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CONTINUUM MECHANICS MODELS OF FRACTAL POROUS MEDIA: INTEGRAL RELATIONS AND EXTREMUM PRINCIPLES

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This paper continues the extension of continuum mechanics and thermodynamics to fractal porous media which are specified by a mass (or spatial) fractal dimension D, a surface fractal dimension d, and a resolution length-scale R. The focus is on a theory based on dimensional regularization, in which D is also the order of fractional integrals employed to state global balance laws. Thus, we first generalize the main integral theorems of continuum mechanics to fractal media: Stokes, Reynolds, and Helmholtz–Żórawski. Then, we review balance equations and recently obtained extensions of several subfields of continuum mechanics to fractal media. This is followed by derivations of extremum and variational principles of elasticity and Hamilton's principle for fractal porous materials. In all the cases, we derive relations which depend explicitly on D, d and R, and which, upon setting $D = 3$ and $d = 2$, reduce to the conventional forms of governing equations for continuous media with Euclidean geometries.

1. Background and motivation

The term fractal was coined by Benoît Mandelbrot in 1975 [Mandelbrot 1982] to denote an object that is "broken" or "fractured" in space and/or time. Basically, a fractal object can be subdivided in parts, each of which is in a deterministic or stochastic sense a reduced-size copy of the whole; this is the famous self-similarity property (1). In general, a fractal also has these features: (2) fine structure at arbitrarily small scales; (3) too irregular to be easily described in traditional Euclidean geometric language; (4) Hausdorff dimension greater than the topological dimension; (5) a simple and recursive definition.[1]

Thus, "mathematical fractals" appear similar at all levels of magnification, and, roughly speaking, they are infinitely complex. Focusing on fractals in space, as opposed to those in time (signals, processes), many natural and man-made objects approximate fractals to a degree: coastlines, porous media, cracks, turbulent flows, clouds, mountains, lightning bolts, brains, snow flakes, melting ice (and other systems at phase transitions). The list is very long, and hence book titles like *Fractals Everywhere* [Barnsley 1993]. Mathematical fractals provide appropriate models for many media for some finite range of length scales, with lower and upper cutoffs.

Concerning materials with fractal geometries, since the late eighties a lot of research has been carried out primarily in condensed matter physics [Feder 1988]. That work has been focused on physics — explaining physical phenomena and properties for materials whose fractal (non-Euclidean) geometry plays a key role. However, a field theory, an analogue of continuum physics and mechanics, has sorely been lacking. Some progress in that respect has recently been made by mathematicians [Kigami 2002;

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[1] Self-similarity does not suffice to characterize fractals: a straight line is formally self-similar but has no other fractal characteristics. On the other hand, space-filling curves such as the Hilbert curve do not satisfy (4).

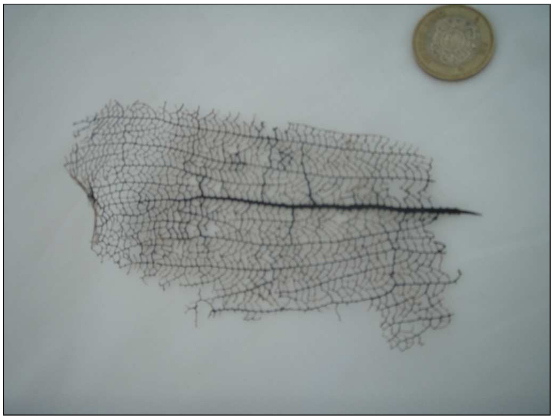

Figure 1. This fern, found on the seashore in Cancún, Mexico, just before PACAM X (January 2008), represents an example of a prefractal. A coin is shown for reference.

Strichartz 2006; Epstein and Śniatycki 2006; Epstein and Adeeb 2008], who began to look at classical problems, like Laplace's or heat equation, on fractal (albeit nonrandom) sets. This approach, in fact very technical from the mathematical analysis standpoint, only begins to offer an avenue to tackle simple initial-boundary value (IBV) problems.

A very different step in the direction of a field theory and IBV problems has recently been taken by Tarasov [2005a, 2005b, 2005c]. He developed continuum-type equations of conservation of mass, linear and angular momenta, and energy for fractals, and, on that basis studied several fluid mechanics and wave problems. The beauty and power of Tarasov's approach relies on a generalization of the Green–Gauss theorem to fractal objects through fractional integrals in Euclidean space, see Section 2.2 below. Another advantage of this approach is that it admits upper and lower cutoffs of fractal scaling, so that one effectively deals with a physical "prefractal", like the one in Figure 1, rather than a purely mathematical fractal lacking cutoffs. It is in that sense that fractals are meant here. In principle, one can then map a mechanics problem of a fractal [which is described by its mass (D) and surface (d) fractal dimensions plus the spatial resolution (R)] onto a problem in Euclidean space in which this fractal is embedded, while having to deal with coefficients explicitly involving D, d and R. Clearly, this has very interesting ramifications for formulating continuum-type mechanics of fractal media, which need to be further explored. The great promise stems from the fact that the conventional requirement of continuum mechanics, the separation of scales, can be removed, yet the partial differential equations may still be employed.

Working in the outlined setting, in this paper we examine the integral theorems of continuum mechanics in the setting of fractal media: Stokes, Reynolds, and Helmholtz–Żórawski. In fact, the second of these leads us to modify Tarasov's fractional material derivative back to a conventional material derivative. As a result, the balance laws of mass, linear momentum, energy as well as the second law of thermodynamics take simpler forms than using Tarasov's interpretation. We also list generalizations of: the Clausius-Duhem inequality, the linear thermoelasticity, the Maxwell–Betti reciprocity, the Hill condition and energy principles, and the averaged equations of turbulence in fractal porous media [Ostoja-Starzewski 2007a, 2007b, 2008a]. This is followed by derivations of extremum and variational principles of elasticity and the Hamilton's principle for fractal porous materials. In all the cases, we obtain relations which depend explicitly on D, d and R, and which, upon setting $D = 3$ and $d = 2$, reduce to conventional (well known) forms of governing equations for continuous media with Euclidean geometries.

2. Continuum mechanics in the setting of fractal media

Integral theorems. We consider fractal porous media in which mass obeys a power law relation

$$m(R) = kR^D, \quad D < 3, \tag{2-1}$$

where R is a box size (or a sphere radius, effectively a length scale of measurement), D is a fractal dimension of mass (based on, say, the box-counting method), and k is a proportionality constant. Equation (2-1) implies that the conventional equation giving mass in a three-dimensional region W (of volume V and boundary ∂W)

$$m(W) = \int_W \rho(r)\, d^3 r \tag{2-2}$$

has to be generalized to

$$m_{3d}(W) = \frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)} \int_W \rho(r)|r - r_0|^{D-3} d^3 r. \tag{2-3}$$

That is, the fractal medium with a noninteger mass dimension D is described using a fractional integral of order D. This interpretation of the fractal (intrinsically discontinuous) medium as a continuum — in the vein of *dimensional regularization* of quantum mechanics [Collins 1984] and involving a Riesz fractional form — is based on a reformulation of the Green–Gauss (or divergence) Theorem

$$\int_{\partial W} f_k n_k dA_d = \int_W c_3^{-1}(D, R)\nabla_k(c_2(d, R) f_k)\, dV_D, \tag{2-4}$$

where f_k is a vector field (in subscript notation) and

$$dA_d = c_2(d, R)\, dA_2 dV_D = c_3(D, R)\, dV_3. \tag{2-5}$$

Here dA_2 and dV_3 are, respectively, the conventional infinitesimal elements of surface and volume in Euclidean space, while dA_2 and dV_3 are the corresponding elements of a fractal's surface and volume. Note that the left-hand side in (2-4) is a fractional integral, equal to a conventional integral $\int_{\partial W} c_2(d, R) f_k n_k dA_2$, while the right-hand side is a fractional integral, equal to a conventional integral $\int_W \operatorname{div}(c_2(d, R) f_k)\, dV_3$. Thus, we can rewrite (2-4) as

$$\int_{\partial W} c_2(d, R) f_k n_k dA_2 = \int_W \nabla_k(c_2(d, R) f_k)\, dV_3, \tag{2-6}$$

and, in fact, extend this theorem to the setting with a jump $[f_k]$ on a surface S across W:

$$\int_{\partial W} c_2(d, R) f_k n_k dA_2 = \int_{W-S} \nabla_k(c_2(d, R) f_k)\, dV_3 + \int_S c_2(d, R)[f_k] n_k dA_2. \tag{2-7}$$

The proof of (2-7) follows the same lines as that in conventional continuum mechanics; dividing the body W into two parts separated by S, applying the Green–Gauss theorem to each part while accounting for the jump from either side, and combining both results. From (2-7) one obtains a special form, sometimes simply called the Green's (or gradient) Theorem,

$$\int_{\partial W} c_2(d,R) f n_k dA_2 = \int_{W-S} \nabla_k (c_2(d,R) f)\, dV_3 + \int_S c_2(d,R)[f] n_k dA_2. \tag{2-8}$$

While (2-4) was derived by Tarasov [2005b], the following form of the Reynolds (transport) Theorem was adopted in an *ad hoc* fashion without a derivation:

$$\frac{d}{dt}\int_W P(x,t)\, dV_D = \int_W \frac{\partial}{\partial t} P dV_D + \int_{\partial W} P v_k dA_d. \tag{2-9}$$

Effectively, (2-4) and (2-9) have led Tarasov to introduce these conventional forms of two key differential operators of continuum mechanics:

$$\begin{aligned} \nabla_k^D f &= c_3^{-1}(D,R)\frac{\partial}{\partial x_k}[c_2(d,R) f] \equiv c_3^{-1}(D,R)\nabla_k[c_2(d,R) f] \Big(\frac{d}{dt}\Big)_D f \\ &= \frac{\partial f}{\partial t} + c(D,d,R) v_k \frac{\partial f}{\partial x_k}, \end{aligned} \tag{2-10}$$

where

$$\begin{aligned} c(D,d,R) &= |\boldsymbol{R}|^{d+1-D}\frac{2^{D-d-1}\Gamma(D/2)}{\Gamma(3/2)\Gamma(d/2)} = c_3^{-1}(D,R) c_2(d,R), \\ c_2(d,R) &= |\boldsymbol{R}|^{d-2}\frac{2^{2-d}}{\Gamma(d/2)}, \qquad c_3(D,R) = |\boldsymbol{R}|^{D-3}\frac{2^{3-D}\Gamma(3/2)}{\Gamma(D/2)}. \end{aligned} \tag{2-11}$$

Now, proceeding in the same vein as with (2-7), we obtain an extension of the Stokes (curl) Theorem

$$\int_{\partial W} \boldsymbol{n} \times \boldsymbol{f} dA_d = \int_W c_3^{-1}(D,R)\, \mathbf{curl}[c_2(d,R)\boldsymbol{f}]\, dV_D. \tag{2-12}$$

To clarify, this involved the steps

$$\begin{aligned} \int_{\partial W} e_{ijk} n_j f_k dA_d &= \int_{\partial W} e_{ijk} n_j f_k c_2(d,R)\, dA_2 = \int_W [e_{ijk} f_k c_2(d,R)],_j\, dV_3 \\ &= \int_W c_3^{-1}(D,R)[c_2(d,R) e_{ijk} f_k],_j\, dV_D. \end{aligned} \tag{2-13}$$

This procedure may now be extended to derive the Reynolds (transport) Theorem for fractal media. Similar to the case of nonfractal media, but focusing on a region W of mass fractal dimension D and bounded by a surface of another fractal dimension d, in the following steps we obtain the time rate of change of an integral involving a spatially distributed quantity f:

$$\begin{aligned} \frac{d}{dt}\int_W f(x,t)\, dV_D &= \frac{d}{dt}\int_W f J\, dV_D^0 = \int_W \frac{d}{dt}[f J]\, dV_D^0 = \int_W [\dot{f} J + P\dot{J}] dV_D^0 \\ &= \int_W [\dot{f} J + f v_{k,k} J] dV_D^0 = \int_W [\dot{f} + f v_{k,k}] dV \\ &= \int_W \Big(\frac{\partial}{\partial t} f + f,_k v_k + f v_{k,k}\Big) dV_D = \int_W \Big(\frac{\partial}{\partial t} f + (f v_k),_k\Big)\, dV_D. \end{aligned} \tag{2-14}$$

Here dV_D^0 refers to the volume of a fractal object in the reference configuration, while J denotes a Jacobian of motion defined in terms of material coordinates. Two observations are relevant at this point:

(i) The final result cannot be written as a sum of a volume integral and a surface integral, the way it is done in (2-9). Nor is it possible to write $\frac{d}{dt}\int_W f(x,t)\,dV_D$ as $\int_W \frac{\partial}{\partial t} f dV_3 + \int_{\partial W} f v_k dA_2$.

(ii) The third line in (2-14) dictates the conventional material derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x_k}. \tag{2-15}$$

This is in contrast to $(d/dt)_D = \partial f/\partial t + c(D,d,R) v_k f_{,k}$, with $c(D,d,R) = c_3^{-1}(D,R)c_2(d,R)$, of Tarasov's equation (2-9). In consequence, in all previous results in mechanics of fractal media, $(d/dt)_D$ is to be simply replaced by the conventional material derivative d/dt, which leads to certain simplifications.

Proceeding in a similar way for the time rate of change of a surface integral involving a quantity Q distributed over a surface ∂W, we find

$$\begin{aligned}\frac{d}{dt}\int_{\partial W} Q(x,t) n_p dA_d &= \int_{\partial W} \dot{Q}(x,t)\, n_p dA_d + \int_{\partial W} Q(x,t)\, \frac{d}{dt}(n_p dA_d) \\ &= \int_{\partial W} \dot{Q}(x,t)\, c_2 n_p dA_2 + \int_{\partial W} Q(x,t)\, (n_p v_{k,k} - n_k v_{k,p}) c_2 dA_2 \\ &= \int_{\partial W} \big([\dot{Q}(x,t) + v_{k,k}\, Q(x,t)]\delta_{pq} - Q(x,t) v_{q,p}\big) n_q c_2 dA_2 \\ &= \int_{\partial W} \big([\dot{Q}(x,t) + v_{k,k}\, Q(x,t)]\delta_{pq} - Q(x,t) v_{q,p}\big) n_q dA_d. \end{aligned} \tag{2-16}$$

This implies that the fractal structure of the medium does not affect the essential conclusion of the Helmholtz–Żórawski lemma.

Balance equations. The results above lead to modified balance equations of fractal media [Tarasov 2005a, 2005b; Ostoja-Starzewski 2007a]:

- the fractional equation of continuity,

$$\dot{\rho} = -\rho \nabla_k^D v_k; \tag{2-17}$$

- the fractional equation of balance of density of momentum,

$$\rho \dot{v}_k = \rho f_k + \nabla_k^D \sigma_{kl}; \tag{2-18}$$

- the fractional equation of balance of density of energy,

$$\rho \dot{u} = \sigma_{kl} d_{kl} - \nabla_i^D q_i; \tag{2-19}$$

- the Clausius-Duhem inequality,

$$\sigma_{ij}^{(d)} d_{ij} + \beta_{ij}^{(d)} \dot{\alpha}_{ij} - \frac{\theta_{,k}\, q_k}{\theta} \geq 0. \tag{2-20}$$

Here σ_{kl} is the Cauchy stress (symmetric according to the balance of angular momentum, employed just like in nonfractal media), $\sigma_{ij}^{(d)}$ and $\beta_{ij}^{(d)}$ are the dissipative stresses, while α_{ij} are the internal parameters in the vein of thermomechanics with internal variables [Ziegler 1983]. One may argue, however, that a complex microstructure of many fractal media, such as exemplified by lattices involving bending moment interactions (see [Limat 1988], for instance), would imply micropolar effects. This subject will require a separate treatment.

3. Some previous results

For completeness, we list several recently obtained generalizations of conventional continuum mechanics results to the setting of fractal media. See [Ostoja-Starzewski 2007a, 2007b, 2008a] for details.

(i) The equation of Fourier-type heat conduction,

$$\rho c_p \dot{\theta} = \frac{\partial}{\partial x_i}\Big(k_{ij}\frac{\partial \theta}{\partial x_j}\Big), \tag{3-1}$$

is generalized for fractal media to (with k being the thermal conductivity)

$$\rho c_p \dot{\theta} = \nabla_i^D\Big(k_{ij}\frac{\partial \theta}{\partial x_j}\Big). \tag{3-2}$$

(ii) The Duhamel equation of linear thermoelasticity,

$$\rho c_p \dot{\theta} = -(3\lambda + 2\mu)\alpha\theta_0\dot{\varepsilon}_{(1)} + \frac{\partial}{\partial x_i}\Big(k_{ij}\frac{\partial \theta}{\partial x_j}\Big),$$

is generalized for fractal media to

$$\rho c_p \dot{\theta} = -(3\lambda + 2\mu)\alpha\theta_0\dot{\varepsilon}_{(1)} + \nabla_i^D\Big(k_{ij}\frac{\partial \theta}{\partial x_j}.\Big).$$

Here λ and μ are the Lamé coefficients, α is the coefficient of thermal expansion and the subscript $_{(1)}$ indicates the first basic invariant of strain.

(iii) The Maxwell–Betti reciprocity relation of linear elasticity,

$$\int_{\partial W} t_i^* u_i \, dA_2 = \int_{\partial W} t_i u_i^* \, dA_2. \tag{3-3}$$

is generalized for fractal media to

$$\int_{\partial W} t_i^* u_i \, dA_d = \int_{\partial W} t_i u_i^* \, dA_d. \tag{3-4}$$

(iv) The Hill condition, which in the classical case reads

$$\overline{\sigma_{ij}\varepsilon_{ij}} = \overline{\sigma}_{ij}\overline{\varepsilon}_{ij}, \tag{3-5}$$

for a fractal medium becomes

$$\overline{c(D,d,R)\sigma_{ij}\varepsilon_{ij}} = \overline{\sigma}_{ij}\overline{\varepsilon}_{ij}. \tag{3-6}$$

(v) The local form of the balance of linear momentum of a turbulent flow, accounting for fluctuations and after averaging is carried out,

$$\rho\left(\frac{d}{dt}\overline{v}_i + \overline{v}_{i,j}\overline{v}_j\right) = -\rho(\overline{v_i' v_j'}),_j + \rho f_k + \overline{\sigma}_{ij},_j\,, \tag{3-7}$$

is modified for a fractal porous medium to

$$\rho\left(\frac{d}{dt}\overline{v}_i + \overline{v}_{i,j}\overline{v}_j\right) = -\rho\nabla_j^D(\overline{v_i' v_j'}) + \rho f_k + \nabla_j^D\overline{\sigma}_{ij}. \tag{3-8}$$

This implies that the Reynolds stress σ_{ij}^* can no longer be simply written as $\sigma_{ij}^* - \rho(\overline{v_i' v_j'})$, but it is a function of D, d, and R.

The same type of approach allows a generalization of a telegraph equation governing the propagation of second sound in a rigid conductor with fractal geometry [Ignaczak and Ostoja-Starzewski 2009].

4. Extremum and variational principles in elasticity

Statically admissible fields. Consider a *statically admissible field* denoted by *. We can then write

$$\int_{\partial W}(t_i^* - t_i)u_i dA_d = \int_{\partial W_t}(t_i^* - t_i)u_i dA_d. \tag{4-1}$$

On account of (2-4) and the fractional equation of static equilibrium of a body without a body force field — a special case of (2-18) — this becomes

$$\int_W c(D, d, R)(\sigma_{ij}^* - \sigma_{ij})\varepsilon_{ij} dV_D. \tag{4-2}$$

Now, at every point in the fractal, strictly speaking prefractal, elastic body, just like in a nonfractal elastic body, this inequality holds

$$(\sigma_{ij}^* - \sigma_{ij})\varepsilon_{ij} < \frac{1}{2}(\sigma_{ij}^*\varepsilon_{ij}^* - \sigma_{ij}\varepsilon_{ij}). \tag{4-3}$$

Thus, (4-1) is written as

$$\frac{1}{2}\int_W c(D, d, R)\sigma_{ij}^*\varepsilon_{ij}^* dV_D - \int_{\partial W_u} t_i^* u_i dA_d > \frac{1}{2}\int_W c(D, d, R)\sigma_{ij}\varepsilon_{ij} dV_D - \int_{\partial W_u} t_i u_i dA_d, \tag{4-4}$$

or, equivalently,

$$\frac{1}{2}\int_W c(D, d, R)\sigma_{ij}^*\varepsilon_{ij}^* dV_D - \int_{\partial W_u} t_i^* u_i dA_d > \frac{1}{2}\int_{\partial W_t} t_i u_i dA_d - \frac{1}{2}\int_{\partial W_u} t_i u_i dA_d. \tag{4-5}$$

On account of (2-5), (4-5) can be written as

$$\frac{1}{2}\int_W c_2(d, R)\sigma_{ij}^*\varepsilon_{ij}^* dV_3 - \int_{\partial W_u} c_2(d, R) t_i^* u_i dA_2 > \frac{1}{2}\int_{\partial W_t} c_2(d, R) t_i u_i dA_2 - \frac{1}{2}\int_{\partial W_u} c_2(d, R) t_i u_i dA_2, \tag{4-6}$$

which means that the expression on the left-hand side takes an absolute minimum value in the actual state.

Next, consider the expression

$$\int_W c(D, d, R) u^*(\sigma^*_{ij}) dV_D - \int_{\partial W_u} t^*_i u_i dA_d, \tag{4-7}$$

where u^* is the potential energy density, a functional of σ^*_{ij}, such that

$$\frac{\partial}{\partial \sigma^*_{ij}} u^*(\sigma^*_{ij}) = \varepsilon^*_{ij}. \tag{4-8}$$

We inquire when the expression (4-7) assumes a stationary value with respect to σ^*_{ij} satisfying the equilibrium condition. First, we can rewrite (4-7) as

$$\int_W c(D, d, R) U^*(\sigma^*_{ij}) dV_D - \int_{\partial W} t^*_i u_i dA_d + \int_{\partial W_t} t^*_i u_i dA_d, \tag{4-9}$$

which becomes

$$\int_W c(D, d, R)[u^*(\sigma^*_{ij}) - \sigma_{ij}\varepsilon_{ij}] dV_D + \int_{\partial W_t} t^*_i u_i dA_d. \tag{4-10}$$

Since $t^*_i = t_i$ on ∂W_t, this has a stationary value when

$$\frac{\partial}{\partial \sigma^*_{ij}} u^*(\sigma^*_{ij}) = \varepsilon_{ij}. \tag{4-11}$$

In view of (4-8), (4-7) has a stationary value when

$$\varepsilon^*_{ij} = \varepsilon_{ij}, \tag{4-12}$$

that is, in the actual state.

Kinematically admissible fields. Consider a *kinematically admissible field* denoted by *. We can then prove

$$\int_{\partial W_t} (u^*_i - u_i) t_i dA_d = \int_W c(D, d, R)(\varepsilon^*_{ij} - \varepsilon_{ij}) \sigma_{ij} dV_D. \tag{4-13}$$

Now, at every point in the prefractal elastic body, just as in a nonfractal elastic body, we have

$$\tfrac{1}{2}(\sigma^*_{ij}\varepsilon^*_{ij} - \sigma_{ij}\varepsilon_{ij}) > (\varepsilon^*_{ij} - \varepsilon_{ij})\sigma_{ij}, \tag{4-14}$$

unless $\sigma^*_{ij} = \sigma_{ij}$, which leads to

$$\int_{\partial W_t} (u^*_i - u_i) t_i dA_d < \frac{1}{2} \int_W c(D, d, R)(\sigma^*_{ij}\varepsilon^*_{ij} - \sigma_{ij}\varepsilon_{ij})\, dV_D \tag{4-15}$$

or

$$\int_{\partial W_t} (u^*_i - u_i) t_i dA_d - \frac{1}{2} \int_W c(D, d, R) \sigma^*_{ij}\varepsilon^*_{ij} dV_D < \int_{\partial W_t} u_i t_i dA_d - \frac{1}{2} \int_W c(D, d, R) \sigma_{ij}\varepsilon_{ij} dV_D. \tag{4-16}$$

Equivalently, we can write

$$\int_{\partial W_t} (u_i^* - u_i) t_i \, dA_d - \frac{1}{2} \int_W c(D, d, R) \sigma_{ij}^* \varepsilon_{ij}^* dV_D < \int_{\partial W_t} u_i t_i \, dA_d - \frac{1}{2} \int_W c(D, d, R) \sigma_{ij} \varepsilon_{ij} dV_D. \quad \text{(4-17)}$$

Again on account of (2-5), this can be restated in terms of conventional integrals in Euclidean space:

$$\int_{\partial W_t} c_2(d, R)(u_i^* - u_i) t_i \, dA_2 - \frac{1}{2} \int_W c_2(d, R) \sigma_{ij}^* \varepsilon_{ij}^* dV_3 < \int_{\partial W_t} c_2(d, R) u_i t_i \, dA_2 - \frac{1}{2} \int_W c_2(d, R) \sigma_{ij} \varepsilon_{ij} dV_3. \quad \text{(4-18)}$$

This means that the expression on the left-hand side takes an absolute maximum value in the actual state.

The relations derived above imply that one can apply the extremum principles of elasticity to fractal bodies, provided extra information is taken into account through D, d and R.

5. Hamilton's principle for a fractal continuum

Just as in continuum mechanics of conventional media (see [Reddy 1984], for example), we begin with the statement of work done on a fractal body at time t by the resultant force in moving through the virtual displacement

$$\int_W \boldsymbol{f} \cdot \delta \boldsymbol{u} \, dV_D + \int_{\partial W_t} \boldsymbol{t} \cdot \delta \boldsymbol{u} \, dA_d - \int_W c(D, d, R) \sigma : \delta \varepsilon \, dV_D \quad \text{(5-1)}$$

(or equivalently $\int_W f_i \delta u_i dV_D + \int_{\partial W_t} t_i \delta u_i dA_d - \int_W c(D, d, R) \sigma_{ij} \delta \varepsilon_{ij} dV_D$), where the third term is dictated by (2-4) and the variation satisfies the conditions

$$\delta \boldsymbol{u} = 0 \text{ on } \partial W_u \text{ for all } t, \qquad \delta \boldsymbol{u}(\boldsymbol{x}, t_1) = \delta \boldsymbol{u}(\boldsymbol{x}, t_2) = 0 \text{ for all } \boldsymbol{x}. \quad \text{(5-2)}$$

At the same time, the work done by the inertia force $m\boldsymbol{a}$ in moving through the virtual displacement $\delta \boldsymbol{u}$ is given by

$$\int_W \rho \frac{\partial \boldsymbol{u}^2}{\partial t^2} \cdot \delta \boldsymbol{u} \, dV_D. \quad \text{(5-3)}$$

Now Newton's second law,

$$\boldsymbol{F} - m\boldsymbol{a} = \boldsymbol{0}, \quad \text{(5-4)}$$

dictates that

$$\int_{t_1}^{t_2} \left(\int_W \rho \frac{\partial u_i^2}{\partial t^2} \delta u_i \, dV_D - \left(\int_W f_i \delta u_i dV_D + \int_{\partial W_t} t_i \delta u_i dA_d - \int_W c(D, d, R) \sigma_{ij} \delta \varepsilon_{ij} dV_D \right) \right) dt = 0. \quad \text{(5-5)}$$

Upon integration of the first term by parts, this is transformed to a form of Hamilton's principle for a continuous medium (without a requirement of a conservative force system or elastic material behavior)

$$-\int_{t_1}^{t_2} \left(\int_W \rho \frac{\partial u_i}{\partial t} \frac{\partial \delta u_i}{\partial t} \, dV_D + \left(\int_W f_i \delta u_i dV_D + \int_{\partial W_t} t_i \delta u_i dA_d - \int_W c(D, d, R) \sigma_{ij} \delta \varepsilon_{ij} dV_D \right) \right) dt = 0. \quad \text{(5-6)}$$

In the special case of a conservative force system and an elastic body, a potential energy of external forces and a strain energy density (dual by a Legendre transformation to u^* of (4-11) and (4-12)) exist, such that

$$\delta V = -\int_W f_i \delta u_i dV_D - \int_{\partial W_t} t_i \delta u_i dA_d, \qquad \frac{\partial}{\partial \varepsilon_{ij}} u = \sigma_{ij}, \qquad u = u(\varepsilon_{ij}). \tag{5-7}$$

Hence (5-6) becomes

$$\int_{t_1}^{t_2} L\, dt = 0, \qquad L = K - \Pi, \tag{5-8}$$

where L is the Lagrangian and Π is the total potential energy.

Finally, proceeding similarly to classical continuum mechanics albeit within the calculus pertinent to fractional integrals, by using integration by parts, the Green–Gauss Theorem (2-4) and the conditions (5-7), leads to

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L(u_k, \dot{u}_k)\, dt \\ &= \int_{t_1}^{t_2} \{ \int_W [\rho \frac{d}{dt} \upsilon_k - \rho f_k + \nabla_l^D \sigma_{kl}] \delta u_i dV_D + \int_{\partial W_t} (t_k - t_k^*) \delta u_i dA_d \}. \end{aligned} \tag{5-9}$$

Given that $\delta \boldsymbol{u}$ is arbitrary for $t \in (t_1, t_2)$ and $\boldsymbol{x} \in W$, as well as on ∂W_t, we find the Euler–Lagrange equations associated with L:

$$\rho \frac{d}{dt} \upsilon_k = \rho f_k + \nabla_l^D \sigma_{kl} \quad \text{in } W, \qquad t_k - t_k^* = 0 \quad \text{on } \partial W. \tag{5-10}$$

In $(5\text{-}10)_1$ we recognize Equation (2-18). Further results, in the setting of elastic and inelastic materials, are given in [Ostoja-Starzewski 2009].

6. Conclusions

The continuum property is desired in providing mathematical descriptions of random heterogeneous microstructures in terms of homogenizing fields. While a number of methods have been developed over the past few decades to justify this in the setting of materials having Euclidean geometries for deterministic as well as random fields (see a review in [Ostoja-Starzewski 2008b]), in the case of fractal (i.e., almost everywhere nondifferentiable) media, novel methods outside classical continuum mechanics have to be employed. As a result, new forms of governing (partial differential) equations are derived where fractal dimensions and spatial resolution appear through explicit coefficients $c(D, d, R)$, $c_2(d, R)$ and $c_3(D, R)$. This indicates that very complex and multiscale materials of both elastic and inelastic type – which have so far been the domain of condensed matter physics, geophysics and biophysics – will soon be open to studies via initial-boundary value problems in the vein conventionally developed for smooth materials. This is made possible thanks to the approach initiated by Tarasov [2005a, 2005b, 2005c], and, in fact, allows one to deal with prefractal media which are commonly seen in nature.

The resulting field equations are more general than those of nonfractal media encountered in conventional continuum mechanics. In the latter case, $D = 3$, $d = 2$, whereby $c(D, d, R) = c_2(d, R) = c_3(D, R) = 1$, so that one recovers conventional forms of transport equations, balance equations and extremum principles of continuum mechanics. Thus, having to handle continuum mechanics of fractal media implies that one has to deal with partial differential equations and/or extremum principles in which the coefficients $c(D, d, R)$, $c_2(d, R)$ and $c_3(D, R)$ pertinent to a given microstructure are present.

The approach developed in this paper has its limitations: (i) spatial homogeneity, which actually allows smoothing at some finite length scale corresponding to the upper cutoff of the prefractal, and (ii) use of the Riesz form of fractional integrals for fractals in higher dimensional case using one integral variable, the radial scalar r, thus restricting the problem to a spherically symmetric one [Roy 2007], which in turn implies local isotropy of material response. The second limitation is removed with the help of a product measure instead of a Riesz measure, and thereby also ensuring that the mechanical approach to continuum mechanics is consistent with the energetic approach [Ostoja-Starzewski and Li 2009]. In that paper we have also extended the fracture mechanics (in terms of the strain energy release rate approach) and the elastodynamic equations of a Timoshenko beam to materials described by fractional integrals involving D, d and R. That line of approach has then allowed us [Li and Ostoja-Starzewski 2009] to specify the geometry configuration of continua via "fractal metric" coefficients, and therefore grasp the local material anisotropy. This then allows development of wave equations in one-, two- and three-dimensional fractal media or micropolar continuum models.

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