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ANISOTROPIC THIN-WALLED BEAM MODELS: A RATIONAL DEDUCTION FROM THREE-DIMENSIONAL ELASTICITY

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In this paper anisotropic thin-walled beam models are rationally deduced from three-dimensional elasticity by means of a constrained approach. Consistent frictionless internal constraints on both stress and strain dual fields are enforced through a modified Hu–Washizu functional obtained by a nonstandard application of Lagrange multipliers. Beam theories accounting for different shear refinement levels are justified, showing that this variational approach enables the development of new refined models, including high-order nonconventional effects and enhancing standard treatments of shear deformation effects. In agreement with the constrained problem, a locally equilibrated approximation of the stress field acting on beam cross-section is recovered in closed form. Finally, cases of laminated thin-walled beams as well as of unilateral conewise constitutive behavior (with special reference to bimodular materials) are investigated.

1. Introduction

Thin-walled beams comprising anisotropic materials, such as thin-walled elements made of composite fiber-reinforced polymer, are increasingly being used as primary and secondary members in structural applications in many areas, including aeronautical, mechanical, biomechanical, and civil engineering. The most common shapes include open cross-sections, such as I-, C-, L-, Z-shaped profiles, usually produced by pultrusion technology.

Because of the beam's specific geometry (two dimensions fairly smaller than the third) the analysis is generally carried out by means of approximate one-dimensional models. However, thin-walled members need treatments different from those adopted for classical rods because they can be very sensitive to shear deformation effects, which challenge both the validity of the classical Bernoulli–Navier assumption on cross-section deformation and Saint-Venant's pure torsion theory [Kollbrunner and Basler 1969].

The first one-dimensional thin-walled beam theory was developed by Vlasov [1961; 1962], who assumed that the section contour is invariant in its plane and that shear stresses in the middle surface of the beam vanish, and so reached a new distribution law of longitudinal stresses in the cross-section (law of sectorial areas). Vlasov's theory includes extension, bending and torsion deformations of isotropic homogeneous beams and describes the torsional shear stress flow as a superposition of two parts: the Saint-Venant primary flow (pure torsion) and a secondary one, associated to the shear stresses induced by the nonuniform warping of the cross-section. In spite of some inherent limitations, this theory represents a general framework in which many authors have developed more refined theoretical and numerical approaches for the analysis of thin-walled beams.

Keywords: thin-walled beams, constrained elasticity.

The static behavior of thin-walled beams has been analyzed for instance in [Bauld and Tzeng 1984; Ascione et al. 2000; De Lorenzis and La Tegola 2003; Erkmen and Mohareb 2006; Lee and Lee 2004; Lee 2005], where different shear refinement levels are included. Bauld and Tzeng [1984] developed a Vlasov-type theory for fiber-reinforced beams with thin-walled open cross-sections, which does not take into account any shear effect. Ascione et al. [2000], improving the kinematics of the classical Vlasov theory, included shear deformability using Timoshenko-type contributions enhanced through a Galerkin approach. De Lorenzis and La Tegola [2003] generalized the exact theory of thin-walled isotropic beams developed by Capurso [1964] to the case of transversely isotropic materials, including the effects of restraints and of concentrated loads. Erkmen and Mohareb [2006], postulating statically admissible stress fields in agreement with those coming out from the Vlasov's theory, proposed a theoretical and numerical torsional analysis for isotropic thin-walled beams including shear deformation effects on the middle surface. Lee [2005] has recently employed a thin-walled beam model for the analysis of laminated beams, based on a first-order shear-deformable beam theory (such as in [Maddur and Chaturvedi 1999]), accounting not only for shear deformation due to flexure but also for nonuniform warping effects (analogously to [Cortínez and Piovan 2002]).

Starting from generalized Vlasov-type theories, including shear effects as well as first- and second-order terms of rotational parameters, stability of composite thin-walled profiles has been analyzed in [Cortínez and Piovan 2002; Fraternali and Feo 2000; Lee and Kim 2001; Saadé et al. 2004; Piovan and Cortínez 2007]. Modified Vlasov-type theories have also been proposed in [Maddur and Chaturvedi 1999; Cortínez and Piovan 2002; Piovan and Cortínez 2005; 2007; Ambrosini et al. 2000] and elsewhere for the theoretical and numerical analysis of the dynamical behavior of such structures.

Nevertheless, the rational deduction and justification of these theories from three-dimensional elasticity and their consistent generalization for anisotropic materials as well as for nonconventional cases (such as laminated beams or unilateral material behavior) can be truly considered as an open task yet. Note that the deduction of thin-walled beam models in a consistent way is not only a speculative issue, but leads to a safer and more complete technical use of these theories.

In the specialized literature, the rational deduction of structural theories is performed mainly through two strategies: the asymptotic method and the constrained approach. The first, started in the mid-seventies through the influence of several works addressing theories of plates and shells, was later applied to rods and thin-walled beams [Trabucho and Viaño 1996; Rodríguez and Viaño 1997; Volovoi et al. 1999; Hamdouni and Millet 2006]. Its main idea is that the three-dimensional solution of the elasticity equations can be approximated through successive terms of a power series, where, for beams, the slenderness ratio (between cross-section diameter and beam length) is taken as a small parameter. Accordingly, under suitable hypotheses that ensure series convergence, different structural theories can be rationally deduced as approximate solutions of an exactly stated problem, varying the series truncation order. The constrained approach, by contrast, is based on a diametrically opposite concept: it looks for an exact solution of a simplified constrained problem, based on approximate representations of the unknown functions. The primary three-dimensional elastostatic problem is reduced to a consistent simplified one (two-dimensional in the case of plates and shells, one-dimensional for beams), enforcing suitable assumptions on strain and/or stress fields as internal frictionless constraints. That approach was successfully employed for deducing classical plate and shell theories [Podio-Guidugli 1989; Bisegna and Sacco 1997b] and theories of beams with solid sections [Lembo and Podio-Guidugli 2001; Maceri and Bisegna 2002]. To justify Kirchhoff–Love plate theory [Podio-Guidugli 1989] and the Timoshenko beam model [Lembo and Podio-Guidugli 2001], a constrained approach was proposed based on strain assumptions and on the concept of constrained material. By contrast, the constrained approach proposed by Bisegna and Sacco [1997b] for deducing classical plate theories maintains the constitutive law given a priori, and involves consistent assumptions on both strain and stress dual fields. Nevertheless, while strain assumptions can be easily identified if the problem is characterized by special geometries, effective and consistent stress assumptions can be sometimes not obvious. In order to overcome this difficulty, Maceri and Bisegna [2002] showed how plates, shells and planar beams theories can be justified enforcing in a consistent way the same constraints on both stress and strain dual fields.

In this paper, this dual-constraint approach is employed for justifying several anisotropic thin-walled beam models, accounting for different levels of shear refinements. To this aim, the Hu–Washizu variational formulation of the three-dimensional elasticity problem is modified using Lagrange multiplier theory [Antman and Marlow 1991]. Because of the simultaneous presence of constraints on dual spaces, a nonstandard application of Lagrange multipliers is required.

As far as homogeneous beams are considered, an anisotropic model which does not account for any shear deformations is deduced first, justifying the model of Bauld and Tzeng [1984] and reducing to the classical Vlasov's theory when isotropic constitutive symmetry is considered. The proposed approach enables to take into account also nonconventional high-order effects, related to thickness and curvature of the cross-section centerline. In detail, generalized expressions for warping function and torsion constant are obtained, whose influence is investigated through simple numerical applications. Afterwards, a first-order shear-deformable thin-walled beam model is rationally derived, justifying models involving Timoshenko's shear effects. This is the case of the model proposed in [Ascione et al. 2000] when Galerkin-type contributions are neglected, and, for homogeneous beams, of the model analyzed in [Maddur and Chaturvedi 1999]. In the framework of the dual-constraint approach, different shear refinement levels can be also accounted for. This is proved including shear effects due to nonuniform warping (e.g., models in [Lee 2005; Cortínez and Piovan 2002]) and drawing some strategies to deduce a new branchwise model, for multibranch beam cross-sections, including branch-depending Timoshenko's and warping shear effects.

A more accurate evaluation of stresses on beam cross-section with respect to those obtained by constitutive relationship represents an issue of great technical interest when, for instance, damage of composite thin-walled beams is addressed. To this aim, in agreement with the constrained problem, approximated locally equilibrated stresses acting on beam cross-section are recovered in closed form.

Finally, the dual-constraint approach is successfully employed to rationally deduce models for the analysis of laminated thin-walled beams as well as of homogeneous beams comprising nonlinear cone-wise elastic materials [Curnier et al. 1995] (with reference to bimodular ones).

2. Dual-constraint approach

In the framework of infinitesimal deformation theory, the equilibrium problem of a three-dimensional body Ω comprising a linearly elastic material can be recast by adopting the Hu–Washizu variational formulation:

Find the displacement field $\mathbf{u}$, *the strain field* $\boldsymbol{\varepsilon}$ *and the stress field* $\boldsymbol{\sigma}$ *that make stationary the functional*

$$\mathcal{W}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})=\int_\Omega \tfrac{1}{2}\mathscr{C}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\,dv-\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\,dv+\int_\Omega \boldsymbol{\sigma}\cdot\hat{\nabla}\mathbf{u}-\int_\Omega \mathbf{b}\cdot\mathbf{u}\,dv\,dv \\ -\int_{\partial_f\Omega}\mathbf{p}\cdot\mathbf{u}\,da-\int_{\partial_u\Omega}\boldsymbol{\sigma}\mathbf{n}_\partial\cdot(\mathbf{u}-\mathbf{u}_0)\,da, \quad (2\text{-}1)$$

where $\hat{\nabla}$ denotes the symmetrical part of the gradient operator, $\cdot$ the inner product, $\mathbf{b}$ the volume forces, $\mathbf{p}$ the surface forces on $\partial_f\boldsymbol{\Omega}$, $\mathbf{u}_0$ the displacement assigned on $\partial_u\boldsymbol{\Omega}=\partial\boldsymbol{\Omega}\backslash\partial_f\boldsymbol{\Omega}$, $\mathbf{n}_\partial$ the outward normal unit vector to $\partial\boldsymbol{\Omega}$, and $\mathscr{C}$ the fourth-order elasticity tensor satisfying major and minor symmetries. The stationary conditions of $\mathcal{W}$ with respect to $\mathbf{u}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ yield equilibrium, compatibility and constitutive equations governing the three-dimensional elastostatic problem for $\boldsymbol{\Omega}$.

When the body is characterized by special geometrical aspect ratios, the three-dimensional problem can be approximated using suitable assumptions on strain and/or stress fields. If these assumptions are regarded as internal frictionless constraints, reactive fields arise and the original difficult three-dimensional elastic problem can be replaced by a constrained problem that can be often solved more easily.

In order to enforce constraints on both strain and stress dual fields, the constrained equilibrium problem can be suitably formulated employing Lagrange multipliers. Physically, Lagrange multipliers represent reactive actions belonging to the dual space of the one where the constrained variable lives: a Lagrange multiplier of a strain constraint identifies a reactive stress, and conversely a multiplier of a stress constraint has the meaning of reactive strain. The consistent representation of such reactive fields arises as a consequence of the enforced constraints and it is not postulated a priori.

As in [Bisegna and Sacco 1997b], we will adopt the following definitions: the total strain field is the symmetrical part of the gradient of the displacement field; the total stress field satisfies the equilibrium equations; elastic stresses and strains are related to each other by the elastic constitutive law; the total stress (or strain) is sum of its elastic and reactive parts.

In order to build up structural theories, one usually imposes a representation law for the displacement field (which is equivalent to imposing constraints on the total strains), and some hypotheses on the stress field at the constitutive law level (i.e., on the elastic stress field).

Let the total strain and the elastic stress fields be constrained to belong to the kernel of the linear (possibly differential) operators $\mathbf{G}$ and $\mathbf{H}$, respectively. Following [Bisegna and Sacco 1997b], these constraints acting on dual spaces can be enforced by introducing the Lagrangian functional

$$\mathcal{L}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma},\boldsymbol{\chi},\boldsymbol{\omega})=\mathcal{W}(\mathbf{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma})-\int_\Omega \boldsymbol{\chi}\cdot\mathbf{G}\boldsymbol{\varepsilon}\,dv-\int_\Omega \boldsymbol{\omega}\cdot\mathbf{H}\boldsymbol{\sigma}\,dv-\int_\Omega \mathbf{G}^*\boldsymbol{\chi}\cdot\mathbf{H}^*\boldsymbol{\omega}\,dv, \quad (2\text{-}2)$$

where the vectors $\boldsymbol{\chi}$, $\boldsymbol{\omega}$ are Lagrange multipliers, $\mathbf{G}^*$ and $\mathbf{H}^*$ denote the adjoint operators of $\mathbf{G}$ and $\mathbf{H}$, respectively. The stationary condition of $\mathcal{L}$ with respect to $\mathbf{u}$ yields the equilibrium equations

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma}+\mathbf{b}&=\mathbf{0} \quad \text{in } \boldsymbol{\Omega}, \\ \boldsymbol{\sigma}\mathbf{n}_\partial&=\mathbf{p} \quad \text{on } \partial_f\boldsymbol{\Omega}; \end{aligned} \quad (2\text{-}3)$$

the one with respect to $\boldsymbol{\varepsilon}$ yields the constitutive equations

$$\boldsymbol{\sigma}+\mathbf{G}^*\boldsymbol{\chi}=\mathscr{C}\boldsymbol{\varepsilon} \quad \text{in } \Omega; \quad (2\text{-}4)$$

the one with respect to $\boldsymbol{\sigma}$ yields the compatibility equations

$$\begin{aligned} \boldsymbol{\varepsilon} + \mathbf{H}^{*}\boldsymbol{\omega} &= \hat{\nabla}\mathbf{u} \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_0 \quad \text{on } \partial_u\Omega; \end{aligned} \tag{2-5}$$

and finally the stationary conditions with respect to the Lagrange multipliers $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$ yield, respectively, the constraint equations

$$\begin{aligned} \mathbf{G}\left(\boldsymbol{\varepsilon} + \mathbf{H}^{*}\boldsymbol{\omega}\right) &= \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{H}\left(\boldsymbol{\sigma} + \mathbf{G}^{*}\boldsymbol{\chi}\right) &= \mathbf{0} \quad \text{in } \Omega. \end{aligned} \tag{2-6}$$

Accordingly, by (2-3), (2-4) and (2-5), $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon} + \mathbf{H}^{*}\boldsymbol{\omega}$ turn out to be the total stress and strain fields, respectively, and $\boldsymbol{\sigma} + \mathbf{G}^{*}\boldsymbol{\chi}$ and $\boldsymbol{\varepsilon}$ the elastic stress and strain fields, respectively. As a consequence, reactive stress and strain fields are $-\mathbf{G}^{*}\boldsymbol{\chi}$ and $\mathbf{H}^{*}\boldsymbol{\omega}$, respectively. Note that the reactive stress field is orthogonal to every admissible total strain field and likewise the reactive strain field is orthogonal to every admissible elastic stress field, as it appears from (2-6).

In this dual-constraint framework, Maceri and Bisegna [2002] have shown that classical plates and planar beams theories can be rationally deduced by assuming that the dual constraints on the elastic stress field are the same as those imposed on the total strain field. In this case, operators $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{HA} = \mathbf{GA}$ for every symmetrical second order tensor $\mathbf{A}$ and Lagrange multipliers $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$ belong to dual vector subspaces characterized by the same dimensions. In this way, once kinematic constraints are chosen, consistent stress assumptions directly arise.

For what follows, it is more useful transforming the functional $\mathscr{L}$ of (2-2) into a potential energy functional, by enforcing a priori satisfied stationary conditions of $\mathscr{L}$ with respect to $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, that is (2-5) and (2-4). Accordingly, in the framework of the dual constraint approach proposed in [Maceri and Bisegna 2002], the functional $\mathscr{L}$ becomes

$$\mathscr{E}(\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\omega}) = \int_{\Omega} \tfrac{1}{2}\mathscr{C}(\hat{\nabla}\mathbf{u} - \mathbf{G}^{*}\boldsymbol{\omega}) \cdot (\hat{\nabla}\mathbf{u} - \mathbf{G}^{*}\boldsymbol{\omega})\, dv - \int_{\Omega} \boldsymbol{\chi} \cdot \mathbf{G}\hat{\nabla}\mathbf{u}\, dv - \int_{\Omega} \mathbf{b}\cdot\mathbf{u}\, dv - \int_{\partial_f\Omega} \mathbf{p}\cdot\mathbf{u}\, da, \tag{2-7}$$

defined on the manifold $\mathbf{u} = \mathbf{u}_0$ on $\partial_u\Omega$. It can be verified that total strain and elastic strain are $\boldsymbol{\Lambda} = \hat{\nabla}\mathbf{u}$ and $\boldsymbol{\varepsilon} = \hat{\nabla}\mathbf{u} - \mathbf{G}^{*}\boldsymbol{\omega}$, respectively, and total stress and elastic stress are $\boldsymbol{\sigma} = \mathscr{C}(\hat{\nabla}\mathbf{u} - \mathbf{G}^{*}\boldsymbol{\omega}) - \mathbf{G}^{*}\boldsymbol{\chi}$ and $\boldsymbol{\sigma}^{(\mathrm{el})} = \mathscr{C}(\hat{\nabla}\mathbf{u} - \mathbf{G}^{*}\boldsymbol{\omega})$, respectively.

It should be noted that $\mathscr{E}$ depends on the reactive fields. In order to obtain a potential-energy functional that does not depend on the Lagrange multipliers it is sufficient to make stationary conditions of $\mathscr{E}$ with respect to $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$ a priori satisfied.

3. Thin-walled beam models

3A. ***Notation.*** Consider a right cylinder beam-like body $\Omega = \mathscr{P} \times \,]-L, L[$, whose length is $2L$ and whose cross-section is an open thin-walled profile $\mathscr{P}$, assumed to be constant along the beam axis-line. Let the global Cartesian frame (O, x, y, z) be assumed with $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the corresponding orthonormal basis and with z-axis parallel to the cylinder axis. Let p the centerline of $\mathscr{P}$ and $2\delta \ll 2L$ the thickness of the wall. Hence, the middle surface of the beam is $M = p \times \,]-L, L[$ and, with a little abuse of notation, the cross-section can be described as $\mathscr{P} = p \times \,]-\delta, \delta[$. The beam is assumed to be in equilibrium when

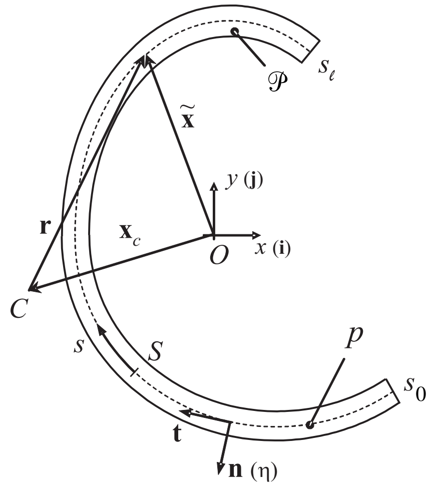

Figure 1. Thin-walled beam cross-section: notation.

volume forces $\mathbf{b}$ act upon Ω, surface tractions $\hat{\mathbf{p}}$ act upon the mantle $\Sigma = \partial\mathcal{P} \times]-L, L[$, and surface forces $\mathbf{p}^+$ and $\mathbf{p}^-$ are given at the ends of the beam $\mathcal{P}^+ = \mathcal{P} \times \{L\}$ and $\mathcal{P}^- = \mathcal{P} \times \{-L\}$, respectively.

As it is customary in thin-walled beam analysis, centerline p can be represented through the parametric equations $\tilde{x} = \tilde{x}(s)$ and $\tilde{y} = \tilde{y}(s)$, where s is a local curvilinear coordinate with origin at an arbitrary point S of p, such that $s \in [s_0, s_\ell]$. Accordingly, at the generic position $\tilde{\mathbf{x}} = \tilde{x}\mathbf{i} + \tilde{y}\mathbf{j}$ of p a local tangent frame may be introduced (Figure 1) by means of tangent and normal unit vectors to p:

$$\mathbf{t}(s) = \frac{d\tilde{x}}{ds}\mathbf{i} + \frac{d\tilde{y}}{ds}\mathbf{j}, \qquad \mathbf{n}(s) = -\frac{d\tilde{y}}{ds}\mathbf{i} + \frac{d\tilde{x}}{ds}\mathbf{j}. \tag{3-1}$$

Hence, position $\mathbf{x}$ of every point in Ω can be represented as

$$\mathbf{x}(s, \eta, z) = \tilde{\mathbf{x}}(s) + \eta\,\mathbf{n}(s) + z\mathbf{k}, \tag{3-2}$$

where $\eta \in [-\delta, \delta]$ is the coordinate along the unit vector $\mathbf{n}$. We assume that thickness variation along p is small, that is $|d\delta/ds| \ll 1$, and hence tangent directions at $\partial\mathcal{P}|_{\eta=\pm\delta}$ can be approximated by $\mathbf{t}(s)$ up to terms of order $|d\delta/ds|$. It should be marked that basis $\{\mathbf{t}, \mathbf{n}, \mathbf{k}\}$ can be superimposed to the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and that derivatives with respect to s of $\mathbf{t}$ and $\mathbf{n}$ are given by the well-known Frenet formulas

$$\frac{d\mathbf{t}}{ds} = \kappa(s)\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\kappa(s)\mathbf{t}, \tag{3-3}$$

where $\kappa(s)$ is the (signed) curvature of p.

Due to the previous positions, integrals over $\mathcal{P}$ can be arranged as

$$\int_{\mathcal{P}} (\,\cdot\,)\,da = \int_{s_0}^{s_\ell} \int_{-\delta}^{\delta} (\,\cdot\,)\,j(s, \eta)\,d\eta\,ds, \tag{3-4}$$

where $j(s, \eta) = 1 - \eta\kappa$ is the Jacobian determinant of the coordinate transformation (3-2) from the global Cartesian frame to the local tangent one. Accordingly, the average of a function $f(s, \eta, z)$ over the thickness is

$$\bar{f}(s,z) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(s,\eta,z)\, j(s,\eta)\, d\eta. \tag{3-5}$$

Moreover, it easy to verify that, taking into account (3-1), the following identity holds:

$$\nabla f = \frac{f_{/1}}{j}\mathbf{t} + f_{/2}\mathbf{n} + f'\,\mathbf{k}. \tag{3-6}$$

Here and in what follows we indicate the partial derivatives of a function f with respect to local coordinates s and η as $f_{/1}$ and $f_{/2}$, respectively, whereas the partial derivative with respect to z is denoted by f'. Where necessary, vector and tensor components are denoted by subscripts and Einstein's summation convention is used, in which case Greek indices imply values in $\{1, 2\}$ and denote components in the plane of $\mathscr{P}$ referred to the local tangent frame (1 standing for the tangent-to-p component, i.e., along s, and 2 for the normal-to-p component, i.e., along η), the index 3 indicates components along the z-axis, and components along in-plane Cartesian axes are explicitly denoted by the subscripts x and y.

It is useful to introduce the following in-plane shear strain and stress vectors:

$$\begin{aligned} \boldsymbol{\gamma} &= 2\varepsilon_{13}\,\mathbf{t} + 2\varepsilon_{23}\,\mathbf{n} = 2\varepsilon_{x3}\,\mathbf{i} + 2\varepsilon_{y3}\,\mathbf{j}, \\ \boldsymbol{\tau} &= \sigma_{13}\,\mathbf{t} + \sigma_{23}\,\mathbf{n} = \sigma_{x3}\,\mathbf{i} + \sigma_{y3}\,\mathbf{j} \end{aligned} \tag{3-7}$$

for both elastic and total fields.

Finally, in this section the beam is assumed to be homogeneous and comprising a linearly elastic material having at least a monoclinic symmetry, with symmetry plane orthogonal to $\mathbf{k}$. Accordingly, $\mathscr{C}_{\alpha\beta\gamma 3} = \mathscr{C}_{\alpha 333} = 0$. This material symmetry includes, for instance, the case of thin-walled profiles pultruded along the z-direction.

3B. *No-shear beam model.* The total strain field is assumed to satisfy Vlasov's assumptions [1961; 1962] on M and the Euler–Bernoulli constraints of classical beam theories:

(i) The in-plane (dilatation and shear) total strain components vanish everywhere on $\mathscr{P}$.

(ii) The shear total strain between the z-axis and $\mathbf{n}(s)$ vanishes everywhere on $\mathscr{P}$.

(iii) The flux through the thickness of the in-plane shear total strain vector (see (3-7)) is zero.

Accordingly, the operator $\mathbf{G}$ is such that

$$\mathbf{G}\boldsymbol{\varepsilon} = \{\varepsilon_{11}\ \ \varepsilon_{22}\ \ \varepsilon_{12}\ \ \bar{\varepsilon}_{13}\ \ \varepsilon_{23}\}^T. \tag{3-8}$$

Following the constrained approach proposed in [Maceri and Bisegna 2002], the dual constraints on the elastic stress field can be directly expressed as follows:

(iv) The elastic stress vector on every plane parallel to the z-axis is parallel to $\mathbf{k}$.

(v) The shear elastic stress along $\mathbf{n}(s)$ vanishes everywhere on $\mathscr{P}$.

(vi) The flux through the thickness of the in-plane shear elastic stress vector is zero.

Accordingly, the functional (2-7) can be written as

$$\mathscr{E}(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\omega})=\frac{1}{2}\int_{\Omega}\big[\mathscr{C}_{\alpha\beta\gamma\delta}(\Lambda_{\alpha\beta}-\omega_{\alpha\beta})(\Lambda_{\gamma\delta}-\omega_{\gamma\delta})+2\mathscr{C}_{\alpha\beta 33}(\Lambda_{\alpha\beta}-\omega_{\alpha\beta})u_3'+\mathscr{C}_{3333}(u_3')^2 +4\mathscr{C}_{\alpha 3\beta 3}(\Lambda_{\alpha 3}-\omega_{\alpha 3})(\Lambda_{\beta 3}-\omega_{\beta 3})\big]\,dv-\int_{\Omega}(\chi_{\alpha\beta}\Lambda_{\alpha\beta}+2\chi_{\alpha 3}\Lambda_{\alpha 3})\,dv-\Pi_{\text{ext}}, \tag{3-9}$$

where $\boldsymbol{\Lambda}=\hat{\nabla}\mathbf{u}$, whose components in the local tangent frame are evaluated through (3-6), and Π_{ext} accounts for external loads:

$$\Pi_{\text{ext}}=\int_{\Omega}(b_\alpha u_\alpha+b_3u_3)\,dv+\int_{\Sigma}(\hat{p}_\alpha u_\alpha+\hat{p}_3u_3)\,d\varrho dz+\int_{\mathscr{P}}(p_\alpha^{\pm}u_\alpha|_{\pm L}+p_3^{\pm}u_3|_{\pm L})\,da, \tag{3-10}$$

where $d\varrho$ is the arc element along $\partial\mathscr{P}$ and the notation $f^{\pm}(\cdot)|_{\pm L}$ means $f^{+}(\cdot)|_{z=L}+f^{-}(\cdot)|_{z=-L}$.

Note that, since constraints (iii) and (vi) are imposed by average condition (3-5), relevant Lagrange multipliers are defined on M, i.e., $\omega_{13}=\omega_{13}(s,z)$, $\chi_{13}=\chi_{13}(s,z)$.

The stationary conditions of the functional (3-9) with respect to $\chi_{\alpha\beta}$ and $\chi_{\alpha 3}$ give, respectively, the following constraints on the displacement field, i.e., on the total strain field:

$$\begin{aligned} \Lambda_{11}&=[u_{1/1}-\kappa u_2]/j=0, & 4\delta\overline{\Lambda}_{13}&=\int_{-\delta}^{\delta}(u_{3/1}+ju_1')\,d\eta=0,\\ \Lambda_{22}&=u_{2/2}=0, & &\\ 2\Lambda_{12}&=[u_{2/1}+\kappa u_1]/j+u_{1/2}=0, & 2\Lambda_{23}&=u_2'+u_{3/2}=0, \end{aligned} \tag{3-11}$$

where $u_1=\mathbf{u}\cdot\mathbf{t}$, $u_2=\mathbf{u}\cdot\mathbf{n}$ and $u_3=\mathbf{u}\cdot\mathbf{k}$.

Integration of (3-11) yields the Cartesian representation formulas for the displacement field:

$$\begin{aligned} u_x(s,\eta,z)&=\mathbf{u}\cdot\mathbf{i}=u_c(z)-\theta(z)[y(s,\eta)-y_c],\\ u_y(s,\eta,z)&=\mathbf{u}\cdot\mathbf{j}=v_c(z)+\theta(z)[x(s,\eta)-x_c],\\ u_3(s,\eta,z)&=\mathbf{u}\cdot\mathbf{k}=w_c(z)-u_c'(z)x(s,\eta)-v_c'(z)y(s,\eta)+\theta'(z)\psi_c(s,\eta) \end{aligned} \tag{3-12}$$

where x_c, y_c are the coordinates of an arbitrary point C in the plane $x-y$, assumed as the pole of the in-plane rigid rotation $\theta(z)$ of $\mathscr{P}$; $u_c(z)$, $v_c(z)$ and $w_c(z)$ are the components of the displacement of C along Cartesian axes; $\psi_c(s,\eta)$ is the warping function referred to C and defined as

$$\psi_c(s,\eta)=\psi_0+\int_0^s\mathbf{r}\cdot\mathbf{n}\,ds-\eta\,\mathbf{r}\cdot\mathbf{t}-\frac{1}{3}\int_0^s\delta^2\kappa\,ds, \tag{3-13}$$

where $\mathbf{r}=[\tilde{x}(s)-x_c]\mathbf{i}+[\tilde{y}(s)-y_c]\mathbf{j}$ is the position vector of a point $\tilde{\mathbf{x}}$ on p with respect to C. If $\int_{\mathscr{P}}\psi_c x\,da=\int_{\mathscr{P}}\psi_c y\,da=0$ then C coincides with the twist-center of $\mathscr{P}$. Moreover, the constant ψ_0 can be evaluated by equating to zero the average value of ψ_c on $\mathscr{P}$, i.e., considering the point S coincident to the sectorial centroid of p. The second and third term in (3-13) represent the middle-contour warping (i.e., the primary warping) and the thickness warping (i.e., the secondary warping), respectively. Furthermore, the last contribution takes into account high-order effects relating to thickness and curvature of p, generalizing the warping expression discussed by Lin and Hsiao [2003] and widely used in the specialized literature; see for example [Lee and Lee 2004; Lee 2005; Cortínez and Piovan 2002; Saadé et al. 2004; Piovan and Cortínez 2005; 2007].

The stationary conditions of the functional $\mathscr{E}$ with respect to $\omega_{\alpha\beta}$ and $\omega_{\alpha 3}$ give, respectively,

$$
\begin{aligned}
\mathscr{C}_{\alpha\beta\gamma\delta}[\Lambda_{\gamma\delta}-\omega_{\gamma\delta}]+\mathscr{C}_{\alpha\beta 33}u_3' &= 0,\\
\int_{-\delta}^{\delta}\mathscr{C}_{\alpha 313}[\Lambda_{\alpha 3}-\omega_{\alpha 3}]\, j(s,\eta)\, d\eta &= 0,\\
\mathscr{C}_{\alpha 323}[\Lambda_{\alpha 3}-\omega_{\alpha 3}] &= 0.
\end{aligned}
\tag{3-14}
$$

Taking into account (3-11), the Lagrange multipliers $\omega_{\alpha\beta}$ and $\omega_{\alpha 3}$ are uniquely determined by (3-14):

$$
\omega_{\alpha\beta}=(\mathscr{C}_{\alpha\beta\gamma\delta})^{-1}\mathscr{C}_{\gamma\delta 33}u_3', \qquad \omega_{13}=0, \qquad \omega_{23}=\mathscr{C}_{1323}\Lambda_{13}/\mathscr{C}_{2323}, \tag{3-15}
$$

where $(\mathscr{C}_{\alpha\beta\gamma\delta})^{-1}$ denotes the inverse tensor of $\mathscr{C}_{\alpha\beta\gamma\delta}$.

Substituting (3-12) and (3-15) into the functional (3-9) and performing integration over $\mathscr{P}$, the potential energy functional for the thin-walled beam in terms of pure displacement unknowns can be written as

$$
\hat{\mathscr{E}}(u_c, v_c, w_c, \theta)=\frac{1}{2}\int_{-L}^{L}\mathbf{D}\mathbf{e}\cdot\mathbf{e}\, dz-\int_{-L}^{L}\mathbf{q}\cdot\hat{\mathbf{s}}\, dz-\mathbf{Q}^{\pm}\cdot\hat{\mathbf{s}}|_{\pm L}, \tag{3-16}
$$

where the vector

$$
\hat{\mathbf{s}}=\{u_c \;\; v_c \;\; w_c \;\; -v_c' \;\; u_c' \;\; \theta \;\; \theta'\}^T \tag{3-17}
$$

collects the generalized displacements,

$$
\mathbf{e}=\{w_c' \;\; -v_c'' \;\; u_c'' \;\; \theta'' \;\; \theta'\}^T \tag{3-18}
$$

denotes the generalized total strain,

$$
\mathbf{q}=\{q_x \;\; q_y \;\; q_z \;\; m_x \;\; m_y \;\; m_z \;\; m_\psi\}^T, \quad \mathbf{Q}^{\pm}=\{Q_x^{\pm} \;\; Q_y^{\pm} \;\; Q_z^{\pm} \;\; M_x^{\pm} \;\; M_y^{\pm} \;\; M_z^{\pm} \;\; M_\psi^{\pm}\}^T \tag{3-19}
$$

indicate, respectively, the generalized distributed and end-located forces acting on the beam, with

$$
\{q_x, q_y, q_z\}=\int_{\mathscr{P}}\{b_x, b_y, b_z\}\, da+\int_{\partial\mathscr{P}}\{\hat{p}_x, \hat{p}_y, \hat{p}_z\}\, d\varrho, \tag{3-20}
$$

$$
\{m_x, m_y, m_\psi\}=\int_{\mathscr{P}}\{y, -x, \psi_c\}b_z\, da+\int_{\partial\mathscr{P}}\{y, -x, \psi_c\}\hat{p}_z\, d\varrho, \tag{3-21}
$$

$$
m_z=\int_{\mathscr{P}}[b_y(x-x_c)-b_x(y-y_c)]\, da+\int_{\partial\mathscr{P}}[\hat{p}_y(x-x_c)-\hat{p}_x(y-y_c)]\, d\varrho, \tag{3-22}
$$

$$
\{Q_x^{\pm}, Q_y^{\pm}, Q_z^{\pm}\}=\int_{\mathscr{P}}\{p_x^{\pm}, p_y^{\pm}, p_z^{\pm}\}\, da, \tag{3-23}
$$

$$
\{M_x^{\pm}, M_y^{\pm}, M_\psi^{\pm}\}=\int_{\mathscr{P}}\{y, -x, \psi_c\}p_z^{\pm}\, da, \qquad M_z^{\pm}=\int_{\mathscr{P}}[p_y^{\pm}(x-x_c)-p_x^{\pm}(y-y_c)]\, da, \tag{3-24}
$$

and finally

$$
\mathbf{D}=\begin{bmatrix}
\hat{\mathscr{C}}_{3333}A & \hat{\mathscr{C}}_{3333}S_x & \hat{\mathscr{C}}_{3333}S_y & \hat{\mathscr{C}}_{3333}S_\psi & 0\\
 & \hat{\mathscr{C}}_{3333}I_x & \hat{\mathscr{C}}_{3333}I_{xy} & \hat{\mathscr{C}}_{3333}I_{x\psi} & 0\\
 & & \hat{\mathscr{C}}_{3333}I_y & \hat{\mathscr{C}}_{3333}I_{y\psi} & 0\\
 & sym & & \hat{\mathscr{C}}_{3333}I_\psi & 0\\
 & & & & \hat{\mathscr{C}}_{1313}J_\theta
\end{bmatrix} \tag{3-25}
$$

is the generalized elasticity matrix, with

$$\{A, S_x, S_y, S_\psi\} = \int_{\mathcal{P}} \{1, y, -x, \psi_c\}\, da, \tag{3-26}$$

$$\{I_x, I_{xy}, I_y, I_{x\psi}, I_{y\psi}, I_\psi\} = \int_{\mathcal{P}} \{y^2, -xy, x^2, y\psi_c, -x\psi_c, \psi_c^2\}\, da, \tag{3-27}$$

$$J_\theta = \int_{\mathcal{P}} [\kappa(\eta^2 - \delta^2/3) - 2\eta]^2 / j^2\, da, \tag{3-28}$$

where J_θ is the torsion constant, which generalizes the one commonly used in classical thin-walled open cross-section beam models, through nonconventional high-order terms depending on curvature of p and thickness.

Note that the coefficients of $\mathbf{D}$ depend on the so-called reduced elastic moduli:

$$\hat{\mathscr{C}}_{1313} = \mathscr{C}_{1313} - \frac{\mathscr{C}_{1323}^2}{\mathscr{C}_{2323}}, \qquad \hat{\mathscr{C}}_{3333} = \mathscr{C}_{3333} - \mathscr{C}_{\alpha\beta 33}(\mathscr{C}_{\alpha\beta\gamma\delta})^{-1}\mathscr{C}_{\gamma\delta 33}. \tag{3-29}$$

As is customary with beam theories, the stress resultants on $\mathcal{P}$ (generalized stresses) are introduced:

$$\mathbf{S} = \{N \ \ C_x \ \ C_y \ \ C_\psi \ \ C_z\}^T = \mathbf{De}, \tag{3-30}$$

where $\mathbf{S}$ is the generalized stress vector, defined in terms of normal force N, bending moments C_x and C_y, (primary) twisting moment C_z and bimoment (warping torque) C_ψ:

$$\{N, C_x, C_y, C_\psi\} = \int_{\mathcal{P}} \{1, y, -x, \psi_c\}\sigma_{33}\, da, \qquad C_z = \int_{\mathcal{P}} (\sigma_{y3}x - \sigma_{x3}y)\, da. \tag{3-31}$$

Assuming that the global Cartesian frame is centered at the centroid of the beam $\mathbf{\Omega}$ and that its x and y axes are principal for $\mathcal{P}$, referring ψ_c to the twisting center for $\mathcal{P}$ and taking S coincident with the sectorial centroid of p, the matrix $\mathbf{D}$ reduces to a diagonal matrix and the functional $\hat{\mathscr{E}}$ of (3-16) can be written as

$$\hat{\mathscr{E}}(u_c, v_c, w_c, \theta) = \mathscr{A}(w_c) + \mathscr{F}_x(v_c) + \mathscr{F}_y(u_c) + \mathscr{T}(\theta), \tag{3-32}$$

where

$$\begin{aligned}
\mathscr{A}(w_c) &= \tfrac{1}{2}\hat{\mathscr{C}}_{3333} A \int_{-L}^{L} (w_c')^2\, dz - \int_{-L}^{L} q_z w_c\, dz - Q_z^{\pm} w_c|_{\pm L}, \\
\mathscr{F}_x(v_c) &= \tfrac{1}{2}\hat{\mathscr{C}}_{3333} I_x \int_{-L}^{L} (v_c'')^2\, dz - \int_{-L}^{L} (q_y v_c - m_x v_c')\, dz - (Q_y^{\pm} v_c|_{\pm L} - M_x^{\pm} v_c'|_{\pm L}), \\
\mathscr{F}_y(u_c) &= \tfrac{1}{2}\hat{\mathscr{C}}_{3333} I_y \int_{-L}^{L} (u_c'')^2\, dz - \int_{-L}^{L} (q_x u_c + m_y u_c')\, dz - (Q_x^{\pm} u_c|_{\pm L} + M_y^{\pm} u_c'|_{\pm L}), \\
\mathscr{T}(\theta) &= \tfrac{1}{2}\hat{\mathscr{C}}_{3333} I_\psi \int_{-L}^{L} (\theta'')^2\, dz + \tfrac{1}{2}\hat{\mathscr{C}}_{1313} J_\theta \int_{-L}^{L} (\theta')^2\, dz - \int_{-L}^{L} (m_z\theta + m_\psi\theta')\, dz \\
&\qquad - (M_z^{\pm}\theta|_{\pm L} + M_\psi^{\pm}\theta'|_{\pm L}).
\end{aligned} \tag{3-33}$$

The functional $\mathscr{A}$ models the extensional problem of the beam, $\mathscr{F}_x$ and $\mathscr{F}_y$ govern the Euler–Bernoulli flexural model of the beam, $\mathscr{T}$ is relevant to the torsional problem. Observe that these functionals are uncoupled and therefore the corresponding governing equations for extensional, flexural and torsional problems are themselves uncoupled.

The governing equations and natural boundary conditions for the extensional problem of the beam arise from the stationary conditions of $\mathcal{A}$ with respect to w_c:

$$\hat{\mathscr{C}}_{3333} A w_c'' + q_z = 0, \qquad \hat{\mathscr{C}}_{3333} A w_c'|_{-L}^{L} = Q_z^{\pm}, \tag{3-34}$$

where $f|_{-L}^{L}$ indicates $f(-L)$ for $z=-L$ or $f(L)$ for $z=L$.

Analogously, the stationary conditions of $\mathscr{F}_x$ and $\mathscr{F}_y$ give governing equations and boundary conditions for flexural problems in the yz and xz planes, respectively:

$$\hat{\mathscr{C}}_{3333} I_x v_c'''' - q_y - m_x' = 0, \qquad \begin{cases} \hat{\mathscr{C}}_{3333} I_x v_c'''|_{-L}^{L} - m_x|_{-L}^{L} = -Q_y^{\pm}, \\ \hat{\mathscr{C}}_{3333} I_x v_c''|_{-L}^{L} = -M_x^{\pm}, \end{cases} \tag{3-35}$$

$$\hat{\mathscr{C}}_{3333} I_y u_c'''' - q_x + m_y' = 0, \qquad \begin{cases} \hat{\mathscr{C}}_{3333} I_y u_c'''|_{-L}^{L} + m_y|_{-L}^{L} = -Q_x^{\pm}, \\ \hat{\mathscr{C}}_{3333} I_y u_c''|_{-L}^{L} = M_y^{\pm}. \end{cases} \tag{3-36}$$

Finally, the stationary conditions of the functional $\mathscr{T}$ yield the governing equations and boundary conditions for the torsional problem:

$$\hat{\mathscr{C}}_{3333} I_\psi \theta'''' - \hat{\mathscr{C}}_{1313} J_\theta \theta'' - m_z + m_\psi' = 0, \qquad \begin{cases} [\hat{\mathscr{C}}_{3333} I_\psi \theta''' - \hat{\mathscr{C}}_{1313} J_\theta \theta' + m_\psi]\big|_{-L}^{L} = -M_z^{\pm}, \\ \hat{\mathscr{C}}_{3333} I_\psi \theta''\big|_{-L}^{L} = M_\psi^{\pm}. \end{cases} \tag{3-37}$$

Solution of (3-34) to (3-37) gives the unknown functions u_c, v_c, w_c, θ from which total, reactive and elastic strain fields can be computed. Therefore, the elastic stress field $\boldsymbol{\sigma}^{(\mathrm{el})} = \mathscr{C}(\hat{\nabla}\mathbf{u} - \mathbf{G}^*\boldsymbol{\omega})$ referred to the local tangent frame turns out to be

$$\begin{aligned} \sigma_{\alpha\beta}^{(\mathrm{el})} &= \sigma_{23}^{(\mathrm{el})} = 0, \\ \sigma_{13}^{(\mathrm{el})} &= \hat{\mathscr{C}}_{1313}\theta'[\kappa(\eta^2 - \delta^2/3) - 2\eta]/j, \\ \sigma_{33}^{(\mathrm{el})} &= \hat{\mathscr{C}}_{3333}(w_c' - u_c'' x - v_c'' y + \theta'' \psi_c). \end{aligned} \tag{3-38}$$

The total stress field coincides with the elastic one only for the component σ_{33}, whereas total stress components $\sigma_{\alpha\beta}$ and $\sigma_{\alpha 3}$ can be not uniquely recovered in a general case from the equilibrium equations (2-3). Nevertheless, (3-30) furnish stress resultants on $\mathscr{P}$ which satisfy global equilibrium. Moreover, as will be showed in the following, a more accurate evaluation of total shear stress components on $\mathscr{P}$ may be recovered from the field equilibrium equation along z.

We emphasize that the so-called reduced constitutive law enters the expression of $\hat{\mathscr{E}}$ and the generalized constitutive law (3-30) by means of the reduced elastic moduli (3-29). The present derivation clearly shows how the appearance of these quantities is a straightforward and rational consequence of constraints on dual fields, without contradiction because they act on fields (total strain and elastic stress) which are not related by constitutive law. In other words, the reduced constitutive law comes out from the procedure adopted and is not a priori enforced by means of a constrained constitutive law.

Note also that when constraints on the total strain field only (i.e., without constraints (iv), (v) and (vi)) are enforced, the functional $\hat{\mathscr{E}}$ contains nonreduced elastic moduli, whereas when condition (iv) is added to total strain assumptions, $\hat{\mathscr{C}}_{3333}$ appears instead of $\mathscr{C}_{3333}$.

Finally, it is easy to verify that the discussed model, rationally deduced by means of the dual-constraint approach, corresponds to that proposed by Bauld and Tzeng [1984] when nonconventional high-order effects relating to curvature κ and thickness are neglected, and it reduces to Vlasov's classical one [1961; 1962] when an isotropic material is considered.

3C. ***First-order shear-deformable beam model.*** Since the wall thickness is small compared with other cross-section dimensions, for the sake of brevity we assume in what follows that

$$\delta \ll \int_p ds, \qquad \delta\kappa \ll 1, \tag{3-39}$$

and then $j \cong 1$.

In order to include first-order shear deformation effects, constraints (ii) and (iii) of Section 3B can be replaced by the following assumptions:

(ii′) The shear total strain between z-axis and $\mathbf{n}(s)$ is constant over $\mathcal{P}$.

(iii′) The flux through the thickness of the in-plane shear total strain vector is constant over $\mathcal{P}$.

The corresponding dual constraints on the elastic stress field are:

(v′) The shear elastic stress along $\mathbf{n}(s)$ is constant over $\mathcal{P}$.

(vi′) The flux through the thickness of the in-plane shear elastic stress vector is constant over $\mathcal{P}$.

Accordingly, the operator $\mathbf{G}$ turns out to be such that

$$\mathbf{G}\boldsymbol{\varepsilon} = \{\varepsilon_{11}\ \ \varepsilon_{22}\ \ \varepsilon_{12}\ \ (2\delta\bar{\varepsilon}_{13})_{/1}\ \ \varepsilon_{23/1}\ \ \varepsilon_{23/2}\}^T. \tag{3-40}$$

Under assumptions (3-39) and recalling (3-3), the following equality can be stated:

$$(2\delta\overline{\Lambda}_{x3})_{/1}\mathbf{i} + (2\delta\overline{\Lambda}_{y3})_{/1}\mathbf{j} \cong (2\delta\overline{\Lambda}_{13})_{/1}\mathbf{t} + (2\delta\overline{\Lambda}_{23})_{/1}\mathbf{n} = \mathbf{0}. \tag{3-41}$$

Therefore, constraints (ii′) and (iii′) are equivalent to requiring that the integral over the thickness of total shear strain between the z-axis and every in-plane direction depends only on z, that is, to the Timoshenko-type kinematic shear assumptions

$$2\overline{\Lambda}_{x3} = \gamma_x(z), \qquad 2\overline{\Lambda}_{y3} = \gamma_y(z). \tag{3-42}$$

Analogously, the dual constraints (v′) and (vi′) can be thought as referring to $\sigma^{(\mathrm{el})}_{x3}$ and $\sigma^{(\mathrm{el})}_{y3}$.

Hence, the functional (2-7) can be written as

$$\begin{aligned}\mathscr{E}(\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\omega}) = \frac{1}{2}\int_\Omega \big\{ &\mathscr{C}_{\alpha\beta\gamma\delta}(\Lambda_{\alpha\beta} - \omega_{\alpha\beta})(\Lambda_{\gamma\delta} - \omega_{\gamma\delta}) \\ &+ 2\mathscr{C}_{\alpha\beta33}(\Lambda_{\alpha\beta} - \omega_{\alpha\beta})u_3' + \mathscr{C}_{3333}(u_3')^2 + 4\mathscr{C}_{1313}[\Lambda_{13} + (2\delta\,\omega_{13})_{/1}]^2 \\ &+ 8\mathscr{C}_{1323}[\Lambda_{13} + (2\delta\,\omega_{13})_{/1}][\Lambda_{23} + \tilde{\omega}_{\alpha3/\alpha}] + 4\mathscr{C}_{2323}[\Lambda_{23} + \tilde{\omega}_{\alpha3/\alpha}]^2 \big\}\, dv \\ &- \int_\Omega [\chi_{\alpha\beta}\Lambda_{\alpha\beta} + 2\chi_{13}(2\delta\overline{\Lambda}_{13})_{/1} + 2\tilde{\chi}_{\alpha3}\Lambda_{23/\alpha}]\, dv - \Pi_{\mathrm{ext}},\end{aligned} \tag{3-43}$$

where $\tilde{\omega}_{\alpha3}$, $\tilde{\chi}_{\alpha3}$, $\omega_{13}(s,z)$, and $\chi_{13}(s,z)$ are the Lagrange multipliers relating to constraints (ii′), (v′), (iii′) and (vi′), respectively. The functional $\mathscr{E}$ is defined on the manifold: $\tilde{\chi}_{\alpha3} = \tilde{\omega}_{\alpha3} = 0$ on Σ and $\omega_{13} = \chi_{13} = 0$ on $\partial p \times]-L, L[$.

The stationary conditions of (3-43) with respect to $\boldsymbol{\chi}$ return constraints on the displacement field:

$$u_{1/1} - \kappa u_2 = 0, \quad u_{2/2} = 0, \quad u_{2/1} + \kappa u_1 + u_{1/2} = 0, \tag{3-44}$$

$$\int_{-\delta}^{\delta} (u_{3/11} + u'_{1/1})\, d\eta = 0, \quad u'_{2/\alpha} + u_{3/2\alpha} = 0, \tag{3-45}$$

whose integration yields the Cartesian representation formulas

$$\begin{aligned} u_x(s,\eta,z) &= u_c(z) - \theta(z)[y(s,\eta) - y_c],\\ u_y(s,\eta,z) &= v_c(z) + \theta(z)[x(s,\eta) - x_c],\\ u_3(s,\eta,z) &= w_c(z) - \phi_y(z)x(s,\eta) + \phi_x(z)y(s,\eta) + \theta'(z)\psi_c(s,\eta), \end{aligned}\tag{3-46}$$

where the positions introduced in (3-12) are assumed to be valid and where

$$\phi_x(z) = \gamma_y(z) - v'_c(z), \quad \phi_y(z) = u'_c(z) - \gamma_x(z). \tag{3-47}$$

The stationary conditions of $\mathscr{E}$ in (3-43) with respect to $\boldsymbol{\omega}$ give the equations

$$\begin{aligned} \mathscr{C}_{\alpha\beta\gamma\delta}[\Lambda_{\gamma\delta} - \omega_{\gamma\delta}] + \mathscr{C}_{\alpha\beta 33}u'_3 &= 0,\\ \left\{\int_{-\delta}^{\delta} \mathscr{C}_{1313}[\Lambda_{13} + (2\delta\,\omega_{13})_{/1}] + \mathscr{C}_{1323}[\Lambda_{23} + \tilde{\omega}_{\alpha 3/\alpha}]\, d\eta\right\}_{/1} &= 0,\\ \{\mathscr{C}_{1323}[\Lambda_{13} + (2\delta\,\omega_{13})_{/1}] + \mathscr{C}_{2323}[\Lambda_{23} + \tilde{\omega}_{\alpha 3/\alpha}]\}_{/\beta} &= 0, \end{aligned}\tag{3-48}$$

from which the Lagrange multipliers $\omega_{\alpha\beta}$, ω_{13} and $\tilde{\omega}_{\alpha 3}$ can be found:

$$\omega_{\alpha\beta} = (\mathscr{C}_{\alpha\beta\gamma\delta})^{-1}\mathscr{C}_{\gamma\delta 33}u'_3, \quad \omega_{13} = \tilde{\omega}_{13} = 0, \quad \tilde{\omega}_{23} = \frac{\mathscr{C}_{1323}}{\mathscr{C}_{2323}}\theta'(\eta^2 - \delta^2). \tag{3-49}$$

Substituting equations (3-46) and (3-49) into the functional (3-43), the potential energy functional $\hat{\mathscr{E}}$ is obtained in terms of pure displacement unknowns $u_c, v_c, w_c, \phi_x, \phi_y, \theta$, and it can be expressed as in (3-16). In this case, the generalized displacements $\hat{\mathbf{s}}$ and generalized strains $\mathbf{e}$ are

$$\hat{\mathbf{s}} = \{u_c \;\; v_c \;\; w_c \;\; \phi_x \;\; \phi_y \;\; \theta \;\; \theta'\}^T, \quad \mathbf{e} = \{w'_c \;\; \phi'_x \;\; \phi'_y \;\; \theta'' \;\; \theta' \;\; \gamma_x \;\; \gamma_y\}^T,$$

and the generalized elastic matrix of the beam, of size 7×7, takes on the form

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{(1)} \end{bmatrix} \tag{3-50}$$

where the 5×5 submatrix $\mathbf{D}^{(0)}$ coincides with the matrix $\mathbf{D}$ of (3-25), derived for the no-shear case, and the symmetrical 2×2 submatrix $\mathbf{D}^{(1)}$ is defined by

$$\begin{aligned} D^{(1)}_{11} &= \mathscr{C}_{1313}\int_{\mathscr{P}} (\tilde{x}_{/1})^2\, da + \mathscr{C}_{2323}\int_{\mathscr{P}} (\tilde{y}_{/1})^2\, da - 2\mathscr{C}_{1323}\int_{\mathscr{P}} \tilde{x}_{/1}\,\tilde{y}_{/1}\, da,\\ D^{(1)}_{12} &= (\mathscr{C}_{1313} - \mathscr{C}_{2323})\int_{\mathscr{P}} \tilde{x}_{/1}\,\tilde{y}_{/1}\, da + \mathscr{C}_{1323}\int_{\mathscr{P}} [(\tilde{x}_{/1})^2 - (\tilde{y}_{/1})^2]\, da,\\ D^{(1)}_{22} &= \mathscr{C}_{2323}\int_{\mathscr{P}} (\tilde{x}_{/1})^2\, da + \mathscr{C}_{1313}\int_{\mathscr{P}} (\tilde{y}_{/1})^2\, da + 2\mathscr{C}_{1323}\int_{\mathscr{P}} \tilde{x}_{/1}\,\tilde{y}_{/1}\, da. \end{aligned}\tag{3-51}$$

Moreover, in this model the generalized stress vector $\mathbf{S} = \mathbf{D}\mathbf{e}$ contains shear resultant forces T_x and T_y, representing static quantities associated to the generalized strains γ_x and γ_y, respectively:

$$\{T_x, T_y\} = \int_{\mathcal{P}} \boldsymbol{\tau} \cdot \{\mathbf{i}, \mathbf{j}\}\, da = \int_{\mathcal{P}} \sigma_{13}\{\tilde{x}_{/1}, \tilde{y}_{/1}\}\, da + \int_{\mathcal{P}} \sigma_{23}\{-\tilde{y}_{/1}, \tilde{x}_{/1}\})\, da. \tag{3-52}$$

Under geometrical assumptions making $\mathbf{D}^{(0)}$ diagonal, the functional $\hat{\mathscr{E}}$ can be written in the form

$$\hat{\mathscr{E}}(u_c, v_c, w_c, \phi_x, \phi_y, \theta) = \mathscr{A}(w_c) + \mathscr{F}(u_c, v_c, \phi_x, \phi_y) + \mathscr{T}(\theta), \tag{3-53}$$

where $\mathscr{A}$ and $\mathscr{T}$ are defined as in (3-33), and the functional $\mathscr{F}$ is

$$\begin{aligned}\mathscr{F}(u_c, v_c, \phi_x, \phi_y) = \tfrac{1}{2}\hat{\mathscr{C}}_{3333}\left[I_x \int_{-L}^{L} (\phi_x')^2\, dz + I_y \int_{-L}^{L} (\phi_y')^2\, dz\right] + \tfrac{1}{2}D_{11}^{(1)} \int_{-L}^{L} (u_c' - \phi_y)^2\, dz \\ + D_{12}^{(1)} \int_{-L}^{L} (u_c' - \phi_y)(v_c' + \phi_x)\, dz + \tfrac{1}{2}D_{22}^{(1)} \int_{-L}^{L} (v_c' + \phi_x)^2\, dz.\end{aligned} \tag{3-54}$$

The stationary conditions of the functional $\hat{\mathscr{E}}$ of (3-53) supply governing equations and natural boundary conditions for the beam, allowing us to compute the unknown functions u_c, v_c, w_c, ϕ_x, ϕ_y, and θ. In detail, the extensional and torsional problems are governed by (3-34) and (3-37), and the stationary conditions of the functional $\mathscr{F}$ yield governing equations of the Timoshenko flexural beam model, which, for the sake of brevity, are not reported here.

The elastic stress field turns out to be

$$\sigma_{\alpha\beta}^{(\mathrm{el})} = 0, \qquad \begin{aligned} \sigma_{13}^{(\mathrm{el})} &= (\mathscr{C}_{1313}\tilde{x}_{/1} - \mathscr{C}_{1323}\tilde{y}_{/1})\gamma_x + (\mathscr{C}_{1313}\tilde{y}_{/1} + \mathscr{C}_{1323}\tilde{x}_{/1})\gamma_y - 2\hat{\mathscr{C}}_{1313}\theta'\eta, \\ \sigma_{23}^{(\mathrm{el})} &= (\mathscr{C}_{1323}\tilde{x}_{/1} - \mathscr{C}_{2323}\tilde{y}_{/1})\gamma_x + (\mathscr{C}_{1323}\tilde{y}_{/1} + \mathscr{C}_{2323}\tilde{x}_{/1})\gamma_y, \\ \sigma_{33}^{(\mathrm{el})} &= \hat{\mathscr{C}}_{3333}(w_c' - \phi_y' x + \phi_x' y + \theta'' \psi_c). \end{aligned} \tag{3-55}$$

The total stress field coincides with the elastic one only for the component σ_{33}, whereas, as for the case of the no-shear model, total stress components $\sigma_{\alpha\beta}$ and $\sigma_{\alpha 3}$ can not be uniquely recovered in a general case from the equilibrium equations (2-3).

We note that the present model corresponds to the one proposed in [Ascione et al. 2000] when Galerkin-type enhancements are neglected and, when applied to homogeneous beams, to the model analyzed in [Maddur and Chaturvedi 1999]. Moreover, it reduces to the no-shear model discussed in Section 3B when γ_x and γ_y are enforced to vanish.

3D. ***A shear refinement.*** The procedure until now employed can be successfully applied for deducing in a consistent way different thin-walled beam models, in which shear deformation effects can be taken into account through different refinement levels. For instance, the kinematics employed in [Lee 2005; Cortínez and Piovan 2002; Piovan and Cortínez 2007] and including shear effects due to nonuniform warping by a first-order warping shear term, can be deduced, under assumptions (3-39), by enforcing on the total strain field, together with (i) and (iii′), the following constraint:

(ii″) The shear total strain between z-axis and $\mathbf{n}(s)$ at every position $(s, \eta) \in \mathcal{P}$ does not depend upon the thickness coordinate η and varies linearly along p.

Constraints (ii$''$) and (iii$'$) are equivalent to prescribing the equalities

$$2\overline{\Lambda}_{xz} = \gamma_x(z) - \gamma_\psi(z)(\tilde{y} - y_c), \qquad 2\overline{\Lambda}_{yz} = \gamma_y(z) + \gamma_\psi(z)(\tilde{x} - x_c), \tag{3-56}$$

where γ_ψ is the first-order warping shear unknown function. In this case, the operator $\mathbf{G}$ is given by

$$\mathbf{G}\boldsymbol{\varepsilon} = \{\varepsilon_{11} \;\; \varepsilon_{22} \;\; \varepsilon_{12} \;\; (2\delta\,\bar{\varepsilon}_{13})_{/1} \;\; \varepsilon_{23/11} \;\; \varepsilon_{23/2}\}^T. \tag{3-57}$$

Thus the admissible displacement field belonging to the kernel of $\mathbf{G}$ can be represented through (3-46) for the components u_x and u_y, and the displacement along beam axis z becomes

$$u_3(s, \eta, z) = w_c(z) - \phi_y(z)x(s, \eta) + \phi_x(z)y(s, \eta) + \phi_z(z)\psi_c(s, \eta), \tag{3-58}$$

where

$$\phi_z(z) = \theta'(z) - \gamma_\psi(z). \tag{3-59}$$

Following the present dual-constraint approach, the constraints on the elastic stress field are directly stated and the potential energy functional is

$$\begin{aligned}\mathscr{E}(\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\omega}) = \frac{1}{2}\int_\Omega \big\{&\mathscr{C}_{\alpha\beta\gamma\delta}(\Lambda_{\alpha\beta} - \omega_{\alpha\beta})(\Lambda_{\gamma\delta} - \omega_{\gamma\delta}) + 2\mathscr{C}_{\alpha\beta 33}(\Lambda_{\alpha\beta} - \omega_{\alpha\beta})u_3' \\ &+ \mathscr{C}_{3333}(u_3')^2 + 8\mathscr{C}_{1323}[\Lambda_{13} + (2\delta\,\omega_{13})_{/1}][\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}] \\ &+ 4\mathscr{C}_{1313}[\Lambda_{13} + (2\delta\,\omega_{13})_{/1}]^2 + 4\mathscr{C}_{2323}[\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}]^2\big\}\,dv \\ &- \int_\Omega [\chi_{\alpha\beta}\Lambda_{\alpha\beta} + 2\chi_{13}(2\delta\overline{\Lambda}_{13})_{/1} + 2(\tilde{\chi}_{13}\Lambda_{23/11} + \tilde{\chi}_{23}\Lambda_{23/2})]\,dv - \Pi_{\text{ext}},\end{aligned} \tag{3-60}$$

defined on the manifold: $\tilde{\chi}_{\alpha 3} = \tilde{\chi}_{23/1} = \tilde{\omega}_{\alpha 3} = \tilde{\omega}_{13/1} = 0$ on Σ, and $\omega_{13} = \chi_{13} = 0$ on $\partial p \times]-L, L[$.

Consistent generalized stress-strain relationship and governing equations can be rationally obtained as in the previous sections, and now the vectors of generalized displacements, strains and stresses become

$$\begin{aligned}\hat{\mathbf{s}} &= \{u_c \;\; v_c \;\; w_c \;\; \phi_x \;\; \phi_y \;\; \theta \;\; \phi_z\}^T, \\ \mathbf{e} &= \{w_c' \;\; \phi_x' \;\; \phi_y' \;\; \phi_z' \;\; \theta' \;\; \gamma_x \;\; \gamma_y \;\; \gamma_\psi\}^T, \\ \mathbf{S} &= \{N \;\; C_x \;\; C_y \;\; C_\psi \;\; C_z \;\; T_x \;\; T_y \;\; C_z^\psi\}^T,\end{aligned} \tag{3-61}$$

where the static quantity C_z^ψ associated to γ_ψ represents the secondary twisting moment, defined as

$$C_z^\psi = \int_{\mathscr{P}} (\sigma_{13}\,\mathbf{r}\cdot\mathbf{t} - \sigma_{23}\,\mathbf{r}\cdot\mathbf{n})\,da. \tag{3-62}$$

Accordingly, generalized elastic matrix $\mathbf{D}$ is expressed as in (3-50), with a symmetrical submatrix $\mathbf{D}^{(1)}$ of size 3×3; the new elements of $\mathbf{D}$ are

$$\begin{aligned}D_{13}^{(1)} &= -\mathscr{C}_{1313}\int_{\mathscr{P}} \mathbf{r}\cdot\mathbf{n}\,\tilde{x}_{/1}\,da + \mathscr{C}_{1323}\int_{\mathscr{P}} (\mathbf{r}\cdot\mathbf{t}\,\tilde{x}_{/1} + \mathbf{r}\cdot\mathbf{n}\,\tilde{y}_{/1})\,da - \mathscr{C}_{2323}\int_{\mathscr{P}} \mathbf{r}\cdot\mathbf{t}\,\tilde{y}_{/1}\,da, \\ D_{23}^{(1)} &= -\mathscr{C}_{1313}\int_{\mathscr{P}} \mathbf{r}\cdot\mathbf{n}\,\tilde{y}_{/1}\,da + \mathscr{C}_{1323}\int_{\mathscr{P}} (\mathbf{r}\cdot\mathbf{t}\,\tilde{y}_{/1} - \mathbf{r}\cdot\mathbf{n}\,\tilde{x}_{/1})\,da + \mathscr{C}_{2323}\int_{\mathscr{P}} \mathbf{r}\cdot\mathbf{t}\,\tilde{x}_{/1}\,da, \\ D_{33}^{(1)} &= \mathscr{C}_{1313}\int_{\mathscr{P}} (\mathbf{r}\cdot\mathbf{n})^2\,da - 2\mathscr{C}_{1323}\int_{\mathscr{P}} (\mathbf{r}\cdot\mathbf{t})(\mathbf{r}\cdot\mathbf{n})\,da + \mathscr{C}_{2323}\int_{\mathscr{P}} (\mathbf{r}\cdot\mathbf{t})^2\,da.\end{aligned} \tag{3-63}$$

The elastic stress field comes out as

$$\sigma_{\alpha\beta}^{(\mathrm{el})}=0,\qquad \begin{aligned}\sigma_{13}^{(\mathrm{el})}&=\hat{\sigma}_{13}^{(\mathrm{el})}-(\mathscr{C}_{1313}\,\mathbf{r}\cdot\mathbf{n}-\mathscr{C}_{1323}\,\mathbf{r}\cdot\mathbf{t})\gamma_{\psi},\\ \sigma_{23}^{(\mathrm{el})}&=\hat{\sigma}_{23}^{(\mathrm{el})}-(\mathscr{C}_{1323}\,\mathbf{r}\cdot\mathbf{n}-\mathscr{C}_{2323}\,\mathbf{r}\cdot\mathbf{t})\gamma_{\psi},\\ \sigma_{33}^{(\mathrm{el})}&=\hat{\mathscr{C}}_{3333}(w_c'-\phi_y'x+\phi_x'y+\phi_z'\psi_c),\end{aligned}\tag{3-64}$$

where $\hat{\sigma}_{\alpha 3}^{(\mathrm{el})}$ is the elastic in-plane shear stress components expressed in (3-55).

The present model reduces, for homogeneous thin-walled beams, to the one employed, for instance, in [Lee 2005; Cortínez and Piovan 2002] and it coincides with the model deduced in Section 3C when γ_ψ is enforced to vanish.

3E. *A branchwise model.* If the thin-walled beam cross-section is subdivided in N_b branches (as it naturally occurs for I, C, L, Z -type profiles) a new and more refined consistent model could be developed by enforcing dual constraints associated to (i), (ii$''$) and (iii$'$) for every branch b. As for layerwise laminated plate models [Bisegna and Sacco 1997a], a branchwise compatible displacement field comes out with usual components u_x and u_y and with u_3 represented as

$$u_3^{(b)}(s,\eta,z)=w^{(b)}(z)-\phi_y^{(b)}(z)x(s,\eta)+\phi_x^{(b)}(z)y(s,\eta)+\phi_z^{(b)}(z)\psi_c(s,\eta).\tag{3-65}$$

The quantities $w^{(b)}(z)$ can be easily expressed in terms of unknown functions $w_c(z)$, $\phi_x^{(b)}(z)$, $\phi_y^{(b)}(z)$ and $\phi_z^{(b)}(z)$ by imposing continuity conditions on u_3 at every centerline joint between contiguous branches. Accordingly, the number of displacement unknown functions is $4+3N_b$ (namely u_c, v_c, w_c, θ, $\phi_x^{(b)}$, $\phi_y^{(b)}$, $\phi_z^{(b)}$, where b varies from 1 to N_b) and the corresponding governing equations can be deduced following the previous variational constrained approach. Further developments will be given in a future paper.

4. Recovering total shear stresses

In many technical problems the primary interest is on the total stress field (elastic plus reactive stress field), that is the stress field which is in equilibrium with the external loads. Indeed, just total stress values should be compared against material's strength in order to prevent failure. Therefore, once the elastic stress field is obtained by means of the dual-constraint approach, the reactive stress field should be determined. Unfortunately, the constraint (i) employed in the previously discussed thin-walled beam models and prescribing the rigid rotation of the beam cross-section in its representation plane, does not enable to recover equilibrated stress components $\sigma_{\alpha\beta}$. Nevertheless, a technical analysis of beam-type structural elements usually involves the stress vector on the plane of $\mathscr{P}$, i.e., only the stress components σ_{33}, $\sigma_{\alpha 3}$.

For all cases under investigation the component $\sigma_{33}^{(\mathrm{el})}$ of the elastic stress field coincides with the total σ_{33}, and an useful estimate of the total shear stress components $\sigma_{\alpha 3}$, much more accurate in a local sense than the elastic ones, can be obtained as follows.

The field and boundary (on Σ) equilibrium equations along z can be written, respectively, as

$$\begin{aligned}\operatorname{Div}\boldsymbol{\tau}+\sigma_{33/3}+b_3=0&\qquad\text{in }\Omega,\\ \boldsymbol{\tau}\cdot\mathbf{n}_\partial=\hat{p}_3&\qquad\text{on }\Sigma,\end{aligned}\tag{4-1}$$

where Div denotes the divergence operator acting on $\mathcal{P}$ and the in-plane total shear stress vector $\boldsymbol{\tau}$ is defined as in (3-7). Due to the stated assumptions (see Section 3A) the outward normal unit vector $\mathbf{n}_\partial$ to the mantle Σ (i.e., to $\partial\mathcal{P}$) can be approximated by means of normal and tangent unit vectors to the center-line p, that is

$$\mathbf{n}_\partial = \begin{cases} \pm\mathbf{n} \ \text{ on } \ \partial\mathcal{P}|_{\eta=\pm\delta}, \\ -\mathbf{t} \ \text{ on } \ \partial\mathcal{P}|_{s=s_0}, \\ \ \ \mathbf{t} \ \text{ on } \ \partial\mathcal{P}|_{s=s_\ell}. \end{cases} \tag{4-2}$$

Consider an internal beam cross-section $\mathcal{P}^i = \mathcal{P} \times \{z^i\}$, where $z^i \in]-L, L[$, and let $s^i \in]s_0, s_\ell[$ to be an internal coordinate on p. Then $\mathcal{P}^i$ can be subdivided into two complementary parts $\mathcal{P}^i_1$ and $\mathcal{P}^i_2$, such that $\mathcal{P}^i = \mathcal{P}^i_1 \cup \mathcal{P}^i_2$ and $\mathcal{P}^i_1 \cap \mathcal{P}^i_2 = \xi$, where ξ denotes the thickness chord at s^i, i.e., $\xi = \{s^i\} \times]-\delta, \delta[$, whose unit normal vector is $\mathbf{t}(s^i)$, outward directed from $\mathcal{P}^i_1$.

Integrating $(4\text{-}1)_1$ over $\mathcal{P}^i_1$, applying the divergence theorem and taking $(4\text{-}1)_2$ into account, the exact average value over the thickness of the total shear stress σ_{13} is obtained as

$$\bar{\sigma}_{13}(s^i, z) = -\frac{1}{2\delta}\left(\int_{\mathcal{P}^i_1} (\sigma_{33/3} + b_3)\, da + \int_{\partial\mathcal{P}^i_1 \setminus \xi} \hat{p}_3 \, d\varrho\right), \tag{4-3}$$

where, expressing σ_{33} via (3-64), we have

$$\int_{\mathcal{P}^i_1} \sigma_{33/3}\, da = \hat{\mathscr{C}}_{3333}(w_c'' A_1 + \phi_x'' S_{x1} + \phi_y'' S_{y1} + \phi_z'' S_{\psi 1}); \tag{4-4}$$

here the quantities A_1 and $S_{\cdot 1}$ are defined as in (3-26) and refer to $\mathcal{P}^i_1$, i.e., as functions of s^i.

Since the wall thickness is small, the reactive part relating to σ_{13} can be identified with its average value over the thickness. Accordingly, the total shear stress σ_{13} can be approximated as

$$\sigma_{13}(s^i, \eta, z) \cong \sigma_{13}^{(\text{el})} + (\bar{\sigma}_{13} - \bar{\sigma}_{13}^{(\text{el})}) = -2\hat{\mathscr{C}}_{1313}\theta' + \bar{\sigma}_{13}. \tag{4-5}$$

Observe that (4-5) satisfies boundary conditions at $\partial\mathcal{P}^i|_{s=s_0}$ and $\partial\mathcal{P}^i|_{s=s_\ell}$ in an integral sense, i.e., considering the resultant over the thickness of distribution $\hat{p}_3$ at $s = s_0$ and $s = s_\ell$. Nevertheless, in the framework of assumptions (3-39) and considering s^i sufficiently far from ∂p, estimate (4-5) can be applied without significant error.

In order to give significant indications about the total stress component σ_{23}, due to (4-3), (4-4) and (4-5), and observing that $\sigma_{13/1} = \bar{\sigma}_{13/1}$, the field equilibrium along z in (4-1) can be written as

$$\sigma_{23/2} + (\sigma_{33/3} - \bar{\sigma}_{33/3}) + (b_3 - \bar{b}_3) - \frac{\hat{p}_3^+ + \hat{p}_3^-}{2\delta} = 0 \qquad \text{in } \Omega \tag{4-6}$$

where $\hat{p}_3^+$ and $\hat{p}_3^-$ stand for the components of surface forces $\hat{\mathbf{p}}$ along z at $\partial\mathcal{P}^i_1|_{\eta=\delta}$ and $\partial\mathcal{P}^i_1|_{\eta=-\delta}$, respectively.

Let the volume force component b_3 be almost linear in η-coordinate or let such an approximation be possible. Using (3-64) and taking into account boundary conditions $\sigma_{23}|_{\eta=\pm\delta} = \pm\hat{p}_3^\pm$, we integrate (4-6) with respect to η to obtain

$$\sigma_{23}(s^i, \eta, z) = \frac{\Xi}{2}(\delta^2 - \eta^2) + \frac{\hat{p}_3^+ + \hat{p}_3^-}{2\delta}\eta + \frac{\hat{p}_3^+ - \hat{p}_3^-}{2} \tag{4-7}$$

with

$$\Xi = \hat{\mathscr{C}}_{3333}[\phi_x''\tilde{x}_{/1} + \phi_y''\tilde{y}_{/1} - \phi_z''\mathbf{r}\cdot\mathbf{t}] + \left.\frac{\partial b_3}{\partial \eta}\right|_{\eta=0}. \tag{4-8}$$

Therefore, in the limit of positions (3-39) and assuming that the components of the surface forces $\mathbf{p}$ acting upon the beam ends $\mathscr{P}^{\pm}$ coincide with the total stress components at $\mathscr{P}^{\pm}$ (minus at $\mathscr{P}^{-}$), equations (4-5) and (4-7) recover the in-plane total shear stress field. However, even when the boundary equilibrium on $\mathscr{P}^{\pm}$ is not locally satisfied, (4-5) and (4-7) can be considered as an useful approximation in the spirit of Saint-Venant's principle [Toupin 1965].

5. Some generalizations

We now turn to some generalized models in the case of laminated thin-walled beam and of nonlinear conewise elastic materials, with particular reference to bimodular ones.

5A. *Laminated thin-walled beams.* Assume the thin-walled beam Ω is formed by perfectly bonded layers and the beam cross-section $\mathscr{P}$ is obtained as the union of N_b perfectly bonded branches $\mathscr{P}^{(b)}$, such that the centerline curve p can be defined as the union of the branch centerlines $p^{(b)}$. Each branch is formed by $N_l^{(b)}$ layers and in what follows any quantity referring to the l-th layer of the b-th branch is marked by superscript (b, l). Shape and dimensions of each branch $\mathscr{P}^{(b)}$ are assumed to be independent on z. Accordingly, the l-th layer of the b-th branch occupies the region

$$\Omega^{(b,l)} = \mathscr{P}^{(b)} \times]-L, L[= p^{(b)} \times]\eta^{(b,l-1)}, \eta^{(b,l)}[\times]-L, L[,$$

with $\eta^{(b,0)} = -\delta^{(b)}$ and $\eta^{(b,N_l^{(b)})} = \delta^{(b)}$, where $2\delta^{(b)}$ is the overall thickness of the b-th branch. Each layer of the laminated beam is assumed to be homogeneous and comprising a linearly elastic material, having at least a monoclinic symmetry with symmetry plane orthogonal to $\mathbf{k}$. This material symmetry enables to model multilayer composite profiles commonly used in civil engineering, as for example fiber-reinforced beams produced with pultrusion technology, characterized by orthotropic symmetry with one of the orthotropy axes coincident with the beam axis-line.

If the first-order shear-deformable beam model including shear effects due to nonuniform warping is considered, constraints (i), (ii$''$) and (iii$'$) (see Sections 3B, 3C and 3D) on the total strain field have to be enforced on Ω. Correspondingly, the dual constraints on the elastic stress field are (iv), (v$''$) and (vi$'$) and they refer to each region $\Omega^{(b,l)}$. Accordingly, the potential energy functional $\mathscr{E}$ is expressed by (3-60), where the integrals over $\mathscr{P}$ have to be regarded as the summation of integrals over $\mathscr{P}^{(b,l)}$, and it is defined on the manifold: $\tilde{\chi}_{\alpha 3} = \tilde{\chi}_{23/1} = 0$ on Σ, $\tilde{\omega}_{\alpha 3} = \tilde{\omega}_{13/1} = 0$ on $\partial\mathscr{P}^{(b,l)} \times]-L, L[$, $\chi_{13} = 0$ on $\partial p \times]-L, L[$, and $\omega_{13} = 0$ on $\partial p^{(b)} \times]-L, L[$.

Following the procedure discussed previously and eliminating the dependency on $\boldsymbol{\omega}$ and $\boldsymbol{\chi}$ in $\mathscr{E}$, the kinematics of the beam is described by equations $(3\text{-}46)_1$, $(3\text{-}46)_2$, and (3-58); the generalized vectors are defined as in (3-19) and (3-61); the elastic stress field for each region $\Omega^{(b,l)}$ is expressed by (3-64) and the nonzero elements of the 8×8 symmetric generalized elastic matrix $\mathbf{D}$ are

$$D_{11} = \sum_b \int_{p^{(b)}} \hat{\mathscr{A}}^{(b)}_{3333}\, ds, \qquad D_{12} = \sum_b \int_{p^{(b)}} (\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{y} + \hat{\mathscr{B}}^{(b)}_{3333}\, \tilde{x}_{/1})\, ds,$$

$$D_{13} = -\sum_b \int_{p^{(b)}} (\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{x} - \hat{\mathscr{B}}^{(b)}_{3333}\, \tilde{y}_{/1})\, ds, \qquad D_{14} = \sum_b \int_{p^{(b)}} (\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{\psi}_c - \hat{\mathscr{B}}^{(b)}_{3333}\, \mathbf{r}\cdot\mathbf{t})\, ds,$$

$$D_{22} = \sum_b \int_{p^{(b)}} [\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{y}^2 + 2\hat{\mathscr{B}}^{(b)}_{3333}\, \tilde{x}_{/1}\, \tilde{y} + \hat{\mathscr{D}}^{(b)}_{3333}(\tilde{x}_{/1})^2]\, ds,$$

$$D_{23} = -\sum_b \int_{p^{(b)}} [\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{x}\tilde{y} + \hat{\mathscr{B}}^{(b)}_{3333}(\tilde{x}\tilde{x}_{/1} - \tilde{y}\tilde{y}_{/1}) - \hat{\mathscr{D}}^{(b)}_{3333}\tilde{x}_{/1}\tilde{y}_{/1}]\, ds,$$

$$D_{24} = \sum_b \int_{p^{(b)}} [\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{y}\tilde{\psi}_c + \hat{\mathscr{B}}^{(b)}_{3333}(\tilde{x}_{/1}\tilde{\psi}_c - \tilde{y}\mathbf{r}\cdot\mathbf{t}) - \hat{\mathscr{D}}^{(b)}_{3333}\tilde{x}_{/1}\mathbf{r}\cdot\mathbf{t}]\, ds,$$

$$D_{33} = \sum_b \int_{p^{(b)}} [\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{x}^2 - 2\hat{\mathscr{B}}^{(b)}_{3333}\tilde{x}\, \tilde{y}_{/1} + \hat{\mathscr{D}}^{(b)}_{3333}(\tilde{y}_{/1})^2]\, ds,$$

$$D_{34} = -\sum_b \int_{p^{(b)}} [\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{x}\, \tilde{\psi}_c - \hat{\mathscr{B}}^{(b)}_{3333}(\tilde{y}_{/1}\, \tilde{\psi}_c + \tilde{x}\mathbf{r}\cdot\mathbf{t}) + \hat{\mathscr{D}}^{(b)}_{3333}\tilde{y}_{/1}\mathbf{r}\cdot\mathbf{t}]\, ds,$$

$$D_{44} = \sum_b \int_{p^{(b)}} [\hat{\mathscr{A}}^{(b)}_{3333}\, \tilde{\psi}_c^2 - 2\hat{\mathscr{B}}^{(b)}_{3333}\tilde{\psi}_c\, \mathbf{r}\cdot\mathbf{t} + \hat{\mathscr{D}}^{(b)}_{3333}(\mathbf{r}\cdot\mathbf{t})^2]\, ds,$$

$$D_{55} = 4\sum_b \int_{p^{(b)}} \hat{\mathscr{D}}^{(b)}_{1313}\, ds, \qquad D_{56} = -2\sum_b \int_{p^{(b)}} \hat{\mathscr{B}}^{(b)}_{1313}\tilde{x}_{/1}\, ds,$$

$$D_{57} = -2\sum_b \int_{p^{(b)}} \hat{\mathscr{B}}^{(b)}_{1313}\tilde{y}_{/1}\, ds, \qquad D_{58} = 2\sum_b \int_{p^{(b)}} \hat{\mathscr{B}}^{(b)}_{1313}\mathbf{r}\cdot\mathbf{n}\, ds,$$

$$D_{66} = \sum_b \int_{p^{(b)}} [\mathscr{A}^{(b)}_{1313}(\tilde{x}_{/1})^2 - 2\mathscr{A}^{(b)}_{1323}\tilde{x}_{/1}\, \tilde{y}_{/1} + \mathscr{A}^{(b)}_{2323}(\tilde{y}_{/1})^2]\, ds,$$

$$D_{67} = \sum_b \int_{p^{(b)}} \{(\mathscr{A}^{(b)}_{1313} - \mathscr{A}^{(b)}_{2323})\tilde{x}_{/1}\, \tilde{y}_{/1} + \mathscr{A}^{(b)}_{1323}[(\tilde{x}_{/1})^2 - (\tilde{y}_{/1})^2]\}\, ds,$$

$$D_{68} = -\sum_b \int_{p^{(b)}} [\mathscr{A}^{(b)}_{1313}\tilde{x}_{/1}\mathbf{r}\cdot\mathbf{n} + \mathscr{A}^{(b)}_{2323}\tilde{y}_{/1}\mathbf{r}\cdot\mathbf{t} - \mathscr{A}^{(b)}_{1323}(\tilde{x}_{/1}\mathbf{r}\cdot\mathbf{t} + \tilde{y}_{/1}\mathbf{r}\cdot\mathbf{n})]\, ds,$$

$$D_{77} = \sum_b \int_{p^{(b)}} [\mathscr{A}^{(b)}_{1313}(\tilde{y}_{/1})^2 + 2\mathscr{A}^{(b)}_{1323}\tilde{x}_{/1}\tilde{y}_{/1} + \mathscr{A}^{(b)}_{2323}(\tilde{x}_{/1})^2]\, ds,$$

$$D_{78} = -\sum_b \int_{p^{(b)}} [\mathscr{A}^{(b)}_{1313}\tilde{y}_{/1}\mathbf{r}\cdot\mathbf{n} + \mathscr{A}^{(b)}_{1323}(\tilde{x}_{/1}\mathbf{r}\cdot\mathbf{n} - \tilde{y}_{/1}\mathbf{r}\cdot\mathbf{t}) - \mathscr{A}^{(b)}_{2323}\tilde{x}_{/1}\mathbf{r}\cdot\mathbf{t}]\, ds,$$

$$D_{88} = \sum_b \int_{p^{(b)}} [\mathscr{A}^{(b)}_{1313}(\mathbf{r}\cdot\mathbf{n})^2 - 2\mathscr{A}^{(b)}_{1323}(\mathbf{r}\cdot\mathbf{n})(\mathbf{r}\cdot\mathbf{t}) + \mathscr{A}^{(b)}_{2323}(\mathbf{r}\cdot\mathbf{t})^2]\, ds,$$

where $\tilde{\psi}_c = \psi_c|_{\eta=0}$ and where the laminate extensional, coupling and bending stiffness coefficients $\mathscr{A}_{.}, \mathscr{B}_{.}, \mathscr{D}_{.}$ are defined by

$$\{\hat{\mathscr{A}}^{(b)}_{.}, \hat{\mathscr{B}}^{(b)}_{.}, \hat{\mathscr{D}}^{(b)}_{.}\} = \sum_l \int_{\eta^{(b,l-1)}}^{\eta^{(b,l)}} \hat{\mathscr{C}}_{.}\{1, \eta, \eta^2\}\, d\eta, \quad \mathscr{A}^{(b)}_{\alpha 3\beta 3} = \sum_l \int_{\eta^{(b,l-1)}}^{\eta^{(b,l)}} \hat{\mathscr{C}}_{\alpha 3\beta 3}\, d\eta. \tag{5-1}$$

This model, in the case of the addressed material symmetry, justifies [Lee 2005] and it is easy to verify that it reduces to the one discussed in the Section 3D, when $N_b = N_l^{(b)} = 1$.

Finally, following the approach discussed in Section 4 and referring, for the sake of brevity, to the case $N_b = 1$, the total shear stresses on $\mathcal{P}^{i(l)}$ can be recovered by means of the equations

$$\begin{aligned}\sigma_{13}^{(l)}(s^i,\eta,z) &\cong \sigma_{13}^{(el,l)} + (\bar{\sigma}_{13}^{(l)} - \bar{\sigma}_{13}^{(el,l)}),\\ \sigma_{23}^{(l)}(s^i,\eta,z) &= \sigma_{23}^{-(l)} + \frac{\sigma_{23}^{+(l)} - \sigma_{23}^{-(l)}}{\eta^{(l)} - \eta^{(l-1)}}(\eta - \eta^{(l-1)}) - \Xi^{(l)}\left(\frac{\eta^2}{2} - \eta\,\bar{\eta}^{(l)} + \frac{\eta^{(l-1)}\eta^{(l)}}{2}\right),\end{aligned} \tag{5-2}$$

where $\Xi^{(l)}$ is as in (4-8), $\sigma_{23}^{+(l)}$ and $\sigma_{23}^{-(l)}$ stand for σ_{23} at $\partial\mathcal{P}^{i(l)}|_{\eta=\eta^{(l)}}$ and $\partial\mathcal{P}^{i(l)}|_{\eta=\eta^{(l-1)}}$ respectively, $\bar{\eta}^{(l)} = (\eta^{(l)} + \eta^{(l-1)})/2$, and

$$\bar{\sigma}_{13}^{(l)} = -\frac{1}{\eta^{(l)} - \eta^{(l-1)}}\left(\int_{\mathcal{P}^{i(l)}} (\sigma_{33/3}^{(l)} + b_3)\,da + \int_{\eta^{(l-1)}}^{\eta^{(l)}} \hat{p}_3|_{s=s_0}\,d\eta + \int_{s_0}^{s^i} (\sigma_{23}^{+(l)} - \sigma_{23}^{-(l)})\,da\right). \tag{5-3}$$

Observing that

$$\sigma_{13/1}^{(l)}(s^i,\eta,z) = -\bar{\sigma}_{33/3}^{(l)} - \bar{b}_3 - \frac{\sigma_{23}^{+(l)} - \sigma_{23}^{-(l)}}{\eta^{(l)} - \eta^{(l-1)}}, \tag{5-4}$$

where here $\bar{f}$ denotes the average value over the l-th layer thickness of f, the closure of (5-2) and (5-3) has to be performed determining $\sigma_{23}^{\pm(l)}$ by means of the continuity conditions

$$\sigma_{23}^{+(l)} = \sigma_{23}^{-(l+1)}, \qquad \sigma_{13/1}^{(l)}|_{\eta^{(l)}} = \sigma_{13/1}^{(l+1)}|_{\eta^{(l)}} \tag{5-5}$$

for l from 1 to N_l and with $\sigma_{23}^{+(N_l)} = \hat{p}_3^+$ and $\sigma_{23}^{+(0)} = \hat{p}_3^-$.

5B. ***Nonlinear conewise elastic materials: the fiber-governed bimodular case.*** The variational dual-constraint approach can be successfully employed also when a thin-walled beam comprises nonlinear elastic materials characterized by a continuous and convex elastic potential (not necessarily differentiable), such as the conewise materials addressed in [Curnier et al. 1995]. Bimodular behavior, which generally may characterize the constitutive response of a number of composite materials, belong to this category. In this case a nonlinear elastic response appears, identified by a linear relationship between stress and strain both in tension and in compression, but with different elastic moduli. We will refer to the special case of fiber-reinforced composite materials, where the bimodularity depends on the sign of the unit elongation in the fiber-direction [Curnier et al. 1995; Bert 1977; Maceri and Sacco 1990; Bisegna et al. 1995].

Let $\boldsymbol{\varepsilon}$ be the symmetric strain tensor and $\mathbf{f}$ the unit vector along the fiber direction. Denoting as $\epsilon_f = \boldsymbol{\varepsilon}\mathbf{f}\cdot\mathbf{f}$ the extension in the fiber direction, the following definitions are introduced:

$$E^+ = \{\boldsymbol{\varepsilon} : \epsilon_f > 0\}, \quad E^o = \{\boldsymbol{\varepsilon} : \epsilon_f = 0\}, \quad E^- = \{\boldsymbol{\varepsilon} : \epsilon_f < 0\}. \tag{5-6}$$

As showed in [Bisegna et al. 1995], a bimodular constitutive law can be deduced in a consistent way by assuming that an elastic potential Φ exists. Accordingly, restrictions of Φ to E^+ and E^- are the potentials for the mappings:

$$\boldsymbol{\varepsilon} \in E^+ \mapsto \mathscr{C}^+\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} \in E^- \mapsto \mathscr{C}^-\boldsymbol{\varepsilon}, \tag{5-7}$$

and the fourth order constitutive tensors $\mathscr{C}^+$ and $\mathscr{C}^-$ (relevant to tension and compression behavior, respectively) are symmetric. Moreover, due to the definition of a conservative (or hyperelastic) material, the potential $\Phi(\boldsymbol{\varepsilon})$ is continuous [Bisegna et al. 1995]. As a consequence, restrictions of Φ to E^+ and E^- can be extended by continuity to E^o, providing the following equality to be satisfied:

$$\mathscr{C}^+\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} = \mathscr{C}^-\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}, \quad \forall\boldsymbol{\varepsilon}\in E^o. \tag{5-8}$$

Equations (5-7) and (5-8) imply that the material strain energy density Φ can be written (omitting constant contributions) in the form:

$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2}[h\mathscr{C}^+ + (1-h)\mathscr{C}^-]\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}, \tag{5-9}$$

where $h(\epsilon_f)$ is the Heaviside function, such that $h=1$ if ϵ_f is positive, $h=0$ otherwise.

Thus, the first term of the Hu–Washizu functional $\mathscr{W}$ (2-1) has to be replaced by the integral over $\boldsymbol{\Omega}$ of $\Phi(\boldsymbol{\varepsilon})$ and then, introducing a Lagrangian functional $\mathscr{L}$ as in (2-2), the potential energy functional $\mathscr{E}$ turns out to be

$$\mathscr{E}(\mathbf{u},\boldsymbol{\chi},\boldsymbol{\omega}) = \int_\Omega \Phi(\hat{\nabla}\mathbf{u} - \mathbf{G}^*\boldsymbol{\omega})\,dv - \int_\Omega \boldsymbol{\chi}\cdot\mathbf{G}\hat{\nabla}\mathbf{u}\,dv - \Pi_{\text{ext}}. \tag{5-10}$$

The stationary conditions of $\mathscr{L}$ with respect to $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$ formally give the same equations (2-3) to (2-6), where now $\mathscr{C}$ has to be regarded as

$$\mathscr{C} = [h\mathscr{C}^+ + (1-h)\mathscr{C}^-]. \tag{5-11}$$

Accordingly, the dual-constraint approach can be employed in the case of bimodular fiber-governed materials.

In this case, the stationary condition of the Lagrangian functional $\mathscr{L}$ (2-2) with respect of $\boldsymbol{\varepsilon}$ yields (2-4) without jumping terms (depending on the difference between $\mathscr{C}^+$ and $\mathscr{C}^-$) as a consequence of the continuity condition (5-8). Moreover, strain along the fiber-direction ϵ_f, whose sign discriminates the unilateral constitutive behavior, has to be considered as deduced from the elastic strain field.

Referring to the first-order shear-deformable beam model accounting for warping shear, when a homogeneous beam comprises a bimodular material at least monoclinic, the generalized elastic matrix $\mathbf{D}$ can be evaluated as in Section 5A taking $\mathscr{P}$ as $\mathscr{P} = \mathscr{P}|_{\mathscr{C}^+}\cup\mathscr{P}|_{\mathscr{C}^-}$. Clearly, this partition of $\mathscr{P}$ needs the preliminary knowledge for every z-coordinate value of the neutral region satisfying $\epsilon_f = 0$. Therefore, a free-boundary problem underlies this formulation and an iterative procedure has to be employed in order to evaluate elements of $\mathbf{D}$.

6. Influence of the curvature

In order to assess the influence of high-order contributions associated with the curvature κ of p (see (3-13) and (3-28)), some results relating to very simple numerical applications are herein presented. In detail, we refer to the no-shear model discussed in Section 3B when it is applied to a pure torsion problem for a thin-walled open section beam, with a circular centerline ($\kappa = const$).

Assuming that $\hat{\mathscr{C}}_{3333}/\hat{\mathscr{C}}_{1313} = 2.5$, the following two cases are analyzed:

- a simply supported beam ($\theta(\pm L) = \theta''(\pm L) = 0$, i.e., warping free) loaded by an uniform twisting moment distribution m_z;

- a cantilever beam ($\theta(-L) = \theta'(-L) = 0$, i.e., warping constrained) loaded by a twisting moment M_z applied at its free end.

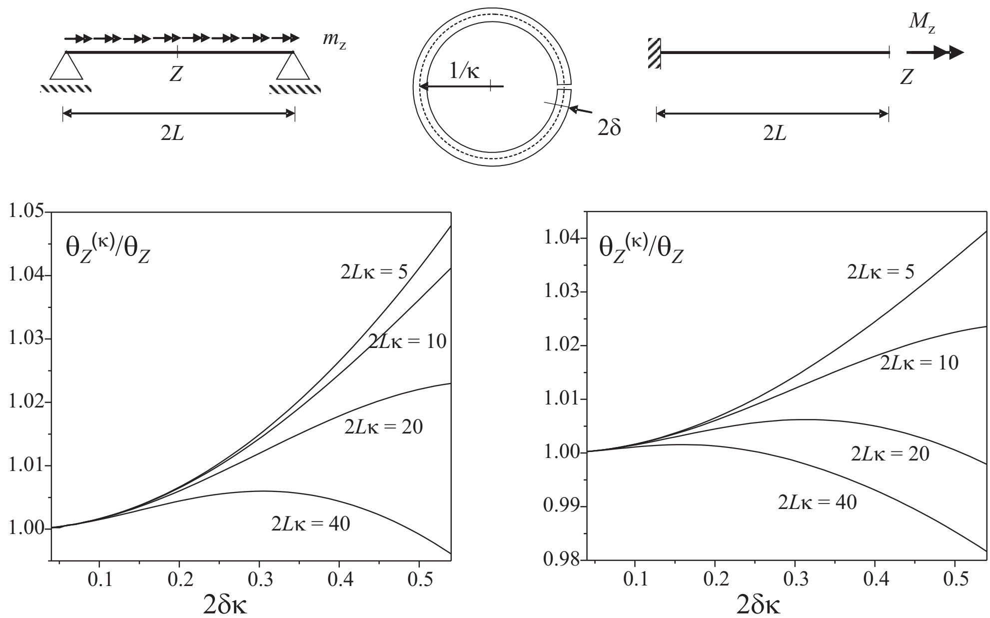

Figure 2. Influence of high-order nonconventional effects relating to curvature κ of p in torsional analysis of a simply supported (on the left) and a cantilever (on the right) thin-walled beam with a circular open cross-section.

Figure 2 sketches the cases under investigation and depicts the ratios between the maximum rotation angle around the z-axis evaluated taking into account curvature effects ($\theta^{(\kappa)}$) versus that computed disregarding curvature effects (θ). It can be noted that, varying the dimensionless thickness $2\delta\kappa$ and the dimensionless beam length $2L\kappa$, the influence of curvature increases when the beam slenderness ratio decreases (i.e., when $L \to \sim 1/\kappa$), amounting to approximately 4–5% and increasing when the simply supported scheme is experienced. Therefore, high-order effects associated to κ and δ might significantly affect, form an engineering point of view, thin-walled beam static response.

7. Concluding remarks

This paper presents a consistent deduction of anisotropic thin-walled beam models from three-dimensional elasticity. Employing a constrained approach and following the strategy first outlined in [Maceri and Bisegna 2002], a modified Hu–Washizu variational formulation has been proposed. This provides a rational and unified foundation for thin-walled beam theories widely used in the recent literature, and allows us to account for different shear refinement levels.

This variational approach enables to take into account high-order nonconventional effects related to curvature of cross-section profile and wall thickness, leading to a generalized warping function and torsional constant. Influence of these effects has been proved through simple numerical applications, highlighting that in some cases it could be not completely negligible form an engineering point of view.

It was shown in the case of a branchwise model for the analysis of multibranch cross-sections that this approach allows the consistent deduction of new refined models. Moreover, due to its variational character, this formulation opens the possibility of building new consistent and refined thin-walled beam finite elements. Finally, a more accurate evaluation of stresses acting on the beam cross-section with respect to those obtained by constitutive relationship has been also proposed. In agreement with the dual-constraint approach and starting from the one-dimensional generalized unknown functions, approximated locally equilibrated stresses are recovered in closed form.

The dual-constraint variational framework has also proved effective for the treatment of laminated pultruded thin-walled beams and nonlinear conewise elastic materials (with special reference to bimodular materials).

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