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FORCED VIBRATIONS OF A NONLINEAR OSCILLATOR WITH WEAK FRACTIONAL DAMPING

YURIY A. ROSSIKHIN, MARINA V. SHITIKOVA AND TATIANA SHCHEGLOVA

This article deals with force-driven vibrations of nonlinear mechanical oscillators whose constitutive equations involve fractional derivatives, defined as fractional powers of the conventional time-derivative operator. This definition of fractional derivatives enables one to analyze approximately the vibratory regimes of the oscillator. The assumption of small fractional derivative terms allows one to use the method of multiple time scales, whereby a comparative analysis of the solutions obtained for different orders of low-level fractional derivatives and disturbing force terms can be carried out. The relationship between the fractional parameter (order of the fractional operator) and nonlinearity manifests itself in full measure when the orders of the small fractional derivative term and of the cubic nonlinearity appearing in the oscillator's constitutive equation coincide.

1. Introduction

Fractional derivatives have been useful in describing, among other things, the frequency-dependent damping behavior of nonlinear structural systems [Padovan and Sawicki 1998; Rossikhin and Shitikova 1997a; 1998; 2000; 2003; 2006; Li et al. 2003; Seredyńska and Hanyga 2005; Nasuno et al. 2006]. Since the methods of integral transformations are unusable in nonlinear problems, different perturbation techniques or numerical methods must be used for investigating vibrations of such nonlinear structures.

The dynamics of the fractionally damped Duffing oscillator has been examined by several authors [Padovan and Sawicki 1998; He 1998; Sheu et al. 2007; Seredyńska and Hanyga 2000; Gao and Yu 2005; Singh and Chatterjee 2006; Wahi and Chatterjee 2004; Chen and Zhu 2009; Atanackovic and Stankovic 2008]. In particular, a Duffing-like oscillator with positive linear stiffness and weak damping defined by a fractional derivative has been studied in [Padovan and Sawicki 1998] using an energy-constrained Lindstedt–Poincaré perturbation procedure that involves a diophantine version of the fractional operator powers. The influence of fractional damping on the frequency amplitude response has been examined when the oscillator is subjected to the action of an external harmonic force.

The case of free vibrations with a half-order Riemann–Liouville fractional derivative was analyzed in [He 1998] using variational iteration method, allowing the author to obtain an approximate analytical solution.

The occurrence and nature of chaotic motion in a single-degree-of-freedom system described by a Duffing-like equation with negative linear stiffness have been studied using different numerical methods in [Sheu et al. 2007], including the use of Caputo-type fractional derivatives. The Galerkin projection

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method and the finite element method were adopted in [Singh and Chatterjee 2006] for solving a similar equation but with the dissipative force modeled via Riemann–Liouville fractional derivatives of half-order, while the possibilities of using other values of the fractional parameter were also discussed.

The method of averaging was applied in [Wahi and Chatterjee 2004] for investigating the equations with a different type of small damping including Riemann–Liouville half-order fractional derivative terms and delayed terms, as well as in [Chen and Zhu 2009] for treating the case of combined external harmonic and parametric white noise excitations.

In [Rossikhin and Shitikova 2009], departing from earlier practice in the literature, we suggested an approximate approach to the analysis of free vibrations of mechanical oscillators whose constitutive equations involve fractional derivatives. The approach is based on the representation of the fractional derivative as a fractional power of the ordinary time derivative operator d/dt, a representation typically given by the equality

$$\left(\frac{d}{dt}\right)^{\gamma} x(t) = D_{+}^{\gamma} x(t) \tag{1-1}$$

(see formula (5.82) of [Samko et al. 1993]), where

$$D_{+}^{\gamma} x(t) = \frac{d}{dt} \int_{-\infty}^{t} \frac{x(t-t')dt'}{\Gamma(1-\gamma)t'^{\gamma}} \tag{1-2}$$

is the Riemann–Liouville fractional time-derivative. Since the lower limit of the integral here is $-\infty$, the equality in (1-1) allows one to use the Liouville representation of the fractional derivative applied to the exponential function:

$$D_{+}^{\gamma} e^{i\omega t} = (i\omega)^{\gamma} e^{i\omega t}. \tag{1-3}$$

This latter formula is no longer valid when the lower limit of integration is 0. For this case there exists another formula (see Appendix for details) based on the Riemann–Liouville fractional derivative

$$D_{0+}^{\gamma} e^{i\omega t} = \frac{d}{dt} \int_{0}^{t} \frac{e^{i\omega(t-t')}dt'}{\Gamma(1-\gamma)t'^{\gamma}} = (i\omega)^{\gamma} e^{i\omega t} + \frac{\sin\gamma\pi}{\pi} \int_{0}^{\infty} \frac{u^{\gamma} e^{-ut} du}{u+i\omega}, \tag{1-4}$$

which turns into (1-3) when $t \to +\infty$.

If one uses the exact formula (1-4) for the fractional differentiation of the exponent, in many cases the integral appearing in (1-4) can also be neglected compared with the first term in the same formula [Rossikhin and Shitikova 1997a; 1997b; 1998; 2000; 2003], since this integral decays rapidly with time. For example, if γ is small, while the frequency ω lies within the range of interest in engineering, the integral on the right-hand side of (1-4) can be ignored, what allows one to use formula (1-3).

Calculations of the magnitude of the fractional parameters carried out on the basis of experimental data [Abdel-Ghaffar and Scanlan 1985] show that this value for suspension bridges is of the order of 0.05–0.1 [Rossikhin and Shitikova 1998; 2008]. The value $\gamma = 0.118$ was reported in [Giovagnoni and Berti 1992] when studying the experimental response of a deformable single-link mechanism, which was realized by means of a brass bar fixed onto a vertical shaft. The fractional parameters $\gamma_1 = 0.1991$ and $\gamma_2 = 0.2499$ were identified in [Schmidt and Gaul 2006] from experimental measurements of a cantilever made of Delrin$^{\text{TM}}$. The value $\gamma = 0.28$ was obtained in [Cooke and Keltie 1987] in a beam impact experiment. A series of experiments measuring the frequency responses of viscoelastic rods of materials like teflon, polyamide, polyurethane, polyvinyl chloride, and polyethylene was reported in

[Schäfer 2000; Schäfer and Seifert 2002], where it was found that the fractional parameter lies in the range of 0.086–0.11. During flexible polyurethane foam modeling via a nonlinear fractional oscillator in [Deng et al. 2003], viscoelastic parameters for automotive seating applications were identified with a fractional parameter equal to 0.019.

The evaluation of the second term in (1-4) is of great importance only during the consideration of linear vibrations, since ignoring this term allows one to solve the equation of linear vibrations of a fractionally damped oscillator

$$\ddot{x} + \omega_0^2 \tau_\sigma^\gamma \left(\frac{d}{dt}\right)^\gamma x(t) + \omega_0^2 x = 0, \tag{1-5}$$

where $\omega_0^2 = E_0 m^{-1}$, E_0 is the spring rigidity, m is the oscillator's mass, and τ_σ is the relaxation time, with the help of the Euler substitution as done in [Rossikhin and Shitikova 2009]:

$$x(t) = Ce^{\lambda T}. \tag{1-6}$$

Indeed, substituting (1-6) into (1-5) we obtain the characteristic equation

$$\lambda^2 + \omega_0^2 \tau_\sigma^\gamma \lambda^\gamma + \omega_0^2 = 0, \tag{1-7}$$

which possesses two complex conjugate roots [Rossikhin and Shitikova 1997b]

$$\lambda_{1,2} = -\alpha \pm i\omega, \tag{1-8}$$

where α and ω are the damping coefficient and the frequency of vibrations, respectively.

The solution of (1-5) with due account for (1-8) can be written as [Rossikhin and Shitikova 2009]

$$x(t) = Ae^{-\alpha t} \cos(\omega t + \varphi), \tag{1-9}$$

where A and φ are arbitrary constants to be determined from the initial conditions.

The Green's function for (1-5) with the second term of (1-4) taken into consideration is written in the form [Rossikhin and Shitikova 1997b]

$$x(t) = A_0(t) + Ae^{-\alpha t} \cos(\omega t + \varphi), \tag{1-10}$$

where $A_0(t)$ is the term governing the drift of the position of equilibrium.

In the present paper, the approach suggested in [Rossikhin and Shitikova 2009] for the analysis of free vibrations of nonlinear mechanical oscillators is generalized to the case of forced vibrations. It will be shown that the second term in (1-4) can altogether be ignored in nonlinear problems, since it does not affect the first approximations to be constructed here using the method of multiple time scales.

The need for studying fractional oscillators is motivated by two reasons: first, engineers often use one-degree-of-freedom models as a first approximation or as a benchmark before preceding to more intricate models or multi-degree-of-freedom structural systems (for example, as the simplest model of a vibration-isolation system [Koh and Kelly 1990; Makris and Constantinou 1991; Hwang and Ku 1997; Aprile et al. 1997; Munshi 1997; Hwang and Hsu 2001; Gusella and Terenzi 2001; Sjöberg and Kari 2003]), and second, the study of vibrations of more complex structures can be reduced to vibrations of a set of fractional oscillators [Giovagnoni and Berti 1992; Rossikhin and Shitikova 2001; 2004; Agrawal 2004; Schäfer and Kempfle 2004].

In all the examples considered below the emphasis will be on investigating the influence of a small external force on vibratory motion, because many of our recent publications have already examined the influence of the order γ of the fractional derivative on nonlinear free damped vibrations of such fractionally damped structures as oscillators [Rossikhin and Shitikova 2009], two-degree-of-freedom mechanical systems [2000], plates [2003; 2006], and suspension bridges [1998; 2008]. We have shown that the fractional parameter plays the role of a structural parameter of the whole system and influences the character of the system's damping coefficient as a function of the natural frequencies of linear vibrations. For example, the power relationships obtained in [Rossikhin and Shitikova 1998; 2008] between the damping coefficient of the system and its natural frequencies of linear vibration correlate well with the experimental data describing the natural frequency dependence of the damping ratio for the Golden Gate suspension bridge [Abdel-Ghaffar and Scanlan 1985]. When the fractional parameter tends to one, i.e., when the fractional derivative transforms into the common derivative with respect to time, the system's damping coefficient does not depend on the natural frequencies of linear vibrations, which is in contradiction with experimental data. Thus, nonlinear viscoelastic models with fractional derivatives with respect to time are to be preferred over models with integral derivatives for describing the damping features of a combined suspension system.

2. Problem formulation

We will consider force-driven vibrations of the Duffing-like oscillator with positive linear stiffness and damping defined by a fractional derivative (1-1):

$$m\ddot{x}(t)+\beta\Big(\frac{d}{dt}\Big)^{\gamma}x(t)+k_1x(t)+k_2x(t)^3=f\cos(\omega t), \tag{2-1}$$

where x, β, k_1, and k_2 are, respectively, the oscillator's displacement, damping coefficient, linear stiffness, and small parameter of nonlinear stiffness, f is the force amplitude, and ω is its frequency.

Dividing (2-1) by the mass and introducing dimensionless values

$$\tilde{t}=t\Omega_0,\quad \tilde{x}=\frac{x}{x_0},\quad \tilde{\omega}=\frac{\omega}{\Omega_0},\quad \tilde{\omega}_0^2=\frac{\omega_0^2}{\Omega_0^2}, \tag{2-2}$$

where

$$\omega_0^2=\frac{k_1}{m},\quad \Omega_0=\sqrt{\frac{g}{l_0}},\quad x_0=\frac{mg}{k_1}=\frac{g}{\omega_0^2}$$

(g being the acceleration of gravity and l_0 the undeformed spring length) yields

$$\ddot{\tilde{x}}+\frac{\beta}{m}\,\Omega_0^{\gamma-2}\Big(\frac{d}{d\tilde{t}}\Big)^{\gamma}\tilde{x}+\tilde{\omega}_0^2\,\tilde{x}+\frac{k_2}{m}\,x_0^2\Omega_0^{-2}\,\tilde{x}^3=\frac{f}{m}\,\Omega_0^{-2}x_0^{-1}\cos(\tilde{\omega}\tilde{t}), \tag{2-3}$$

which we then turn into the dimensionless form of Equation (2-1):

$$\ddot{\tilde{x}}+\varepsilon^k\mu\,\Big(\frac{d}{d\tilde{t}}\Big)^{\gamma}\tilde{x}+\tilde{\omega}_0^2\,\tilde{x}+\tilde{k}_2\,\tilde{x}^3=\varepsilon^{k+1}F\cos(\tilde{\omega}\tilde{t})\quad (k=1\text{ or }2), \tag{2-4}$$

where

$$\varepsilon^k\mu=\frac{\beta}{m}\,\Omega_0^{\gamma-2},\quad \tilde{k}_2=\frac{k_2}{m}\,x_0^2\Omega_0^{-2},\quad \varepsilon^{k+1}F=\frac{f}{m}\,\Omega_0^{-2}x_0^{-1}.$$

Here ε is a small parameter which is of the same order of magnitude as the amplitudes, and μ and F are finite values. The choice of k in (2-4) depends on the order of smallness of the exciting force amplitude and viscosity coefficient.

To lighten the notation, tildes over dimensionless values will be omitted henceforth.

We will assume that the linear natural frequency ω_0 is approximately equal to the frequency of the external excitation ω, i.e.,

$$\omega_0 \approx \omega. \tag{2-5}$$

3. Method of solution

An approximate solution of (2-4) for small amplitudes varying weakly with time can be represented by an expansion in terms of different time scales in the following form [Nayfeh 1973]:

$$x(t) = \varepsilon x_1(T_0, T_1, T_2, \dots) + \varepsilon^2 x_2(T_0, T_1, T_2, \dots) + \varepsilon^3 x_3(T_0, T_1, T_2, \dots) + \cdots \tag{3-1}$$

Here, $T_n = \varepsilon^n t$ $(n = 0, 1, 2, \dots)$ are new independent variables, among them: $T_0 = t$ is a fast scale, characterizing motions with ω and the natural frequency ω_0, and $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are slow scales characterizing the modulations of the amplitude and phase.

Recall that the first, the second and fractional derivatives are defined by

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \qquad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 \left(D_1^2 + 2D_0 D_2\right) + \cdots,$$

$$\left(\frac{d}{dt}\right)^{\gamma} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)^{\gamma} = D_+^{\gamma} + \varepsilon \gamma D_+^{\gamma-1} D_1 + \tfrac{1}{2}\varepsilon^2 \gamma \left((\gamma-1) D_+^{\gamma-2} D_1^2 + 2 D_+^{\gamma-1} D_2\right) + \cdots,$$

where $D_n = \partial/\partial T_n$, and D_+^{γ}, $D_+^{\gamma-1}$, $D_+^{\gamma-2}, \dots$ are the Riemann–Liouville fractional time derivatives:

$$D_+^{\gamma-n} x = \frac{d}{dt} \int_{-\infty}^{t} \frac{x(t-t')\,dt'}{\Gamma(1-\gamma+n)\,t'^{\gamma-n}} \quad (n = 0, 1, 2, \dots).$$

Using this and substituting (3-1) into (2-4), after equating the coefficients at equal powers of ε, we are led to a set of recurrence equations to various orders:

to order ε:

$$D_0^2 x_1 + \omega_0^2 x_1 = 0, \tag{3-2}$$

to order ε^2:

$$D_0^2 x_2 + \omega_0^2 x_2 = -2 D_0 D_1 x_1 - \mu(2-k) D_+^{\gamma} x_1 + (2-k) F \cos \omega T_0 \tag{3-3}$$

to order ε^3:

$$D_0^2 x_3 + \omega_0^2 x_3 = -2 D_0 D_1 x_2 - \left(D_1^2 + 2 D_0 D_2\right) x_1 - \mu(2-k) D_+^{\gamma} x_2 \\ - \mu\gamma(2-k) D_+^{\gamma-1} D_1 x_1 - \mu(k-1) D_+^{\gamma} x_1 - k_2 x_1^3 + (k-1) F \cos \omega T_0. \tag{3-4}$$

The general solution of (3-2) has the form

$$x_1 = A_1(T_1, T_2) e^{i\omega_0 T_0} + \bar{A}_1(T_1, T_2) e^{-i\omega_0 T_0}, \tag{3-5}$$

where A_1 and $\bar{A}_1$ are yet unknown complex conjugate functions.

For further analysis we need to specify the order of weak damping and external excitation.

3A. ***Viscosity of the order of ε.*** Consider first the case where the viscosity is of the order of ε. Then (2-4) reduces to

$$\ddot{x}+\varepsilon\mu\left(\frac{d}{dt}\right)^{\gamma}x+\omega_0^2\,x+k_2\,x^3=\varepsilon^2 F\cos(\omega t). \tag{3-6}$$

Substituting (3-5) in the right-hand side of (3-3) with $k=1$ and taking (1-4) into account, we obtain

$$D_0^2x_2+\omega_0^2x_2=-2i\omega_0\left(\tfrac{1}{2}(i\omega_0)^{\gamma-1}\mu A_1+D_1A_1\right)e^{i\omega_0T_0}+2i\omega_0\left(\tfrac{1}{2}(-i\omega_0)^{\gamma-1}\mu\bar{A}_1+D_1\bar{A}_1\right)e^{-i\omega_0T_0}$$
$$+\tfrac{1}{2}F(e^{i\omega T_0}+e^{-i\omega T_0})-\mu A_1\,\frac{\sin\gamma\pi}{\pi}\int_0^\infty\frac{u^\gamma e^{-uT_0}du}{u+i\omega_0}-\mu\bar{A}_1\,\frac{\sin\gamma\pi}{\pi}\int_0^\infty\frac{u^\gamma e^{-uT_0}du}{u-i\omega_0}. \tag{3-7}$$

The functions $\exp(\pm i\omega_0T_0)$ on the right-hand side of (3-7) produce secular terms, so the coefficients affecting these functions must be made to vanish. Taking (2-5) into account, we have as a result

$$D_1A_1+\tfrac{1}{2}(i\omega_0)^{\gamma-1}\mu A_1-\frac{F}{4i\omega_0}=0, \tag{3-8}$$

whence it follows that

$$A_1(T_1,T_2)=a_1(T_2)\exp\left(-\tfrac{1}{2}(i\omega_0)^{\gamma-1}\mu T_1\right)+\frac{F}{2\mu(i\omega_0)^\gamma}, \tag{3-9}$$

where $a_1(T_2)$ is yet unknown function.

In view of (3-9), Equation (3-7) takes on the form

$$D_0^2x_2+\omega_0^2x_2=-\mu\,\frac{\sin\gamma\pi}{\pi}\int_0^\infty u^\gamma e^{-uT_0}\left(A_1\frac{u-i\omega_0}{u^2+\omega_0^2}+\bar{A}_1\frac{u+i\omega_0}{u^2+\omega_0^2}\right)du. \tag{3-10}$$

Writing the function $A_1(T_1,T_2)$ defined by (3-9) and its complex conjugate $\bar{A}_1(T_1,T_2)$ as

$$A_1=a+ib,\qquad \bar{A}_1=a-ib \tag{3-11}$$

and substituting (3-11) into (3-10) yields

$$D_0^2x_2+\omega_0^2x_2=-2\mu\,\frac{\sin\gamma\pi}{\pi}f(T_0,T_1,T_2), \tag{3-12}$$

where

$$f(T_0,T_1,T_2)=\int_0^\infty\frac{u^\gamma(au+b\omega_0)}{u^2+\omega_0^2}\,e^{-uT_0}du. \tag{3-13}$$

Then the solution of (3-12) has the form

$$x_2=A_2(T_1,T_2)e^{i\omega_0T_0}+\bar{A}_2(T_1,T_2)e^{-i\omega_0T_0}+C(T_0,T_1,T_2)e^{i\omega_0T_0}+\overline{C}(T_0,T_1,T_2)e^{-i\omega_0T_0}, \tag{3-14}$$

where A_2 and $\bar{A}_2$, and C and $\overline{C}$ are yet unknown complex conjugate functions. The first two terms of (3-14) represent the general solution of the homogeneous part of (3-12), while the second pair of terms is the particular solution of the inhomogeneous equation (3-12).

Substituting the particular solution from (3-14) in (3-12), we find

$$C(T_0,T_1,T_2)=-2\mu\,\frac{\sin\gamma\pi}{\pi}\int_0^{T_0}e^{-2i\omega_0T_0'}dT_0'\int_0^{T_0'}f(T_0'',T_1,T_2)e^{i\omega_0T_0''}dT_0''. \tag{3-15}$$

Substituting (3-5), (3-9), (3-14), and (3-15) in the right-hand side of (3-4) with $k=1$, we are led to the equation for determining x_3. Eliminating the terms that produce secular terms, we obtain the solvability condition

$$D_2 a_1(T_2) + a_1\left(\tfrac{1}{8}\mu^2(1-2\gamma)(i\omega_0)^{2\gamma-3} - \kappa\,\frac{3k_2F^2}{4\mu^2\omega_0^{2\gamma+1}}\right) = 0,$$

whence it follows that

$$a_1 = a_{11}\exp\left(-\tfrac{1}{8}\mu^2(1-2\gamma)(i\omega_0)^{2\gamma-3} + \kappa\,\frac{3k_2F^2}{4\mu^2\omega_0^{2\gamma+1}}\right)T_2, \tag{3-16}$$

where a_{11} is a constant to be determined from the initial conditions and $\kappa = i\cos 2\gamma\pi + \sin 2\gamma\pi$.

Considering (3-16), the coefficients a and b appearing in the expression (3-13) of $f(T_0, T_1, T_2)$ take the form

$$\begin{aligned} a(T_1,T_2) = {} & a_{11}\exp\left(-\tfrac{1}{2}\mu T_1\omega_0^{\gamma-1}\sin\left(\tfrac{1}{2}\gamma\pi\right) + \tfrac{1}{8}\mu^2 T_2(1-2\gamma)\omega_0^{2\gamma-3}\sin\gamma\pi\right) \\ & \times\cos\left(\tfrac{1}{2}\mu T_1\omega_0^{\gamma-1}\sin\left(\tfrac{1}{2}\gamma\pi\right) + \tfrac{1}{8}\mu^2 T_2(2\gamma-1)\omega_0^{2\gamma-3}\cos\gamma\pi + \frac{3k_2F^2}{4\mu^2\omega_0^{2\gamma+1}}\,T_2\cos 2\gamma\pi\right) \\ & + \frac{F}{\mu\omega_0^{\gamma}}\cos\frac{\gamma\pi}{2}, \end{aligned} \tag{3-17}$$

$$\begin{aligned} b(T_1,T_2) = {} & a_{11}\exp\left(-\tfrac{1}{2}\mu T_1\omega_0^{\gamma-1}\sin\left(\tfrac{1}{2}\gamma\pi\right) + \tfrac{1}{8}\mu^2 T_2(1-2\gamma)\omega_0^{2\gamma-3}\sin\gamma\pi\right) \\ & \times\sin\left(\tfrac{1}{2}\mu T_1\omega_0^{\gamma-1}\sin\left(\tfrac{1}{2}\gamma\pi\right) + \tfrac{1}{8}\mu^2 T_2(2\gamma-1)\omega_0^{2\gamma-3}\cos\gamma\pi + \frac{3k_2F^2}{4\mu^2\omega_0^{2\gamma+1}}\,T_2\cos 2\gamma\pi\right) \\ & - \frac{F}{\mu\omega_0^{\gamma}}\sin\frac{\gamma\pi}{2}, \end{aligned} \tag{3-18}$$

Combining (3-9) and (3-16) with (3-5) yields

$$\begin{aligned} x_1 = {} & \left[a_{11}\exp\left(-\tfrac{1}{8}\,\mu^2(1-2\gamma)(i\omega_0)^{2\gamma-3} + \kappa\,\frac{3k_2F^2}{4\mu^2\omega_0^{2\gamma+1}}\right)T_2\exp\left(-\tfrac{1}{2}(i\omega_0)^{\gamma-1}\mu T_1\right) + \frac{F}{2\mu(i\omega_0)^{\gamma}}\right] \\ & \times\exp(i\omega_0 T_0) + \text{c.c.}, \end{aligned} \tag{3-19}$$

where c.c. stands for the complex conjugate to the preceding terms.

Reference to (3-19) shows that the second term of formula (1-4) does not affect the solution within the limits of this approximation.

Limiting ourselves to the first term in (3-1) with due account for (3-19), we find the solution of (3-6) in the form

$$x = \varepsilon\left(a_0 e^{-\alpha t}\cos\Omega t + \frac{F}{\mu\omega_0^{\gamma}}\cos\left(\omega_0 t - \tfrac{1}{2}\gamma\pi\right)\right), \tag{3-20}$$

where we have introduced the quantities $a_0 = 2a_{11}$,

$$\alpha = \tfrac{1}{2}\varepsilon\mu\omega_0^{\gamma-1}\sin\tfrac{1}{2}\gamma\pi\left(1 + \tfrac{1}{2}\varepsilon\mu(2\gamma-1)\omega_0^{\gamma-2}\cos\tfrac{1}{2}\gamma\pi\right) - \varepsilon^2\,\frac{3k_2F^2}{4\mu^2\omega_0^{2\gamma+1}}\sin 2\gamma\pi, \tag{3-21}$$

and

$$\Omega = \omega_0\left(1 + \tfrac{1}{2}\varepsilon\mu\omega_0^{\gamma-2}\cos\tfrac{1}{2}\gamma\pi + \tfrac{1}{8}\varepsilon^2\mu^2(2\gamma-1)\omega_0^{2(\gamma-2)}\cos\gamma\pi + \varepsilon^2\frac{3k_2F^2}{4\mu^2\omega_0^{2(\gamma+1)}}\cos 2\gamma\pi\right). \tag{3-22}$$

Reference to (3-20) shows that the solution involves two parts: the first corresponds to the damping vibrations and describes the transient process, while the second one is nondamping in character and describes forced vibrations with the frequency of the exciting force and with the phase difference depending on the fractional parameter γ. Note that in the first term of (3-20) the amplitude of the external force F does not affect the damping coefficient α (3-21), while it weakly influences the nonlinear frequency Ω of vibrations (3-22).

When $\gamma = 1$, Equation (3-20) goes over into the equation describing vibrations of the viscoelastic Duffing oscillator with ordinary Kelvin–Voigt constitutive relations, i.e.,

$$x = \varepsilon\left\{a_0 e^{-\varepsilon\mu t/2}\cos\omega_0\left[1 - \frac{\varepsilon^2\mu^2}{4\omega_0^2}\left(\frac{1}{2} - \frac{3k_2F^2}{\mu^4\omega_0^2}\right)\right]t + \frac{F}{\mu\omega_0}\cos\left(\omega_0 t - \frac{\pi}{2}\right)\right\}. \tag{3-23}$$

It can be noted that if in the right-hand part of (3-6) one takes $\sin\omega t$ instead of the cosine function, then the first term in the solution remains unchanged, while in the second term of (3-20) or (3-23) the cosine function should be simply substituted with the sine function.

3B. *Viscosity of the order of ε^2.* Now let us consider vibrations of a nonlinear oscillator putting $k = 2$ in the equation of motion (2-4):

$$\ddot{x} + \varepsilon^2\mu\left(\frac{d}{dt}\right)^{\gamma}x + \omega_0^2 x + k_2x^3 = \varepsilon^3 F\cos\omega t. \tag{3-24}$$

Substituting (3-5) into the right-hand side of (3-3) with $k = 2$, we obtain

$$D_0^2x_2 + \omega_0^2x_2 = -2i\omega_0 D_1A_1\exp(i\omega_0T_0) + \text{c.c.} \tag{3-25}$$

To eliminate circular terms in (3-25), it is necessary to vanish to zero the coefficient standing at $\exp(i\omega_0T_0)$, i.e.,

$$D_1A_1(T_1, T_2) = 0,$$

whence it follows that A_1 is T_1-independent.

Then the general solution of (3-25) has the form

$$x_2 = A_2(T_1, T_2)e^{i\omega_0T_0} + \bar{A}_2(T_1, T_2)e^{-i\omega_0T_0}. \tag{3-26}$$

Substituting (3-5) and (3-26) in the right-hand side of (3-4) with $k = 2$ and considering formula (1-4) and condition (2-5), we are led to this equation for determining x_3:

$$D_0^2x_3 + \omega_0^2x_3 = -2i\omega_0D_1A_2\exp(i\omega_0T_0) - k_2A_1^3\exp(3i\omega_0T_0)$$
$$-\left(2i\omega_0D_2A_1 + \mu(i\omega_0)^{\gamma}A_1 + 3k_2A_1^2\bar{A}_1 - \tfrac{1}{2}F\right)\exp(i\omega_0T_0) - \mu A_1\frac{\sin\gamma\pi}{\pi}\int_0^{\infty}\frac{u^{\gamma}e^{-uT_0}du}{u + i\omega_0} + \text{c.c.} \tag{3-27}$$

From (3-27) it is evident that its last term does not generate secular terms and thus does not affect the solution constructed thereafter.

Eliminating secular terms in (3-27), we obtain the solvability conditions

$$D_1 A_2(T_1, T_2) = 0, \tag{3-28}$$

$$2i\omega_0 D_2 A_1 + \mu (i\omega_0)^\gamma A_1 + 3k_2 A_1^2 \bar{A}_1 - \tfrac{1}{2} F = 0. \tag{3-29}$$

From (3-28) it follows that A_2 is independent of T_1.

We multiply (3-29) by $\bar{A}_1$ and write its complex conjugate. Separately adding and subtracting together the two conjugate equations, we find

$$3k_2\omega_0^{-1} A_1^2 \bar{A}_1^2 + i(\bar{A}_1 D_2 A_1 - A_1 D_2 \bar{A}_1) + 2\mu\omega_0^{\gamma-1} A_1 \bar{A}_1 \cos\tfrac{\pi}{2}\gamma - \frac{F}{4\omega_0}(A_1 + \bar{A}_1) = 0, \tag{3-30}$$

$$i(\bar{A}_1 D_2 A_1 + A_1 D_2 \bar{A}_1) + 2\mu i\omega_0^{\gamma-1} A_1 \bar{A}_1 \sin\tfrac{\pi}{2}\gamma + \frac{F}{4\omega_0}(A_1 - \bar{A}_1) = 0. \tag{3-31}$$

Representing the function $A_1(T_2)$ in the polar form

$$A_1 = a\exp(i\varphi),$$

we obtain from (3-30) and (3-31)

$$\dot{\varphi} - \frac{1}{2}\,\delta - \frac{3k_2}{2\omega_0} a^2 + \frac{1}{4\omega_0}\, F a^{-1}\cos\varphi = 0, \qquad (a^2)^{\dot{}} + s a^2 + \frac{1}{2\omega_0} F a \sin\varphi = 0, \tag{3-32}$$

where the superscript dot denotes the T_2-derivative, $\delta = \mu\omega_0^{\gamma-1}\cos\frac{\pi}{2}\gamma$, and $s = \mu\omega_0^{\gamma-1}\sin\frac{\pi}{2}\gamma$.

Dividing the second equation in (3-32) by a we obtain

$$\dot{a} + \tfrac{1}{2} s a + \frac{1}{4\omega_0}\, F\sin\varphi = 0 \tag{3-33}$$

and then integrating (3-33), we obtain

$$a = \left(a_0 - \frac{F}{4\omega_0}\int_0^{T_2} e^{sT_2/2}\sin\big(\varphi(T_2)\big)\,dT_2\right)e^{-sT_2/2} \tag{3-34}$$

To obtain the equation for determining the function $\varphi(T_2)$, rewrite $(3\text{-}32)_2$ as

$$(\ln a^2)^{\dot{}} = -s - \frac{F}{2\omega_0}\, a^{-1}\sin\varphi, \tag{3-35}$$

multiply it by $\cos\varphi$ and add it to $(3\text{-}32)_1$ multiplied by $-\sin\varphi$. Considering (3-35), as a result we obtain

$$(\cos\varphi)^{\dot{}} - \left(a_0 - \frac{F}{4\omega_0}\int_0^{T_2} e^{sT_2/2}\sqrt{1-\cos^2\varphi}\, dT_2\right)^{-1} \frac{F}{4\omega_0}\, e^{sT_2/2}\cos\varphi\,\sqrt{1-\cos^2\varphi}$$
$$+\left(a_0 - \frac{F}{4\omega_0}\int_0^{T_2} e^{sT_2/2}\sqrt{1-\cos^2\varphi}\, dT_2\right)^2 \frac{3k_2}{2\omega_0}\, e^{-sT_2}\,\sqrt{1-\cos^2\varphi} + \tfrac{1}{2}\,\delta\,\sqrt{1-\cos^2\varphi} = 0. \tag{3-36}$$

Integrating (3-36), we find the T_2-dependence of $\cos\varphi$, and then substituting the function $\sin\varphi(T_2)$ thus found in (3-35), we can obtain T_2-dependence of a.

To find the functions $\varphi(T_2)$ and $a(T_2)$, we use another approach. Dividing $(3\text{-}32)_1$ by (3-33), we get

$$\frac{d\varphi}{da}=\frac{F(4\omega_0)^{-1}a^{-1}\cos\varphi-3k_2(2\omega_0)^{-1}a^2-2^{-1}\delta}{2^{-1}sa+F(4\omega_0)^{-1}\sin\varphi} \tag{3-37}$$

or

$$\tan\chi(F)=\frac{d\varphi}{d(\ln a)}=\frac{F(4\omega_0)^{-1}\exp(-\ln a)\cos\varphi-3k_2(2\omega_0)^{-1}\exp(2\ln a)-2^{-1}\delta}{2^{-1}s+F(4\omega_0)^{-1}\exp(-\ln a)\sin\varphi} \tag{3-38}$$

From (3-38) first we find the function $\varphi(a)$ and substituting it in (3-33), we can determine the function $a(T_2)$, hence, $\varphi[a(T_2)]=\varphi(T_2)$.

If on the right-hand part of (3-24) one takes $\sin\omega t$ instead of the cosine function, (3-38) takes on the form

$$\tan\chi(F)=\frac{d\varphi}{d(\ln a)}=\frac{F(4\omega_0)^{-1}\exp(-\ln a)\sin\varphi-3k_2(2\omega_0)^{-1}\exp(2\ln a)-2^{-1}\delta}{2^{-1}s+F(4\omega_0)^{-1}\exp(-\ln a)\cos\varphi} \tag{3-39}$$

3B1. *The case of free vibrations.* At $F=0$, the system (3-32) is reduced to the form

$$\dot\varphi-\tfrac12\delta-\frac{3k_2}{2\omega_0}\,a^2=0,\qquad \left(a^2\right)^{\cdot}+sa^2=0. \tag{3-40}$$

Integration yields

$$a^2=a_0^2e^{-sT_2},\qquad \varphi=\frac12\,\delta T_2-\frac{3k_2}{2\omega_0 s}\,a_0^2e^{-sT_2}+\varphi_0, \tag{3-41}$$

where a_0 and φ_0 are the initial magnitudes of a and φ, respectively. Eliminating T_2 from (3-41), we find

$$\varphi=-\frac{\delta}{s}\,\ln\frac{a}{a_0}-\frac{3k_2}{2\omega_0 s}(a^2-a_0^2)+\varphi_0, \tag{3-42}$$

or

$$G(\ln a,\varphi)=\tfrac12\varphi s+\tfrac12\delta\ln a+\frac{3k_2}{4\omega_0}\exp(2\ln a)=G_0(\ln a_0,\varphi_0) \tag{3-43}$$

This relationship can be interpreted as the stream-function for the phase fluid moving in the plane with the coordinates $\ln a$, φ. Really, considering (3-40), it is followed from (3-43) that the components of the vector of the phase fluid motion $\vec{\mathbf{V}}\left\{v_{\ln a}=(\ln a)^{\cdot},\, v_\varphi=\dot\varphi\right\}$ are determined by the formulas

$$(\ln a)^{\cdot}=-\frac{\partial G}{\partial\varphi},\qquad \dot\varphi=\frac{\partial G}{\partial(\ln a)}. \tag{3-44}$$

Since

$$dG=\frac{\partial G}{\partial\varphi}\,d\varphi+\frac{\partial G}{\partial(\ln a)}\,d(\ln a)=0,$$

we obtain, taking into account (3-44),

$$-v_{\ln a}d\varphi+v_\varphi d(\ln a)=0,\quad\text{or}\quad \frac{d\varphi}{v_\varphi}=\frac{d(\ln a)}{v_{\ln a}}. \tag{3-45}$$

This is the equation of the streamline.

In the case of free vibrations, it follows from (3-38) that $\tan\chi|_{F=0}$ defines the angle of inclination of the tangent to the streamline. If a is small, then

$$\tan\chi(0)=-\frac{\delta}{s}=-\cot\frac{\gamma\pi}{2}\quad\text{or}\quad \chi=\frac{\pi}{2}+\frac{\gamma\pi}{2}. \tag{3-46}$$

For large a from (3-38) it follows that

$$\tan\chi(0) \to -\infty \quad \text{or} \quad \chi \to \frac{\pi}{2}+. \tag{3-47}$$

The streamlines for the oscillator with natural frequency $\omega_0 = 1$ are presented in Figure 1 in the semilogarithmic coordinates $\ln a, \varphi$. The top pane shows streamlines constructed for various G_0 and fixed $\gamma = 0.5$ and $k_2 = 0.1$. On the bottom left we have $G_0 = 0$ and $k_2 = 0.1$ fixed and varying γ. The bottom right pane presents the streamlines constructed for the fixed $G_0 = 0$ and $\gamma = 0.5$, while k_2 is used as the parameter.

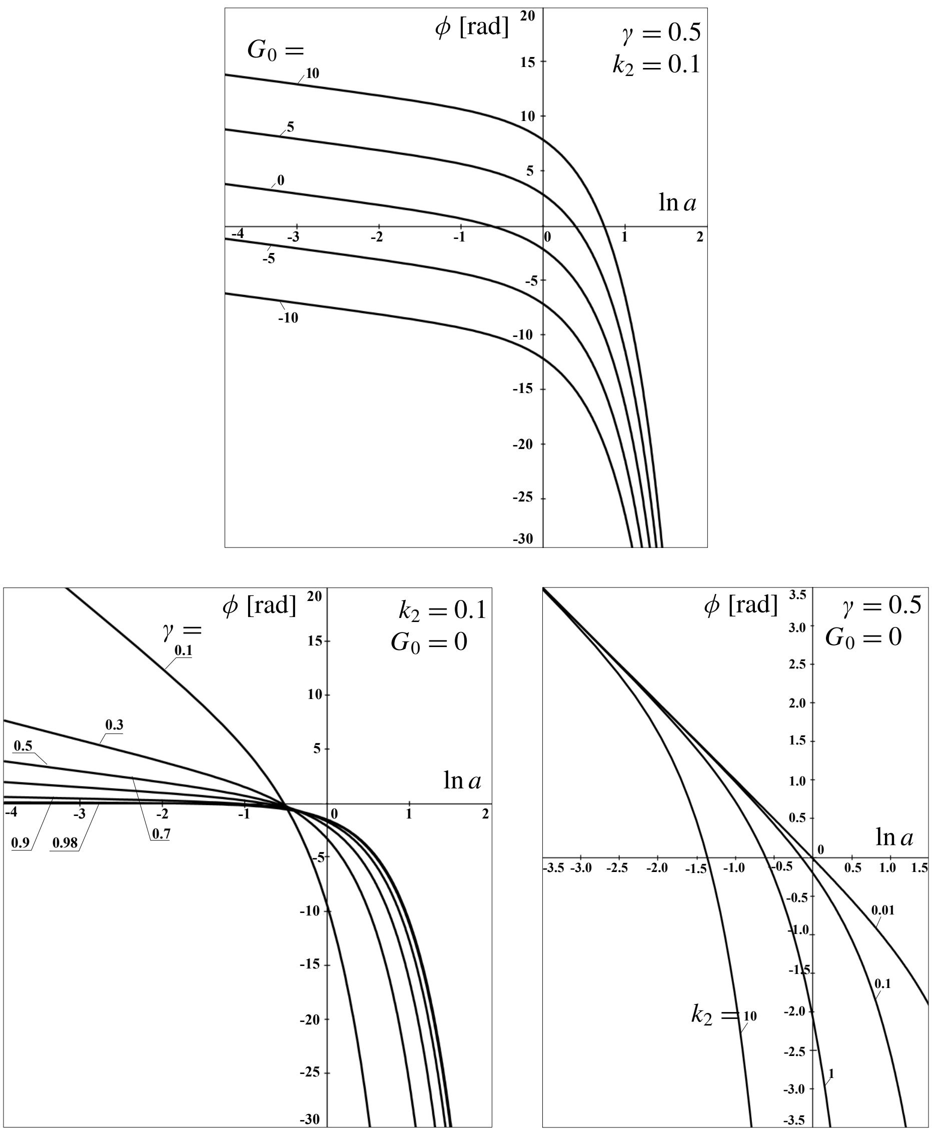

Figure 1. Streamlines of the phase fluid corresponding to different values of G_0 (top), fractional parameter γ (bottom left), and nonlinear stiffness coefficient k_2 (bottom right).

The asymptotic character of the curves in Figure 1 is verified by relationships (3-46) and (3-47). In the first and last panes of the figure, the fractional parameter γ is fixed for all curves. Therefore their left branches are asymptotically parallel in the first case (Figure 1, top) due to the different values of G_0, and they tend to the same asymptote in the latter case (Figure 1, bottom right) since the value of G_0 is shared. Further, from (3-46) and the bottom left pane of the figure it is seen that, for small values of a, the left branches of the curves approach infinitely close the vertical and horizontal axes as $\gamma \to 1$ and $\gamma \to 0$, respectively. Thus the curves in Figure 1 show that the various parameters have different effects upon the behavior of the curves.

The solution for nonlinear free damped vibrations has the form

$$x = \varepsilon a_0 e^{-\alpha t/2} \cos\left(\Omega t + \varphi_0 - d a_0^2 e^{-\alpha t}\right), \tag{3-48}$$

where

$$\alpha = \varepsilon^2 \mu \omega_0^{\gamma-1} \sin \tfrac{\pi}{2}\gamma, \quad d = \tfrac{3}{2} k_2 \left(\mu \omega_0^{\gamma} \sin \tfrac{\pi}{2}\gamma\right)^{-1}, \quad \Omega = \omega_0\big(1 + \tfrac{1}{2}\varepsilon^2 \mu \omega_0^{\gamma-2} \cos \tfrac{\pi}{2}\gamma\big).$$

In the particular case when $\gamma = 1$, the solution takes the form

$$x = \varepsilon a_0 e^{-\varepsilon^2 \mu t/2} \cos\left(\omega_0 t + \varphi_0 - \frac{3k_2}{2\mu\omega_0}\, a_0^2 e^{-\varepsilon^2 \mu t}\right), \tag{3-49}$$

3B2. *The case of small force amplitude.* If the amplitude F of the external force is small, Equation (3-37) takes the form

$$\frac{d\varphi}{da} = -\frac{\delta}{s}\, a^{-1} - \frac{3k_2}{\omega_0 s}\, a + F f(a, \varphi), \tag{3-50}$$

where

$$f(a, \varphi) = \frac{\cos\varphi}{2\omega_0 s a^2} + \frac{\sin\varphi}{\omega_0 s^2 a^2} \left(\frac{3k_2 a^2}{2\omega_0} + \frac{1}{2}\,\delta\right).$$

Integrating (3-50) yields

$$\varphi = -\frac{\delta}{s}\, \ln\frac{a}{a_0} - \frac{3k_2}{2\omega_0 s}(a^2 - a_0^2) + \varphi_0 + F \int_{a_0}^{a} f[a, \varphi(a)]\, da, \tag{3-51}$$

or

$$\tfrac{1}{2}\,\varphi s + \tfrac{1}{2}\,\delta \ln a + \frac{3k_2}{4\omega_0} e^{2\ln a} = G_0 + \tfrac{1}{2}\, sF \int_{a_0}^{a} f[a, \varphi(a)]\, da. \tag{3-52}$$

From this relationship it is evident that the streamlines do not remain unchanged; they vary even for small external forces.

3B3. *The case of finite force amplitude.* To investigate the influence of a finite exciting force $F \cos\omega t$ (or $F \sin\omega t$) on the character of the oscillator's vibratory motions, let us choose some streamline and assume that at $t = 0$ a phase fluid point lie somewhere on this line. If the external force $F \cos\omega t$ or $F \sin\omega t$ acts on the oscillator beginning from the moment $t = 0$, the phase fluid point under consideration moves, according to (3-38) or (3-39), respectively, along a trajectory that does not coincide with the chosen streamline. We can take another point on the same streamline and calculate the trajectory according to the same equations, and so on. If after some instant of time we connect with a curve the points thus found, lying on the different trajectories, and compare this curve to the reference streamline, we can judge by the departure of one line from the other the character of the transient vibratory motion occurring in the mechanical system after the external force begins to act.

Relationships (3-38) and (3-39) define the angle of inclination of the tangent to the trajectory of the phase fluid point. For small a from (3-38) it follows that

$$\tan\chi = \cot\varphi \quad \text{or} \quad \chi = \frac{\pi}{2} - \varphi \quad \text{or} \quad \chi = \frac{3\pi}{2} - \varphi, \tag{3-53}$$

while from (3-39) for small a we have

$$\tan\chi = \tan\varphi \quad \text{or} \quad \chi = \varphi \quad \text{or} \quad \chi = \pi + \varphi. \tag{3-54}$$

The tangent vectors to the trajectories of motion for the phase fluid points (i.e., the polarization vector of motion of the given system) at different points of the streamline with the parameters $G_0 = 0$, $\gamma = 0.5$, and $k_2 = 0.1$ are presented in Figure 2, at the instant the force $F\cos\omega t$ or $F\sin\omega t$ begins to take effect ($T_2 = 0$).

Examination of the left half of the figure shows that under the action of the force $F\cos\omega t$ the polarization vector executes a vibrational motion with a decrease in $\ln a$: first it rotates counterclockwise until attaining the maximal angle $\chi = 164.5°$ at the point with the coordinates $\ln a_0 = -1.18$, $\varphi_0 = 1.18$, and it then begins to rotate in a clockwise direction. From the right half of the figure it is seen that for the oscillator driven by the force $F\sin\omega t$ the polarization vector executes a counterclockwise rotational motion with the decrease in $\ln a$: the angle χ increases monotonically starting from 90° and tending to make a complete turn as $a_0 \to 0$.

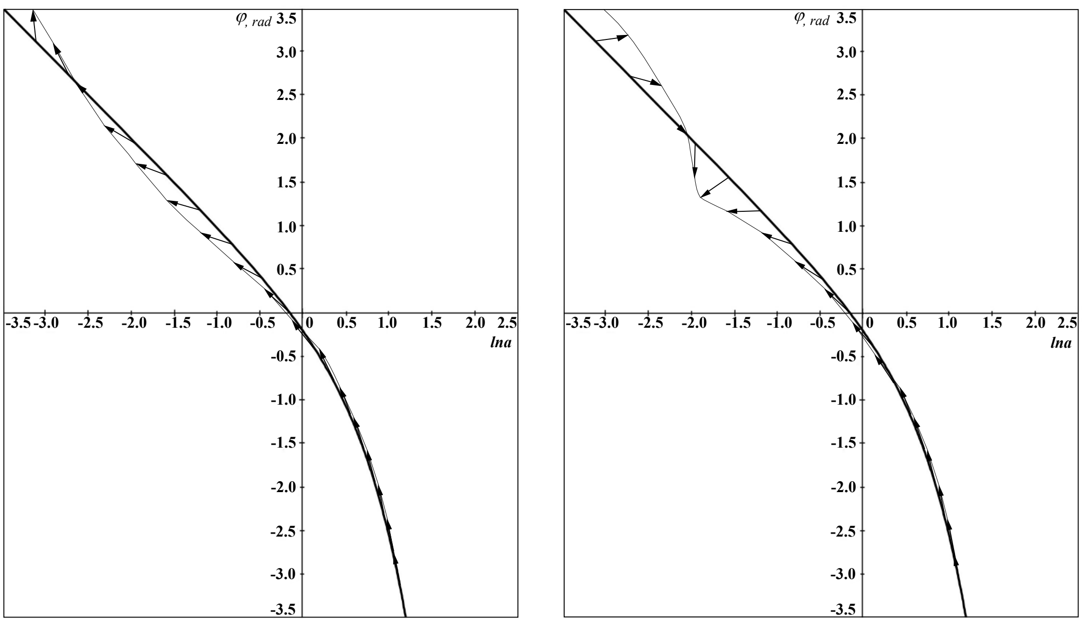

Figure 2. Directions of the polarization vectors of the system's vibrational motions at the instant the force $F\cos\omega t$ (left) or $F\sin\omega t$ (right) takes effect, for $F = 0.5$.

Figure 2 makes it clear that at the instant the external exciting force begins to take effect on the system the points of the phase fluid leave the stream line either to the left or to the right of it and start to move along their own trajectories. In other words, the stream lines disappear the moment the force is applied.

4. Conclusion

The engineering analytical approach proposed in [Rossikhin and Shitikova 2009] for the approximate analysis of the dynamic behavior of linear and nonlinear fractional oscillators has been generalized to the case of forced vibrations. It allows the authors to analyze the force-driven vibrations of a fractional oscillator of Duffing type at different low-level orders of damping and external force terms using the method of multiple time scales.

In the case of viscosity of the order of ε and the external exciting force of the order of ε^2, it has been shown that the solution involves two parts, where the first term corresponds to damping vibrations and describes the transient process, while the second one is nondamping in character and describes forced vibrations with the frequency of the exciting force and with a phase difference depending on the fractional parameter γ.

In the case of free vibrations with a weak damping term of the order of ε^2, the nonlinear fractional oscillator performs steady-state vibrations, which are in compliance with the phase fluid motion in the phase plane along the streamlines in the direction of decreasing amplitude of vibrations. At the instant the small external force of the order of ε^3 begins to take effect, vibrations of the nonlinear fractional oscillator go over into transient ones, leading to the disappearance of the stream lines, while the phase fluid points lying on the streamlines at the moment of the force application start to follow their own phase trajectories.

It has been shown that the integral term in formula (1-4) does not affect the solution of either problem within the chosen approximation framework, so it is sufficient to use formula (1-3) regardless of the values of the driving frequency or the fractional parameter.

Appendix

It is well known [Janke et al. 1960; Abramowitz and Stegun 1964] that many special functions exist in two equivalent representations: as incomplete integrals and as infinite power series.

To show the validity of formula (1-4), we apply the Laplace transformation to the expression $x(t) = D_{0+}^{\gamma} e^{i\omega t}$, obtaining

$$\bar{x}(p) = \frac{p^{\gamma}}{p - i\omega}. \tag{4-1}$$

Applying the Mellin–Fourier inversion formula

$$x(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p^{\gamma} e^{pt}}{p - i\omega}\, dp, \tag{4-2}$$

to go back to the time domain, and using the integration contour presented in Figure 3, we find

$$x(t) = \sum_k \operatorname{res}\big(\bar{x}(p_k) e^{p_k t}\big) + \frac{1}{2\pi i} \int_0^{\infty} \big(\bar{x}(ue^{-i\pi}) - \bar{x}(ue^{i\pi})\big) e^{-ut}\, du. \tag{4-3}$$

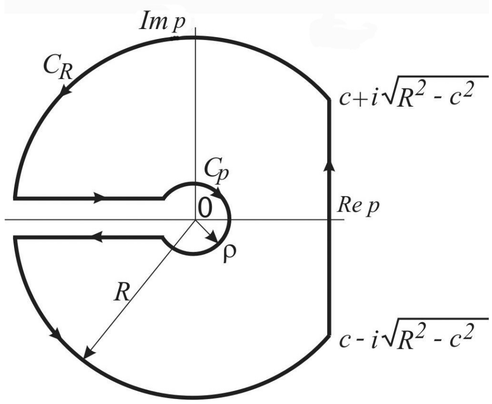

Figure 3. Integration contour used in the proof of (1-4).

Formula (1-4) follows immediately from (4-3).

For the function $x(t)$, it is possible to obtain another representation in the form of an infinite series. For this purpose, we rewrite formula (4-1) as

$$\bar{x}(p) = \frac{p^{\gamma-1}p}{p - i\omega} = \frac{p^{\gamma-1}}{1 - i\omega p^{-1}} \tag{4-4}$$

Now express this last fraction as the sum of an infinite descending geometric progression with initial term $p^{\gamma-1}$ and ratio $i\omega p^{-1}$:

$$\frac{p^{\gamma-1}}{1 - i\omega p^{-1}} = \sum_{n=1}^{\infty} (i\omega)^{n-1} p^{\gamma-n} = \sum_{n=0}^{\infty} (i\omega)^n p^{\gamma-n-1}. \tag{4-5}$$

The inversion of formula (4-5) gives

$$x(t) = t^{\nu} \sum_{n=0}^{\infty} \frac{Z^n}{\Gamma(n+1+\nu)} = \frac{t^{\nu}}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{Z^n}{\nu(\nu+1)\dots(\nu+n)}, \tag{4-6}$$

where $\nu = -\gamma$ and $Z = i\omega t$. We next use the equalities [Janke et al. 1960]

$$\sum_{n=0}^{\infty} \frac{Z^n}{\nu(\nu+1)\dots(\nu+n)} = e^Z Z^{-\nu} \gamma(\nu, Z)$$

and

$$\gamma^*(\nu, Z) = \frac{Z^{-\nu}}{\Gamma(\nu)} \gamma(\nu, Z),$$

where $\gamma(\nu, Z)$ is the incomplete gamma function [Janke et al. 1960; Abramowitz and Stegun 1964], finally we obtain

$$x(t) = t^{\nu} e^Z \gamma^*(\nu, Z) = t^{-\gamma} e^{i\omega t} \gamma^*(-\gamma, i\omega t). \tag{4-7}$$

Formula (4-7) coincides (apart from notation) with that discussed in [Miller and Ross 1993].

In the present paper, the first representation for the function $x(t)$, resulting in formula (1-4), has been adopted, since it is more convenient and physically admissible for engineering applications.

Formulas (1-3) and (1-4) can also be easily obtained from the similar expressions for fractional integrals presented in [Samko et al. 1993, Tables 9.1 and 9.2].

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YURIY A. ROSSIKHIN: yar@vgasu.vrn.ru
Department of Structural Mechanics, Voronezh State University of Architecture and Civil Engineering, 84, 20-letija Oktjabrja Street, Voronezh 394006, Russia

MARINA V. SHITIKOVA: shitikova@vmail.ru
Department of Structural Mechanics, Voronezh State University of Architecture and Civil Engineering, 84, 20-letija Oktjabrja Street, Voronezh 394006, Russia

TATIANA SHCHEGLOVA: laguna@vmail.ru
Department of Structural Mechanics, Voronezh State University of Architecture and Civil Engineering, 84, 20-letija Oktjabrja Street, Voronezh 394006, Russia