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THE EFFECT OF INFINITESIMAL DAMPING ON NONCONSERVATIVE DIVERGENCE INSTABILITY SYSTEMS

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The present work discuss the local dynamic asymptotic stability of 2-DOF weakly damped nonconservative systems under follower compressive loading in regions of divergence, using the Liénard–Chipart stability criterion. Individual and coupling effects of the mass and stiffness distributions on the local dynamic asymptotic stability in the case of infinitesimal damping are examined. These autonomous systems may either be subjected to compressive loading of constant magnitude and varying direction (follower) with infinite duration or be completely unloaded. Attention is focused on regions of divergence (static) instability of systems with positive definite damping matrices. The aforementioned mass and stiffness parameters combined with the algebraic structure of positive definite damping matrices may have under certain conditions a tremendous effect on the Jacobian eigenvalues and thereafter on the local dynamic asymptotic stability of these autonomous systems. It is also found that contrary to conservative systems local dynamic asymptotic instability may occur, strangely enough, for positive definite damping matrices before divergence instability, even in the case of infinitesimal damping (failure of Ziegler's kinetic criterion).

1. Introduction

The importance of damping on the local dynamic asymptotic stability of nonconservative systems was recognized long ago [Ziegler 1952; Nemat-Nasser and Herrmann 1966; Crandall 1970]. Particular attention was given to nonconservative discrete systems under follower load (autonomous systems) which may lose their stability either via flutter (vibrations of continuously increasing amplitude) or via divergence (static) instability depending on the region of variation of the nonconservativeness loading parameter.

The local dynamic stability of such autonomous nonconservative damped systems is governed by the *matrix-vector* differential equation [Kounadis 2006; 2007]

$$\boldsymbol{M}\ddot{q} + \boldsymbol{C}\dot{q} + \boldsymbol{V}q = 0, \tag{1}$$

where the dot denotes differentiation with respect to time t, $q(t)$ is an n-dimensional state vector with coordinates $q_i(t)$ $(i = 1, \ldots, n)$, and $\boldsymbol{M}$ and $\boldsymbol{C}$ are $n \times n$ real symmetric matrices, while $\boldsymbol{V}$ is an asymmetric matrix if the nonconservativeness loading parameter η is different from one ($\eta = 1$ corresponds to a conservative load). Specifically, the matrix $\boldsymbol{M}$, associated with the total kinetic energy of the system, is a function of the concentrated masses m_i $(i = 1, \ldots, n)$, and is always positive definite; $\boldsymbol{C}$, whose elements are the damping coefficients c_{ij} $(i, j = 1, \ldots, n)$, may be positive definite, positive semidefinite, as in the case of pervasive damping [Zajac 1964; 1965], or indefinite [Laneville and Mazouzi 1996; Sygulski

Keywords: nonconservative divergence, follower load, infinitesimal damping mass, Liénard–Chipart criterion, asymptotic instability.

1996]; $\boldsymbol{V}$ is a generalized stiffness matrix with coefficients k_{ij} $(i, j = 1, \dots, n)$, whose elements V_{ij} are also linear functions of η and of a suddenly applied external load λ of constant magnitude with varying direction (partial follower load defined by η) and infinite duration [Kounadis 1999], that is, $V_{ij} = V_{ij}(\lambda; k_{ij}, \eta)$. Apparently, due to this type of loading the system under discussion is autonomous. The static instability or buckling loads λ_i^c $(i = 1, \dots, n)$ are obtained by setting to zero the determinant of the stiffness asymmetric $(\eta \neq 1)$ matrix $\boldsymbol{V}(\lambda; k_{ij}, \eta)$:

$$V = |\boldsymbol{V}(\lambda; k_{ij}, \eta)| = 0. \tag{2}$$

This clearly yields an n-th degree algebraic equation in λ for given values of k_{ij} and η. Assuming distinct critical states the determinant of the matrix $\boldsymbol{V}(\lambda; k_{ij}, \eta)$ is positive for $\lambda < \lambda_1^c$, zero for $\lambda = \lambda_1^c$, and negative for $\lambda > \lambda_1^c$.

The boundary between flutter and divergence instability is obtained by solving with respect to λ and η the system of algebraic equations [Kounadis 1997]

$$V = \frac{\partial V}{\partial \lambda} = 0 \tag{3}$$

for given stiffness parameters k_{ij} $(i, j = 1, \dots, n)$.

We established in [Kounadis 2006; 2007] the conditions under which the above autonomous dissipative systems under step loading of constant magnitude and direction (conservative load) with infinite duration may exhibit dynamic bifurcation modes of instability before divergence, that is, for $\lambda < \lambda_1^c$, when infinitesimal damping is included. These dynamic bifurcational modes may occur through either a degenerate Hopf bifurcation (leading to periodic motion around centers) or a generic Hopf bifurcation (leading to periodic attractors or to flutter). These unexpected findings (implying failure of Ziegler's kinetic criterion and other singularity phenomena) may occur for a certain combination of values of the mass (primarily) and stiffness distributions of the system in connection with a positive semidefinite or an indefinite damping matrix [Kounadis 2006; 2007].

The question now arises whether there are combinations of values of these parameters (the mass and stiffness distributions) which, in connection with positive definite damping matrices, may lead to dynamic bifurcational modes of instability when the system is nonconservative due to a partial follower compressive load associated with the nonconservativeness parameter η. Only cases of divergence instability occurring for suitable values of η are considered. Namely, pseudoconservative systems are considered which are subjected to nonconservative circulatory forces, being therefore essentially nonconservative systems [Huseyin 1978]. Systems exhibiting flutter are called Ziegler circulatory, although in this terminology pseudoconservative systems are not distinguished [Ziegler 1952]. Attention is focused mainly on infinitesimal damping which may have a tremendous effect on the system's divergence instability. Such local dynamic instability will be sought through Liénard–Chipart's set of asymptotic stability criteria [Gantmacher 1959; 1970] which are elegant and more readily employed than the well known Routh–Hurwitz stability criteria. The local dynamic asymptotic stability of these systems using the above criteria is also discussed if there is no loading $(\lambda = 0)$.

In addition to the above main objective of this work, some new cases when the above autonomous systems are loaded by the aforementioned type of step follower compressive load will be also discussed by analyzing 2-degree of freedom (DOF) systems for which a lot of numerical results are available. Finally, the conditions for the existence of a double purely imaginary root (eigenvalue) are properly discussed.

2. Basic equations

The solution of (1) can be sought in the form

$$\boldsymbol{q} = \boldsymbol{r} e^{\rho t}, \tag{4}$$

where ρ is in general a complex number (eigenvalue) and $\boldsymbol{r}$ is a complex vector independent of time t.

Introducing $\boldsymbol{q}$ from (4) into (1) we get

$$(\rho^2 \boldsymbol{M} + \rho \boldsymbol{C} + \boldsymbol{V})\boldsymbol{r} = 0. \tag{5}$$

For given matrices $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{V}$ solutions of (5) are related to the Jacobian eigenvalues $\rho = \rho(\lambda)$ obtained by setting the determinant to zero, so

$$|\rho^2 \boldsymbol{M} + \rho \boldsymbol{C} + \boldsymbol{V}| = 0; \tag{6}$$

expansion of the determinant gives the characteristic (secular) equation for an n-DOF system

$$\rho^{2n} + \alpha_1 \rho^{2n-1} + \cdots + \alpha_{2n-1}\rho + \alpha_{2n} = 0, \tag{7}$$

where the real coefficients α_i ($i = 1, \ldots, 2n$) are determined by means of the Bôcher formula [Pipes and Harvill 1970]. The eigenvalues ρ_j ($j = 1, \ldots, 2n$) of (7) are, in general, complex conjugate pairs $\rho_j = \nu_j \pm \mu_j i$ (where ν_j and μ_j are real numbers and $i = \sqrt{-1}$), with corresponding complex conjugate eigenvectors $\boldsymbol{r}_j$ and $\bar{\boldsymbol{r}}_j$ ($j = 1, \ldots, n$). Since $\rho_j = \rho_j(\lambda)$, clearly $\nu_j = \nu_j(\lambda)$, $\mu_j = \mu_j(\lambda)$, $\boldsymbol{r}_j = \boldsymbol{r}_j(\lambda)$, and $\bar{\boldsymbol{r}}_j = \bar{\boldsymbol{r}}_j(\lambda)$. Thus, the solutions of (1) are of the form

$$A e^{\nu_j t} \cos \mu_j t, \qquad B e^{\nu_j t} \sin \mu_j t, \tag{8}$$

where constants A and B are determined from the initial conditions. Solutions (7) are bounded, tending to zero as $t \to \infty$, if all eigenvalues of (7) have negative real parts, that is, when $\nu_j < 0$ for all j. In this case the algebraic polynomial (7) is called a Hurwitz polynomial (since all its roots have negative real parts) and the origin ($\boldsymbol{q} = \dot{\boldsymbol{q}} = 0$) of the system is asymptotically stable.

Regarding the criteria for asymptotic stability it is worth mentioning the following. Consider the more general case of a polynomial in z with real coefficients α_i ($i = 0, 1, \ldots, n$)

$$f(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n = 0 \quad (\alpha_0 > 0), \tag{9}$$

for which we will seek the necessary and sufficient conditions so that all its roots have negative real parts.

Denoting by z_κ ($\kappa = 1, \ldots, m$) the real and by $r_j \pm i s_j$ ($j = 1, \ldots, (n-m)/2$; $i = \sqrt{-1}$) the complex roots of (9) we can arrange for all these complex roots to lie to the left of the imaginary axis:

$$z_\kappa < 0, \qquad r_j < 0 \quad \Big(\kappa = 1, \ldots, m;\ j = 1, \ldots, \frac{n-m}{2}\Big). \tag{10}$$

Then one can write

$$f(z) = \alpha_0 \prod_{\kappa=1}^{m} (z - z_\kappa) \prod_{j=1}^{n-m} (z^2 - 2rz + r_j^2 + s_j^2). \tag{11}$$

Since due to inequality (10) each term in the last part of (11) has positive coefficients, it is deduced that all coefficients of (9) are also positive. However, this (meaning $\alpha_i > 0$ for all i with $\alpha_0 > 0$) is a necessary but by no means sufficient condition for all roots of (9) to lie in the left half-plane ($\mathrm{Re}(z) < 0$).

The Routh–Hurwitz criterion [Gantmacher 1959; 1970] gives necessary and sufficient conditions for asymptotic stability, that is, for all roots of (9) to have negative real parts; the conditions are

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \dots, \quad \Delta_n > 0, \tag{12}$$

where

$$\Delta_1 = \alpha_1, \quad \Delta_2 = \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_0 & \alpha_2 \end{bmatrix}, \quad \Delta_3 = \begin{bmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_0 & \alpha_2 & \alpha_4 \\ 0 & \alpha_1 & \alpha_3 \end{bmatrix}, \quad \dots, \quad \Delta_n = \begin{bmatrix} \alpha_1 & \alpha_3 & \alpha_5 & & \cdots & \\ \alpha_0 & \alpha_2 & \alpha_4 & & \cdots & \\ 0 & \alpha_1 & \alpha_3 & & \cdots & \\ 0 & \alpha_0 & \alpha_2 & \alpha_4 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & & & & \alpha_i \end{bmatrix} \tag{13}$$

with $\alpha_\kappa = 0$ for $\kappa > n$. The last equality yields $\Delta_n = \alpha_n \Delta_{n-1}$.

Note that when the necessary conditions $\alpha_i > 0$ (for all i) hold, the inequalities (17) are *not* independent. For instance, for $n = 4$ the Routh–Hurwitz conditions reduce to the single inequality $\Delta_3 > 0$, for $n = 5$ they reduce to $\Delta_2 > 0$ and $\Delta_4 > 0$, while for $n = 6$ they reduce again to two inequalities, $\Delta_3 > 0$ and $\Delta_5 > 0$. This case was discussed by Liénard and Chipart who established the following elegant criterion for asymptotic stability [Gantmacher 1970].

The Liénard–Chipart stability criterion. Necessary and sufficient conditions for all roots of the real polynomial $f(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n = 0$ ($\alpha_0 > 0$) to have negative real parts can be given in any one of the following forms:

$$\alpha_n > 0, \quad \alpha_{n-2} > 0, \quad \dots, \quad \text{with} \begin{cases} \text{either} & \Delta_1 > 0, \quad \Delta_3 > 0, \quad \dots, \\ \text{or} & \Delta_2 > 0, \quad \Delta_4 > 0, \quad \dots, \end{cases} \tag{14}$$

or

$$\alpha_n > 0, \quad \alpha_{n-1} > 0, \quad \alpha_{n-3} > 0, \quad \dots, \quad \text{with} \begin{cases} \text{either} & \Delta_1 > 0, \quad \Delta_3 > 0, \quad \dots, \\ \text{or} & \Delta_2 > 0, \quad \Delta_4 > 0, \quad \dots. \end{cases} \tag{15}$$

This stability criterion was rediscovered by Fuller [1968].

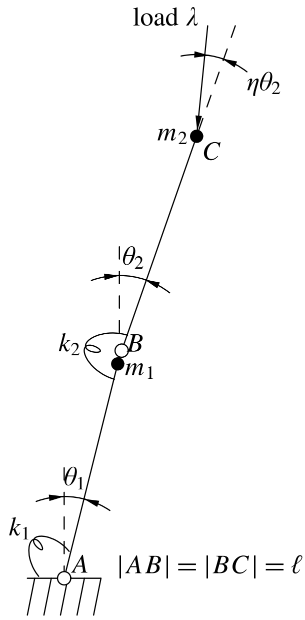
In this study attention is focused on 2-DOF nonconservative (due to partial follower compressive loading) dissipative systems, whose characteristic equation (7) is written as follows:

$$\rho^4 + \alpha_1 \rho^3 + \alpha_2 \rho^2 + \alpha_3 \rho + \alpha_4 = 0 \quad (\alpha_0 = 1). \tag{16}$$

According to the last criterion all roots of (16) have negative real parts provided that $\alpha_4 > 0$, $\alpha_2 > 0$, and $\Delta_3 = \alpha_3(\alpha_1\alpha_2 - \alpha_3) - \alpha_1^2 \alpha_4 > 0$. Clearly, from the last inequality it follows that $\alpha_3 > 0$. Hence, the positivity of α_1 and α_3 was assured via the above conditions ($\alpha_4 > 0$, $\alpha_2 > 0$, $\Delta_1 > 0$, and $\Delta_3 > 0$).

3. Mathematical analysis

Consider the cantilevered dissipative spring model with 2 DOFs under a partial follower compressive tip load which is shown on the next page. Subsequently we will examine in detail the effect of a violation of one or more of the conditions of the Liénard–Chipart criterion on its local dynamic asymptotic stability. The response of this dynamic model carrying two concentrated masses is studied when it is either loaded under a suddenly applied load of constant magnitude and varying direction with infinite duration or completely unloaded. Such autonomous dissipative systems with positive definite damping matrices and particularly with infinitesimal damping are properly investigated. If at least one root of the secular equation (16) has a positive real part the corresponding solution — see (8) — will contain an exponentially increasing function with time, and the system will become dynamically asymptotically unstable.



The seeking of an imaginary root of the secular equation (16) which represents a borderline between dynamic stability and instability is a first, important step in our discussion. Clearly, an imaginary root gives rise to an oscillatory motion of the form $e^{i\mu t}$ ($i = \sqrt{-1}$, μ real number) around the trivial state. However, the existence of at least one multiple imaginary root of the κ-th order of multiplicity leads to a solution containing functions of the form $e^{i\mu t}, te^{i\mu t}, \dots, t^{\kappa-1}e^{i\mu t}$, which increase with time. Hence, the multiple imaginary root on the imaginary axis denotes local dynamic instability. The discussion of such a situation is also another objective of this study.

The nonlinear equations of motion for the 2-DOF nonconservative model of the figure with rigid links of equal length ℓ are given by [Kounadis 1997]

$$\begin{aligned}(1+m)\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1+\theta_2) - \dot{\theta}_2^2\sin(\theta_1-\theta_2) + c_{11}\dot{\theta}_1 + c_{12}\dot{\theta}_2 + V_1 = 0,\\ \ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2) + c_{22}\dot{\theta}_2 + c_{12}\dot{\theta}_1 + V_2 = 0,\end{aligned} \tag{17}$$

where

$$V_1 = (1+k)\theta_1 - \theta_2 - \lambda\sin\bigl(\theta_1 + (\eta-1)\theta_2\bigr), \qquad V_2 = \theta_2 - \theta_1 - \lambda\sin\eta\theta_2,$$

η is the nonconservativeness loading parameter, and

$$m = \frac{m_1}{m_2}, \qquad k = \frac{k_1}{k_2}, \qquad \lambda = \frac{P\ell}{k_2}.$$

Linearization of Equation (17) after setting

$$\Theta = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = e^{\rho t}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix} = e^{\rho t}\phi$$

gives $(\rho^2\boldsymbol{M}+\rho\boldsymbol{C}+\boldsymbol{V})\phi=0$, where

$$M=\begin{bmatrix} m_{11} & m_{12}\\ m_{12} & m_{22}\end{bmatrix}=\begin{bmatrix} 1+m & 1\\ 1 & 1\end{bmatrix},\qquad C=\begin{bmatrix} c_{11} & c_{12}\\ c_{12} & c_{22}\end{bmatrix},$$
$$V=\begin{bmatrix} V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}=\begin{bmatrix} k+1-\lambda & -1-\lambda(\eta-1)\\ -1 & 1-\lambda\eta\end{bmatrix}. \tag{18}$$

In the case of a positive definite damping matrix of Rayleigh viscous type $c_{11}=c_1+c_2$, $c_{12}=-c_2$, and $c_{22}=c_2$, where c_i $(i=1,2)$ is the damping coefficient of the i-th bar.

The static buckling equation, $\det V=0$, leads to

$$\eta\lambda^2-\eta(k+2)\lambda+k=0, \tag{19}$$

whose lowest root is the first buckling load λ_1^c equal to

$$\lambda_1^c=\tfrac{1}{2}\left(k+2-\sqrt{(k+2)^2-4k/\eta}\right)\quad(\eta\neq 0). \tag{20}$$

For real roots the discriminant Δ of (19) must be greater or equal to zero ($\Delta\geq 0$) which yields

$$\eta\geq\frac{4k}{(k+2)^2}. \tag{21}$$

For instance, for $k=1$ it follows that static instability occurs for $\eta\geq 4/9$ and flutter instability for $\eta<4/9$. The coefficients of the characteristic equation (16) are given by

$$\begin{aligned}
\alpha_1&=\frac{1}{m}[(1+m)c_{22}+c_{11}-2c_{12}],\\
\alpha_2&=\frac{1}{m}[(1+m)(1-\lambda\eta)+3+k-\lambda+\lambda(\eta-1)+|c|],\\
\alpha_3&=\frac{1}{m}\{c_{11}(1-\lambda\eta)+c_{22}(1+k-\lambda)+[2+\lambda(\eta-1)]c_{12}\},\\
\alpha_4&=\frac{1}{m}[\eta\lambda^2-\eta(k+2)\lambda+k]=\frac{1}{m}\det V,
\end{aligned} \tag{22}$$

where $|c|=\det C$.

The region of existence of adjacent equilibria (region of divergence instability) is related to static bifurcations with two distinct critical loads obtained via $\alpha_4=0$ or (19). The boundary between the region of existence and nonexistence of adjacent equilibria is defined by

$$\alpha_4=\frac{d\alpha_4}{d\lambda}=0, \tag{23}$$

which due to relations (22) gives

$$\eta_0=\frac{4k}{(k+2)^2},\qquad \lambda_0=\frac{k+2}{2}. \tag{24}$$

This is a double (compound) branching point related to a double root of (19) with respect to λ. Considering the function $\eta=\eta(\lambda,k)$ the necessary condition for an extremum $\partial\eta/\partial\lambda=\partial\eta/\partial k=0$ yields $\lambda_0=2$ and $k_0=2$ implying $\eta_0=\frac{1}{2}$. Note that η_0 is the maximum distance of the double branching point from the λ^c-axis (curve η versus λ^c). Two characteristic curves are considered, $k<2$ and $k>2$. It is clear that

$\lambda_0^c \to 1$ and $\eta_0 \to 0$ as $k \to 0$, whereas for $k > 2\lambda_0^c \to \infty$, $\eta_0 \to 0$ as $k \to \infty$. It is easy now to establish the locus of the double branching points in the plane of $\eta - \lambda^c$ (see Figure 1), being independent of m. Note that for $k \to 0$ or $k \to \infty$ the region of flutter instability disappears.

Subsequently, the Liénard–Chipart criterion for asymptotic stability is used, which is more simple and efficient than that of Routh–Hurwitz. Clearly, if one of the conditions ($15\alpha, b$) is violated there is *no* asymptotic stability. We will apply this criterion for the above 2-DOF cantilevered model ($n = 4$, $\alpha_0 = 1$) in the case of a positive definite damping matrix for which one can show that $m > 0$ always implies $\alpha_1 > 0$. Now consider the case of the Rayleigh positive definite viscous damping matrix in the region of divergence stability, that is, for $\eta \geq \eta_0 = 4k(k+2)^2$. Then $c_{11} = c_1 + c_2$, $c_{12} = c_{21} = -c_2$, and $c_{22} = c_2$

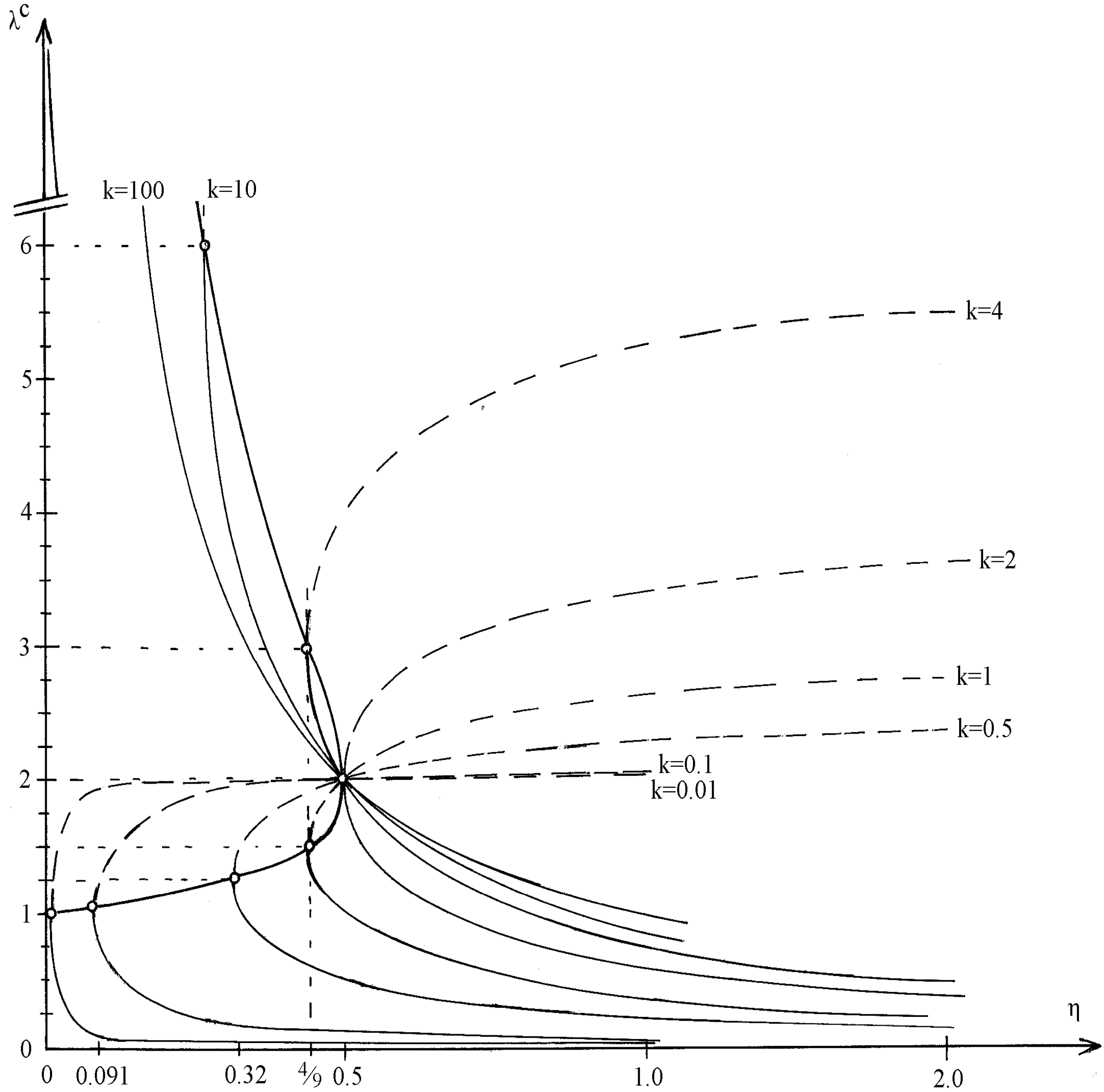


Figure 1. Locus of double branching points (λ_0^c, η_0).

($c_i > 0$, $i = 1, 2$), and relations (22) become

$$\alpha_1 = \frac{1}{m}[c_1 + (4+m)c_2], \qquad \alpha_2 = \frac{1}{m}[m+k+4+c_1c_2 - \lambda(\eta m+2)]$$
$$\alpha_3 = \frac{1}{m}[c_1(1-\lambda\eta) + c_2(k-2\lambda\eta)], \qquad \alpha_4 = \frac{1}{m}[\eta\lambda^2 - \eta(k+2)\lambda + k]. \tag{25}$$

According to the first set of conditions (14) we have

$$\alpha_4 > 0, \qquad \alpha_2 > 0, \qquad \Delta_1 = \alpha_1 > 0, \qquad \Delta_3 > 0, \tag{26}$$

where

$$\Delta_3 = \begin{vmatrix} \alpha_1 & \alpha_3 & 0 \\ 1 & \alpha_2 & \alpha_4 \\ 0 & \alpha_1 & \alpha_3 \end{vmatrix} = \alpha_3(\alpha_1\alpha_2 - \alpha_3) - \alpha_1^2\alpha_4. \tag{27}$$

From (25) it follows that $\alpha_1 > 0$. Since $\alpha_4 = \det(V/m)$ ($m > 0$) one may consider the following cases regarding the interval of variation of λ:

$$\left.\begin{array}{lll} \text{For } \lambda < \lambda_1^c & \Rightarrow & \det V > 0 \text{ and hence } \alpha_4 > 0, \\ \text{For } \lambda_1^c < \lambda < \lambda_2^c & \Rightarrow & \det V < 0 \text{ and hence } \alpha_4 < 0, \\ \text{For } \lambda \geq \lambda_2^c & \Rightarrow & \det V > 0 \text{ and hence } \alpha_4 > 0. \end{array}\right\} \tag{28}$$

Considering always the region of divergence instability, $\eta \geq 4k/(k+2)^2$, and keeping in mind the interval of values of λ the following cases of violation of conditions (26) are discussed:

First case: $\alpha_4 > 0$ (for $\lambda < \lambda_1^c$), $\alpha_2 < 0$, and $\Delta_3 > 0$. In view of (27), clearly $\Delta_3 > 0$ implies $\alpha_3 < 0$ (since always $\alpha_1 > 0$) or due to relation (25)

$$c_1(1-\lambda\eta) + c_2(k-2\lambda\eta) < 0. \tag{29}$$

Since $c_1, c_2 > 0$ the quantities $1-\lambda\eta$ and $k-2\lambda\eta$ must be of opposite sign. Inequality (29) can always be satisfied for suitable values of $c_i > 0$ ($i = 1, 2$). Subsequently one can find suitable values for k, η, and m for which

$$\lambda < \lambda_1^c = \tfrac{1}{2}\big(k+2-\sqrt{(k+2)^2 - 4k/\eta}\big)$$

is also consistent with $\alpha_2 < 0$. The important conclusion which then can be drawn is that a local dynamic asymptotic instability in regions of divergence (for λ less than the first buckling load) may occur in the case of a positive definite damping matrix. This is excluded in the case of conservative loading ($\eta = 1$), as shown in [Kounadis 2006; 2007].

More specifically one can establish to the following proof: In view of (25), the condition $\alpha_2 < 0$ implies

$$\lambda > \frac{m+k+4+c_1c_2}{\eta m+2}, \tag{30}$$

which must be consistent with (20),

$$\lambda < \tfrac{1}{2}\big(k+2-\sqrt{(k+2)^2 - 4k/\eta}\big). \tag{31}$$

One can show that there are values of λ for which both inequalities (30) and (31) are satisfied for $\eta \geq 4k/(k+2)^2$, $m > 0$, $k > 0$, and $c_i > 0$ $(i = 1, 2)$. For example for $k = 5$, $m = 8$, $c_1 = 0.001$, and $c_2 = 0.00013$ we get $\eta \geq 4k/(k+2)^2 = 20/49 = 0.408163265$. Choosing $\eta = 0.41$ we obtain $\lambda_1^c = 3.26574$, as well as

$$\lambda > \frac{m+k+4+c_1c_2}{\eta m+2} = 3.219697.$$

For $\lambda = 3.26 < \lambda_1^c = 3.26574$, we find: $\alpha_1 = 0.00032$, $\alpha_2 = -0.0266$, $\alpha_3 = -4.26 \times 10^{-6}$, $\alpha_4 = 0.0001395$, and $m^3\Delta_3 = 1.96 \times 10^{-9} \approx 0$. Figure 2 shows, for these values of parameters α_i $(i = 1, \ldots, 4)$, a large amplitude chaotic-like response in the $(\theta_2, \dot{\theta}_2)$ phase plane. Hence, for $3.26 \leq \lambda \leq 3.26574$, the damped autonomous system exhibits local asymptotic instability before divergence for a positive definite damping matrix (with coefficients practically zero) of the Rayleigh viscous type. This is an unexpected finding which does not occur for the same system under conservative $(\eta = 1)$ tip load [Kounadis 2006; 2007].

Second case: $\alpha_4 < 0$ (for $\lambda_1^c < \lambda < \lambda_2^c$), $\alpha_2 > 0$, and $\Delta_3 > 0$. In view of (25), the condition $\alpha_2 > 0$ implies

$$\lambda < \frac{m+k+4+c_1c_2}{\eta m+2}, \quad \text{and hence} \quad \lambda_1^c < \frac{m+k+4+c_1c_2}{\eta m+2} < \lambda_2^c, \tag{32}$$

or, due to (19),

$$\tfrac{1}{2}\left(k+2-\sqrt{(k+2)^2-4k/\eta}\right) < \frac{m+k+4+c_1c_2}{\eta m+2} < \tfrac{1}{2}\left(k+2+\sqrt{(k+2)^2-4k/\eta}\right). \tag{33}$$

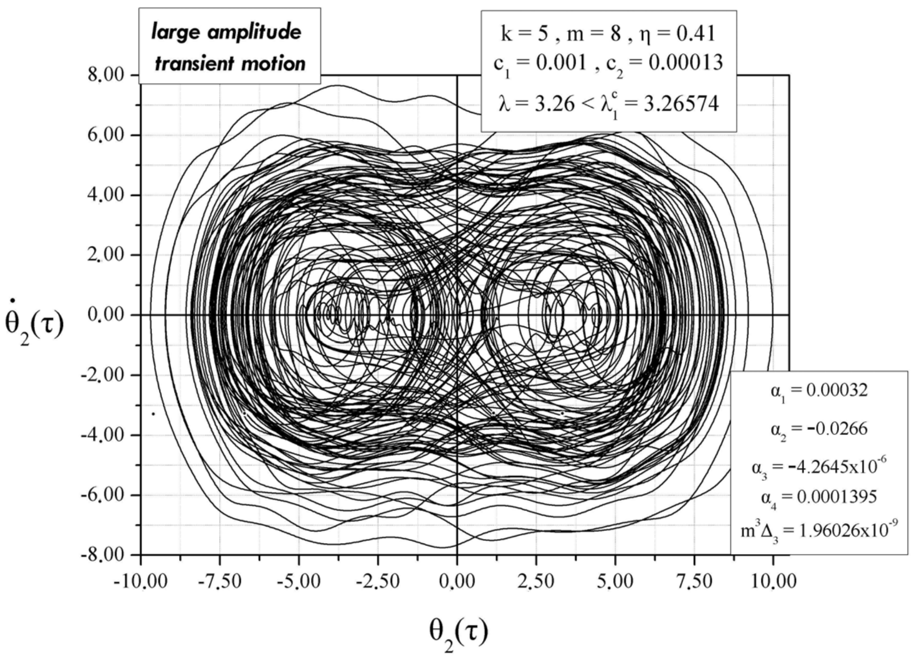


Figure 2. Phase-plane response $(\theta_2(\tau)$ versus $\dot{\theta}_2(\tau))$ for a cantilever with parameters $k = 5$, $\eta = 0.41$, $m = 8$, $c_1 = 0.001$, $c_2 = 0.00013$, and $\lambda = 3.26 < \lambda_1^c = 3.26574$. The model is locally dynamically unstable exhibiting large amplitude chaotic motion.

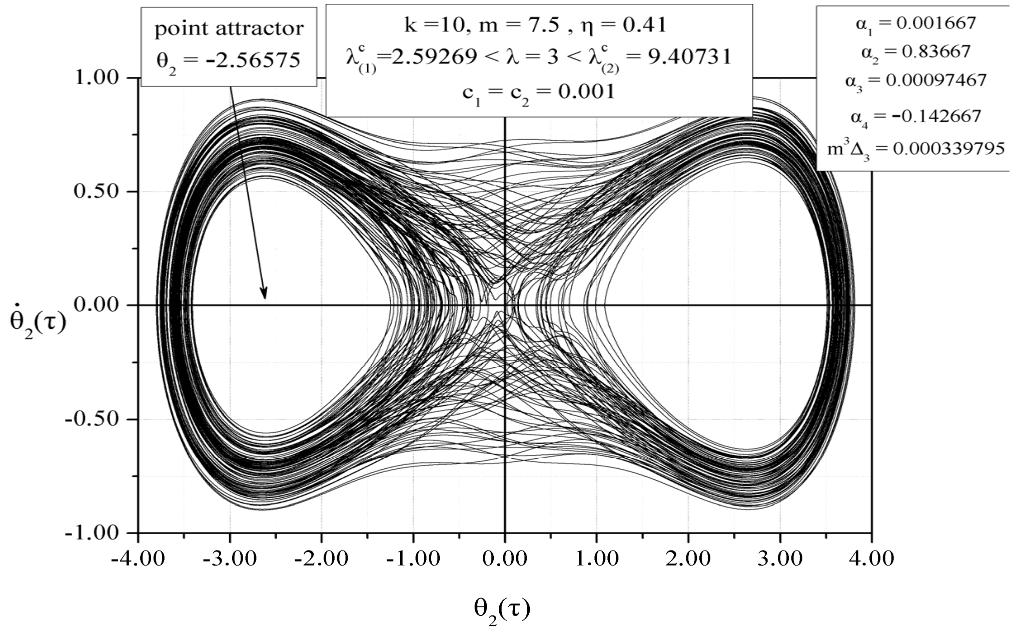


Figure 3. Phase-plane response $\left(\theta_2(\tau) \text{ versus } \dot{\theta}_2(\tau)\right)$ for a cantilever with parameters $k = 10$, $\eta = 0.41$, $m = 7.5$, $c_1 = c_2 = 0.001$, and $\lambda_1^c = 2.59269 < \lambda = 3 < \lambda_2^c = 9.40731$. The model exhibits large amplitude chaotic motion which is finally captured by the left stable equilibrium point acting as an attractor.

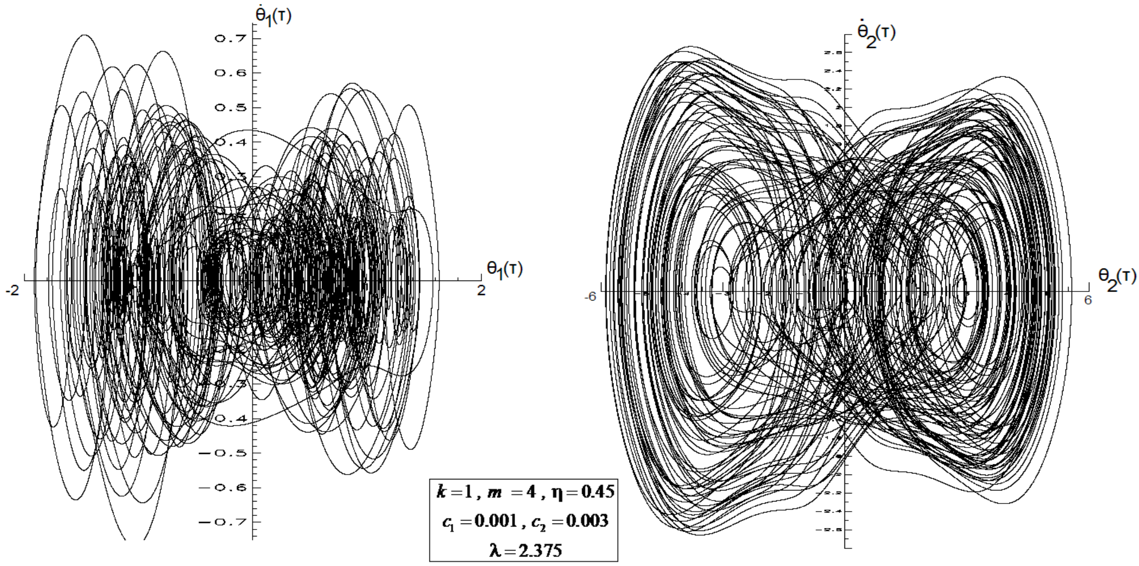


Figure 4. Phase-plane response $\left(\theta_i(\tau) \text{ versus } \dot{\theta}_i(\tau),\ i = 1, 2\right)$ for a cantilever with parameters $k = 1$, $\eta = 0.45$, $m = 4$, $c_1 = 0.001$, $c_2 = 0.003$, and $\lambda = 2.37 > (k + m + 4 + c_1 c_2)/(\eta m + 2) = 2.36842$. The model is locally dynamically unstable exhibiting large amplitude chaotic motion.

For instance, if $k = 10$ then $\eta \geq 4k/(k+2)^2 = 0.2777777$. Choosing $\eta = 0.41$, $c_1 = c_2 = 0.001$, and $m = 7.5$ inequality (33) yields $2.59269 < 4.23645 < 9.40731$. For $\lambda = 3$ we get: $\alpha_1 = 0.001667$, $\alpha_2 = 0.83667$, $\alpha_3 = 0.00097467$, $\alpha_4 = -0.142667$, and $m^3\Delta^3 = 3.39795 \times 10^{-4}$.

As was anticipated the system is locally dynamically asymptotically unstable. However, a nonlinear dynamic analysis will show that the system is globally stable. This is so, because the cantilever under statically applied load exhibits postbuckling strength and hence the postbuckling stable equilibria act as point attractors. Figure 3 shows, corresponding to the given parameters α_i $(i = 1, \dots, 4)$, the motion in the $(\theta_2, \dot{\theta}_2)$ phase plane, which after large amplitude vibrations is finally captured by the left stable equilibrium point (of the cantilever) acting as point attractor.

Third case: $\alpha_4 > 0$ (for $\lambda > \lambda_2^c$), $\alpha_2 < 0$, and $\Delta_3 > 0$. Clearly $\alpha_2 < 0$ and $\Delta_3 > 0$ imply $\alpha_3 < 0$. Inequality $\alpha_2 < 0$ due to relations (25) yields

$$\lambda > \frac{k+m+4+c_1c_2}{\eta m+2}. \tag{34}$$

We must also have

$$\lambda > \lambda_2^c = \tfrac{1}{2}\big(k+2+\sqrt{(k+2)^2-4k/\eta}\big). \tag{35}$$

One can readily show that both (34) and (35) can be satisfied for various values of λ and of the parameters $m > 0$, $k > 0$, $c_i > 0$ $(i = 1, 2)$, and $\eta \geq 4k/(k+2)^2$.

For instance, for $m = 4$, $c_1 = 0.001$, $c_2 = 0.003$, and $k = 1$ implying $\eta = 4/9$, after choosing $\eta = 0.45$ we obtain $\lambda \geq (k+m+4+c_1c_2)/(\eta m+2) = 2.36842$ and $\lambda_2^c = 1.66666$. Hence, for $\lambda = 2.375$ we have local asymptotic instability. Figure 4 shows, corresponding to these values of the parameters, the $(\theta_1, \dot{\theta}_1)$ and $(\theta_2, \dot{\theta}_2)$ phase plane responses similar to those presented by Sophianopoulos et al. [2002] using the same cantilever model.

Fourth case. $\alpha_4 > 0$ for $\lambda < \lambda_1^c$, $\alpha_2 > 0$, and $\Delta_3 \leq 0$. The condition $\Delta_3 = 0$ (being necessary for a Hopf bifurcation) yields

$$\alpha_3(\alpha_1\alpha_2 - \alpha_3) - \alpha_1^2\alpha_4 = 0, \tag{36}$$

which due to $\alpha_1 > 0$ implies also $\alpha_3 > 0$. For instance, if $k = 1$ then $\eta = 4k/(k+2)^2 = 4/9$. Subsequently choosing $\eta = 0.45$ we obtain $\lambda_1^c = \frac{1}{2}(k+2-\sqrt{(k+2)^2-4k/\eta}) = 1.3333$. Take $\lambda = 1.2$, $m = 1$, $c_1 = 0.001$, and $c_2 = 0.0036$, which yield $\alpha_1 = 0.019$, $\alpha_2 = 3.06$, $\alpha_3 = 0.000172$, $\alpha_4 = 0.028$, and $\Delta_3 = -1.3749 \times 10^{-7}$. Figure 5, on the basis of these values of parameters α_i $(i = 1, \dots, 4)$, shows periodic motion around centers in the $(\theta_1, \dot{\theta}_1)$, whose final amplitude depends on the initial conditions.

Equation (36) is the necessary condition for the existence of a pair of purely imaginary roots of the characteristic equation (16). This case is associated either with a degenerate Hopf bifurcation or with a generic Hopf bifurcation [Kounadis 2006; 2007].

Using (22), we reduce (36) to a second-degree algebraic equation in λ:

$$A\lambda^2 + B\lambda + \Gamma = 0, \tag{37}$$

where

$$
\begin{aligned}
A &= m[\eta c_{11}+c_{22}-c_{12}(\eta-1)]^2+\eta[(1+m)c_{22}+c_{11}-2c_{12}]^2 \\
&\qquad -(\eta m+2)[(1+m)c_{22}+c_{11}-2c_{12}][\eta c_{11}+c_{22}-c_{12}(\eta-1)], \\
B &= [(1+m)c_{22}+c_{11}-2c_{12}]\big\{(\eta m+2)[c_{11}+c_{22}(1+k)+2c_{12}]+(4m+k+|c|)[\eta c_{11}+c_{22}-c_{12}(\eta-1)]\big\} \\
&\qquad -2m[c_{11}+c_{12}(1+k)+2c_{12}][\eta c_{11}+c_{22}-c_{12}(\eta-1)]-\eta(k+2)[(1+m)c_{22}+c_{11}-2c_{12}]^2, \\
\Gamma &= m[c_{11}+c_{22}(1+k)+2c_{12}]^2+k[(1+m)c_{22}+c_{11}-2c_{12}]^2 \\
&\qquad -[(1+m)c_{22}+c_{11}-2c_{12}](4+m+k+|c|)[c_{11}+c_{22}(1+k)+2c_{12}].
\end{aligned}
\tag{38}
$$

Unlike A and B, the coefficient Γ is independent of η.

For λ to be real the discriminant $\mathfrak{D}=B^2-4A\Gamma$ of (37) must be nonnegative. If $\mathfrak{D}>0$, the quadratic equation has two unequal roots; if $\mathfrak{D}=0$, it has a double root, equal to $\lambda_H=-B/2A$. Note also that the intersection between the curve of (37) and the curve of the first static load λ_1^c, corresponds to a dynamic coupled flutter-divergence bifurcation.

***The case* $\boldsymbol{\lambda=0}$.** The most important particular case is when $\lambda=0$, implying $\Gamma=\Gamma(k,m,c_{ij})=0$; then all the coefficients of the characteristic equation (16) given in relations (22) or (25) are independent of

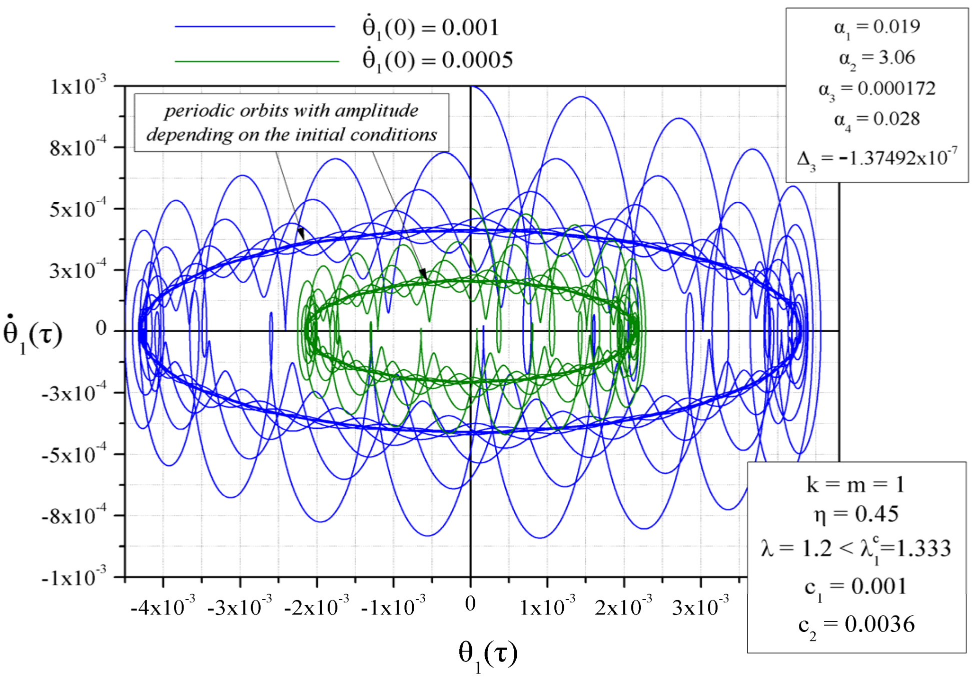


Figure 5. Phase-plane response $(\theta_1(\tau)$ versus $\dot\theta_1(\tau))$ for a cantilever with parameters $k=1$, $\eta=0.45$, $m=1$, $c_1=0.001$, $c_2=0.0036$, and $\lambda=1.2<\lambda_1^c=1.33333$. The model is locally dynamically unstable exhibiting periodic motion around centers, whose final amplitude depends on the initial conditions.

η. Thus Γ is the same as for a conservative load ($\eta = 1$). Strangely enough, the unloaded cantilever, although statically stable, is dynamically locally unstable under any small disturbances!

Conditions for a double imaginary root. For a double imaginary root the first derivative of the secular equation (16) must also be zero, which yields $4\rho^3 + 3\alpha_1\rho^2 + 2\alpha_2\rho + \alpha_3 = 0$. Inserting $\rho = \mu i$ into this equation, where μ is real, yields $\mu^2 = \frac{1}{2}\alpha_2 = \alpha_3/3\alpha_1$ and thus $\alpha_3 = \frac{3}{2}\alpha_1\alpha_2$. Since $\rho = \mu i$ must also be a root of (16) we obtain $\mu^2 = \alpha_3/\alpha_1$, which implies $\alpha_3 = \frac{1}{2}\alpha_1\alpha_2$. This is consistent with the previous expression $\alpha_3 = \frac{3}{2}\alpha_1\alpha_2$ only when $\alpha_3 = 0$ due to either $\alpha_1 = 0$ (which is excluded for a positive definite damping matrix) or $\alpha_2 = 0$ (which is also excluded since it implies $\mu = 0$). Hence, if the damping matrix $\boldsymbol{C}$ is positive definite and of Rayleigh viscous type ($c_{11} = c_1 + c_2$, $c_{12} = c_{21} = -c_2$ and $c_{22} = c_2$ with c_1 and c_2 both positive) then the case of a double imaginary root is excluded [Sophianopoulos et al. 2008]. Note also that in this case the expressions of A, B, and Γ are simplified as follows:

$$
\begin{aligned}
A &= \eta[m\eta(c_1 + 2c_2)]^2 + [c_1 + (m+4)c_2]^2 - (c_1 + 2c_2)(2 + \eta m)[c_1 + (m+4)c_2],\\
B &= [c_1 + (m+4)c_2]\big\{(\eta m + 2)(c_1 + c_2 k) + \eta(4 + m + k + c_1 c_2)(c_1 + 2c_2)\big\}\\
&\qquad - 2m\eta(c_1 + c_2 k)(c_1 + 2c_2) - \eta(k+2)[c_1 + (m+4)c_2]^2,\\
\Gamma &= m(c_1 + c_2 k)^2 + k[c_1 + (m+4)c_2]^2 - [c_1 + (m+4)c_2](4 + m + k + c_1 c_2)(c_1 + k c_2).
\end{aligned}
\tag{39}
$$

4. Conclusions

The coupling effect of the mass and stiffness distributions of a 2-DOF cantilevered model under partial follower compressive load at its tip in connection with (mainly) infinitesimal positive definite damping is discussed in detail in regions of divergence stability. For the local dynamic asymptotic stability of such autonomous systems attention is focused on the violation of the Liénard–Chipart asymptotic stability criterion. The most important findings of this study are:

- The geometric locus of the double branching points (η_0, λ_0^c) corresponding to various values of k is established via the relations η versus λ^c. The locus is independent of the mass m, whose effect on dynamic instability is of paramount importance. Note that for $k \to 0$ or $k \to \infty$ the region of flutter tends to zero. The intersection between the curve (37) and curve λ_1^c corresponds with a coupled fluttered-divergence instability bifurcation.
- The Liénard–Chipart, a more elegant and readily employed stability criterion than that of Routh–Hurwitz, brought into light new types of dynamic bifurcations.
- The mass and stiffness distributions combined with a positive definite negligibly small damping matrix, strangely enough, may have a considerable effect on the local dynamic asymptotic stability prior to divergence. Similar phenomena may occur in conservative systems, but only in the cases of positive semidefinite or indefinite damping matrices [Kounadis 2006; 2007].
- The model under partial follower tip load (step load of constant-magnitude and varying direction with infinite duration) under certain conditions may exhibit a divergent (unbounded) motion before divergence in the case of a positive definite negligibly small damping matrix at a certain value of the external load. This is a completely unexpected result.

- The cantilevered model when unloaded (although being statically stable) under certain conditions becomes dynamically locally unstable to any small disturbance which is also an unexpected finding.
- The case of a double imaginary root in the case of a positive definite damping matrix is excluded.

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