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In the present paper, we investigate the spatial behavior of transient and steady-state solutions for the problem of bending applied to a linear Mindlin-type plate model; the plate is supposed to be made of a material characterized by rhombic isotropy, with the elasticity tensor satisfying the strong ellipticity condition. First, using an appropriate family of measures, we show that the transient solution vanishes at distances greater than cT from the support of the given data on the time interval $[0, T]$, where c is a characteristic material constant. For distances from the support less than cT, we obtain a spatial decay estimate of Saint-Venant type. Then, for a plate whose middle section is modelled as a (bounded or semiinfinite) strip, a family of measures is used to obtain an estimate describing the spatial behavior of the amplitude of harmonic vibrations, provided that the frequency is lower than a critical value.

A list of symbols can be found on page 337.

1. Introduction

Mechanical structures involving elastic plates are useful in a wide range of technical applications and have been the subject of many studies, such as [Green and Naghdi 1967; Naghdi 1971; Lagnese and Lions 1988]. Initially, authors considered the plate model based on Kirchhoff's elastic strain-displacement relations, completely neglecting the effects of transverse shear forces [Nowinski 1978; Lagnese and Lions 1988]. Then, increasingly refined models were introduced, taking into account not only the deflection of the middle section, but also transverse shear deformations. The theory of elastic plates based on the Mindlin model has been developed by Constanda [1990] for the elastostatic bending of a thin slab, including the effects of transverse shear deformation. Furthermore, a dynamic model for small deformations of a thin thermoelastic plate was developed by Schiavone and Tait [1993].

The Reissner–Mindlin and Kirchhoff–Love models are the two most common models of a thin linear elastic plate. It is often remarked in the engineering literature that the Reissner–Mindlin model is more accurate, particularly for thin plates and when transverse shear strain plays a significant role [Hughes 1987]; in fact, both Mindlin [1951] and Reissner [1945; 1947] independently proposed theories which also include the effects of transverse shear deformation. Arnold et al. [2002] showed that the Reissner–Mindlin plate bending model has a wider range of applicability than the Kirchhoff–Love model. Under the assumption of constant body force density in the transverse direction, they proved that the Reissner–Mindlin model solution converges to the three-dimensional linearly elastic solution in the relative energy norm for the full range of surface loads. Fabrizio and Chiriță [2004] studied the deformations of a linear viscoelastic plate, including transverse shear deformations.

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The state of bending for a transversely isotropic plate of Mindlin-type has been analyzed by Passarella and Zampoli [2009a; 2009b]. The first of these papers contains a uniqueness result and a Galerkin representation of the solution, without positive definiteness assumptions on the elasticity tensor. Furthermore, under the hypotheses of positive definiteness of the elasticity tensor, the authors prove a Gurtin type variational theorem and a minimum principle. In [Passarella and Zampoli 2009b], previous hypotheses are relaxed and, assuming strong ellipticity of the elasticity tensor, results about the spatial behavior of transient and steady-state solutions are established.

With regard to the spatial behavior of solutions, Saint-Venant's principle plays a central role in the theory and applications of elasticity. A comprehensive review of research on the spatial behavior of solutions is given by [Toupin 1965; Knowles 1966; Flavin 1974; Horgan and Knowles 1983; Gregory and Wan 1985; Horgan 1989; 1996; Mielke 1988].

For genuine dynamic problems of elasticity, useful information on the spatial behavior of solutions can be furnished by the well-known domain of influence theorem as presented by Gurtin [1972] and ascribed to Wheeler and Sternberg [1968]. According to Gurtin the domain of influence of the given external data at time T consists of the set D_T of all points in the body that can be reached by signals propagating from the support $\widehat{D}_T$, on the time interval $[0, T]$, with speeds equal to or less than the maximum speed of propagation c. Then the domain of influence theorem shows that on $[0, T]$ the externally given data have no effect on points outside of D_T.

In the present paper we study a bending state of a rhombic system modelled as a (either bounded or unbounded) Mindlin-type plate, under a strong ellipticity condition on the elasticity tensor. In this context, we use the properties of rhombic systems widely investigated by Gurtin [1972] and Chiriță et al. [2007]. Such systems, together with transversely isotropic materials, show a good applicability to mathematical models of plates that take into account transverse shear deformations (see [Paroni et al. 2006] for details). In particular, in the context of the linear theory, we investigate the spatial behavior of transient and steady-state solutions. This investigation is concerned with some properties of the solutions of the evolution equations ruling the displacement and rotation fields, supposing the plate to be initially at rest. Moreover, time-dependent displacements and rotation fields are imposed on its boundary. These fields on the boundary induce disturbances that propagate in the interior of the plate.

Outline of paper. In Section 2, we derive the evolution equations for the two-dimensional theory and state assumptions concerning the strong ellipticity of the elasticity tensor.

In Section 3, considering the spatial support of data contained in a bounded domain D_T^*, we define a set of appropriate line-integral measures $\mathcal{I}_\kappa$ and establish some estimates in order to obtain the domain of influence and the spatial decay of transient solutions away from D_T^*. Using these estimates and following the time-weighted power function method [Chiriță and Ciarletta 1999; 2003; Ciarletta et al. 2005], we determine the speed of propagation of mechanical disturbances from D_T^*, as well as the spatial decay of solutions.

In Section 4, we treat a more specific mechanical problem, concerning the steady-state vibrations of a (either bounded or semiinfinite) strip. Starting from an idea of Ciarletta et al. [2005] and Chiriță [1995], we introduce a set of appropriate line-integral measures $\mathcal{J}_\kappa$ associated with the amplitude of time-harmonic vibrations. We obtain a differential inequality describing the behavior of steady-state solutions under the hypothesis that the frequency of harmonic vibrations is lower than a certain critical value.

2. Formulation of the problem

In this paper, we study the behavior of a homogeneous body filled by a material that occupies at time $t = 0$ a right cylinder $\bar{B}$ of height $2h$ with (bounded or unbounded) cross-section $\bar{\Sigma}$ and smooth lateral boundary Π. We call B and Σ the interiors of $\bar{B}$ and $\bar{\Sigma}$; we choose the rectangular Cartesian coordinate frame in such a way that Ox_1x_2 is the middle plane of the cylinder and thus its faces are located at $x_3 = \pm h$. We also suppose that Σ is a simply connected region, $\partial\Sigma$ is the boundary of Σ, and $h \ll \operatorname{diam}\Sigma$. Latin subscripts (unless otherwise specified) range over the integers $\{1, 2, 3\}$, whereas Greek subscripts are confined to the range $\{1, 2\}$; summation over repeated subscripts is implied. Superposed dots, or subscripts preceded by a comma, mean a partial derivative with respect to time or to the corresponding Cartesian variables. When needed, vector fields are represented by bold-type letters. We will disregard regularity questions, simply assuming a degree of smoothness sufficient to ensure analysis is valid.

For each point $(x_1, x_2, x_3) \in \bar{B}$ and $t \in [0, \infty)$, a bending state for such an elastic cylinder is characterized by

$$u_\alpha(x_1, x_2, x_3, t) = -u_\alpha(x_1, x_2, -x_3, t) \qquad \text{and} \qquad u_3(x_1, x_2, x_3, t) = u_3(x_1, x_2, -x_3, t),$$

where u_i are the components of the displacement vector that vary smoothly with respect to x_3. Nevertheless, as with the study carried out in [Passarella and Zampoli 2009a; 2009b], we restrict our attention to the bending state characterized by

$$u_\alpha(x_1, x_2, x_3, t) = x_3 v_\alpha(x_1, x_2, t) \qquad \text{and} \qquad u_3(x_1, x_2, x_3, t) = w(x_1, x_2, t),$$

and we assume that the components f_i of the body force vector obey the relations

$$f_\alpha(x_1, x_2, x_3, t) = -f_\alpha(x_1, x_2, -x_3, t), \qquad f_3(x_1, x_2, x_3, t) = f_3(x_1, x_2, -x_3, t).$$

The behavior of thin plates of uniform thickness is described by the equations of motion

$$\tfrac{1}{3}h^2 M_{\beta\alpha,\beta} - \tau_{\alpha 3} + H_\alpha = \tfrac{1}{3}h^2 \varrho \ddot{v}_\alpha, \qquad \tau_{\beta 3,\beta} + F = \varrho \ddot{w}, \qquad \text{on } \bar{\Sigma} \times (0, \infty) \tag{2-1}$$

and the constitutive equations

$$\begin{aligned} M_{ij}(x_1, x_2, t) &= \frac{3}{2h^3} \int_{-h}^{h} x_3 C_{ijkl} u_{k,l}(x_1, x_2, x_3, t)\, dx_3, \\ \tau_{ij}(x_1, x_2, t) &= \frac{1}{2h} \int_{-h}^{h} C_{ijkl} u_{k,l}(x_1, x_2, x_3, t)\, dx_3, \end{aligned} \tag{2-2}$$

where ϱ is the reference mass density, C_{ijkl} are the components of the elasticity tensor obeying the usual symmetry relations, and

$$\begin{aligned} F(x_1, x_2, t) &= \frac{1}{2h} \int_{-h}^{h} f_3(x_1, x_2, x_3, t)\, dx_3 + \frac{1}{h} C_{33kl} u_{k,l}(x_1, x_2, h, t), \\ H_\alpha(x_1, x_2, t) &= \frac{1}{2h} \int_{-h}^{h} x_3 f_\alpha(x_1, x_2, x_3, t)\, dx_3 + C_{3\alpha kl} u_{k,l}(x_1, x_2, h, t). \end{aligned}$$

Suppose that

$$H_\alpha = 0, \qquad F = 0, \qquad \text{on } \bar{\Sigma} \times [0, \infty). \tag{2-3}$$

In what follows, we consider a homogeneous rhombic elastic material, with the group $\mathscr{C}_3$ generated[1] by $\boldsymbol{R}^{\pi}_{\boldsymbol{e}_3}$ and $\boldsymbol{R}^{\pi}_{\boldsymbol{e}_2}$ characterized by (see, for example, [Gurtin 1972])

$$\begin{aligned} C_{1123} = C_{1131} = C_{1112} = C_{2223} = C_{2231} = C_{2212} = 0,\\ C_{3323} = C_{3331} = C_{3312} = C_{2331} = C_{2312} = C_{3112} = 0. \end{aligned} \tag{2-4}$$

Using these equations and defining as usual

$$\begin{aligned} &c_{11} = C_{1111}, \quad c_{22} = C_{2222}, \quad c_{33} = C_{3333}, \quad c_{44} = C_{2323}, \quad c_{55} = C_{1313},\\ &c_{66} = C_{1212}, \quad c_{12} = C_{1122}, \quad c_{23} = C_{2233}, \quad c_{31} = C_{3311}, \end{aligned}$$

the constitutive equations (2-2) for rhombic materials become

$$M_{11} = c_{11}v_{1,1} + c_{12}v_{2,2}, \quad M_{22} = c_{12}v_{1,1} + c_{22}v_{2,2}, \quad M_{12} = M_{21} = c_{66}(v_{1,2} + v_{2,1}), \tag{2-5}$$

$$\tau_{13} = \tau_{31} = c_{55}(w_{,1} + v_1), \quad \tau_{23} = \tau_{32} = c_{44}(w_{,2} + v_2). \tag{2-6}$$

Now, we restrict our attention to the class of rhombic materials having a strongly elliptic elasticity tensor, that is,

$$C_{ijkl}m_i m_k n_j n_l > 0 \tag{2-7}$$

for all nonzero vectors (m_1, m_2, m_3) and (n_1, n_2, n_3). As it is possible to see in [Gurtin 1972], this hypothesis on the elasticity tensor is weaker than positive definiteness. Chiriţă et al. [2007] prove that for rhombic systems the strong ellipticity condition (2-7) is equivalent to

$$\begin{aligned} &c_{11}n_1^2m_1^2 + c_{22}n_2^2m_2^2 + c_{33}n_3^2m_3^2 + c_{66}(n_1m_2 + n_2m_1)^2 + c_{44}(n_3m_2 + n_2m_3)^2\\ &\qquad + c_{55}(n_1m_3 + n_3m_1)^2 + 2c_{12}n_1m_1n_2m_2 + 2c_{23}n_2m_2n_3m_3 + 2c_{31}n_3m_3n_1m_1 > 0, \end{aligned} \tag{2-8}$$

and this relation implies the conditions

$$\begin{aligned} &c_{11} > 0, \quad c_{22} > 0, \quad c_{33} > 0, \quad c_{44} > 0, \quad c_{55} > 0, \quad c_{66} > 0,\\ &|c_{12} + c_{66}| < c_{66} + \sqrt{c_{11}c_{22}}, \quad |c_{13} + c_{55}| < c_{55} + \sqrt{c_{11}c_{33}}, \quad |c_{23} + c_{44}| < c_{44} + \sqrt{c_{22}c_{33}}. \end{aligned}$$

3. Transient solutions: decay estimate of Saint-Venant type

In this section, we establish results describing the spatial behavior of solutions of the given data on the interval $[0, T]$ under the strong ellipticity condition on the elasticity tensor. To this end, we can rewrite the system of (2-1), (2-3), (2-5), and (2-6) in terms of (v_α, w) as

$$\begin{aligned} \tfrac{1}{3}h^2[c_{11}v_{1,11} + c_{66}v_{1,22} + (c_{12} + c_{66})v_{2,12}] - c_{55}(w_{,1} + v_1) = \tfrac{1}{3}h^2\varrho\ddot{v}_1,\\ \tfrac{1}{3}h^2[c_{66}v_{2,11} + c_{22}v_{2,22} + (c_{12} + c_{66})v_{1,12}] - c_{44}(w_{,2} + v_2) = \tfrac{1}{3}h^2\varrho\ddot{v}_2,\\ c_{55}(w_{,11} + v_{1,1}) + c_{44}(w_{,22} + v_{2,2}) = \varrho\ddot{w}, \end{aligned} \tag{3-1}$$

or, equivalently, as

$$\tfrac{1}{3}h^2\widetilde{M}_{\beta\alpha,\beta} - \tau_{\alpha 3} = \tfrac{1}{3}\varrho h^2\ddot{v}_\alpha, \qquad \tau_{\beta 3,\beta} = \varrho\ddot{w}, \tag{3-2}$$

[1] $\boldsymbol{R}^{\theta}_{\boldsymbol{u}}$ is the orthogonal tensor corresponding to a right-handed rotation through the angle θ about an axis of unit vector $\boldsymbol{u}$.

where $\widetilde{M}_{\beta\alpha}$ represents the family of tensors depending on the parameter κ through

$$\begin{aligned} \widetilde{M}_{11} &= c_{11}v_{1,1} + (c_{66}-\kappa)v_{2,2}, & \widetilde{M}_{12} &= (c_{12}+\kappa)v_{1,2} + c_{66}v_{2,1}, \\ \widetilde{M}_{21} &= c_{66}v_{1,2} + (c_{12}+\kappa)v_{2,1}, & \widetilde{M}_{22} &= (c_{66}-\kappa)v_{1,1} + c_{22}v_{2,2}, \end{aligned} \tag{3-3}$$

and $\tau_{\beta 3}$ is defined by Equation (2-6). The family of tensors $\widetilde{M}_{\beta\alpha}$ has a nonvanishing skew-symmetric part, depending on the skew-symmetric part of $v_{\alpha,\beta}$.

Let $\mathcal{P}$ be the problem defined by (2-6), (3-2), (3-3) and the initial-boundary conditions

$$\begin{aligned} v_\alpha(x_1, x_2, 0) &= 0, & w(x_1, x_2, 0) &= 0 && \text{on } \Sigma, \\ \dot{v}_\alpha(x_1, x_2, 0) &= 0, & \dot{w}(x_1, x_2, 0) &= 0 && \text{on } \Sigma, \\ v_\alpha &= \breve{v}_\alpha, & w &= \breve{w} && \text{on } L \times [0, +\infty), \\ v_\alpha &= 0, & w &= 0 && \text{on } L_C \times [0, +\infty), \end{aligned}$$

where L and L_C are disjoint and complementary nonempty subsets of $\partial\Sigma$ and $\mathcal{D} = (\breve{v}_\alpha, \breve{w})$ is the set of assigned fields representing the external data of the problem in question.

We define the support $\widehat{D}_T$ of external data $\mathcal{D}$ on the time interval $[0, T]$ as the set of points

$$\boldsymbol{x} = (x_1, x_2) \in L$$

for which there exists $\tau \in [0, T]$ such that $\breve{v}_\alpha(\boldsymbol{x}, \tau) \neq 0$ or $\breve{w}(\boldsymbol{x}, \tau) \neq 0$. In other words, for $\boldsymbol{x} \in \partial\Sigma \setminus \widehat{D}_T$, we have

$$\breve{v}_\alpha(\boldsymbol{x}, t) = 0 \quad \text{and} \quad \breve{w}(\boldsymbol{x}, t) = 0 \qquad \text{for all } t \in [0, T]. \tag{3-4}$$

For convenience, we assume that $\widehat{D}_T$ is a nonempty bounded set and that D_T^* is the smallest regular subcurve of L including $\widehat{D}_T$. We consider the sets

$$D_r = \big\{\boldsymbol{x} \in \overline{\Sigma} : D_T^* \cap S(\boldsymbol{x}, r) \neq \varnothing\big\}, \qquad \Sigma_r = \overline{\Sigma} \setminus D_r, \qquad \Sigma(r_1, r_2) = \Sigma_{r_2} \setminus \Sigma_{r_1},$$

where $r > 0$, $r_2 \leq r_1$, and $S(\boldsymbol{x}, r)$ is the closed ball with radius r and center at $\boldsymbol{x}$, that is,

$$S(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^2 : |\boldsymbol{y} - \boldsymbol{x}| \leq r\}.$$

Then, $L_r = \partial\Sigma_r \cap \Sigma$ is the subcurve of $\partial\Sigma_r$ that lies inside the inner part of $\overline{\Sigma}$, and whose unit normal vector $\boldsymbol{n}$ is directed towards the interior of Σ_r (or, equivalently, towards the exterior of D_r).

We also agree that, for $r = 0$, D_0 and L_0 coincide with D_T^*, and the normal vector $\boldsymbol{n}$ to L_0 is directed towards the interior of Σ.

For any positive parameter λ, we introduce the function

$$\mathcal{I}_\kappa(r, t) = -\int_0^t \int_{L_r} e^{-\lambda s}\big[\tau_{\beta 3}(s)\dot{w}(s) + \tfrac{1}{3}h^2 \widetilde{M}_{\beta\alpha}(s)\dot{v}_\alpha(s)\big]n_\beta \, dl \, ds, \quad r \geq 0, \quad t \in [0, T]; \tag{3-5}$$

for what follows, it is useful to remark that

$$\mathcal{I}_\kappa(r, 0) = 0 \quad \text{for all } r \geq 0. \tag{3-6}$$

From direct differentiation of (3-5) with respect to the variable t, we get

$$\frac{\partial \mathcal{I}_\kappa}{\partial t}(r,t) = -\int_{L_r} e^{-\lambda t}\big[\tau_{\beta 3}(t)\dot{w}(t) + \tfrac{1}{3}h^2\widetilde{M}_{\beta\alpha}(t)\dot{\upsilon}_\alpha(t)\big]n_\beta\, dl. \tag{3-7}$$

On the other hand, from the definition of Σ_r, from (3-4), and from the divergence theorem, we have

$$\mathcal{I}_\kappa(r_1,t) - \mathcal{I}_\kappa(r_2,t) = -\int_0^t\int_{\Sigma(r_1,r_2)} e^{-\lambda s}\big[\tau_{\beta 3}(s)\dot{w}(s) + \tfrac{1}{3}h^2\widetilde{M}_{\beta\alpha}(s)\dot{\upsilon}_\alpha(s)\big]_{,\beta}\, d\sigma\, ds.$$

Using (2-6) and (3-2)–(3-5), we obtain, for $r_2 < r_1$,

$$\begin{aligned}
&\mathcal{I}_\kappa(r_1,t)-\mathcal{I}_\kappa(r_2,t)\\
&\quad=-\frac{1}{2}\int_0^t\int_{\Sigma(r_1,r_2)} e^{-\lambda s}\frac{\partial}{\partial s}\Big\{\varrho\big[\dot{w}^2(s)+\tfrac{1}{3}h^2\dot{\upsilon}_\alpha(s)\dot{\upsilon}_\alpha(s)\big]+c_{55}\big(w_{,1}(s)+\upsilon_1(s)\big)^2+c_{44}\big(w_{,2}(s)+\upsilon_2(s)\big)^2\\
&\qquad\qquad+\tfrac{1}{3}h^2\big[c_{11}\upsilon_{1,1}^2(s)+c_{22}\upsilon_{2,2}^2(s)+2(c_{66}-\kappa)\upsilon_{1,1}(s)\upsilon_{2,2}(s)\\
&\qquad\qquad\quad+c_{66}\big(\upsilon_{2,1}^2(s)+\upsilon_{1,2}^2(s)\big)+2(c_{12}+\kappa)\upsilon_{2,1}(s)\upsilon_{1,2}(s)\big]\Big\}\, d\sigma\, ds.
\end{aligned} \tag{3-8}$$

Introducing

$$\mathcal{T} = \tfrac{1}{2}\varrho\big(\dot{w}^2 + \tfrac{1}{3}h^2\dot{\upsilon}_\alpha\dot{\upsilon}_\alpha\big), \qquad \mathcal{W} = \tfrac{1}{2}\big(\mathcal{W}_0 + \tfrac{1}{3}h^2\mathcal{W}_1 + \tfrac{1}{3}h^2\mathcal{W}_2\big), \tag{3-9}$$

and

$$\begin{aligned}
&\mathcal{W}_0 = c_{55}(w_{,1}+\upsilon_1)^2 + c_{44}(w_{,2}+\upsilon_2)^2, \qquad \mathcal{W}_1 = c_{11}\upsilon_{1,1}^2 + c_{22}\upsilon_{2,2}^2 + 2(c_{66}-\kappa)\upsilon_{1,1}\upsilon_{2,2},\\
&\mathcal{W}_2 = c_{66}(\upsilon_{2,1}^2+\upsilon_{1,2}^2) + 2(c_{12}+\kappa)\upsilon_{2,1}\upsilon_{1,2},
\end{aligned} \tag{3-10}$$

it follows from (3-8) that

$$\mathcal{I}_\kappa(r_1,t) - \mathcal{I}_\kappa(r_2,t) = -\int_0^t\int_{\Sigma(r_1,r_2)} e^{-\lambda s}\frac{\partial}{\partial s}[\mathcal{T}(s)+\mathcal{W}(s)]\, d\sigma\, ds \qquad \text{if } r_2 < r_1. \tag{3-11}$$

Now, it is possible to show that $\mathcal{I}_\kappa$ is continuously differentiable with respect to r; in fact, considering

$$\begin{aligned}
\lim_{r_1\to r_2}\frac{\mathcal{I}_\kappa(r_1,t)-\mathcal{I}_\kappa(r_2,t)}{r_1-r_2} &= -\lim_{r_1\to r_2}\frac{1}{r_1-r_2}\int_0^t\int_{\Sigma(r_1,r_2)} e^{-\lambda s}\frac{\partial}{\partial s}[\mathcal{T}(s)+\mathcal{W}(s)]\, d\sigma\, ds\\
&= -\int_0^t\int_{L_r} e^{-\lambda s}\frac{\partial}{\partial s}[\mathcal{T}(s)+\mathcal{W}(s)]\, d\sigma\, ds,
\end{aligned}$$

and integrating by parts with respect to the time variable s, we arrive at

$$\frac{\partial \mathcal{I}_\kappa}{\partial r}(r,t) = -\int_{L_r} e^{-\lambda t}[\mathcal{T}(t)+\mathcal{W}(t)]\, dl - \lambda\int_0^t\int_{L_r} e^{-\lambda s}[\mathcal{T}(s)+\mathcal{W}(s)]\, dl\, ds. \tag{3-12}$$

Let $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be the matrices associated with the quadratic forms $\mathcal{W}_0$, $\mathcal{W}_1$, and $\mathcal{W}_2$, respectively. When the strong ellipticity condition (2-8) is valid and κ satisfies the relation

$$\max\big\{-c_{66}-c_{12},\, c_{66}-\sqrt{c_{11}c_{22}}\big\} < \kappa < \min\big\{c_{66}-c_{12},\, c_{66}+\sqrt{c_{11}c_{22}}\big\}, \tag{3-13}$$

the eigenvalues of $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$, which are

$$c_{55}, \quad c_{44}, \quad \tfrac{1}{2}\big[c_{11}+c_{22} \pm \sqrt{(c_{11}-c_{22})^2+4(c_{66}-\kappa)^2}\,\big], \quad c_{66} \pm |c_{12}+\kappa|,$$

are all strictly positive. In what follows, we denote with μ_κ and η_κ the smallest and largest eigenvalues of $\mathscr{A}_0$, $\mathscr{A}_1$, and $\mathscr{A}_2$, respectively. These are

$$\mu_\kappa = \min\Big\{c_{44}, c_{55}, \tfrac{1}{2}\big[c_{11}+c_{22}-\sqrt{(c_{11}-c_{22})^2+4(c_{66}-\kappa)^2}\big], c_{66}-|c_{12}+\kappa|\Big\},$$
$$\eta_\kappa = \max\Big\{c_{44}, c_{55}, \tfrac{1}{2}\big[c_{11}+c_{22}+\sqrt{(c_{11}-c_{22})^2+4(c_{66}-\kappa)^2}\big], c_{66}+|c_{12}+\kappa|\Big\}.$$

If $\varrho > 0$ and (2-8) and (3-13) hold, then $\mathscr{T}$ and $\mathscr{W}$ are positive definite; thus, we show through (3-12) that $\mathscr{I}_\kappa(r,t)$ is a nonincreasing function with respect to r, that is,

$$\mathscr{I}_\kappa(r_1,t) \le \mathscr{I}_\kappa(r_2,t) \qquad \text{if } r_2 < r_1. \tag{3-14}$$

If the plate is bounded and $\ell = \max_{\boldsymbol{x}\in\bar{\Sigma}}\big\{\min_{\boldsymbol{y}\in D_T^*}\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\big\}$, we obtain from the definition of L_r that $L_\ell = \varnothing$ and, consequently,

$$\mathscr{I}_\kappa(\ell,t) = 0, \qquad t\in[0,T]; \tag{3-15}$$

moreover, (3-11), (3-14), and (3-15) imply that

$$0 \le \mathscr{I}_\kappa(r,t) = \int_0^t \int_{\Sigma(\ell,r)} e^{-\lambda s}\frac{\partial}{\partial s}[\mathscr{T}(s)+\mathscr{W}(s)]\,d\sigma\,ds \quad \text{for } r\le\ell. \tag{3-16}$$

In this case, it is obvious that $\Sigma(\ell,r) = \Sigma_r$. It is possible to obtain a relation similar to (3-16) for an unbounded plate. To this end, we estimate $\mathscr{I}_\kappa(r,t)$ and $\partial\mathscr{I}_\kappa(r,t)/\partial t$ in terms of $\partial\mathscr{I}_\kappa(r,t)/\partial r$.

Theorem 3.1. *Let $\varrho > 0$ and suppose the hypotheses* (2-8) *and* (3-13) *hold. Let (v_α, w) be a solution of the initial-boundary value problem $\mathscr{P}$ and $\widehat{D}_T$ the bounded support of external data $\mathscr{D}$ on the time interval $[0,T]$, then the function $\mathscr{I}_\kappa$ satisfies the first-order differential inequalities*

$$|\mathscr{I}_\kappa(r,t)| + \frac{c_\kappa}{\lambda}\frac{\partial\mathscr{I}_\kappa}{\partial r}(r,t) \le 0, \qquad \Big|\frac{\partial\mathscr{I}_\kappa}{\partial t}(r,t)\Big| + c_\kappa\frac{\partial\mathscr{I}_\kappa}{\partial r}(r,t) \le 0, \qquad \textit{with } c_\kappa = \sqrt{\frac{\eta_\kappa}{\varrho}}. \tag{3-17}$$

Proof. Under the hypotheses (2-8) and (3-13), we can observe that

$$0 \le k_m^{(\alpha)}(\gamma_1^2+\gamma_2^2) \le \mathscr{F}[\mathscr{A}_\alpha;\boldsymbol{\gamma},\boldsymbol{\gamma}] \le k_M^{(\alpha)}(\gamma_1^2+\gamma_2^2), \tag{3-18}$$

where $k_m^{(\alpha)}$ and $k_M^{(\alpha)}$ are the smallest and largest eigenvalues of $\mathscr{W}_\alpha$, the functional $\mathscr{F}$ is given by

$$\mathscr{F}[\mathscr{A}_\alpha;\boldsymbol{\varphi},\boldsymbol{\gamma}] := \boldsymbol{\varphi}\cdot\mathscr{A}_\alpha\boldsymbol{\gamma},$$

with the variables $\boldsymbol{\varphi} = (\varphi_1,\varphi_2)$ and $\boldsymbol{\gamma} = (\gamma_1,\gamma_2)$. In particular,

$$k_m^{(1)} = \tfrac{1}{2}\big[c_{11}+c_{22}-\sqrt{(c_{11}-c_{22})^2+4(c_{66}-\kappa)^2}\,\big], \quad k_m^{(2)} = c_{66}-|c_{12}+\kappa|,$$
$$k_M^{(1)} = \tfrac{1}{2}\big[c_{11}+c_{22}+\sqrt{(c_{11}-c_{22})^2+4(c_{66}-\kappa)^2}\,\big], \quad k_M^{(2)} = c_{66}+|c_{12}+\kappa|.$$

Through the Schwarz inequality and Equations (3-3) and (3-10), we get

$$\mathscr{F}[\mathscr{A}_q;\boldsymbol{\varphi},\boldsymbol{\gamma}] \le \big[\mathscr{F}[\mathscr{A}_q;\boldsymbol{\varphi},\boldsymbol{\varphi}]\big]^{1/2}\big[\mathscr{F}[\mathscr{A}_q;\boldsymbol{\gamma},\boldsymbol{\gamma}]\big]^{1/2} \tag{3-19}$$

and

$$\begin{aligned} &\mathcal{F}\big[\mathcal{A}_1;\, \hat{\boldsymbol{\gamma}}^{(1)}, \hat{\boldsymbol{\gamma}}^{(1)}\big] = {}^{\circ}\mathcal{W}_1, \qquad && \mathcal{F}\big[\mathcal{A}_1;\, \widetilde{\boldsymbol{M}}^{(1)}, \hat{\boldsymbol{\gamma}}^{(1)}\big] = \widetilde{M}_{11}^2 + \widetilde{M}_{22}^2, \\ &\mathcal{F}\big[\mathcal{A}_2;\, \hat{\boldsymbol{\gamma}}^{(2)}, \hat{\boldsymbol{\gamma}}^{(2)}\big] = {}^{\circ}\mathcal{W}_2, && \mathcal{F}\big[\mathcal{A}_2;\, \widetilde{\boldsymbol{M}}^{(2)}, \hat{\boldsymbol{\gamma}}^{(2)}\big] = \widetilde{M}_{21}^2 + \widetilde{M}_{12}^2, \end{aligned} \tag{3-20}$$

where

$$\hat{\boldsymbol{\gamma}}^{(1)} = (v_{1,1}, v_{2,2}), \quad \widetilde{\boldsymbol{M}}^{(1)} = (\widetilde{M}_{11}, \widetilde{M}_{22}), \quad \hat{\boldsymbol{\gamma}}^{(2)} = (v_{1,2}, v_{2,1}), \quad \widetilde{\boldsymbol{M}}^{(2)} = (\widetilde{M}_{21}, \widetilde{M}_{12}).$$

Now, collecting (3-10) and (3-18)–(3-20), we deduce that

$$\begin{aligned} \widetilde{M}_{11}^2 + \widetilde{M}_{22}^2 &\le \mathcal{F}^{1/2}\big[\mathcal{A}_1;\, \widetilde{\boldsymbol{M}}^{(1)}, \widetilde{\boldsymbol{M}}^{(1)}\big]{}^{\circ}\mathcal{W}_1^{1/2} \le \big[k_M^{(1)}(\widetilde{M}_{11}^2 + \widetilde{M}_{22}^2)\big]^{1/2}{}^{\circ}\mathcal{W}_1^{1/2}, \\ \widetilde{M}_{21}^2 + \widetilde{M}_{12}^2 &\le \mathcal{F}^{1/2}\big[\mathcal{A}_2;\, \widetilde{\boldsymbol{M}}^{(2)}, \widetilde{\boldsymbol{M}}^{(2)}\big]{}^{\circ}\mathcal{W}_2^{1/2} \le \big[k_M^{(2)}(\widetilde{M}_{21}^2 + \widetilde{M}_{12}^2)\big]^{1/2}{}^{\circ}\mathcal{W}_2^{1/2}. \end{aligned} \tag{3-21}$$

On the other hand, from (2-6) and (3-10) it follows that

$$0 \le k_m^{(0)}{}^{\circ}\mathcal{W}_0 \le \tau_{\beta 3}\tau_{\beta 3} \le k_M^{(0)}{}^{\circ}\mathcal{W}_0, \tag{3-22}$$

where $k_m^{(0)} = \min\{c_{44}, c_{55}\}$ and $k_M^{(0)} = \max\{c_{44}, c_{55}\}$. Equations (3-9), (3-21), and (3-22) allow us to show easily that

$$\tau_{\beta 3}\tau_{\beta 3} + \tfrac{1}{3}h^2\widetilde{M}_{\beta\alpha}\widetilde{M}_{\beta\alpha} \le k_M^{(0)}{}^{\circ}\mathcal{W}_0 + \tfrac{1}{3}h^2 k_M^{(1)}{}^{\circ}\mathcal{W}_1 + \tfrac{1}{3}h^2 k_M^{(2)}{}^{\circ}\mathcal{W}_2 \le 2\eta_\kappa{}^{\circ}\mathcal{W}. \tag{3-23}$$

Next, the Cauchy–Schwarz and arithmetic-geometric mean inequalities lead to

$$\Big|\big(\tau_{\beta 3}\dot{w} + \tfrac{1}{3}h^2\widetilde{M}_{\beta\alpha}\dot{v}_\alpha\big)n_\beta\Big| \le \frac{\varepsilon\varrho}{2}\big(\dot{w}^2 + \tfrac{1}{3}h^2\dot{v}_\alpha\dot{v}_\alpha\big) + \frac{1}{2\varepsilon\varrho}\big(\tau_{\beta 3}\tau_{\beta 3} + \tfrac{1}{3}h^2\widetilde{M}_{\beta\alpha}\widetilde{M}_{\beta\alpha}\big). \tag{3-24}$$

Finally, we use the estimates (3-23) and (3-24) in (3-5) and (3-7) in order to obtain

$$|\mathcal{I}_\kappa(r,t)| \le \varepsilon\int_0^t\int_{L_r} e^{-\lambda s}\Big[\mathcal{T}(s) + \frac{\eta_\kappa}{\varepsilon^2\varrho}{}^{\circ}\mathcal{W}(s)\Big]\,dl\,ds, \tag{3-25}$$

$$\left|\frac{\partial\mathcal{I}_\kappa}{\partial t}(r,t)\right| \le \varepsilon\int_{L_r} e^{-\lambda t}\Big[\mathcal{T}(t) + \frac{\eta_\kappa}{\varepsilon^2\varrho}{}^{\circ}\mathcal{W}(t)\Big]\,dl. \tag{3-26}$$

Recalling (3-12) and setting $\varepsilon = c_\kappa$ in (3-25) and (3-26), we derive the differential inequalities (3-17). □

Using Theorem 3.1, a result similar to (3-16) may be shown for an unbounded plate. In this case, the variable r ranges in $[0, \infty)$. Thanks to $(3\text{-}17)_2$ and for any pair (r_0, t_0) such that $t_0 \in [0, T]$ and $r_0 \ge c_\kappa t_0$, we can see that the functions $\mathcal{I}_\kappa(r,\ t_0 + (r - r_0)/c_\kappa)$ and $\mathcal{I}_\kappa(r,\ t_0 - (r - r_0)/c_\kappa)$ are nonincreasing with respect to r. This feature, together with (3-6), implies

$$\begin{aligned} &\mathcal{I}_\kappa(r_0, t_0) \le \mathcal{I}_\kappa(r_0 - c_\kappa t_0, 0) = 0, \qquad && r_0 \ge r_0 - c_\kappa t_0, \\ &\mathcal{I}_\kappa(r_0, t_0) \ge \mathcal{I}_\kappa(r_0 + c_\kappa t_0, 0) = 0, && r_0 + c_\kappa t_0 \ge r_0, \end{aligned} \quad \Longrightarrow \quad \lim_{r_0\to\infty}\mathcal{I}_\kappa(r_0, t_0) = 0. \tag{3-27}$$

Taking into account (3-11), (3-14), and (3-27), it follows that

$$0 \le \mathcal{I}_\kappa(r,t) = \lim_{r_0\to\infty}\int_0^t\int_{\Sigma(r_0,r)} e^{-\lambda s}\frac{\partial}{\partial s}[\mathcal{T}(s) + {}^{\circ}\mathcal{W}(s)]\,d\sigma\,ds, \tag{3-28}$$

and it is obvious that $\lim_{r_0\to\infty}\Sigma(r_0, r) = \Sigma_r$.

With the help of (3-16) and (3-28), and integrating by parts with respect to s, it is possible to prove, for bounded or unbounded plates, the following theorem.

Theorem 3.2. *Let the hypotheses of Theorem 3.1 be still valid. The function $\mathcal{I}_\kappa(r,t)$ is nonnegative and*

$$0 \le \mathcal{I}_\kappa(r,t) = \int_{\Sigma_r} e^{-\lambda t}[\mathcal{T}(t)+\mathcal{W}(t)]\,d\sigma + \lambda\int_0^t\int_{\Sigma_r} e^{-\lambda s}[\mathcal{T}(s)+\mathcal{W}(s)]\,d\sigma\,ds. \tag{3-29}$$

Now, since $\mathcal{I}_\kappa(r,t)$ is a nonnegative function, we can write from $(3\text{-}17)_1$

$$\frac{\partial}{\partial r}\Big[\exp\Big(\frac{\lambda}{c_\kappa}r\Big)\mathcal{I}_\kappa(r,t)\Big] \le 0 \quad\Longrightarrow\quad \mathcal{I}_\kappa(r,t) \le \exp\Big(-\frac{\lambda}{c_\kappa}r\Big)\mathcal{I}_\kappa(0,t).$$

On the other hand, $(3\text{-}17)_2$ implies that the function $\mathcal{I}_\kappa(c_\kappa t,t)$ is nonincreasing with respect to t. This characteristic, together with (3-6), implies

$$\mathcal{I}_\kappa(c_\kappa t,t) \le \mathcal{I}_\kappa(0,0) = 0, \qquad t \ge 0. \tag{3-30}$$

We deduce from (3-14) and (3-29) that

$$0 \le \mathcal{I}_\kappa(r,t) \le \mathcal{I}_\kappa(c_\kappa t,t), \qquad r \ge c_\kappa t. \tag{3-31}$$

Thus, (3-29)–(3-31) imply

$$\int_{\Sigma_r} e^{-\lambda t}[\mathcal{T}(t)+\mathcal{W}(t)]\,d\sigma + \lambda\int_0^t\int_{\Sigma_r} e^{-\lambda s}[\mathcal{T}(s)+\mathcal{W}(s)]\,d\sigma\,ds = 0 \quad \text{for} \quad r \ge c_\kappa t. \tag{3-32}$$

When $\mathcal{T}$ and $\mathcal{W}$ are positive definite, (3-32) is valid if and only if $\mathcal{T}$ and $\mathcal{W}$ are null in Σ_r for any $r \ge c_\kappa t$.

The results obtained up to now lead us to formulate the following theorem about the spatial decay.

Theorem 3.3. *Let the hypotheses of Theorem 3.1 be still valid; for each fixed $t \in [0,T]$, we have*

$$\mathcal{I}_\kappa(r,t) \le \exp\Big(-\frac{\lambda}{c_\kappa}r\Big)\mathcal{I}_\kappa(0,t) \quad \textit{for} \quad 0 \le r < c_\kappa t, \qquad \mathcal{I}_\kappa(r,t) = 0 \quad \textit{for} \quad r \ge c_\kappa t. \tag{3-33}$$

It is easy to observe that, for $t \in [0,T]$ and for each $r \ge c_\kappa t$, (3-29) and $(3\text{-}33)_2$ lead to

$$\int_{\Sigma_r} e^{-\lambda t}[\mathcal{T}(t)+\mathcal{W}(t)]\,d\sigma + \lambda\int_0^t\int_{\Sigma_r} e^{-\lambda s}[\mathcal{T}(s)+\mathcal{W}(s)]\,d\sigma\,ds = 0.$$

Consequently, when $\varrho > 0$ and $\mathcal{W}$ is positive definite, we obtain $\dot{v}_\alpha = \dot{w} = 0$ on $\Sigma_r \times [0,T]$, and through homogeneous initial conditions, we get

$$v_\alpha = w = 0 \qquad \text{on } \Sigma_r \times [0,T]. \tag{3-34}$$

Following [Gurtin 1972], we depict the domain of influence of the externally given data at time T as the set of the points of $\bar{\Sigma}$ that can be reached by signals propagating from the support $\hat{D}_T$ on the time interval $[0,T]$, with speeds equal to or less than the maximum speed of propagation

$$c = \sqrt{\frac{\eta}{\varrho}} \qquad \text{with} \quad \eta = \inf_\kappa \eta_\kappa,$$

where κ satisfies the relation (3-13). In fact, for a bounded or unbounded plate, we can show by means of (3-34) that on $[0,T]$ the externally given data have no effect on points outside of D_{cT}.

Lemma 3.4 (domain of influence). *Let the hypotheses of Theorem 3.1 be still valid and let (v_α, w) be a solution of initial-boundary value problem $\mathcal{P}$. Then, $v_\alpha = w = 0$, on $\Sigma_{cT} \times [0,T]$.*

We can easily prove the following uniqueness result, valid for a bounded or unbounded plate.

Lemma 3.5 (uniqueness). *Let the hypotheses of Theorem 3.1 hold. There exists at most one solution for the initial-boundary value problem $\mathcal{P}$.*

Proof. Thanks to the linearity of the problem, we have only to show that the null data imply a null solution. Let $(\tilde{v}_\alpha, \tilde{w})$ be a solution corresponding to the null data. If we choose $T_1 > 0$, since the set $\hat{D}_T = \varnothing$ for each $T \in (0, T_1)$ and the function $\mathcal{F}_\kappa(r,t) = 0$, we can conclude that: $v_\alpha = w = 0$, on $\bar{\Sigma} \times [0, T_1]$. □

4. Steady-state solutions: decay estimate of Saint-Venant type

Throughout this section, the cross-section $\overline{\Sigma}$ is a rectangular strip, and the problem of steady-state vibrations is studied assuming that (v_α, w) are separable with respect to space and time variables and that the time dependence is periodic, that is $v_\alpha = \mathrm{Re}[\zeta_\alpha(x_1, x_2)e^{i\omega t}]$, $w = \mathrm{Re}[\psi(x_1, x_2)e^{i\omega t}]$, where $\mathrm{Re}[f]$ represents the real part of f, $\omega \in \mathbb{R}^+$ is the prescribed frequency of oscillations, and ζ_α and ψ are complex functions. We choose a Cartesian frame of reference such that the middle section of the plate is defined by $\overline{\Sigma} = [0, \ell_1] \times [0, \ell_2] \subset \mathbb{R}^2$, where ℓ_1 and ℓ_2 are some positive constants and ℓ_1 can also tend to infinity. Moreover, we impose prescribed harmonic vibrations on the end of the strip located at $x_1 = 0$.

Using the equations of motion (3-1), we conclude that the amplitude (ζ_α, ψ) satisfies

$$\begin{aligned}
\tfrac{1}{3}h^2[c_{11}\zeta_{1,11} + c_{66}\zeta_{1,22} + (c_{12} + c_{66})\zeta_{2,12}] - c_{55}(\psi_{,1} + \zeta_1) + \varrho\tfrac{1}{3}h^2\omega^2\zeta_1 = 0,\\
\tfrac{1}{3}h^2[c_{66}\zeta_{2,11} + c_{22}\zeta_{2,22} + (c_{12} + c_{66})\zeta_{1,12}] - c_{44}(\psi_{,2} + v_2) + \varrho\tfrac{1}{3}h^2\omega^2\zeta_2 = 0,\\
c_{55}(\psi_{,11} + \zeta_{1,1}) + c_{44}(\psi_{,22} + \zeta_{2,2}) + \varrho\omega^2\psi = 0,
\end{aligned}$$

or, equivalently, it satisfies the system

$$\tfrac{1}{3}h^2\Gamma_{\beta\alpha,\beta} - \chi_\alpha + \varrho\tfrac{1}{3}h^2\omega^2\zeta_\alpha = 0, \qquad \chi_{\beta,\beta} + \varrho\omega^2\psi = 0, \tag{4-1}$$

with

$$\begin{aligned}
\Gamma_{11} &= c_{11}\zeta_{1,1} + (c_{66} - \kappa)\zeta_{2,2}, & \Gamma_{12} &= (c_{12} + \kappa)\zeta_{1,2} + c_{66}\zeta_{2,1},\\
\Gamma_{21} &= c_{66}\zeta_{1,2} + (c_{12} + \kappa)\zeta_{2,1}, & \Gamma_{22} &= (c_{66} - \kappa)\zeta_{1,1} + c_{22}\zeta_{2,2},
\end{aligned} \tag{4-2}$$

and

$$\chi_1 = c_{55}(\psi_{,1} + \zeta_1), \qquad \chi_2 = c_{44}(\psi_{,2} + \zeta_2). \tag{4-3}$$

In particular, using (4-1) we can easily show that

$$\tfrac{1}{3}h^2\Gamma_{1\alpha,1} = -\tfrac{1}{3}h^2\Gamma_{2\alpha,2} + \chi_\alpha - \varrho\tfrac{1}{3}h^2\omega^2\zeta_\alpha, \qquad \chi_{1,1} = -\chi_{2,2} - \varrho\omega^2\psi. \tag{4-4}$$

Let $\mathcal{P}_0$ be the problem defined by (4-1)–(4-3) and by the boundary conditions

$$\begin{aligned}
&\zeta_\alpha(x_1, 0) = 0, \quad \psi(x_1, 0) = 0, \quad \zeta_\alpha(x_1, \ell_2) = 0, \quad \psi(x_1, \ell_2) = 0, && x_1 \in [0, \ell_1],\\
&\zeta_\alpha(\ell_1, x_2) = 0, \quad \psi(\ell_1, x_2) = 0, \quad \zeta_\alpha(0, x_2) = \breve{\zeta}_\alpha(x_2), \quad \psi(0, x_2) = \breve{\psi}(x_2), && x_2 \in [0, \ell_2],
\end{aligned} \tag{4-5}$$

where $\breve{\zeta}_\alpha$ and $\breve{\psi}$ are prescribed continuous functions such that

$$\breve{\zeta}_\alpha(0) = 0, \qquad \breve{\psi}(0) = 0, \qquad \breve{\zeta}_\alpha(\ell_2) = 0, \qquad \breve{\psi}(\ell_2) = 0.$$

Using a superposed bar for complex conjugation and defining the segment $L_{x_1}=\{(x_1,x_2): x_2\in[0,\ell_2]\}$, we introduce the function

$$\mathcal{J}_\kappa(x_1)=\int_{L_{x_1}} 2\,\mathrm{Re}\big[\overline{\chi}_1\psi+\tfrac{1}{3}h^2\overline{\Gamma}_{1\alpha}\zeta_\alpha\big]dx_2,\qquad x_1\in[0,\ell_1]. \tag{4-6}$$

Differentiating $\mathcal{J}_\kappa$, we obtain

$$\mathcal{J}'_\kappa(x_1)=\int_{L_{x_1}} 2\,\mathrm{Re}\big[\overline{\chi}_{1,1}\psi+\overline{\chi}_1\psi_{,1}+\tfrac{1}{3}h^2(\overline{\Gamma}_{1\alpha,1}\zeta_\alpha+\overline{\Gamma}_{1\alpha}\zeta_{\alpha,1})\big]dx_2. \tag{4-7}$$

Using integration by parts, boundary conditions (4-5), and Equations (4-3), (4-4), and (4-7), we arrive at

$$\mathcal{J}'_\kappa(x_1)=\int_{L_{x_1}} 2\big[\widehat{\mathcal{W}}-\varrho\omega^2\big(\bar{\psi}\psi+\tfrac{1}{3}h^2\bar{\zeta}_\alpha\zeta_\alpha\big)\big]dx_2, \tag{4-8}$$

where

$$\begin{aligned}
\widehat{\mathcal{W}}&=\widehat{\mathcal{W}}_0+\tfrac{1}{3}h^2\widehat{\mathcal{W}}_1+\tfrac{1}{3}h^2\widehat{\mathcal{W}}_2,\\
\widehat{\mathcal{W}}_0&=c_{55}(\bar{\zeta}_1+\bar{\psi}_{,1})(\zeta_1+\psi_{,1})+c_{44}(\bar{\zeta}_2+\bar{\psi}_{,2})(\zeta_2+\psi_{,2}),\\
\widehat{\mathcal{W}}_1&=c_{11}\bar{\zeta}_{1,1}\zeta_{1,1}+c_{22}\bar{\zeta}_{2,2}\zeta_{2,2}+(c_{66}-\kappa)(\bar{\zeta}_{1,1}\zeta_{2,2}+\zeta_{1,1}\bar{\zeta}_{2,2}),\\
\widehat{\mathcal{W}}_2&=c_{66}(\zeta_{2,1}\bar{\zeta}_{2,1}+\zeta_{1,2}\bar{\zeta}_{1,2})+(c_{12}+\kappa)(\bar{\zeta}_{2,1}\zeta_{1,2}+\zeta_{2,1}\bar{\zeta}_{1,2}).
\end{aligned}$$

Again, as in the previous section, the matrices associated with $\widehat{\mathcal{W}}_0$, $\widehat{\mathcal{W}}_1$, and $\widehat{\mathcal{W}}_2$ are $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$, respectively. When the relations (2-8) and (3-13) are satisfied, we have

$$\widehat{\mathcal{W}}\ge\mu_\kappa\big[(\bar{\zeta}_\alpha+\bar{\psi}_{,\alpha})(\zeta_\alpha+\psi_{,\alpha})+\tfrac{1}{3}h^2\bar{\zeta}_{\alpha,\beta}\zeta_{\alpha,\beta}\big]>0. \tag{4-9}$$

Taking into account the well-known membrane problem, we can write

$$\frac{\pi^2}{\ell_2^2}\int_{L_{x_1}}\bar{\zeta}_\alpha\zeta_\alpha\,dx_2\le\int_{L_{x_1}}\bar{\zeta}_{\alpha,2}\zeta_{\alpha,2}\,dx_2,\qquad \frac{\pi^2}{\ell_2^2}\int_{L_{x_1}}\bar{\psi}\,\psi\,dx_2\le\int_{L_{x_1}}\bar{\psi}_{,2}\psi_{,2}\,dx_2, \tag{4-10}$$

and considering (4-8) and (4-9) we arrive at

$$\mathcal{J}'_\kappa(x_1)\ge\int_{L_{x_1}}2\mu_\kappa\Big[(\bar{\zeta}_\alpha+\bar{\psi}_{,\alpha})(\zeta_\alpha+\psi_{,\alpha})+\tfrac{1}{3}h^2\bar{\zeta}_{\alpha,\beta}\zeta_{\alpha,\beta}-\frac{\varrho\omega^2\ell_2^2}{\mu_\kappa\pi^2}\big(\bar{\psi}_{,2}\psi_{,2}+\tfrac{1}{3}h^2\bar{\zeta}_{\alpha,2}\zeta_{\alpha,2}\big)\Big]dx_2. \tag{4-11}$$

Using relations (4-10), we can observe that

$$\frac{6\ell_2^2}{\pi^2h^2}\int_{L_{x_1}}\big[(\bar{\zeta}_2+\bar{\psi}_{,2})(\zeta_2+\psi_{,2})+\tfrac{1}{6}h^2\bar{\zeta}_{2,2}\zeta_{2,2}\big]dx_2\ge\frac{\ell_2^2}{\pi^2}\int_{L_{x_1}}\zeta_{2,2}\bar{\zeta}_{2,2}\,dx_2\ge\int_{L_{x_1}}\zeta_2\bar{\zeta}_2\,dx_2. \tag{4-12}$$

On the other side, Schwarz and arithmetic-geometric mean inequalities imply

$$(\bar{\zeta}_2+\bar{\psi}_{,2})(\zeta_2+\psi_{,2})\ge(1-\varepsilon)\zeta_2\bar{\zeta}_2+\Big(1-\frac{1}{\varepsilon}\Big)\psi_{,2}\bar{\psi}_{,2}\qquad\text{for every }\varepsilon>0,$$

and in particular, for $\varepsilon = 2$, $(\bar{\zeta}_2 + \bar{\psi}_{,2})(\zeta_2 + \psi_{,2}) \geq -\zeta_2\bar{\zeta}_2 + \frac{1}{2}\psi_{,2}\bar{\psi}_{,2}$; thus,

$$\int_{L_{x_1}} \left[(\bar{\zeta}_2 + \bar{\psi}_{,2})(\zeta_2 + \psi_{,2}) + \tfrac{1}{6}h^2\bar{\zeta}_{2,2}\zeta_{2,2}\right] dx_2 \geq \int_{L_{x_1}} (\bar{\zeta}_2 + \bar{\psi}_{,2})(\zeta_2 + \psi_{,2})\, dx_2 \geq \int_{L_{x_1}} \left(-\bar{\zeta}_2\zeta_2 + \tfrac{1}{2}\bar{\psi}_{,2}\psi_{,2}\right) dx_2. \tag{4-13}$$

Combining (4-12) and (4-13) we obtain

$$\int_{L_{x_1}} \left[(\bar{\zeta}_2 + \bar{\psi}_{,2})(\zeta_2 + \psi_{,2}) + \tfrac{1}{6}h^2\bar{\zeta}_{2,2}\zeta_{2,2}\right] dx_2 \geq \frac{h^2\pi^2}{2(6\ell_2^2 + \pi^2h^2)} \int_{L_{x_1}} \bar{\psi}_{,2}\psi_{,2}\, dx_2. \tag{4-14}$$

Moreover, (4-11) and (4-14) imply

$$\mathcal{J}'_\kappa(x_1) \geq \int_{L_{x_1}} \mu_\kappa \left\{ 2(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + \frac{h^2\pi^2}{6\ell_2^2 + \pi^2h^2}\left[1 - \frac{2\varrho\ell_2^2\omega^2}{\pi^2\mu_\kappa}\left(1 + \frac{6\ell_2^2}{\pi^2h^2}\right)\right]\bar{\psi}_{,2}\psi_{,2} + \tfrac{1}{3}h^2\left[2\bar{\zeta}_{1,1}\zeta_{1,1} + 2\bar{\zeta}_{2,1}\zeta_{2,1} + 2\left(1 - \frac{\varrho\ell_2^2\omega^2}{\pi^2\mu_\kappa}\right)\bar{\zeta}_{1,2}\zeta_{1,2} + \left(1 - \frac{2\varrho\ell_2^2\omega^2}{\pi^2\mu_\kappa}\right)\bar{\zeta}_{2,2}\zeta_{2,2}\right]\right\} dx_2. \tag{4-15}$$

If $\varrho > 0$, (2-8) and (3-13) hold and

$$\omega < \omega_m \qquad \text{with} \qquad \omega_m = \left[\frac{2\varrho\ell_2^2}{\pi^2\mu_\kappa}\left(1 + \frac{6\ell_2^2}{\pi^2h^2}\right)\right]^{-1/2}. \tag{4-16}$$

The frequency ω_m depends on the geometrical properties of the plate, namely h and ℓ_2. We can see that μ_κ is strictly positive and the critical frequency ω_m is such that

$$0 < 1 - \frac{\omega^2}{\omega_m^2} = \min\left\{1 - \frac{2\varrho\ell_2^2\omega^2}{\pi^2\mu_\kappa}\left(1 + \frac{6\ell_2^2}{\pi^2h^2}\right), 1 - \frac{\varrho\ell_2^2\omega^2}{\pi^2\mu_\kappa}, 1 - \frac{2\varrho\ell_2^2\omega^2}{\pi^2\mu_\kappa}\right\}. \tag{4-17}$$

Then, (4-15) and (4-17) lead to

$$\mathcal{J}'_\kappa(x_1) \geq \int_{L_{x_1}} \mu_\kappa\left[p_1(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + p_2\bar{\psi}_{,2}\psi_{,2} + \tfrac{1}{3}h^2(p_3\bar{\zeta}_{1,1}\zeta_{1,1} + p_4\bar{\zeta}_{2,1}\zeta_{2,1} + p_5\bar{\zeta}_{1,2}\zeta_{1,2} + p_6\bar{\zeta}_{2,2}\zeta_{2,2})\right] dx_2, \tag{4-18}$$

where $p_1, \ldots, p_6$ represent the following strictly positive coefficients:

$$p_1 = p_3 = p_4 = 2, \qquad p_2 = \frac{h^2\pi^2}{6\ell_2^2 + h^2\pi^2}\left(1 - \frac{\omega^2}{\omega_m^2}\right), \qquad p_5 = 2p_6 = 2\left(1 - \frac{\omega^2}{\omega_m^2}\right).$$

Under the above hypotheses, $\mathcal{J}_\kappa$ is a nondecreasing function and we can prove the following theorem.

Theorem 4.1. *Let the hypotheses of Theorem 3.1 be still valid and let the frequency of harmonic vibrations ω be lower than the critical value ω_m, as in* (4-16)*. The function $\mathcal{J}_\kappa$ is such that*

$$\left|\mathcal{J}_\kappa(x_1)\right| \leq \int_{L_{x_1}} \frac{\ell_2}{\pi}\left[P_1(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + P_2\bar{\psi}_{,2}\psi_{,2} + \tfrac{1}{3}h^2(P_3\bar{\zeta}_{1,1}\zeta_{1,1} + P_4\bar{\zeta}_{2,1}\zeta_{2,1} + P_5\bar{\zeta}_{1,2}\zeta_{1,2} + P_6\bar{\zeta}_{2,2}\zeta_{2,2})\right] dx_2, \tag{4-19}$$

where

$$P_1 = c_{55}, \quad P_2 = c_{55}, \quad P_3 = c_{11}, \quad P_4 = c_{66}, \quad P_5 = (c_{11} + c_{66} + c_{12}), \quad P_6 = (2c_{66} + c_{12}). \tag{4-20}$$

Furthermore, $\mathcal{J}_\kappa$ satisfies the inequality

$$\mu_\kappa \big|\mathcal{J}_k(x_1)\big| \le \nu \mathcal{J}_k'(x_1) \qquad \textit{with} \quad \nu = \max_{i=1,\dots,6} \frac{\ell_2 P_i}{\pi p_i}. \tag{4-21}$$

We underline that, as with ω_m, ν also depends on the geometrical properties of the plate.

Proof. Considering (4-2)–(4-6), we arrive at

$$\mathcal{J}_\kappa(x_1) = \int_{L_{x_1}} 2\,\mathrm{Re}\big\{c_{55}(\bar\zeta_1 + \bar\psi_{,1})\psi + \tfrac{1}{3}h^2\big[c_{11}\bar\zeta_{1,1}\zeta_1 + (c_{66}-\kappa)\bar\zeta_{2,2}\zeta_1 + (c_{12}+\kappa)\bar\zeta_{1,2}\zeta_2 + c_{66}\bar\zeta_{2,1}\zeta_2\big]\big\}\,dx_2. \tag{4-22}$$

Using the arithmetic-geometric and Schwarz inequalities and taking into account (4-10), we can show that

$$\begin{aligned}
\textstyle\int_{L_{x_1}} \mathrm{Re}[(\bar\zeta_1 + \bar\psi_{,1})\psi]\,dx_2 &\le \frac{\ell_2}{\pi}\big(\textstyle\int_{L_{x_1}}(\bar\zeta_1 + \bar\psi_{,1})(\zeta_1 + \psi_{,1})\,dx_2\big)^{1/2}\big(\textstyle\int_{L_{x_1}}\bar\psi_{,2}\psi_{,2}\,dx_2\big)^{1/2} \\
&\le \frac{\ell_2}{2\pi}\big[\textstyle\int_{L_{x_1}}(\bar\zeta_1 + \bar\psi_{,1})(\zeta_1 + \psi_{,1})\,dx_2 + \int_{L_{x_1}}\psi_{,2}\bar\psi_{,2}\,dx_2\big], \\
\textstyle\int_{L_{x_1}} \mathrm{Re}[\bar\zeta_{1,1}\zeta_1]\,dx_2 &\le \frac{\ell_2}{\pi}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{1,1}\zeta_{1,1}\,dx_2\big)^{1/2}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{1,2}\zeta_{1,2}\,dx_2\big)^{1/2} \le \frac{\ell_2}{2\pi}\textstyle\int_{L_{x_1}}\bar\zeta_{1,\alpha}\zeta_{1,\alpha}\,dx_2, \\
\textstyle\int_{L_{x_1}} \mathrm{Re}[\bar\zeta_{2,2}\zeta_1]\,dx_2 &\le \frac{\ell_2}{\pi}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{2,2}\zeta_{2,2}\,dx_2\big)^{1/2}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{1,2}\zeta_{1,2}\,dx_2\big)^{1/2} \le \frac{\ell_2}{2\pi}\textstyle\int_{L_{x_1}}\bar\zeta_{\alpha,2}\zeta_{\alpha,2}\,dx_2, \\
\textstyle\int_{L_{x_1}} \mathrm{Re}[\bar\zeta_{1,2}\zeta_2]\,dx_2 &\le \frac{\ell_2}{\pi}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{1,2}\zeta_{1,2}\,dx_2\big)^{1/2}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{2,2}\zeta_{2,2}\,dx_2\big)^{1/2} \le \frac{\ell_2}{2\pi}\textstyle\int_{L_{x_1}}\bar\zeta_{\alpha,2}\zeta_{\alpha,2}\,dx_2, \\
\textstyle\int_{L_{x_1}} \mathrm{Re}[\bar\zeta_{2,1}\zeta_2]\,dx_2 &\le \frac{\ell_2}{\pi}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{2,1}\zeta_{2,1}\,dx_2\big)^{1/2}\big(\textstyle\int_{L_{x_1}}\bar\zeta_{2,2}\zeta_{2,2}\,dx_2\big)^{1/2} \le \frac{\ell_2}{2\pi}\textstyle\int_{L_{x_1}}\bar\zeta_{2,\alpha}\zeta_{2,\alpha}\,dx_2.
\end{aligned} \tag{4-23}$$

By means of (4-22) and (4-23), we obtain the inequality

$$\begin{aligned}
\big|\mathcal{J}_\kappa(x_1)\big| \le \int_{L_{x_1}} \frac{\ell_2}{\pi}\Big\{&c_{55}(\bar\zeta_1 + \bar\psi_{,1})(\zeta_1 + \psi_{,1}) + c_{55}\bar\psi_{,2}\psi_{,2} \\
&+ \tfrac{1}{3}h^2\big[c_{11}\bar\zeta_{1,\alpha}\zeta_{1,\alpha} + (c_{66}-\kappa)\bar\zeta_{\alpha,2}\zeta_{\alpha,2} + (c_{12}+\kappa)\bar\zeta_{\alpha,2}\zeta_{\alpha,2} + c_{66}\bar\zeta_{2,\alpha}\zeta_{2,\alpha}\big]\Big\}\,dx_2,
\end{aligned}$$

which, through (4-20), leads to (4-19).

Since μ_κ and $p_1, \dots, p_6$ are strictly positive, the definition of ν_κ and (4-19) allow us to write

$$\begin{aligned}
\big|\mathcal{J}_\kappa(x_1)\big| \le \int_{L_{x_1}} \nu\big[&p_1(\bar\zeta_1 + \bar\psi_{,1})(\zeta_1 + \psi_{,1}) + p_2\bar\psi_{,2}\psi_{,2} \\
&+ \tfrac{1}{3}h^2(p_3\bar\zeta_{1,1}\zeta_{1,1} + p_4\bar\zeta_{2,1}\zeta_{2,1} + p_5\bar\zeta_{1,2}\zeta_{1,2} + p_6\bar\zeta_{2,2}\zeta_{2,2})\big]dx_2.
\end{aligned} \tag{4-24}$$

Finally, (4-18) and (4-24) prove that (4-21) is satisfied. □

For what follows, it is useful to introduce the class of steady-state vibrations (ζ_α, ψ) for which

$$\mathscr{E}(x_1) = \int_{S_{x_1}} \frac{\ell_2}{\pi}\Big[P_1(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + P_2\psi_{,2}\bar{\psi}_{,2} + \tfrac{1}{3}h^2(P_3\zeta_{1,1}\bar{\zeta}_{1,1} + P_4\bar{\zeta}_{2,1}\zeta_{2,1} + P_5\bar{\zeta}_{1,2}\zeta_{1,2} + P_6\bar{\zeta}_{2,2}\zeta_{2,2})\Big]da < \infty, \quad (4\text{-}25)$$

where $S_{x_1} = \overline{\Sigma} \cap [x_1, \infty)\times[0, \ell_2]$. From a direct differentiation of $\mathscr{E}(x_1)$, we get

$$-\mathscr{E}'(x_1) = \int_{L_{x_1}} \frac{\ell_2}{\pi}\Big[P_1(\bar{\zeta}_1 + \bar{\psi}_{,1})(\zeta_1 + \psi_{,1}) + P_2\psi_{,2}\bar{\psi}_{,2} + \tfrac{1}{3}h^2(P_3\zeta_{1,1}\bar{\zeta}_{1,1} + P_4\zeta_{2,1}\bar{\zeta}_{2,1} + P_5\zeta_{1,2}\bar{\zeta}_{1,2} + P_6\zeta_{2,2}\bar{\zeta}_{2,2})\Big]dx_2. \quad (4\text{-}26)$$

As a consequence of (4-19) and (4-26), it follows that

$$\big|\mathscr{J}_\kappa(x_1)\big| \le -\mathscr{E}'(x_1), \quad (4\text{-}27)$$

so $\mathscr{E}(x_1)$ is a nonincreasing function.

When the strip $\overline{\Sigma}$ is bounded, the hypothesis (4-25) is trivially satisfied; Equations (4-5) and (4-27) imply

$$\mathscr{E}'(\ell_1) = 0 \quad \Rightarrow \quad \mathscr{J}_\kappa(\ell_1) = 0. \quad (4\text{-}28)$$

On the other side, if the strip $\overline{\Sigma}$ is semiinfinite and the hypothesis (4-25) is satisfied, then (4-26) and (4-27) lead to

$$\lim_{x_1\to\infty} \mathscr{E}'(x_1) = 0 \quad \Rightarrow \quad \lim_{x_1\to\infty} \mathscr{J}_\kappa(x_1) = 0. \quad (4\text{-}29)$$

Now, provided that the frequency of harmonic vibrations is lower than the critical value ω_m, we can establish the following theorem.

Theorem 4.2. *Let the hypotheses of Theorem 4.1 be still valid and let* $\{\zeta_\alpha, \psi\}$ *be the steady-state vibrations for which* (4-25) *holds. Then, the function* $\mathscr{J}_\kappa$ *satisfies a decay estimate of Saint-Venant type*:

$$0 \le -\mathscr{J}_\kappa(x_1) \le -\mathscr{J}_\kappa(0)\exp\Big(-\frac{\mu_\kappa}{\nu}x_1\Big). \quad (4\text{-}30)$$

Proof. Whether the considered strip is bounded or not, from hypothesis (4-16) on the frequency ω and from the nondecreasing property of $\mathscr{J}_\kappa$, (4-28) and (4-29) imply

$$\mathscr{J}_\kappa(x_1) \le 0. \quad (4\text{-}31)$$

We can also remark, from the definitions of coefficients $P_1, \dots, P_6$ and ν, that ν is strictly positive. Then, (4-21) and (4-31) allow us to write

$$\mathscr{J}'_\kappa(x_1) + \frac{\mu_\kappa}{\nu}\mathscr{J}_\kappa(x_1) \ge 0 \quad \Rightarrow \quad \frac{d}{dx_1}\Big[\mathscr{J}_\kappa(x_1)\exp\Big(\frac{\mu_\kappa}{\nu}x_1\Big)\Big] \ge 0. \quad (4\text{-}32)$$

It is easy to observe that (4-32) leads to the estimate (4-30) with exponential decay factor μ_κ/ν. □

Remark. The investigation performed is based on an assumption concerning the strong ellipticity of the elasticity tensor. The results obtained under such an hypothesis are thus valid also for classes of particular materials characterized by special properties, like negative Poisson's ratio and negative stiffness (auxetic or antirubber materials). These particular structures (see, for example, [Park and Lakes 2007]) expand laterally when stretched, in contrast to ordinary materials.

List of symbols

	Symbol	Page
Right cylinder of height $2h$	$\bar{B}$	325
Interiors of $\bar{B}$ and $\bar{\Sigma}$	B, Σ	325
Rectangular Cartesian coordinate frame	Ox_1x_2	325
Variable functions characterizing a bending state	v_α, w	325
Constitutive equations	M_{ij}, τ_{ij}	325
Family of tensors depending on a parameter κ	$\tilde{M}_{ij}$	327
Definition of initial-boundary value problem	$\mathcal{P}$	327
External data	$\mathcal{D}$	327
Support of external data	$\hat{D}_T$	327
Set defined on the plate	D_r, Σ_r, L_r	327
Integral function	$\mathcal{I}_\kappa$	327
Quadratic forms	$\mathcal{T}$, $\mathcal{W}$, $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$	328
Matrices associated with the quadratic forms $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$	$\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$	328
Smallest and largest eigenvalues of $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$	μ_κ, η_κ	329
Smallest and largest eigenvalues of $\mathcal{W}_\alpha$	$k_m^{(\alpha)}$, $k_M^{(\alpha)}$	329
Constant characteristic of the material	c_κ	329
Variable functions characterizing steady-state vibrations	ζ_α, ψ	332
Frequency of oscillations	ω	332
Dimensions of the middle section of the plate	ℓ_1, ℓ_2	332
Constitutive equations for problem of steady-state vibrations	Γ_{ij}, χ_i	332
Set defined on the plate for steady-state vibrations	L_{x_1}	333
Integral function for steady-state vibrations	$\mathcal{J}_\kappa$	333
Complex quadratic forms	$\widehat{\mathcal{W}}$, $\widehat{\mathcal{W}}_0$, $\widehat{\mathcal{W}}_1$, $\widehat{\mathcal{W}}_2$	333
Estimate coefficients	$p_1, \ldots, p_6$	334
Estimate coefficients	$P_1, \ldots, P_6$	335
Constant characteristic of the material	ν	335
Integral function	$\mathcal{E}$	336
Set defined on the plate for steady-state vibrations	S_{x_1}	336

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