Journal of
Mechanics of
Materials and Structures

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Silvano Tizzi

Volume 4, Nº 3

March 2009

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 4, No. 3, 2009

INFLUENCE OF THE NONLINEAR AERODYNAMIC FORCE COMPONENTS ON A VIBRATING BEAM, SIMPLY SUPPORTED OR CLAMPED AT BOTH ENDS, EXPOSED TO A HIGH SUPERSONIC AIRFLOW ALONG ITS AXIAL DIRECTION

SILVANO TIZZI

In this work we analyze the case of a vibrating beam, simply supported or clamped at both ends, under the effect of a high supersonic airflow along its axial direction. A complete aerodynamic model of the *piston theory*, which also takes into account the nonlinear components of the distributed aerodynamic transversal force, is used. The postcritical flutter behavior and its influence on the vibration state solutions of a fluttering beam without aerodynamic damping have been studied. This paper focuses particularly on the effects of these nonlinear aerodynamic forces on three frequencies, which are useful in characterizing the postcritical flutter solution set of the undamped beam in the whole frequency range: the minimum frequency, the frequency where the change of the modal shape with lower amplitude occurs, and the frequency corresponding to the solution with minimum amplitude of the vibration mode. Special attention has been given to the influence on the solution of the vibrating undamped beam with minimum modal amplitude, whose frequency is the most important among the three mentioned above; in fact, in the neighborhood of this particular solution, there exists the flutter state of the vibrating damped beam in limit cycle conditions.

Three different schemes, two of them semianalytical (based on the classical and well known Rayleigh–Ritz and Galerkin methods) and one of them numerical (based on the finite element method), have been herein exploited, as in the author's previous papers, where beam flutter models with linear aerodynamic analysis were used. The good agreement between the results obtained by the three methods corroborates their effectiveness.

More sophisticated models have been herein set up, considering that a more accurate analysis is necessary than in previous cases, where the aerodynamic numerical model was limited to within the framework of the quasisteady linearized piston theory, both for the coupling component between odd and even order vibrating modes, and for the aerodynamic damping component.

The results obtained enable us to assess quantitatively the influence of these nonlinear aerodynamic forces on the postcritical beam flutter behavior, and particularly on the undamped beam solution with minimum amplitude of the vibration mode.

1. Introduction

For many years steady and unsteady aerodynamic theory for aeroelastic panels flutter computations has received a lot of interest, often in connection with high supersonic speeds. It is useful to recall the main authors who developed studies for flutter analysis of panels exposed to a high supersonic flow.

Keywords: vibrating beam, nonlinear, aerodynamic force, supersonic airflow, flutter.

Lighthill [1953] first proposed a *piston theory*, which was proved to be an efficient and powerful tool for panel flutter analysis. It can be used to calculate the pressure on an airfoil in steady or unsteady motion with remarkable accuracy, even under nonisentropic conditions, whenever the flight Mach Number M_∞ has such an order of magnitude that $M_\infty^2 \gg 1$. This piston theory is quite attractive for flutter studies due to its simplicity in comparison with other supersonic theories.

This theory has been discussed by Ashley and Zartarian [1956], who made suggestions for future research based on this new efficient aerodynamic tool, with particular regard to areas where computational labor can be reduced without losing the necessary accuracy.

Morgan et al. [1958] analyzed some of the theories for two dimensional oscillatory wing structures, which could be applied for flutter computations with high Mach Number. The results obtained by the various aerodynamic theories have been compared for their flutter prediction in various Mach Number ranges. Also some possible refinements of the piston theory have been proposed for high Mach Numbers.

The heritage of the studies of these authors enables us to know the complete expression of the aerodynamic transverse distributed force acting on a beam, which makes it possible to determine its flutter dynamic response after appropriate approximations.

The objective of this paper is to investigate the effects of the nonlinear components of the aerodynamic transverse force on the permanent postcritical solutions of a fluttering beam without air damping. Since the fluttering beam solution in limit cycle conditions, derived by the aerodynamic model with damping, is very near to the undamped vibrating beam state with minimum amplitude, these effects could also influence the damped beam aeroelastic vibration.

Three different schemes have been exploited for the flutter computational work, as in the case of the beam flutter analysis with linearized and idealized piston theory [Tizzi 1994; 2003]. First a numerical procedure [Tizzi 1994; 1996; 2003] which arises from the Rayleigh–Ritz method [Kantorovich and Krylov 1964, pp. 258–303; Mikhlin 1964, pp. 74–125 and 448–490; Reddy 1986, pp. 258–285] has been used, together with the finite element method (FEM) [Weaver and Johnston 1984, pp. 1–102; Reddy et al. 1988, pp. 41–89; Qin et al. 1993]. By knowing the structural and inertial forces potential functional and the aerodynamic generalized force, it has been possible to apply the Lagrange equations [Pars 1968, pp. 28–89] and derive the generalized governing equation in time, for which appropriate time-integration algorithms exist.

Then the Galerkin method [Kantorovich and Krylov 1964; Mikhlin 1964; Tizzi 1994; 2003] was employed in the case of a simply supported beam, as in Dowell's model [1966; 1967], to validate the results of Ritz and FEM procedures.

The effectiveness of the three methods is apparent from the good accordance of the results obtained by the three different simulation approaches.

An analysis of the results achieved has been necessary to point out the influence of the nonlinear aerodynamic force components on particular frequency parameters characterizing the undamped beam solution set in the whole frequency range. It is fundamental to evaluate the effects of these nonlinear forces on the solution with minimum amplitude of the modal shape, considering that the flutter solution in limit cycle conditions of the damped vibrating beam lies in its neighborhood.

Studies on the influence of the nonlinear aerodynamic terms on the postcritical limit cycle of fluttering panels have also been developed by other authors [McIntosh 1973; Smith and Morino 1976]. However, investigations into the effects of these nonlinear aerodynamic components, in the presence of nonlinear structural forces, on the postcritical flutter solutions of a beam in an airflow without damping, have been herein performed. Moreover, a fluttering beam with both simply supported and clamped ends has been considered; this is useful because the boundary conditions of panels actually employed in aerospace structures often lie between the two supposed ones in the analyzed cases.

Index of notation

A_s beam cross-section area

a_m nondimensional modal shape amplitude (maximum transverse displacement along the beam, divided by length of beam)

a_∞ speed of sound

b_w beam width

$a_{ij}, c_{ij}, b_{ijkl}, d_{ijkl}$ coefficients determined by integrals in Ritz and FEM models

$c_{i_p i_a}$ generic coefficient of the nondimensional flexural deflection series expansion in the generic i_e-th element of FEM model

E Young's modulus

F_a resultant of the aerodynamic forces acting on both sides of the beam per unit length

$F_i^{(a)}$ generalized aerodynamic force acting on the i-th degree of freedom

$f_i(\xi)$ generic trial describing function of the nondimensional flexural deflection through the beam length

h beam thickness

I flexural moment of inertia

J_{sccc}, J_{sscc} particular integrals utilized in the Galerkin method

k_{ij} stiffness matrix elements

k^*_{ij} linear structural and aerodynamic forces resultant matrix elements

L beam length

M_∞ Mach number

m_{ij} mass matrix elements

N whole number of the degrees of freedom

N_E whole number of the elements in FEM model

p_∞ unperturbed air pressure

P_a force per unit axial length acting on a side of the beam profile

$Q_{i_p}^{(i_e)}$ generic degree of freedom in the i_e-th element of FEM model

q dynamic pressure

t beam state evolution time

T_o reference time

U_∞ airflow speed

u, w beam points axial and flexural displacements, respectively

W nondimensional flexural displacement

W_i generic coefficients of the nondimensional flexural displacement series expansion

W_{i_e} nondimensional flexural displacement on a generic $(i_e + 1)$-th section S_{i_e} of FEM model

x beam axial coordinate

Greek symbols

α nondimensional beam axial parameter

β, γ_m nondimensional Mach numbers parameters

γ ratio between the specific heat at constant pressure and volume, respectively

γ', γ'' nondimensional aerodynamic damping coefficients

λ nondimensional mass distribution parameter

θ_{i_e} rotation parameter in the generic $(i_e + 1)$-th section S_{i_e} of FEM model

μ mass per unit length

ξ nondimensional axial coordinate of the beam

ξ_{i_e} nondimensional axial coordinate of the $(i_e + 1)$-th section S_{i_e} of the FEM model

ξ_n normalized axial coordinate of a beam element in the FEM model

ρ_∞ unperturbed air density

σ dimensional dynamic pressure parameter

σ_d dimensionless dynamic pressure parameter

τ nondimensional time

φ_{ij} integral of the product between the first derivatives of the generic describing functions $f_i(\xi)$ and $f_j(\xi)$

χ normal speed down- or up-wash

ω angular frequency

ω_d dimensionless angular frequency

Special symbols

$\mathcal{H}_2(S)$ square summable functions space

S definition domain of the describing functions

$\boldsymbol{B}, \boldsymbol{D}$ utilized matrices in the generalized form equation

$\boldsymbol{F}$ structural and linear aerodynamic forces matrix

$\mathcal{L}$ extended Lagrangian functional

$\boldsymbol{M}$ mass matrix

$\mathcal{T}$ kinetic energy

$\mathcal{U}$ strain energy

$\boldsymbol{W}, \boldsymbol{Z}$ column vectors of the unknown variables and their time first derivatives

$\boldsymbol{W}^{(3)}$ column vector with elements the triple products between unknown variables

$\boldsymbol{W}^{(3d)}$ column vector with elements the triple products between two unknown variables and a time first derivative of a third variable

Subscripts and superscripts

d subscript indicating dimensionless parameters

i, j, k, l subscripts indicating functions and unknown variables coefficients of the flexural displacement series expansion

i_e subscript indicating the $(i_e + 1)$-th section S_{i_e} of the FEM model

(i_e) superscript indicating the i_e-th element of FEM model

i_p, j_p, k_p, l_p subscripts characterizing the unknown variables and the coefficients of the flexural displacement series expansion in the i_e-th element of FEM model

nd subscript indicating nondimensional variables

∞ subscript indicating the unperturbed airflow

2. Mathematical formulation

A vibrating beam exposed to a high supersonic flow along the x axis, previously analyzed with the linear aerodynamic model [Tizzi 1994; 2003], is considered and drawn in Figure 1.

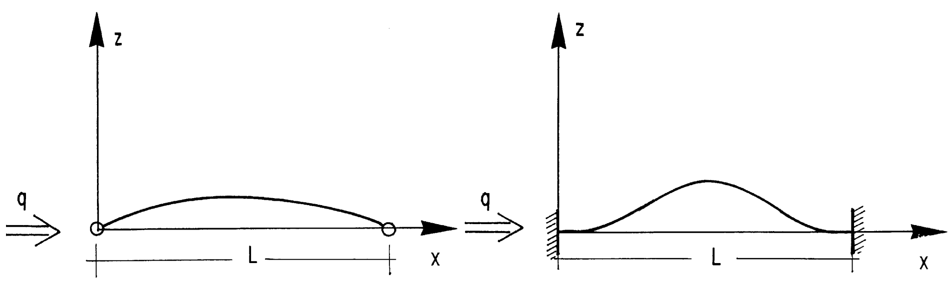

Figure 1. Beam, simply supported or clamped at both ends, exposed to a high supersonic flow.

The focus of the analysis developed here is limited to the nonlinear aerodynamic force components of piston theory, given that the terms of all structural and linear aerodynamic force components have been sufficiently illustrated. The exact expression of the pressure acting on a beam element (one-dimensional flow) is [Lighthill 1953; Ashley and Zartarian 1956; Morgan et al. 1958]:

$$\frac{p}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}\frac{\chi}{a_\infty}\right)^{2\gamma/(\gamma-1)}, \tag{1}$$

where $\gamma = c_p/c_v$ is the ratio between the specific heats at constant pressure and volume (c_p and c_v); χ is the normal down- or up-wash, that is, the component of the fluid velocity in the z-direction, normal to the beam profile; $a_\infty = \sqrt{\gamma p_\infty/\rho_\infty}$ is the speed of sound; and p_∞ and ρ_∞ are the pressure and density of the unperturbed airflow. This pressure can be expressed in terms of series expansion function elements versus χ, as follows:

$$\frac{p}{p_\infty} = 1 + \gamma\frac{\chi}{a_\infty} + \frac{1}{4}\gamma(\gamma+1)\left(\frac{\chi}{a_\infty}\right)^2 + \frac{1}{12}\gamma(\gamma+1)\left(\frac{\chi}{a_\infty}\right)^3 + \cdots. \tag{2}$$

The dimensional dynamic pressure parameter σ is also introduced:

$$\sigma = \frac{2q}{\beta}b_w, \tag{3}$$

where $\beta = \sqrt{M_\infty^2 - 1}$, $M_\infty = U_\infty/a_\infty$ is the Mach number, $q = \frac{1}{2}\rho_\infty U_\infty^2$ is the dynamic pressure, U_∞ is the airflow speed, and b_w is the beam width. As in [Tizzi 2003], the aerodynamic expressions of an infinite plate along the third, not considered y axis are applied, together with the structural constitutive relations of a beam; this hypothesis can be accepted if the beam width b_w is much greater than L.

Since

$$\sqrt{M_\infty^2 - 1} \cong M_\infty, \tag{4}$$

if the Mach number is high enough, it is also true that

$$\sigma = \frac{2q}{\beta}b_w \cong \gamma p_\infty M_\infty b_w, \tag{5}$$

in view of the previously introduced expressions of q, M_∞ and a_∞.

Thus the force per unit axial length acting on each side of the beam profile can be evaluated from (2), and, by virtue of (5) and the expression for M_∞, it can be written as

$$P_a = \Delta p b_w = \sigma\left[\frac{\chi}{U_\infty} + \frac{\gamma+1}{4}M_\infty\left(\frac{\chi}{U_\infty}\right)^2 + \frac{\gamma+1}{12}M_\infty^2\left(\frac{\chi}{U_\infty}\right)^3 + \cdots\right], \tag{6}$$

where $\Delta p = p - p_\infty$ is the pressure variation with respect to the unperturbed static conditions $p = p_\infty$.

The analysis has been developed only for symmetric cases, that is, both sides of the beam profile are exposed to the same airflow, and consequently only the odd powers give a contribution. In fact, taking into account that the normal speed component χ has opposite values on the upper and lower side, their effects sum-up for the odd powers and vanish for the even ones. Thus the resultant of the aerodynamic forces acting on both sides per unit length can be written as

$$F_a = 2\sigma\left[\frac{\chi}{U_\infty} + \frac{\gamma+1}{12}M_\infty^2\left(\frac{\chi}{U_\infty}\right)^3 + \cdots\right]. \tag{7}$$

The normal speed component due to the profile dynamics can be expressed as

$$\chi = U_\infty\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}, \tag{8}$$

where w is the beam point's flexural displacement.

Substituting (8) into (7) gives the new expression of the resultant aerodynamic distributed force:

$$F_a = 2\sigma\left[\frac{\partial w}{\partial x} + \gamma_m\left(\frac{\partial w}{\partial x}\right)^3 + \cdots\right] + 2\sigma\left\{\frac{1}{U_\infty}\frac{\partial w}{\partial t} + \gamma_m\left[3\left(\frac{\partial w}{\partial x}\right)^2\frac{1}{U_\infty}\frac{\partial w}{\partial t} + \cdots\right] + \cdots\right\}, \tag{9}$$

where

$$\gamma_m = \frac{\gamma+1}{12}M_\infty^2 \tag{10}$$

and the subsequent terms in the series expansion can be neglected. This is formed by two components: F_{a1}, containing only spatial derivatives, which is the coupling element between odd and even vibrating modes, and which exists also without damping; and F_{a2}, containing also time derivatives, which give rise to the aerodynamic dissipative force and damping. This second component is not considered for the undamped vibrating beam solution.

The linear component of the distributed aerodynamic force in (9) equals that obtained in [Bisplinghoff and Ashley 1975, pp. 416–437; Tizzi 2003], except for the presence of the ratio $(M_\infty^2 - 2)/(M_\infty^2 - 1)$ before the time derivative $\partial w/\partial t$, which is approximately equal to unity for high Mach numbers.

It is necessary to recall the vibration-governing equation of the fluttering beam [Tizzi 2003]:

$$EI\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} - EA_s\frac{1}{2}\overline{\left(\frac{\partial w}{\partial x}\right)^2}\frac{\partial^2 w}{\partial x^2} + F_a = 0, \tag{11}$$

where $A_s = b_w h$ is the cross-sectional area, h is the beam thickness, E is the Young's modulus, μ is the distributed mass per unit length, and $I = Eb_w h^3/12$ is the flexural moment of inertia.

Furthermore, $\overline{(\partial w/\partial x)^2}$ has been previously defined as the mean square value of the flexural displacement first axial derivative over the whole beam length; the third term of (11), containing this mean square value, corresponds to the nonlinear component of the transverse structural force, due to the beam axial stretching.

The axial inertia effects are being neglected, as in the previous analysis with a linearized aerodynamic model, considering that the axial vibration frequencies are higher than the corresponding ones of the flutter vibration, and so the axial vibration frequency range is different from the flutter frequency range.

Substitution of (9) into (11), with the approximation assumptions, leads to the following governing equation:

$$EI\frac{\partial^4 w}{\partial x^4} + \mu\frac{\partial^2 w}{\partial t^2} - EA_s\frac{1}{2}\overline{\left(\frac{\partial w}{\partial x}\right)^2}\frac{\partial^2 w}{\partial x^2} + 2\sigma\left[\frac{\partial w}{\partial x} + \gamma_m\left(\frac{\partial w}{\partial x}\right)^3\right] + 2\sigma\left\{\frac{1}{U_\infty}\frac{\partial w}{\partial t} + \gamma_m\left[3\left(\frac{\partial w}{\partial x}\right)^2\frac{1}{U_\infty}\frac{\partial w}{\partial t}\right]\right\} = 0, \tag{12}$$

which can be reformulated in dimensionless form as

$$\frac{\partial^4 W}{\partial \xi^4} + \lambda \frac{\partial^2 W}{\partial \tau^2} + \sigma_d \left[\frac{\partial W}{\partial \xi} + \gamma_m \left(\frac{\partial W}{\partial \xi} \right)^3 \right] - \frac{\alpha}{2} \overline{\left(\frac{\partial W}{\partial \xi} \right)^2} \frac{\partial^2 W}{\partial \xi^2} + \left[\gamma' + \gamma'' \left(\frac{\partial W}{\partial \xi} \right)^2 \right] \frac{\partial W}{\partial \tau} = 0, \tag{13}$$

where the flexural displacement and the axial coordinate have been reformulated in nondimensional form, and other dimensionless parameters have been introduced:

$$W(\xi, \tau) = \frac{w(x,t)}{L}, \quad \xi = \frac{x}{L}, \quad \lambda = \frac{\mu L^4}{EIT_o^2}, \quad \tau = \frac{t}{T_o}, \tag{14}$$

$$\sigma_d = \frac{2\sigma L^3}{EI}, \quad \alpha = \frac{A_s L^2}{I}, \quad \gamma' = \frac{2\sigma}{U_\infty} \frac{L^4}{EIT_o}, \quad \gamma'' = 3\gamma' \gamma_m, \tag{15}$$

and T_o is a reference time.

The third and fifth terms in (13) refer to the nondimensional equivalent form of the distributed aerodynamic force expression in (9):

$$(F_a)_{nd} = \sigma_d \left[\frac{\partial W}{\partial \xi} + \gamma_m \left(\frac{\partial W}{\partial \xi} \right)^3 \right] + \left[\gamma' + \gamma'' \left(\frac{\partial W}{\partial \xi} \right)^2 \right] \frac{\partial W}{\partial \tau}. \tag{16}$$

Einstein's summation convention for repeated indices will be adopted in all the following expressions.

A series expansion for $W(\xi, \tau)$ in terms of function elements can be chosen:

$$W(\xi, \tau) = W_i(\tau) f_i(\xi), \quad i = 1, 2, \ldots, N, \tag{17}$$

where each coefficient $W_i(\tau)$ is a Lagrangian degree of freedom, and $f_i(\xi)$ are polynomials describing functions, belonging to the space of the square summable functions $H_2(S)$, defined in the domain S (which in this case is the whole beam length). These satisfy only the geometric boundary conditions, as in the Ritz and FEM methods. The meaning of the coefficients $W_i(\tau)$ and the describing functions $f_i(\xi)$ are illustrated in the electronic supplement to this paper.

Equation (13) can be transformed into its generalized equivalent form by the variational principle [Pars 1968], if the Lagrangian functional is introduced:

$$\mathcal{L} = \mathcal{T} - \mathcal{U}, \tag{18}$$

where $\mathcal{T}$ is the kinetic energy and $\mathcal{U}$ is the strain energy. Thus the generic generalized i-th governing equation can be written in the classical Lagrangian form:

$$\frac{d(\partial \mathcal{L} / \partial \dot{W}_i)}{d\tau} - \frac{\partial \mathcal{L}}{\partial W_i} + F_i^{(a)} = 0, \quad i = 1, 2, \ldots, N, \tag{19}$$

where $F_i^{(a)}$ is the generalized aerodynamic force acting on the i-th degree of freedom. It is possible to set a correspondence between each term of this generalized equation with each one of (13).

The potential strain energy expression has been already determined [Tizzi 2003]:

$$\mathcal{U} = \frac{1}{2} k_{ij} W_i W_j + \frac{\alpha}{8} \varphi_{ij} \varphi_{kl} W_k W_l W_i W_j, \quad i, j, k, l = 1, 2, \ldots, N, \tag{20}$$

where

$$\varphi_{ij} = \int_0^1 \frac{\partial f_i(\xi)}{\partial \xi} \frac{\partial f_j(\xi)}{\partial \xi} d\xi. \tag{21}$$

The stiffness matrix elements k_{ij} have been previously evaluated [Tizzi 1994]; obviously $k_{ij}\ddot{W}_j$ is the generalized linear structural force acting on the i-th degree of freedom, which corresponds to the first term in (13). The elements of the second term at the second member of (20), corresponding to the nonlinear contribution to the structural strain energy, are also known.

The expression for the generalized aerodynamic force $F_i^{(a)}$ acting on the i-th degree of freedom, corresponding to the third and fifth terms in (13), can be obtained by the use of the series expansion (17) in (16), and projecting the whole equation (16) onto the generic function element $f_i(\xi)$. Thus

$$F_i^{(a)} = a_{ij}W_j + \gamma_m b_{ijkl} W_k W_l W_j + \gamma' c_{ij}\dot{W}_j + \gamma'' d_{ijkl} W_k W_l \dot{W}_j,$$
$$j, k, l = 1, 2, \dots, N, \quad i = 1, 2, \dots, N, \tag{22}$$

where the coefficients a_{ij} and c_{ij} are well known from the previous analysis [Tizzi 2003]:

$$a_{ij} = \sigma_d \int_0^1 f_i(\xi)\frac{\partial f_j(\xi)}{\partial \xi} d\xi, \quad c_{ij} = \int_0^1 f_i(\xi) f_j(\xi) d\xi, \tag{23}$$

and the newly introduced ones are defined as

$$b_{ijkl} = \sigma_d \int_0^1 f_i(\xi)\frac{\partial f_j(\xi)}{\partial \xi}\frac{\partial f_k(\xi)}{\partial \xi}\frac{\partial f_l(\xi)}{\partial \xi} d\xi, \quad d_{ijkl} = \sigma_d \int_0^1 f_i(\xi) f_j(\xi)\frac{\partial f_k(\xi)}{\partial \xi}\frac{\partial f_l(\xi)}{\partial \xi} d\xi. \tag{24}$$

The knowledge of the describing functions of the series in (17) allows us to determine the coefficients b_{ijkl} and d_{ijkl}; see the electronic supplement.

The mass matrix elements m_{ij} in the kinetic energy expression,

$$\mathcal{T} = \tfrac{1}{2} m_{ij}\dot{W}_i\dot{W}_j, \quad \dot{W}_i = \partial W_i/\partial \tau, \quad i, j = 1, 2, \dots, N, \tag{25}$$

have also been previously evaluated [Tizzi 1994]. It is true that

$$m_{ij} = \lambda c_{ij}, \tag{26}$$

where the nondimensional coefficient λ has been defined in (14). It is well known that $-m_{ij}\ddot{W}_j$ is the generalized inertial force acting on the i-th degree of freedom, corresponding to the second term of (13).

If the expressions of the strain and kinetic energy in (20) and (25), along with the expression of the generalized aerodynamic force in (22), are substituted into (19), in view of the Lagrangian functional expression in (18), it is possible to achieve the equivalent generalized form of (13), corresponding to the generic i-th degree of freedom, as follows:

$$\Big[k^*_{ij} + \frac{\alpha}{2}\varphi_{ij}(\varphi_{kl}W_k W_l)\Big]W_j + \gamma_m b_{ijkl}W_k W_l W_j + m_{ij}\ddot{W}_j + \gamma' c_{ij}\dot{W}_j + \gamma'' d_{ijkl}W_k W_l \dot{W}_j = 0,$$
$$j, k, l = 1, 2, \dots, N, \quad i = 1, 2, \dots, N. \tag{27}$$

The matrix elements k^*_{ij}, referring to both structural and aerodynamic linear forces, can be written as

$$k^*_{ij} = k_{ij} + a_{ij}. \tag{28}$$

Thus $k^*_{ij} W_j$ is the generalized linear structural-aerodynamic force acting on the same i-th degree of freedom, which corresponds to the first term and the first part of the third term of (13).

Furthermore, the term $(\varphi_{kl} W_k W_l)$ between brackets in (27) is the mean square value $\overline{(\partial W/\partial\xi)^2}$ of the first derivative $\partial W/\partial\xi$ over the whole beam length, which has been already introduced in (11). This is the reason for which it has been written separately in round brackets. As mentioned above, it takes into account the beam axial stretching, which gives rise to the nonlinear component of the structural transverse force, as in the fourth term of (13).

The term containing the coefficient γ_m in (27) is equivalent to the second part of the third term of (13), and the terms containing γ' and γ'' are equivalent to the ones with the same coefficients in (13).

The system of the generalized governing equations (27), in view of (26), can also be written in matrix form:

$$\boldsymbol{Z} = \dot{\boldsymbol{W}}, \quad \dot{\boldsymbol{Z}} = -\boldsymbol{M}^{-1}\boldsymbol{F}\boldsymbol{W} - \gamma_m \boldsymbol{M}^{-1}\boldsymbol{B}\boldsymbol{W}^{(3)} - \frac{\gamma'}{\lambda}\boldsymbol{Z} - \gamma''\boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{W}^{(3d)}, \tag{29}$$

where: $\boldsymbol{W}$ and $\boldsymbol{Z}$ are the column vectors of the coefficients W_j and their first derivatives $\dot{W}_j$ versus time τ, respectively; $\boldsymbol{F}$ is the matrix whose elements are:

$$f_{ij} = k^*_{ij} + \frac{\alpha}{2}(\varphi_{kl} W_k W_l)\varphi_{ij}; \tag{30}$$

$\boldsymbol{M}$ is the mass matrix; $\boldsymbol{B}$ is a matrix with dimensions $N \times N^3$, whose elements are $b_{ij_{t3}} = b_{ijkl}$ (j_{t3} is the contraction of the three indices jkl and obviously $j_{t3} = 1, 2, \ldots, N^3$); $\boldsymbol{W}^{(3)}$ is the column vector with dimensions N^3, whose elements are the triple products $p_{jt3} = W_k W_l W_j$ between the coefficients of the series expansion in (17); $\boldsymbol{D}$ is a matrix whose elements are $d_{ij_{t3}} = d_{ijkl}$ and with the same dimensions of $\boldsymbol{B}$; and $\boldsymbol{W}^{(3d)}$ is a column vector with the same dimensions of $\boldsymbol{W}^{(3)}$, whose elements are the triple products $q_{jt3} = W_k W_l \dot{W}_j$.

Equations (27) and (29) are the same as in the previous analysis with linearized aerodynamic forces, except for the presence of the terms with b_{ijkl} and d_{ijkl} in (27), and the corresponding matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ in (29), referring to the nonlinear contribution of the piston theory to the aerodynamic forces. The system (29) can be integrated in time by appropriate algorithms.

In the case of a simply supported beam it is easy to apply also the Galerkin method, as in Dowell's model [Dowell 1966; 1967]. The advantages of the Galerkin procedure arise from the diagonal form of the mass matrix $\boldsymbol{M}$, due to the orthogonality between different describing function elements; see the electronic supplement for details.

3. Applications and results

The analysis of the results is limited to the regular solutions with repetitive dynamic characteristics, obtained by giving particular starting conditions ($\tau = 0$) to the vibrating system. There are also many more spurious solutions, which are quite irregular and without any particular meaning.

The first case considered is the simply supported beam. The ratio between the length L and the thickness h is assumed to be 100.

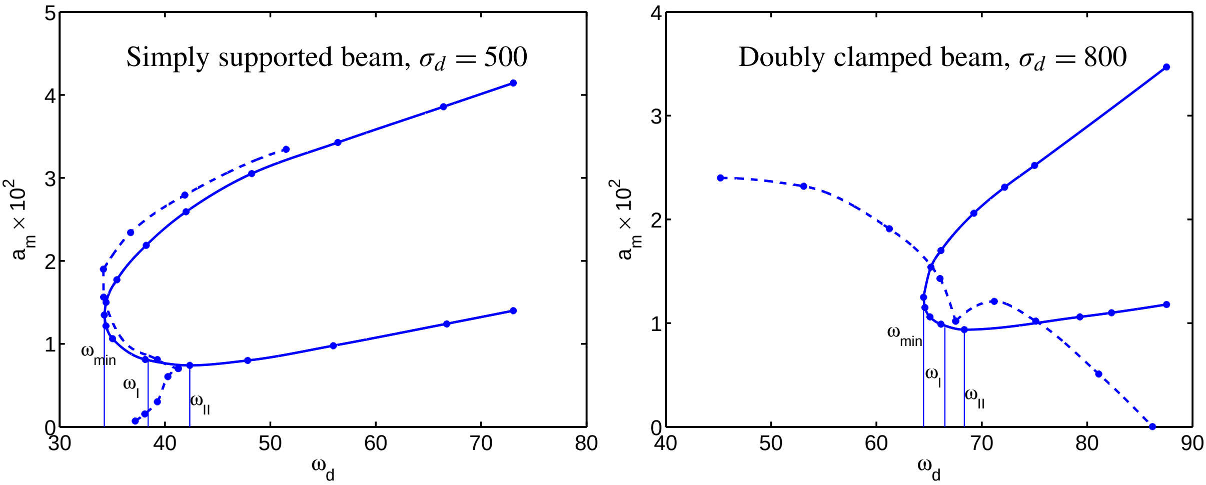

Figure 2. Solid curves: modal shape amplitude a_m versus dimensionless frequency ω_d for two undamped beam systems (Table 1 lists the values). Dashed curves: evolution of the damped beam vibration state towards the limit cycle in each of the two systems (Table 2). All values take into account the nonlinear aerodynamic force components.

All the characteristic behavior features of the nondimensional modal shape amplitude a_m as a function of the dimensionless frequency

$$\omega_d = \omega \sqrt{\frac{\mu L^4}{EI}} \tag{31}$$

(where ω is the angular frequency) are known from previous analyses [Tizzi 2003]. These features are illustrated by the solid curve in Figure 2, left, in the case of a simply supported undamped beam for $\sigma_d = 500$. (See also the top half of Table 1.) In fact this graph is similar to the one obtained without the nonlinear aerodynamic force components.

Three frequency values are particularly important and useful in characterizing this curve: the minimum frequency $\omega_{\min}$; the frequency ω_I, which separates the frequency range where lower amplitude modal shapes have only one half-wave from the range where these shapes have two half-waves; and the frequency ω_{II}, where the modal shape amplitude reaches its minimum.

The dashed lines in Figure 2 show the dynamic evolution of the fluttering beam towards the limit cycle conditions (still taking into account aerodynamic damping), for two different starting conditions. Clearly the dynamic solution of the fluttering beam in limit cycle conditions, also in the presence of nonlinear aerodynamic forces, lies in the neighborhood of the undamped beam solution with minimum amplitude. For this reason this particular solution is of paramount importance, and ω_{II} plays a privileged role, along with its corresponding modal shape amplitude value.

To see the influence of the nonlinear aerodynamic components of the piston theory of a high supersonic idealized flow. it is necessary to show the functional dependence of these frequencies on the dynamic pressure, with and without nonlinear aerodynamic forces. Particular attention must be given to these nonlinear forces' effects on ω_{II} and the corresponding modal amplitude behavior versus σ_d.

Simply supported, undamped beam, $\sigma_d = 500$							
ω_d	a_m	ω_d	a_m	ω_d	a_m	ω_d	a_m
73.06	0.04144	42.00	0.02592	34.24	0.01348	42.35	0.00741
66.43	0.03858	38.22	0.02188	34.40	0.01216	47.84	0.00800
56.40	0.03427	35.43	0.01772	35.02	0.01063	55.97	0.00978
48.23	0.03052	34.41	0.01501	38.13	0.00812	66.74	0.0124
						73.06	0.0140

Doubly clamped, undamped beam, $\sigma_d = 800$							
ω_d	a_m	ω_d	a_m	ω_d	a_m	ω_d	a_m
87.53	0.0347	66.12	0.0170	65.08	0.0106	82.31	0.0110
75.01	0.0252	65.17	0.0154	66.12	0.0099	87.53	0.0118
72.17	0.0231	64.47	0.0125	68.34	0.0094		
69.25	0.0206	64.60	0.0115	79.31	0.0106		

Table 1. Modal amplitude a_m versus dimensionless frequency ω_d, taking into account nonlinear aerodynamic force components.

Simply supported beam						Doubly clamped beam					
τ	ω_d	a_m	τ	ω_d	a_m	τ	ω_d	a_m	τ	ω_d	a_m
Upper dashed curve			Lower dashed curve			Left dashed curve			Right dashed curve		
0.	51.5	0.0334	0.	37.18	0.00071	0.	45.20	0.0240	0.	86.20	0.00002
0.16	41.88	0.0279	0.16	38.08	0.0016	0.43	53.10	0.0232	0.47	81.10	0.0051
0.30	36.74	0.0234	0.24	39.27	0.0030	0.85	61.23	0.0191	2.63	75.12	0.0102
0.47	34.15	0.0190	1.33	40.27	0.0061	1.47	66.03	0.0143	5.33	71.19	0.0121
0.56	34.17	0.0156	1.81	41.28	0.0070	2.53	67.52	0.0102	9.81	67.52	0.0102
1.18	39.27	0.0081									
2.37	41.24	0.0070									

Table 2. Modal amplitude versus dimensionless frequency ω_d at various times, taking into account aerodynamic damping and nonlinear aerodynamic force components.

In some of the following figures the frequency values, derived both by the idealized and linearized beam model and by the nonlinear approach, are drawn together. This is useful for having an overall picture of all the possible beam flutter solutions in pre- and postcritical conditions, obtained by the two different approaching models. It is important to emphasize the convergence of the nonlinear model solutions towards those of the linear approach, as the dynamic pressure, and consequently the flutter modal amplitude, diminishes. The solutions coincide when this amplitude vanishes, considering that the influence of the nonlinear components of the acting forces can be neglected as the vibrating modes tend to disappear.

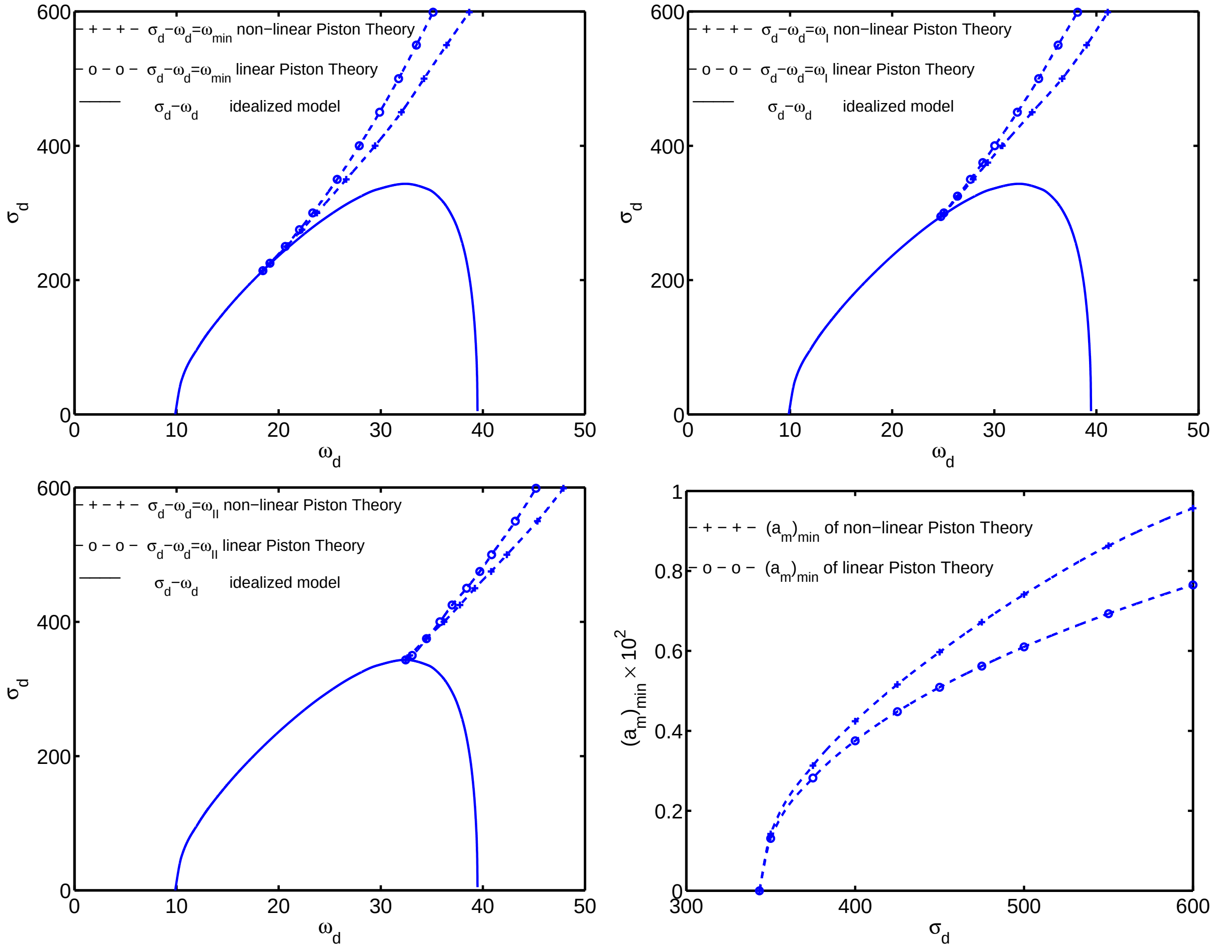

Figure 3. Frequencies $\omega_{\min}$ (top left), ω_I (top right) and ω_{II} (bottom left) versus σ_d for the simply supported, undamped beam, with and without nonlinear aerodynamic force components (dashed lines). The horseshoe-shaped solid curve shows the dependence of σ_d on ω_d for the linearized model (see text). Also shown is the minimum amplitude $(a_m)_{\min}$ as a function of σ_d for the same beam system (bottom right).

Figure 3, top left, shows the behavior of the minimum frequency $\omega_{\min}$ of the undamped solutions versus the dynamic pressure dimensionless parameter σ_d; the two types of dashed lines indicate the presence or absence of nonlinear aerodynamic forces in the calculation. The solid line describes the classical and well known dependence of σ_d on ω_d for the linearized beam flutter simulation model, where only the linear components of both structural and aerodynamic forces are considered.

The dependence of the frequencies ω_I and ω_{II} on σ_d is sketched in the next two parts of Figure 3, with the same conventions. Finally, the bottom right part of the figure is a graph of the minimum amplitude $(a_m)_{\min}$ versus σ_d, with and without nonlinear aerodynamic forces. The bottom two graphs in Figure 3 are particularly important for the reasons mentioned above. See also the top half of Table 3.

Simply supported, undamped beam								
σ_d	$\omega_{\min}$		ω_I		ω_{II}		$(a_m)_{\min} \times 10^2$	
	L	NL	L	NL	L	NL	L	NL
214	18.46	18.46						
225	19.15	19.18						
250	20.65	20.76						
275	22.04	22.28						
294.6			24.76	24.76				
300	23.34	23.75	25.06	25.08				
325			26.39	26.51				
343.356					32.43	32.43	0.	0.
350	25.73	26.62	27.66	27.93	32.95	33.02	0.13	0.14
375			28.87	29.36	34.40	34.48	0.28	0.31
400	27.90	29.44	30.04	30.79	35.79	36.19	0.37	0.42
425					36.99	37.75	0.45	0.52
450	29.89	32.02	32.26	33.71	38.41	39.22	0.51	0.60
475					39.70	40.81	0.56	0.67
500	31.75	34.24	34.35	36.64	40.86	42.35	0.61	0.74
550	33.49	36.44	36.25	39.04	43.18	45.34	0.69	0.86
600	35.12	38.67	38.16	41.11	45.20	47.89	0.76	0.96

Doubly clamped, undamped beam								
σ_d	$\omega_{\min}$		ω_I		ω_{II}		$(a_m)_{\min} \times 10^2$	
	L	NL	L	NL	L	NL	L	NL
449	35.53	35.53	35.53	35.53				
500	38.16	19.18	38.17	38.60				
550	40.54	41.80	40.60	42.00				
600	42.76	45.15	42.89	45.83				
636.569					52.36	52.36	0.	0.
650	44.85	49.44	45.06	50.25				
655					53.36	54.01	0.20	0.30
675					54.48	56.13	0.28	0.45
700	46.83	53.69	47.14	54.69	55.69	58.76	0.36	0.59
725					56.96	61.27	0.42	0.70
750	48.71	58.91	49.13	59.91	58.19	63.70	0.48	0.80
775					59.41	65.88	0.53	0.87
800	50.50	64.47	51.05	66.50	60.57	68.34	0.57	0.94

Table 3. Values of $\omega_{\min}$, ω_I, ω_{II} and $(a_m)_{\min}$ versus the dimensionless dynamic pressure σ_d, without (L) and with (NL) nonlinear components of the aerodynamic forces for the undamped beam.

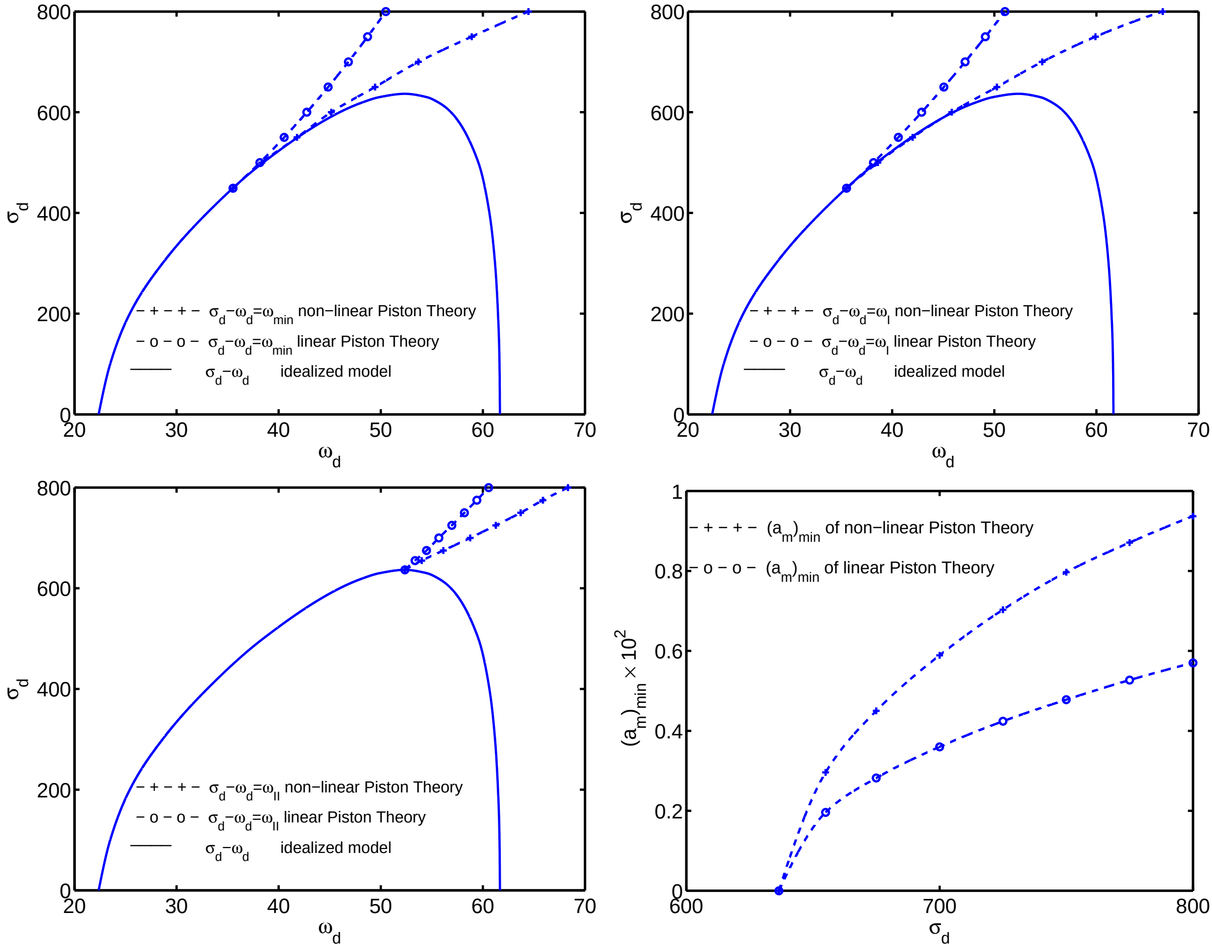

Figure 4. Frequencies $\omega_{\min}$ (top left), ω_I (top right) and ω_{II} (bottom left) versus σ_d for the doubly clamped, undamped beam, with and without nonlinear aerodynamic force components (dashed lines). The horseshoe-shaped solid curve shows the dependence of σ_d on ω_d for the linearized model (see text). Also shown is the minimum amplitude $(a_m)_{\min}$ as a function of σ_d for the same beam system (bottom right).

We next display the results obtained in the case of a beam clamped at both ends. The ratio between the length L and the thickness h is taken to be 110. Figure 2, right, shows the modal amplitude a_m behavior of the fluttering undamped solution versus ω_d for $\sigma_d = 800$ (also described in the bottom half of Table 1), together with the vibrating damped beam state evolution towards the limit cycle point, which lies in the neighborhood of the representative point of the undamped solution with minimum amplitude. The two dashed lines correspond to two different starting values of the fluttering beam with damping. Figure 4, organized in the same way as Figure 3, shows the dependence of $\omega_{\min}$, ω_I, ω_{II} and $(a_m)_{\min}$ on σ_d for the undamped beam clamped at both ends, again with and without taking into account the nonlinear components of the aerodynamic forces. See also the bottom half of Table 3.

4. Conclusions

On the basis of the results achieved, some concluding remarks can be made.

(1) In both cases considered, the minimum frequency value $\omega_{\min}$ is higher compared to the corresponding one derived by the linearized piston theory, which means that the frequency range of the undamped solutions decreases with respect to the case of the linearized aerodynamic model, although this reduction is very limited.

(2) Limited decrease of the resistance to the flutter phenomenon is introduced by the nonlinear aerodynamic force contributions, considering that the minimum value of the undamped vibration modal amplitude $(a_m)_{\min}$, which is very close to the limit cycle amplitude value, increases; but this relative growth is not considerable.

(3) Undoubtedly these nonlinear aerodynamic forces influence the whole postcritical flutter behavior of a beam under the effects of a high supersonic flow at both sides, but, unless the dynamic pressure grows very much over its critical value of the linear model, these effects are limited. Hence the linear aerodynamic analysis of the piston theory is sufficient to describe, within acceptable limits, this postcritical behavior.

(4) The results obtained don't predict a threat to the stability, because in the cases considered, the airflow speed doesn't exceed overmuch the limit critical value of the linearized model (limited Mach number), as in [McIntosh 1973; Smith and Morino 1976]. For very high values of M_∞ the destabilizing effect of aerodynamic nonlinearities is predominant and the limit cycle becomes unstable. However the presence of particular initial conditions (such as those induced by a gust) can lead to instability and chaos even before the dynamic pressure reaches its critical value [Dessi et al. 2002]. The presence of shock waves in transonic flight could also instigate instability in the flutter phenomenon.

(5) The dynamic analysis of the fluttering beam has been limited to the case of symmetrically distributed aerodynamic forces, which is worse, in terms of fluttering beam stability, than the case of a high supersonic flow acting only on one side of the vibrating beam. Therefore all considerations regarding the fluttering stability are very likely valid for asymmetric cases as well.

(6) This study of postcritical behavior has been limited to a one-dimensional panel, as in the linearized piston theory case, but similar conclusions are likely valid for two-dimensional panels.

(7) This dynamic analysis can also be extended to cases not considered here.

Acknowledgements

The author is grateful to Sandro Mileti for helping to improve his English.

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Received 16 Apr 2008. Revised 12 Sep 2008. Accepted 13 Nov 2008.

SILVANO TIZZI: s.tizzi@caspur.it
University of Rome "La Sapienza", Aerospace and Astronautics Engineering Department, Via Eudossiana 16, Rome, 00184, Italy,