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ON THE METHOD OF VIRTUAL POWER IN CONTINUUM MECHANICS

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Dedicated to the memory of Paul Germain

The method of virtual power is generally used to produce balance equations for nontraditional continua such as continua with various types of microstructure. Here I show that the expression of the internal power can be deduced from that of the external power using a general invariance requirement due to Noll and a generalized version of Cauchy's tetrahedron theorem. In other words, the measures of deformation and stress, as well as the balance equations, are determined by the expression chosen for the external power and by the invariance assumptions. A pair of examples taken from the literature shows that both ingredients are essential for defining a specific class of continua.

1. Introduction

In classical continuum mechanics, the balance laws of linear and angular momentum determine an integral identity, the equation of virtual power. On this identity is based the weak formulation of the problems of motion and of equilibrium. A more recent choice is to consider the equation of virtual power as primitive, and to deduce from it the balance equations. A reason for this alternative approach is the difficulty met in formulating generalized versions of the balance equations, appropriate to nonclassical continua. The method of virtual power of Germain [1972] is just a formalization of this new approach, and his papers [Germain 1973a; 1973b] show how the method applies to some specific nonclassical continua, such as second-gradient and micropolar continua.

To describe nonclassical continua within the classical approach is possible, at the price of introducing supplementary balance equations. For example Capriz [1989] introduces just one supplementary equation, the equation of balance of micromomentum, and with it he succeeds in describing a variety of microstructures.

Both approaches have some inconvenience. When introducing new balance equations, it is not clear how many they should be, and which should be their motivation. For example, Capriz [1989, Section 8] says that the supplementary equation is just a plausible form for the balance of micromomentum, justified by the analogy with the form of the balance equation of linear momentum. Clearly, we are far from the status of fundamental laws of mechanics attributed to the classical balance equations.

Even worse, when some form of the equation of virtual power is assumed as primitive, the expressions of the external and internal power are not really arbitrary, due to the general belief that the classical

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balance equations should hold anyway. Thus, there is a tacit preselection of the possible forms of the equation, in spite of its claimed status of a postulate.

Here I follow a different approach, first proposed by Noll [1974] in the context of classical continua. This approach is based on two assumptions: one on the nature of the external actions, and one on the invariance of the power under changes of observer. The first assumption determines the expression of the external power, and the two assumptions together determine the balance equations. With the aid of the divergence theorem, one obtains an expression of the internal power, which consists of a volume integral involving the inner products of a certain number of internal forces by the corresponding generalized deformations.[1]

In this way, the two basic assumptions determine a class of continua. Within each class, specific materials can be characterized by ad hoc constitutive assumptions, such as the existence of energy functions, of dissipation potentials, or of explicit relations between internal forces and generalized deformations. Both assumptions are essential in determining a class of continua. Indeed, the same form of the external power can be associated with different invariance requirements, dictated by the differences in the physical nature of the microstructures. This determines different classes of continua, each one with its own set of balance equations. An example is provided in Section 5.

Besides the genuine balance equations coming from the invariance requirements, it is sometimes convenient to introduce pseudo-balance equations, which are just mathematical devices transforming area integrals into volume integrals. Though in the literature the two are frequently mixed together, it is important to keep them separated. A clear distinction between genuine balance equations and pseudo-balance equations is one of the purposes of the present work.

Section 2 shows the application of the proposed method to classical continua. In the next sections 3 and 4 the same method is applied to higher-gradient continua and to micropolar continua, respectively. The last section deals with two examples taken from strain-gradient plasticity. In them, the expression of the external power is the same as in a micropolar continuum, while the invariance assumptions are different.

2. Classical continua

For a classical continuum occupying a three-dimensional region Ω of space, the assumed system of external actions is a pair (b, s) formed by distance actions and contact actions, where $b = b(x)$ is a vector field on Ω representing the volume density of the body forces, and $s = s(\Pi, x)$ is a system of vector fields acting at the boundary points x of each part Π of Ω, and representing the surface density of the contact force at x.[2] The corresponding expression for the external power is

$$\mathcal{P}(\Pi, v) = \int_{\Pi} b \cdot v \, dV + \int_{\partial\Pi} s \cdot v \, dA \,, \tag{1}$$

[1] Instead of making assumptions on the nature of the external actions, one might fix a priori the structure of the set $\mathcal{V}$ of the virtual displacements, and then define the external power by duality, that is, by identifying it with the most general continuous linear functional on $\mathcal{V}$. Whether or not to consider forces as primitive is a philosophical matter which has long been debated, see Jammer [1957]. Whatever is the preference, the procedure proposed here applies.

[2] Throughout this paper it is assumed that all vector, or tensor, fields defined on the pair (Π, x) are bouded almost everywhere, and that the associated fluxes $\int_{\partial\Pi} s \, dA$ are additive on regions Π with pairwise disjoint interiors [Gurtin and Martins 1976].

where $v = v(x)$ is a field of virtual displacements[3] on Ω, and dV and dA are the volume measure and the area measure, respectively.

The assumption that $\mathcal{P}$ is invariant under changes of observer is expressed by the equation

$$\mathcal{P}(\Pi, v) = \mathcal{P}(\Pi, v + c + \omega \times x)\,, \tag{2}$$

to be satisfied for all parts Π of Ω and for all pairs of vectors c, ω. In view of the linear dependence of $\mathcal{P}$ on v, this is true if and only if

$$\mathcal{P}(\Pi, c) = \mathcal{P}(\Pi, \omega \times x) = 0 \tag{3}$$

for all Π and for all vectors c, ω. That is, if and only if

$$\int_\Pi b\, dV + \int_{\partial\Pi} s\, dA = 0\,, \qquad \int_\Pi x \times b\, dV + \int_{\partial\Pi} x \times s\, dA = 0\,, \tag{4}$$

for all Π. These are the balance equations of the linear and angular momentum for a classical continuum.[4]

It has been proven by Noll [1959] that, as a consequence of the first balance equation, s depends on Π only through the exterior unit normal n to $\partial\Pi$ at x: $s(\Pi, x) = s(n, x)$. Then from Cauchy's tetrahedron theorem one deduces the linearity of the dependence on n, that is, the existence of a tensor T, the Cauchy stress, such that $s(n, x) = T(x)\, n$, as well as the local forms of the two balance equations

$$\operatorname{div} T + b = 0\,, \qquad T = T^T\,. \tag{5}$$

Substituting into the expression of the external power and using the divergence theorem, one gets

$$\begin{aligned} \mathcal{P}(\Pi, v) &= -\int_\Pi \operatorname{div} T \cdot v\, dV + \int_{\partial\Pi} Tn \cdot v\, dA \\ &= \int_\Pi T \cdot \nabla^S v\, dV\,, \end{aligned}$$

where $\nabla^S v$ is the symmetric part of ∇v. The integral on the right is the internal power. In it, the integrand function is the product of an internal force, the Cauchy stress, by a virtual generalized deformation, the symmetric part of the gradient of the virtual displacement. As we see, in the proposed approach there are not indeed two virtual powers, internal and external, but rather a single virtual power with two different expressions, the first of which is given a priori, while the second is deduced from the first, using the invariance assumptions.

[3]The fields v are usually called *virtual velocities*. I prefer *displacements*, since *velocity* suggests that physical time is somehow involved. For consistency, I should then speak of *virtual work* instead of *virtual power*, but the latter term is so generally used that I don't dare go back to the more classical *virtual work*.

[4]In the past, an obstacle against this very simple deduction of the balance equations was that the power of the inertial actions is not indifferent under Galilean changes of observer. Following ideas originating from Mach's criticism of the Newtonian concept of absolute space, Noll [1974] ruled out this difficulty by regarding inertia as the distance action "between the bodies in the solar system and the masses occupying the rest of the universe" [Truesdell and Noll 1965, Sect. 18]. Accordingly, the power of the inertial actions has to be invariant under changes of observer involving the whole universe, and not just a part of it, as it is tacitly done when referring the change of observer to our planet or to a part of it, instead of to the whole universe.

3. Higher-gradient continua

Let us add to the virtual power (1) two terms in ∇v

$$\mathcal{P}(\Pi, v) = \int_\Pi (b \cdot v + B \cdot \nabla v)\, dV + \int_{\partial\Pi} (s \cdot v + S \cdot \nabla v)\, dA\,, \tag{6}$$

one associated with a second-orderer tensor field $B = B(x)$ of body double-forces, and one with a system $S = S(\Pi, x)$ of second-order tensor fields representing surface double-tractions. From the divergence theorem

$$\int_\Pi B \cdot \nabla v\, dV = -\int_\Pi \operatorname{div} B \cdot v\, dV + \int_{\partial\Pi} Bn \cdot v\, dA\,. \tag{7}$$

it follows that the power generated by a field of body double-forces is equal to the power generated by a field $b^* = -\operatorname{div} B$ of body forces, plus a system $s^* = Bn$ of surface tractions. Because the term $B \cdot \nabla v$ does not change substantially the expression of the virtual power, for simplicity it will be neglected in what follows.

For the power generated by S, we first observe that the invariance assumption $(3)_1$ still yields the balance equation $(4)_1$, so Equation (6) reduces to

$$\mathcal{P}(\Pi, v) = \int_\Pi T \cdot \nabla v\, dV + \int_{\partial\Pi} S \cdot \nabla v\, dA\,. \tag{8}$$

Then from the invariance assumption $(3)_2$ one gets

$$\int_\Pi T^W dV + \int_{\partial\Pi} S^W dA = 0\,, \tag{9}$$

with T^W and S^W the skew-symmetric parts of T and S, respectively. This equation is formally identical with Equation $(4)_1$, except for the fact that the integrand functions are now second-order tensors. Using Noll's theorem on the dependence on the normal and Cauchy's tetrahedron theorem[5] one deduces the existence of a third-order tensor $\mathbb{T}^W$, skew-symmetric with respect to the first two indices, such that

$$S^W = \mathbb{T}^W n\,, \qquad S^W_{ij} = \mathbb{T}^W_{ijk} n_k\,. \tag{10}$$

Substituting Equation (10) into the balance equation (9) and then using the divergence theorem one gets

$$\operatorname{div} \mathbb{T}^W + T^W = 0\,, \qquad \mathbb{T}^W_{ijk,k} + T^W_{ij} = 0\,. \tag{11}$$

Therefore,

$$\begin{aligned}
\int_{\partial\Pi} S^W \cdot \nabla v\, dA &= \int_{\partial\Pi} \mathbb{T}^W n \cdot \nabla v\, dA \\
&= \int_\Pi (\mathbb{T}^W \cdot \nabla\nabla v + \operatorname{div} \mathbb{T}^W \cdot \nabla v)\, dV \\
&= \int_\Pi \left(\mathbb{T}^W \cdot \nabla\nabla v - T^W \cdot \nabla v\right) dV\,,
\end{aligned}$$

[5] For an extension of Cauchy's theorem to second-order tensors see Appendix A.

and the expression (8) of the virtual power reduces to

$$\mathcal{P}(\Pi, v) = \int_{\Pi} \big(\mathbb{T}^{W} \cdot \nabla\nabla v + T^{S} \cdot \nabla v\big)\, dV + \int_{\partial\Pi} S^{S} \cdot \nabla v \, dA \,. \tag{12}$$

As we see, the right-hand side has not yet the form of a single volume integral, except in the special case of S skew-symmetric, which will be considered separately at the end of this section. To transform the area integral into a volume integral, we introduce the field of the body forces $\widetilde{B}$ defined by

$$\widetilde{B}(x) = -\lim_{\delta\to 0} \frac{\int_{\partial\mathcal{B}(x,\delta)} S^{S} dA}{V(\mathcal{B}(x,\delta))} \,, \tag{13}$$

where $\mathcal{B}(x,\delta)$ is the three-dimensional ball of radius δ centered at x, and $\widetilde{B}$ is symmetric because S^{S} is symmetric.[6] By integration over a part Π of the body, we get the pseudo-balance equation

$$\int_{\Pi} \widetilde{B}\, dV + \int_{\partial\Pi} S^{S}\, dA = 0 \,. \tag{14}$$

I point out that this is not a proper balance equation, since it does not come from an invariance assumption. It comes instead from the definition (13), which is instrumental to transforming a surface integral into a volume integral. In fact, again from Noll's and Cauchy's theorems one deduces the existence of a third-order tensor $\mathbb{T}^{S}$, symmetric with respect to the first two indices, such that $S^{S} = \mathbb{T}^{S} n$, and substituting into the pseudo-balance equation and using the divergence theorem, the local pseudo-balance equation

$$\operatorname{div} \mathbb{T}^{S} + \widetilde{B} = 0 \tag{15}$$

follows. Then the area integral in (12) transforms as

$$\begin{aligned}
\int_{\partial\Pi} S^{S} \cdot \nabla v \, dA &= \int_{\partial\Pi} \mathbb{T}^{S} n \cdot \nabla v \, dA \\
&= \int_{\Pi} \big(\mathbb{T}^{S} \cdot \nabla\nabla v + \operatorname{div} \mathbb{T}^{S} \cdot \nabla v\big)\, dV \\
&= \int_{\Pi} \big(\mathbb{T}^{S} \cdot \nabla\nabla v - \widetilde{B} \cdot \nabla v\big)\, dV \,,
\end{aligned}$$

and after setting

$$\mathbb{T} := \mathbb{T}^{S} + \mathbb{T}^{W}, \tag{16}$$

one has $S = \mathbb{T}\, n$, $\operatorname{div} \mathbb{T} + T^{W} + \widetilde{B} = 0$, and the virtual power (12) takes the form

$$\mathcal{P}(\Pi, v) = \int_{\Pi} \big(\mathbb{T} \cdot \nabla\nabla v + (T^{S} - \widetilde{B}) \cdot \nabla v\big)\, dV \,. \tag{17}$$

We see that the internal forces are the third-order tensor $\mathbb{S}$ and the second-order tensor $T^{S} - \widetilde{B}$, and that the generalized deformations are the first two gradients of v. If we agree to label a class of continua after the highest gradient of v which appears among the generalized deformations, then the classical

[6]For the regularity assumptions on S^{S} ensuring the existence almost everywhere of the limit $\widetilde{B}$, see Gurtin and Martins [1976, Theorem 7].

continuum is a first-gradient continuum, and the continuum defined by (6) and (3) is a second-gradient continuum.

A special subclass of second-gradient continua is the one in which the external action S is skew-symmetric. As said before, in this case the area term in (12) vanishes, and therefore there is no pseudo-balance equation. Moreover, the power $S \cdot \nabla v$ can be given the form $m \cdot \omega$, where m and ω are the vectors associated with S and with twice the skew-symmetric part of ∇v, respectively:

$$m_i = \frac{1}{2}\, e_{ijk} S_{kj}\,, \qquad \omega_i = e_{ijk} v_{k,j}\,. \tag{18}$$

Because ω measures the local rotation at x, the vector m can be identified with a surface couple. Then, since S skew-symmetric implies $\mathbb{T} = \mathbb{T}^W$ and $\mathbb{T}^S = 0$, the first product in (17) takes the form

$$\mathbb{T} \cdot \nabla\nabla v = \mathbb{T}^W \cdot \nabla\nabla v = M \cdot \nabla\omega\,, \qquad \mathbb{T}_{ijk} v_{i,jk} = M_{ij} e_{ihk} v_{h,kj} = M_{ij}\omega_{i,j}\,, \tag{19}$$

where M is the second-order tensor associated with $\mathbb{T}^W$ [7]

$$M_{ij} = \frac{1}{2}\, e_{ihk}\, \mathbb{T}^W_{khj}\,. \tag{20}$$

In conclusion, for S skew-symmetric the two expressions (6), (17) of the virtual power simplify into

$$\int_\Pi b \cdot v \, dV + \int_{\partial\Pi} (s \cdot v + m \cdot \omega)\, dV\,, \qquad \int_\Pi (M \cdot \nabla\omega + T^S \cdot \nabla v)\, dV\,. \tag{21}$$

The foregoing analysis can be easily generalized to higher-gradient continua. Notice that for all such continua there are no invariance assumptions besides the classical assumptions (3).

4. Micropolar continua

In micropolar continua, with each point x is associated a finite set of vectors $\alpha \mapsto d^\alpha = d^\alpha(x)$, called the *directors*, each with fixed length and variable orientation. The virtual power consists of the two terms appearing in Equation (1) plus two extra terms, representing the power of the body and surface director forces $\beta^\alpha, \sigma^\alpha$, both multiplied by virtual changes ν^α of the directors[8]

$$\mathcal{P}(\Pi, v, \nu^\alpha) = \int_\Pi (b \cdot v + \beta^\alpha \cdot \nu^\alpha)\, dV + \int_{\partial\Pi} (s \cdot v + \sigma^\alpha \cdot \nu^\alpha)\, dA\,. \tag{22}$$

While the first of the invariance axioms (3) remains unchanged, the second now requires the invariance of the virtual power under simultaneous rigid rotations of the body and of the directors. That is,

$$\mathcal{P}(\Pi, c, 0) = \mathcal{P}(\Pi, \omega \times x, \omega \times d^\alpha) = 0, \tag{23}$$

[7] In view of Equation $(18)_1$, M can be seen as the unique second-order tensor such that Me is the vector associated with $\mathbb{T}^W e$ for all vectors e.

[8] Here and in the following, the repeated indices α are summed.

for all Π and for all vectors c, ω. The corresponding balance equations are

$$\begin{aligned} \int_\Pi b\, dV + \int_{\partial\Pi} s\, dA &= 0\,, \\ \int_\Pi (x \times b + d^\alpha \times \beta^\alpha)\, dV + \int_{\partial\Pi} (x \times s + d^\alpha \times \sigma^\alpha)\, dA &= 0\,, \end{aligned} \tag{24}$$

respectively. From the first equation still follows the existence of a Cauchy stress T such that $s = Tn$ and $\operatorname{div} T + b = 0$, and by substitution into the second equation one gets

$$\int_\Pi (T^W + (\beta^\alpha \otimes d^\alpha)^W)\, dV + \int_{\partial\Pi} (\sigma^\alpha \otimes d^\alpha)^W\, dA = 0\,. \tag{25}$$

Using again Noll's theorem on the dependence of the normal and Cauchy's tetrahedron theorem, one deduces the existence of a third-order tensor field $\mathbb{T}^W_{ijk}$, skew-symmetric with respect to the first two indices, such that $(\sigma^\alpha \otimes d^\alpha)^W = \mathbb{T}^W n$, from which one gets the following local form of the second balance equation

$$\operatorname{div} \mathbb{T}^W + T^W + (\beta^\alpha \otimes d^\alpha)^W = 0\,. \tag{26}$$

Then,

$$\begin{aligned} \int_\Pi T^W \cdot \nabla v\, dV &= -\int_\Pi (\operatorname{div} \mathbb{T}^W + (\beta^\alpha \otimes d^\alpha)^W) \cdot \nabla v\, dV \\ &= \int_\Pi (\mathbb{T}^W \cdot \nabla\nabla v - (\beta^\alpha \otimes d^\alpha)^W \cdot \nabla v)\, dV - \int_{\partial\Pi} (\sigma^\alpha \otimes d^\alpha)^W \cdot \nabla v\, dA\,, \end{aligned}$$

and the expression (22) of the virtual power transforms as

$$\begin{aligned} \mathcal{P}(\Pi, v, \nu^\alpha) &= \int_\Pi (T \cdot \nabla v + \beta^\alpha \cdot \nu^\alpha)\, dV + \int_{\partial\Pi} \sigma^\alpha \cdot \nu^\alpha\, dA \\ &= \int_\Pi \big(\mathbb{T}^W \cdot \nabla\nabla v + (T^S - (\beta^\alpha \otimes d^\alpha)^W) \cdot \nabla v + \beta^\alpha \cdot \nu^\alpha\big)\, dV \\ &\quad + \int_{\partial\Pi} \big(\sigma^\alpha \cdot \nu^\alpha - (\sigma^\alpha \otimes d^\alpha)^W \cdot \nabla v\big)\, dA \\ &= \int_\Pi \big(\mathbb{T}^W \cdot \nabla\nabla v + T^S \cdot \nabla v + \beta^\alpha \cdot (\nu^\alpha - \nabla^W v\, d^\alpha)\big)\, dV \\ &\quad + \int_{\partial\Pi} \sigma^\alpha \cdot (\nu^\alpha - \nabla^W v\, d^\alpha)\, dA\,. \end{aligned}$$

The vectors

$$\varphi^\alpha = \nu^\alpha - \nabla^W v\, d^\alpha \tag{27}$$

represent the local relative rotations between the directors and the body. As in the preceding section, to transform the area integral into a volume integral we introduce a pseudo-balance equation. We define

$$\widetilde{\beta}^\alpha(x) = -\lim_{\delta\to 0} \frac{\int_{\partial B(x,\delta)} \sigma^\alpha dA}{V(B(x,\delta))}\,, \tag{28}$$

so that

$$\int_{\Pi} \widetilde{\beta}^{\alpha}\, dV + \int_{\partial\Pi} \sigma^{\alpha}\, dA = 0\,. \tag{29}$$

By the tetrahedron theorem, for each α there is a second-order tensor Σ^{α} such that

$$\sigma^{\alpha} = \Sigma^{\alpha} n\,, \qquad \operatorname{div} \Sigma^{\alpha} + \widetilde{\beta}^{\alpha} = 0\,. \tag{30}$$

Thus,

$$\begin{aligned}\int_{\partial\Pi} \sigma^{\alpha}\cdot\varphi^{\alpha}\, dA &= \int_{\partial\Pi} \Sigma^{\alpha} n\cdot\varphi^{\alpha}\, dA\\ &= \int_{\Pi} (\operatorname{div} \Sigma^{\alpha}\cdot\varphi^{\alpha} + \Sigma^{\alpha}\cdot\nabla\varphi^{\alpha})\, dV\\ &= \int_{\Pi} (-\widetilde{\beta}^{\alpha}\cdot\varphi^{\alpha} + \Sigma^{\alpha}\cdot\nabla\varphi^{\alpha})\, dV\,,\end{aligned}$$

and the virtual power takes the form

$$\mathcal{P}(\Pi, v, \nu^{\alpha}) = \int_{\Pi} \big(\mathbb{T}^{W}\cdot\nabla\nabla v + T^{S}\cdot\nabla v + \Sigma^{\alpha}\cdot\nabla\varphi^{\alpha} + (\beta^{\alpha} - \widetilde{\beta}^{\alpha})\cdot\varphi^{\alpha}\big)\, dV\,. \tag{31}$$

For this class of micropolar continua, the measures of stress are the tensors $\mathbb{T}^{W}$, T^{S}, Σ^{α} and $(\beta^{\alpha} - \widetilde{\beta}^{\alpha})$. They are subject to the balance equations $(5)_1$, (26) and to the pseudo-balance equation $(30)_2$. The measures of deformation are ∇v, φ^{α} and their first gradients. In particular, if the external actions β^{α} and σ^{α} are balanced, that is, if Equation (29) is satisfied by $\widetilde{\beta}^{\alpha} = \beta^{\alpha}$ for all Π, the last term in Equation (31) disappears.

5. Two examples from strain-gradient plasticity

For continua described by internal state variables one can still assume a virtual power of the form (22), where now ν^{α} are virtual variations of the state variables. They may be scalars, vectors, or tensors of any order, according to the nature of the corresponding state variable. The physical nature of the state variables also determines the appropriate invariance assumptions under changes of observer. Thus, in a sense, the invariance assumptions are a part of the definition of a state variable.

In strain-gradient plasticity, a configuration of a body is characterized by the deformation gradient F and by a state variable, the plastic distortion, identified with the plastic part of the strain gradient in its decomposition $F = F^{e}F^{p}$.[9] A virtual variation of plastic distortion is a second-order tensor, which I will denote by L.[10] The assumed expression of the virtual power is

$$\mathcal{P}(\Pi, v, L) = \int_{\Pi} b\cdot v\, dV + \int_{\partial\Pi} (s\cdot v + S\cdot L)\, dA\,. \tag{32}$$

It includes a contact action S associated with L, while the corresponding distance action is neglected. While for a micropolar continuum the virtual power is assumed to be invariant under simultaneous rigid

[9] Usually, a plastic continuum is regarded as a classical continuum subject to appropriate constitutive assumptions. Strain-gradient plasticity deals with a continuum with a microstructure described by the plastic distortion.

[10] It is usually supposed that L is traceless, but I shall ignore this restriction.

rotations of the body and of the directors, here the virtual power is assumed to be invariant under rigid body rotations, associated with a null change of plastic distortion.[11] Then we have the conditions

$$\mathcal{P}(\Pi, c, 0) = \mathcal{P}(\Pi, \omega \times x, 0) = 0\,, \tag{33}$$

and the resulting balance equations are the equations (4) of the classical continuum instead of the equations (24) of the micropolar continuum. But there is a third condition, which comes from the invariance of power under a rigid rotation of the intermediate configuration.[12] It requires that the virtual power associated with a rigid virtual rotation of plastic strain be zero

$$\mathcal{P}(\Pi, 0, W) = 0\,, \tag{34}$$

for all skew-symmetric tensors W.

From the first two invariance assumptions one again deduces the existence of a tensor field T such that $s = Tn$, $\operatorname{div} T + b = 0$ and $T = T^T$, so that (32) becomes

$$\mathcal{P}(\Pi, v, L) = \int_\Pi T \cdot \nabla^S v \, dV + \int_{\partial\Pi} S \cdot L \, dA\,. \tag{35}$$

From (34) it follows that

$$\int_{\partial\Pi} S^W \, dA = 0\,, \tag{36}$$

and from here, using again Noll's and Cauchy's theorems, one deduces the existence of a third-order tensor field $\mathbb{T}^W$, skew-symmetric with respect to the first two indices, such that $S^W = \mathbb{T}^W n$ and $\operatorname{div} \mathbb{T}^W = 0$. Therefore,

$$\int_{\partial\Pi} S^W \cdot L \, dA = \int_{\partial\Pi} \mathbb{T}^W n \cdot L \, dA = \int_\Pi \mathbb{T}^W \cdot \nabla L \, dV\,,$$

and the virtual power reduces to

$$\mathcal{P}(\Pi, v, L) = \int_\Pi (\mathbb{T}^W \cdot \nabla L + T \cdot \nabla^S v) \, dV + \int_{\partial\Pi} S^S \cdot L \, dA\,. \tag{37}$$

Once again, to eliminate the area integral a pseudo-balance equation is required. By defining a symmetric tensor $\widetilde{B}$ as in (13), we deduce the existence of a third-order tensor $\mathbb{T}^S$, symmetric with respect to the

[11] A change of observer transforms the deformation gradient F into QF, with Q a proper rotation. The equation $QF = QF^e F^p$ then suggests that we choose the intermediate configuration determined locally by F^p in such a way that F^e and F^p transform into QF^e and F^p, respectively. With this choice, the intermediate configuration is left unchanged by a change of observer. Then $L = 0$, and the condition Equation $(33)_2$ follows. But there is nothing wrong in allowing the intermediate configuration to rotate, see the following footnote.

[12] A rigid rotation R of the intermediate configuration transforms F^p into RF^p, leaving F unchanged. This corresponds to $v(x) = 0$ and $L(x) = W$ constant and skew-symmetric. Because this is just a change of orientation of an auxiliary configuration, the power involved must be zero.

first two indices, such that $S^s = \mathbb{T}^s n$ and $\operatorname{div}\mathbb{T}^s + \widetilde{B} = 0$. Then the area integral transforms as

$$\begin{aligned}\int_{\partial\Pi} S^s \cdot L\,dA &= \int_{\partial\Pi} \mathbb{T}^s n \cdot L\,dA \\ &= \int_{\Pi} \big(\mathbb{T}^s \cdot \nabla L + \operatorname{div}\mathbb{T}^s \cdot L\big)\,dV \\ &= \int_{\Pi} \big(\mathbb{T}^s \cdot \nabla L - \widetilde{B} \cdot L^s\big)\,dV\,,\end{aligned} \tag{38}$$

and the virtual power takes the form

$$\mathcal{P}(\Pi, v, L) = \int_{\Pi} (\mathbb{T} \cdot \nabla L + T \cdot \nabla^s v - \widetilde{B} \cdot L^s)\,dV\,, \tag{39}$$

with $\mathbb{T} = \mathbb{T}^s + \mathbb{T}^w$, $S = \mathbb{T}n$, and $\operatorname{div}\mathbb{T} + \widetilde{B} = 0$. This is the only case considered in this paper, in which a third balance equation appears. As discussed above, the supplementary equation is motivated by the invariance of the virtual power under a rigid rotation of the intermediate configuration. The local form of this balance equation is $\operatorname{div}\mathbb{T}^w = 0$, while $\operatorname{div}\mathbb{T}^s + \widetilde{B} = 0$ is the pseudo-balance equation used to transform the area integral in (37) into a volume integral.

Notice that if the external action S^s is self-balanced, that is, if Equation (14) is satisfied by $\widetilde{B} = 0$ for every Π, the term $\tilde{B} \cdot L^s$ disappears from (39). This occurs, for example, if S is skew-symmetric. But in this case there is no pseudo-balance equation, and the virtual power (39) further reduces to

$$\mathcal{P}(\Pi, v, L) = \int_{\Pi} (\mathbb{T}^w \cdot \nabla L + T \cdot \nabla^s v)\,dV\,. \tag{40}$$

Let me discuss two examples taken from Gurtin and Anand [2005] and Gurtin [2004]. In the former, the authors assume the external power (32) and an expression of the internal power which, in the present notation, takes the form

$$\mathcal{P}_i(\Pi, v, L) = \int_{\Pi} \big(\mathbb{T} \cdot \nabla L + T \cdot \nabla^s v + (T^p - T) \cdot L\big)\,dV\,. \tag{41}$$

They also assume that L and T^p are symmetric, and that $\mathbb{T}$ is symmetric with respect to the first two indices. By equating (32) and (41), they obtain the equation of virtual power, from which they deduce the classical balance equations (5), plus the relation $S = \mathbb{T}\,n$ and the microforce balance equation

$$\operatorname{div}\mathbb{T} + T - T^p = 0\,. \tag{42}$$

If we compare (39) and (41), we see that they coincide if $T^p - T = -\widetilde{B}$, that is, if $\operatorname{div}\mathbb{T} + \widetilde{B} = 0$. Due to the symmetry of $\widetilde{B}$ and $\mathbb{T}$, this is true if and only if our pseudo-balance equation $\operatorname{div}\mathbb{T}^s + \widetilde{B} = 0$ holds. The expression (39) is slightly more general than (41), since it does not require the symmetry of L, T^p, and $\mathbb{T}$. These symmetry assumptions can be regarded as restrictions defining a special subclass of continua. This example shows that there is no need of assuming an expression for the internal power, since (41) follows from (32) under the appropriate invariance assumptions.

In [Gurtin 2004] the external power is taken as in (32) and the internal power is assumed to be

$$\mathcal{P}_i(\Pi, v, L) = \int_\Pi \big(P \cdot \operatorname{curl} L + T \cdot \nabla^s v + (T^p - T) \cdot L\big)\, dV, \tag{43}$$

respectively. The tensor L is assumed to be nonsymmetric with the purpose of accounting for the power dissipated by the Burgers tensor, whose virtual change is measured by $\operatorname{curl} L$. From the equation of virtual power he obtains the classical balance equations (5), the microforce balance equation

$$-(\operatorname{curl} P^T)^T + T - T^p = 0\,, \tag{44}$$

and the relation $S = \mathbb{T}\, n$, with

$$\mathbb{T}_{ijk} = e_{kjh}\, P_{hi}\,. \tag{45}$$

Then

$$\operatorname{div} \mathbb{T} = -(\operatorname{curl} P^T)^T = T^p - T\,, \qquad \mathbb{T} \cdot \nabla L = P \cdot \operatorname{curl} L\,, \tag{46}$$

and the expressions (43) and (39) of the internal power coincide if $T^p - T = -\widetilde{B}$, that is, if $\operatorname{div} \mathbb{T} + \widetilde{B} = 0$. But this is exactly what is required by the third balance equation $\operatorname{div} \mathbb{T}^w = 0$ and by the pseudo-balance equation $\operatorname{div} \mathbb{T}^s + \widetilde{B} = 0$. Thus, (43) is the expression of the internal power for the subclass of continua obtained from the expression (32) of the virtual power, the invariance assumptions (33), (34), and the supplementary assumption (45).

Appendix A: Proof of the Cauchy theorem for second-order tensors

Here I give a quick proof of Cauchy's theorem for the balance equation

$$\int_\Pi B\, dV + \int_{\partial\Pi} S\, dA = 0\,, \tag{A.1}$$

where $B = B(x)$ and $S = S(\Pi, x)$ are second-order tensor fields. For every fixed vector e, the balance Equation $(4)_1$ holds with $b = Be$ and $s = Se$. Then by Cauchy's theorem there is a second-order tensor $T(e)$ such that

$$Se = T(e)\, n\,, \qquad \operatorname{div} T(e) + Be = 0\,. \tag{A.2}$$

These equations show that the dependence of T on e is linear. That is, there is a third-order tensor $\mathbb{T}$ such that[13]

$$\mathbb{T}[e] = T(e) \qquad\qquad \mathbb{T}_{ijk} e_j = T(e)_{ik}\,. \tag{A.3}$$

Then from Equation (A.2),

$$\begin{aligned} Se &= (\mathbb{T}[e])\, n & \qquad S_{ij} e_j &= \mathbb{T}_{ijk} e_j n_k\,,\\ \operatorname{div}(\mathbb{T}[e]) + Be &= 0 & \qquad \mathbb{T}_{ijk,k} e_j + B_{ij} e_j &= 0\,, \end{aligned} \tag{A.4}$$

and, from the arbitrariness of e,

$$\begin{aligned} S &= \mathbb{T}\, n & \qquad S_{ij} &= \mathbb{T}_{ijk} n_k\,,\\ \operatorname{div} \mathbb{T} + B &= 0 & \qquad \mathbb{T}_{ijk,k} + B_{ij} &= 0\,. \end{aligned} \tag{A.5}$$

[13] The choice of the summed index in $(A.3)_2$ is arbitrary.

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