvability conditions for this problem. Integrating by parts the partial differential equations (3-31) solved by \( u_2 \), both in \( E_1 \) and in \( E_2 \), adding the two contributions, and using (3-32), we get
Thus, we obtain
\[
\left( \kappa_0 \frac{\partial}{\partial t} + \sigma_0 \right) \Delta_x u_0 = 2 \int_\Gamma [\mathcal{D} \nabla_x u_1 \cdot v] \, d\sigma - \int_\Gamma [\mathcal{D} \nabla_x u_1] \, d\sigma = \int_\Gamma [\mathcal{D} \nabla_x u_1 \cdot v] \, d\sigma ,
\] (3-37)
where
\[
\kappa_0 = \kappa_1 |E_1| + \kappa_2 |E_2| ; \quad \sigma_0 = \sigma_1 |E_1| + \sigma_2 |E_2| .
\] (3-38)

Then, we substitute the representation (3-29) into Equation (3-37) and, after simple algebra, obtain the homogenized equation for \( u_0 \) in \( \Omega \times (0, +\infty) \) as
\[
- \text{div} \left( K \nabla_x u_{0t} + A \nabla_x u_0 + \int_0^t B(t-\tau) \nabla_x u_0 (\cdot, \tau) \, d\tau - \mathcal{F} \right) = 0 ,
\] (3-39)
where the matrices \( K, A, B(t) \), and the vector \( \mathcal{F}(x,t) \) are defined as follows:
\[
K = \kappa_0 I + \int_\Gamma v \otimes [\kappa \chi^0(y)] \, d\sigma ,
\]
(3-40)
\[
A = \sigma_0 I + \int_\Gamma v \otimes [\kappa \chi^1(y, 0) + \sigma \chi^0(y)] \, d\sigma ,
\]
(3-41)
\[
B(t) = \int_\Gamma v \otimes [(\mathcal{D} \chi^1)(y, t)] \, d\sigma ,
\]
(3-42)
\[
\mathcal{F}(x, t) = \int_\Gamma [\mathcal{D} \mathcal{F} (\chi^0 \cdot G_0(x)), S_1(x) + [\chi^0 \cdot G_0(x)](y, t)] v \, d\sigma .
\]
(3-43)

Equation (3-39) is complemented with the initial condition
\[
\nabla_x u_{0|t=0} = G_0 , \quad \text{on} \ \Omega .
\]
(3-44)

Finally, integrating Equation (3-39) over time, changing the order in the double integral that results, and using (3-44), we obtain also the following formulation
\[
- \text{div} \left( K \nabla_x u_0 + \int_0^t \left( A + \int_0^t B(\tau) \, d\tau \right) \nabla_x u_0 (\cdot, s) \, ds - KG_0 - \int_0^t \mathcal{F}(\cdot, \tau) \, d\tau \right) = 0 ,
\] (3-45)
which shows that the homogenized equation has exactly the form of an equation with memory of the type derived in [Amar et al. 2003; 2004b] and studied in [Amar et al. 2004a].

### 4. Time-exponential asymptotic decay: proof of Theorem 2.1

The case \( \beta > 0 \) is quite simple. We introduce the space
\[
H^1_\kappa (\Omega) := \{ v \in L^2(\Omega) : v|_{\Omega^\kappa_i} \in H^1(\Omega^\kappa_i), \ i = 1, 2; \ v = 0 \text{ on } \partial \Omega \} .
\] (4-1)
It turns out that, for all \( v \in H^1_\varepsilon(\Omega) \),
\[
\int_\Omega \sigma \nabla v^2 \, dx + \frac{\beta}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \, d\sigma \geq \lambda \left( \int_{\Omega} \frac{\kappa}{2} |\nabla v|^2 \, dx + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \, d\sigma \right),
\]
(4.2)
for \( \lambda = \min\{2\sigma_1/\kappa_1, 2\sigma_2/\kappa_2, 2\beta/\alpha\} \). Taking \( v = u_\varepsilon(\cdot, t) \) in the previous estimate and using equations (2.9), (2.10), and the differential version of Gronwall’s Lemma, we obtain
\[
\int_{\Omega} \frac{\kappa}{2} |\nabla u_\varepsilon(\cdot, t)|^2 \, dx + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [u_\varepsilon(\cdot, t)]^2 \, d\sigma \leq \gamma e^{-\lambda t}, \quad \text{a.e. in } (0, +\infty),
\]
(4.3)
and (2.12) follows from Poincaré’s inequality (Lemma 4.1).

Now we consider the case \( \beta = 0 \). We introduce the space \( \tilde{H}^{1/2}(\Gamma^\varepsilon) \subseteq H^{1/2}(\Gamma^\varepsilon) \) of the functions which have a null average over each connected component of \( \Gamma^\varepsilon \), that is, on \( \varepsilon(\Gamma^\varepsilon + z) \), for each \( z \) belonging to the set \( Z^N_\varepsilon \) defined in (2.8). We decompose the initial datum \( S_\varepsilon(x) \) in (2.6) as \( S_\varepsilon(x) = \tilde{S}_\varepsilon(x) + \bar{S}_\varepsilon(x) \), where
\[
\tilde{S}_\varepsilon(x) = \int_{\varepsilon(\Gamma^\varepsilon + z)} S_\varepsilon \, d\sigma =: C_\varepsilon z \quad \text{on each } \varepsilon(\Gamma^\varepsilon + z), \quad z \in Z^N_\varepsilon;
\]
(4.4)
and the initial datum \( G_\varepsilon(x) \) in (2.5) as \( G_\varepsilon(x) = \tilde{G}_\varepsilon(x) + \bar{G}_\varepsilon(x) \), where \( \tilde{G}_\varepsilon(x) = 0 \) and \( \bar{G}_\varepsilon(x) = G_\varepsilon(x) \). Accordingly, the solution \( u_\varepsilon \) to Problem (2.1)–(2.6) is decomposed as \( \bar{u}_\varepsilon + \tilde{u}_\varepsilon \). Clearly,
\[
\bar{u}_\varepsilon(x, t) = \begin{cases} 0 & \text{for } (x, t) \in \Omega^\varepsilon_2 \times (0, +\infty), \\ -C_\varepsilon z & \text{for } (x, t) \in (\varepsilon(E_1 + z)) \times (0, +\infty), \quad z \in Z^N_\varepsilon. \end{cases}
\]
(4.5)
Using the previous equation, we compute
\[
\int_\Omega |\bar{u}_\varepsilon|^2 \, dx = \sum_{z \in Z^N_\varepsilon} \int_{\varepsilon(E_1 + z)} |\bar{u}_\varepsilon|^2 \, dx = \varepsilon^N |E_1| \sum_{z \in Z^N_\varepsilon} \left| \int_{\varepsilon(\Gamma^\varepsilon + z)} S_\varepsilon \, d\sigma \right|^2.
\]
(4.6)
On the other hand, by Hölder’s inequality, we estimate
\[
\sum_{z \in Z^N_\varepsilon} \left| \int_{\varepsilon(\Gamma^\varepsilon + z)} S_\varepsilon \, d\sigma \right|^2 \leq \frac{\gamma}{\varepsilon^{N-1}} \int_{\Gamma^\varepsilon} S_\varepsilon^2 \, d\sigma.
\]
(4.7)
Hence, as a consequence of (2.10), it follows that
\[
\|\bar{u}_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \varepsilon,
\]
(4.8)
where \( C \) is a constant independent of \( \varepsilon \).

In order to obtain an estimate for \( \tilde{u}_\varepsilon \), we introduce the space
\[
\tilde{H}^1_\varepsilon(\Omega) := \{ v \in H^1_\varepsilon(\Omega) : [v] \in \tilde{H}^{1/2}(\Gamma^\varepsilon) \},
\]
(4.9)
and, using Lemma 4.2 and Remark 4.3 below, we compute, for every \( v \in \tilde{H}^1_\varepsilon(\Omega) \),
\[
\int_{\Omega} \sigma|\nabla v|^2 \, dx \geq \sum_{z \in \mathbb{Z}^N} \int_{\varepsilon(y+z)} \sigma|\nabla v|^2 \, dx \geq \frac{\alpha \tilde{\lambda}}{\varepsilon} \sum_{z \in \mathbb{Z}^N} \int_{\varepsilon(y+z)} [v]^2 \, d\sigma = \frac{\alpha \tilde{\lambda}}{\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \, d\sigma .
\]
where \( \tilde{\lambda} \) is defined in (4-15) and is independent of \( \varepsilon \). Hence,
\[
\int_{\Omega} \sigma|\nabla v|^2 \, dx \geq \lambda \left( \int_{\Omega} \frac{\kappa}{2} |\nabla v|^2 \, dx + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [v]^2 \, d\sigma \right),
\]
for \( \lambda = (\max\{\kappa_1/(2\sigma_1), \kappa_2/(2\sigma_2)\} + 1/(2\tilde{\lambda}))^{-1} \).

On the other hand, reasoning as in Section 2 and using (4-4) and (2-10), we get that \( \tilde{u}_\varepsilon \) satisfies the energy estimate
\[
\int_{\Omega} \frac{\kappa}{2} |\nabla \tilde{u}_\varepsilon(x,t)|^2 \, dx + \int_0^t \int_{\Omega} \sigma|\nabla \tilde{u}_\varepsilon(x,t)|^2 \, dx \, dt + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\tilde{u}_\varepsilon(x,t)]^2 \, d\sigma < \gamma .
\]
Hence, by using (4-11) written for \( \tilde{u}_\varepsilon(\cdot,t) \) and the differential version of Gronwall’s Lemma, we obtain
\[
\int_{\Omega} \frac{\kappa}{2} |\nabla \tilde{u}_\varepsilon(\cdot,t)|^2 \, dx + \frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [\tilde{u}_\varepsilon(\cdot,t)]^2 \, d\sigma \leq \gamma e^{-\lambda t} , \quad \text{a.e. in } (0, +\infty),
\]
and (2-11) follows from Poincaré’s inequality (Lemma 4.1) and (4-8).

**Lemma 4.1** (Poincaré’s inequality [Hummel 2000; Amar et al. 2004b]). Let \( v \) belong to the space \( H^1_\varepsilon(\Omega) \) introduced in Equation (4-1). Then,
\[
\int_{\Omega} v^2 \, dx \leq C \left\{ \int_{\Omega} |\nabla v|^2 \, dx + \varepsilon^{-1} \int_{\Gamma^\varepsilon} [v]^2 \, d\sigma \right\} .
\]
Here \( C \) depends only on \( \Omega \) and \( E \).

**Lemma 4.2** [Amar et al. 2008]. Set \( \tilde{H}^1(Y) := \{ v \in L^2(\Omega) : v|_{E_i} \in H^1(E_i), i = 1, 2, [v] \in \tilde{H}^{1/2}(\Gamma) \} \), where \( \tilde{H}^{1/2}(\Gamma) \) is comprised of the functions of \( H^{1/2}(\Gamma) \) with null integral average. Then,
\[
\tilde{\lambda} := \min_{v \in \tilde{H}^1(Y), [v] \neq 0} \frac{\int_Y \sigma|\nabla v|^2 \, dy}{\alpha \int_{\Gamma^\varepsilon} [v]^2 \, d\sigma} > 0 .
\]

**Remark 4.3** [Amar et al. 2008]. The change of variables \( y = x/\varepsilon \) applied to Equation (4-15) yields
\[
\min_{v \in \tilde{H}^1(\varepsilon Y), [v] \neq 0} \frac{\int_{\varepsilon Y} \sigma|\nabla v|^2 \, dx}{\alpha \int_{\varepsilon \Gamma^\varepsilon} |v|^2 \, d\sigma} = \tilde{\lambda} > 0 ,
\]
where \( \tilde{H}^1(\varepsilon Y) := \{ v \in L^2(\varepsilon Y) : v|_{\varepsilon E_i} \in H^1(\varepsilon E_i), i = 1, 2, [v] \in \tilde{H}^{1/2}(\varepsilon \Gamma) \} \), \( \tilde{H}^{1/2}(\varepsilon \Gamma) \) is comprised of the functions of \( H^{1/2}(\varepsilon \Gamma) \) with null integral average, and \( \tilde{\lambda} \) is the positive constant introduced in Lemma 4.2.
5. A counterexample

As pointed out in the Introduction, the structure of (3-39) is not enough to imply that the solution exponentially decays to zero, nor does it imply the solution’s boundedness, even if an exponentially decaying memory kernel and source are considered. Indeed, let \( \Omega = (-1, 1) \), \( \mu > 0 \), \( a > 0 \), \( b \in \mathbb{R} \), and \( f(x), h(x) \) be smooth functions. Consider the problem

\[
\begin{aligned}
- \left( u_{0xt} + au_{0x} + b \int_0^t e^{-\mu(t-\tau)} u_{0x}(x, \tau) \, d\tau + f(x) e^{-\mu t} \right) \bigg|_{x} = 0, \\
u_0(\pm 1, 0) = 0, \\
u_{0x}(x, 0) = h(x).
\end{aligned}
\]  

(5-1)

Multiplying the previous equation by \( e^{\mu t} \), we obtain

\[
\begin{aligned}
u_{0xx} e^{\mu t} + au_{0xx} e^{\mu t} + b \int_0^t e^{\mu t} u_{0xx}(x, \tau) \, d\tau = f'(x).
\end{aligned}
\]  

(5-2)

Setting \( v(x, t) = u_{0xx} e^{\mu t} \) and differentiating with respect to \( t \), Equation (5-2) can be rewritten as

\[
v_{tt} + (a - \mu)v_t + bu = 0,
\]

which must be complemented with the initial conditions

\[
\begin{aligned}
u(x, 0) &= h'(x), \\
v_t(x, 0) &= f'(x) + (\mu - a)h'(x).
\end{aligned}
\]

This last equation has an explicit solution (if \( (\mu - a)^2 - 4b > 0 \)) of the form,

\[
v(x, t) = C_1(x) \exp \left( \frac{\mu - a + \sqrt{(\mu - a)^2 - 4b}}{2} t \right) + C_2(x) \exp \left( \frac{\mu - a - \sqrt{(\mu - a)^2 - 4b}}{2} t \right),
\]

where \( C_1(x) \) and \( C_2(x) \) are easily determined by using the initial conditions, thus implying that

\[
u_{0xx}(x, t) = C_1(x) \exp \left( \frac{-\mu - a + \sqrt{(\mu - a)^2 - 4b}}{2} t \right) + C_2(x) \exp \left( \frac{-\mu - a - \sqrt{(\mu - a)^2 - 4b}}{2} t \right).
\]

Hence, \( u_0 \) can be obtained by integrating twice with respect to \( x \) and using the previous mentioned boundary conditions.

Note that in general, if \( b \) is negative and \( -b > \mu a \), the first exponential tends to infinity as \( t \to +\infty \). With the exception of particular choices of the initial data, \( C_1 \) is different from zero, and hence solutions to Problem (5-1) do not, in general, decay exponentially in time.

References


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