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EFFECTIVE MEDIUM THEORIES FOR WAVE PROPAGATION IN TWO-DIMENSIONAL RANDOM INHOMOGENEOUS MEDIA

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Two effective medium models for two-dimensional scalar wave propagation in random inhomogeneous media are examined in a single theoretical framework. It is shown how the hypotheses and self-consistency conditions in these models are mathematically formulated. As a special case, a two-phase composite in which circular cylindrical inclusions are embedded in a continuous matrix is considered. Numerical calculations are performed for such composites with different combinations of constituent properties in the frequency range up to $ka = 10$, where geometric optic behavior starts appearing. The models mutually deviate when the motion of inclusions is relatively large, such as at the resonance scattering of the inclusions. Otherwise, deviations in the low-frequency regime ($ka < 1$) are negligible and those at high frequencies are also strikingly small. The same facts are observed for two composites having very different constituent properties and in the high-frequency limit.

1. Introduction

Theoretical prediction of the effective properties of inhomogeneous materials is of fundamental importance in materials research since virtually every material is inhomogeneous at smaller scales. For the prediction of the dynamic effective properties of such materials, analysis of the interaction of propagating waves with inhomogeneities, that is, the multiple scattering of waves, is needed. However, except for some simple cases in which multiple scattering effects are sufficiently small, the complete treatment of such a problem is quite difficult for the mathematical and physical reasons described in [Frisch 1968], and thus an approximate solution is sought.

There are two typical approaches to this approximation: the direct and indirect. In the direct approach, the multiple scattering solution for a set of scatterers is first found and then the solution is averaged (the effective field) for all possible configurations of the scatterer distribution. This approach, however, ends up with an infinite hierarchy of integral equations in which each order contains more statistical information than those preceding [Lax 1952]. This infinite hierarchy shows that an exact solution of the problem is prohibitively veiled. To truncate the infinite hierarchy, the rigorous perturbation method [Karal and Keller 1964] can be used, but this approach is limited to the case of weak scattering [Frisch 1968]. In the case of strong scattering, the quasicrystalline approximation (QCA) of [Lax 1952] is often used as an explicit closure approximation while the alternatives are methods based on the stochastic variational principles [Willis 1981; Weaver 1985]. The QCA is relevant to the cases where the total field can be approximated by the sum of single and double scattering fields and thus often fails in a dense scatterer system [Kim 2010].

Keywords: effective medium theory, wave propagation, composite materials, multiple scattering.

One of the indirect approaches, the so-called effective medium theory (EMT), also known as the self-consistent theory, is a group of approximation methods that are formulated in a few steps of thought experiment. One distinctive point in this theory is that multiple scattering is seen to occur in the effective medium [Kim et al. 1995; Choy 1999]. In most effective medium type formulations, the single scattering approximation is adopted and thus the correlations between scatterers are neglected. However, by solving the associated scattering problem in the effective medium, the average multiple scattering effect is taken into account in an implicit way. In spite of the fact that this hypothesis is not theoretically justifiable, the EMT has been successfully applied to predicting the effective properties (not the effective fields) of inhomogeneous materials. A few variants have been set forward which are based on slightly different hypotheses. One is the coherent potential approximation in solid-state physics [Soven 1967; Velicky et al. 1968], which has been regarded as a reliable theory in predicting alloy properties. A quasistatic version for predicting mechanical properties of composites has been proposed by Berryman [1979; 1980]. A dynamic generalization of this theory has been made for two and three-dimensional problems [Kim et al. 1995; Kim 1996]. As shown in [Kim et al. 1995], this theory was very successful in predicting the effective wave speed and attenuation in random particulate composites for wide ranges of frequency and volume fraction. A second variant is the dynamic extension [Sabina and Willis 1988; Bussink et al. 1995; Kanaun and Levin 2003; Kanaun et al. 2004] of the static self-consistent theories of [Budiansky 1965; Hill 1965]. The major difference between these two theories lies in how the roles of constituents are treated, which will be discussed later in this paper. While the direct approach faces a dead end in that the QCA incorporated with an exact pair-correlation function [Varadan et al. 1985] often fails to predict correct effective dynamic properties [Kim 2010] and an analysis of higher order scattering is extremely difficult, opportunities seem to exist in the indirect approaches that offer a tractable scheme on which a more rigorous formalism can be easily built [Martin 2006].

In this paper, a comparative theoretical and numerical study is conducted for the effective medium models of Kim [1996] and Kanaun and Levin [2003], which are called here EMT-1 and EMT-2, respectively. Following [Kanaun and Levin 2003], a theoretical formalism is elaborated and a clear physical meaning is given in each step of the derivations. Horizontally polarized shear (SH) waves propagating in a two-phase composite, in which circular cylindrical inclusions are randomly distributed in a continuous matrix, are considered. Numerical calculations are performed for composites with different combinations of constituent properties in a wide frequency range. Although SH wave propagation in a two-dimensional composite with circular inclusions is considered here for the computational simplicity, similar conclusions are expected for other types of waves and inclusions.

2. Preliminaries

Let us consider an infinite elastic medium that contains a large number (N) of two-dimensional elastic inclusions. The inclusions occupy a region Ω that consists of discrete subregions Ω_i, so that $\Omega = \sum_{i=1}^{N} \Omega_i$. Let μ_1 and ρ_1 be the shear modulus and the mass density of the host medium and μ_2 and ρ_2 be those of the inclusions. The shear modulus μ and mass density ρ of the entire medium may be written:

$$\mu(\boldsymbol{r}) = \mu_1 + \Delta\mu\Theta(\boldsymbol{r}), \qquad \rho(\boldsymbol{r}) = \rho_1 + \Delta\rho\Theta(\boldsymbol{r}), \tag{1}$$

where $\boldsymbol{r}$ is the position vector, $\Delta\mu = (\mu_2 - \mu_1)$, $\Delta\rho = (\rho_2 - \rho_1)$, and $\Theta(\boldsymbol{r})$ is a step function, $\Theta(\boldsymbol{r}) = 1$ if $\boldsymbol{r} \in \Omega$ and $\Theta(\boldsymbol{r}) = 0$ if $\boldsymbol{r} \notin \Omega$. Now suppose that a time-harmonic source, $S(\boldsymbol{r}, t) = s(\boldsymbol{r})e^{-i\omega t}$, occupying the region Ω_s in the host medium, is generating an antiplane body force. The antiplane time-harmonic displacement field $u(\boldsymbol{r})e^{-i\omega t}$ in this medium satisfies the following equation of motion:

$$\nabla \cdot [\mu(\boldsymbol{r})\nabla u(\boldsymbol{r})] + \rho(\boldsymbol{r})\omega^2 u(\boldsymbol{r}) = s(\boldsymbol{r}). \tag{2}$$

The Green's function for the homogeneous matrix that satisfies the boundary condition at infinity is the solution of the following equation:

$$\mu_1 \nabla^2 G(\boldsymbol{r}-\boldsymbol{r}') + \rho_1\omega^2 G(\boldsymbol{r}-\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{3}$$

$$G(\boldsymbol{r}-\boldsymbol{r}') = \frac{i}{4\mu_1} H_0(k_1|\boldsymbol{r}-\boldsymbol{r}'|), \tag{4}$$

where δ is the Dirac delta function in the two-dimensional space, $H_0(x)$ is the Hankel function of the first kind and order zero, and $k_1 = \omega/c_1$ is the wavenumber associated with the shear wave speed c_1. A two-dimensional Fourier transform pair is defined

$$f(\boldsymbol{k}) = (2\pi)^{-2} \int f(\boldsymbol{r}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r}, \qquad f(\boldsymbol{r}) = \int f(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k}, \tag{5}$$

where $\boldsymbol{k}$ denotes the wave vector with components (k_x, k_y), $d\boldsymbol{r} = dxdy$, and $d\boldsymbol{k} = dk_x dk_y$. Forward Fourier transformation of Equation (3) yields the following relation between the operator $A(\boldsymbol{k}_1)$ and the Green's function in the wavenumber domain: $(\mu_1 k_1^2 - \rho_1\omega^2)G(\boldsymbol{k}_1) = A(\boldsymbol{k}_1)G(\boldsymbol{k}_1) = 1$, where $\boldsymbol{k}_1$ is the wave vector associated with the wavenumber k_1.

Substituting (1) into (2), one obtains $\mu_1\nabla^2 u + \rho_1\omega^2 u = s - [\nabla\cdot(\Delta\mu\nabla u) + \Delta\rho\omega^2 u]\Theta$. Using the Green's function in (3), one gets the integral equation for the total field

$$u(\boldsymbol{r}) = u^{\text{in}}(\boldsymbol{r}) + \int_\Omega \Delta\mu(\boldsymbol{r}')\nabla G(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{\varepsilon}(\boldsymbol{r}') + \Delta\rho(\boldsymbol{r}')\omega^2 G(\boldsymbol{r}-\boldsymbol{r}')u(\boldsymbol{r}')d\boldsymbol{r}', \tag{6}$$

where the first term, given as $u^{\text{in}}(\boldsymbol{r}) = \int_{\Omega_s} G(\boldsymbol{r}-\boldsymbol{r}')s(\boldsymbol{r}')d\boldsymbol{r}'$, is the incident wave, the second term represents the scattered waves, and $\boldsymbol{\varepsilon} = \nabla u$ is the strain vector field in the inclusion. The prime sign in the integral in (6) is used to denote the field quantities in the inclusions and the derivatives in the integral are accordingly differentiation with respect to the primed variables. Assuming that the source is located at infinity (far from the region of interest), the incident wave is regarded as a plane wave:

$$u^{\text{in}}(\boldsymbol{r}) = U e^{i\boldsymbol{k}_1^i\cdot\boldsymbol{r}}, \tag{7}$$

where $\boldsymbol{k}_1^i/k_1$ is the unit vector in the direction of incidence and the Fourier transform of (7) is

$$u^{\text{in}}(\boldsymbol{k}_1) = (2\pi)^{-2} U\delta(\boldsymbol{k}_1 - \boldsymbol{k}_1^i). \tag{8}$$

Since the plane wave satisfies the wave equation without a source in a finite domain, the following holds: $A(\boldsymbol{k}_1)u^{\text{in}}(\boldsymbol{k}_1) = 0$. Using the far-field asymptotic expression of the Green's function, the far-field scattering pattern is obtained as an integral of the displacement and strain in the inclusion:

$$u^{\text{sc}} \sim \sqrt{\frac{2}{\pi k_1 r}} e^{i(k_1 r - \pi/4)} f(\boldsymbol{k}_1), \quad f(\boldsymbol{k}_1) = -\frac{1}{8}\int_\Omega [\Delta\mu i\boldsymbol{k}_1\cdot\nabla u(\boldsymbol{r}') - \Delta\rho\omega^2 u(\boldsymbol{r}')] \exp(-i\boldsymbol{k}_1\cdot\boldsymbol{r}')d\boldsymbol{r}'. \tag{9}$$

Due to the generalized optical theorem for the scattering in an absorbing medium [Kim 2003a; 2003b; Kim and Lee 2009; 2010], the total cross section is

$$\gamma = -\frac{4}{k_1'} \mathrm{Re}[f(\boldsymbol{k}_1^i)], \tag{10}$$

where $f(\boldsymbol{k}_1^i)$ is the forward scattering amplitude and k_1' is the real part of the complex wavenumber k_1. Equation (10) states that the total power loss during the process of scattering is proportional to the forward scattering amplitude.

3. The models

EMTs commonly require the following three steps: first, finding the approximate average displacement and strain fields in a representative inclusion by solving the single scattering problem in the effective medium with yet-unknown properties; second, embedding the inclusions with these average fields in a homogeneous medium (the original or the effective medium) in which averaging over composition and geometry is to be performed; finally, obtaining expressions for the effective properties applying a self-consistency condition. The second step is often called the self-consistent embedding and the self-consistency condition requires the equivalence of the average field to a plane wave that is assumed to propagate in the effective medium. Different EMTs use different averaging schemes in which the roles of the constituent materials are treated differently.

In order to obtain the approximate average fields, both the EMT-1 and EMT-2 start with:

Hypothesis 1. Every inclusion in the composite behaves as an isolated inclusion embedded in a homogeneous medium having the effective properties of the composite. The field acting on this inclusion is a plane wave propagating in the effective medium [Kim et al. 1995; Kanaun and Levin 2003].

By this hypothesis, the original multiple scattering problem defined in the host medium is reduced to a single scattering problem defined in the effective medium. The integral equation (6) for the scattering by a single representative inclusion in the effective medium is written

$$\bar{u}(\boldsymbol{r}) = \bar{u}^{\mathrm{in}}(\boldsymbol{r}) + \int_{\Omega_i} \overline{\Delta\mu}_2 \nabla\bar{G}(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{\varepsilon}(\boldsymbol{r}') + \overline{\Delta\rho}_2\omega^2\bar{G}(\boldsymbol{r}-\boldsymbol{r}')u(\boldsymbol{r}')\,d\boldsymbol{r}', \tag{11}$$

where the overbar ($\bar{\ }$) is used to denote material properties and physical quantities in the effective medium. For example, the incident wave is $\bar{u}^{\mathrm{in}}(\boldsymbol{r}) = Ue^{i\bar{\boldsymbol{k}}^i\cdot\boldsymbol{r}}$, where $\bar{\boldsymbol{k}}^i$ is the wave vector of the incident wave in the effective medium, $\bar{k} = |\bar{\boldsymbol{k}}^i|$ is the wavenumber associated with the effective shear wave speed $\bar{c}$, $\overline{\Delta\mu}_2 = (\mu_2 - \bar{\mu})$, and $\overline{\Delta\rho}_2 = (\rho_2 - \bar{\rho})$. The region Ω_i is now the area occupied by the single representative inclusion. Its location is not prescribed yet since it is a random variable and later the scattered field obtained from (11) is averaged over all possible locations. Averages are also performed over the shape and orientation of the inclusions. If the effective medium is presumed to be isotropic and homogeneous, the location of Ω_i is permitted to be everywhere in the medium, which states the translational invariance.

To realize this averaging process, let us consider a plane wave incident on the representative inclusion located at $\boldsymbol{r}_i$ in the global coordinate system, and place the origin of a local coordinate system (x_1, y_1) at the mass center of the inclusion. Then, the displacement and strain fields described in the global coordinate system are related to those fields in the local coordinate system ($\hat{\bar{u}}(\boldsymbol{r}_1)$ and $\hat{\bar{\boldsymbol{\varepsilon}}}(\boldsymbol{r}_1)$) in response to the incident wave with unit magnitude ($\hat{\bar{u}}^{\text{in}}(\boldsymbol{r}_1) = Ue^{i\bar{k}^i\cdot\boldsymbol{r}_1}$):

$$\bar{u}(\boldsymbol{r}) = \hat{\bar{u}}(\boldsymbol{r}_1)e^{-i\bar{k}^i\cdot\boldsymbol{r}_1}\bar{u}^{\text{in}}(\boldsymbol{r}) \equiv \bar{\Lambda}_u(\boldsymbol{r}_1)\bar{u}^{\text{in}}(\boldsymbol{r}), \qquad \boldsymbol{r} \in \Omega_i, \tag{12}$$

$$\bar{\boldsymbol{\varepsilon}}(\boldsymbol{r}) = \hat{\bar{\boldsymbol{\varepsilon}}}(\boldsymbol{r}_1)e^{-i\bar{k}^i\cdot\boldsymbol{r}_1}\bar{u}^{\text{in}}(\boldsymbol{r}) = \nabla\hat{\bar{u}}(\boldsymbol{r}_1)e^{-i\bar{k}^i\cdot\boldsymbol{r}_1}\bar{u}^{\text{in}}(\boldsymbol{r}) \equiv \bar{\boldsymbol{\Lambda}}_\varepsilon(\boldsymbol{r}_1)\bar{u}^{\text{in}}(\boldsymbol{r}), \qquad \boldsymbol{r} \in \Omega_i, \tag{13}$$

where $\boldsymbol{r}_1 = \boldsymbol{r} - \boldsymbol{r}_i$ is the position vector in the local coordinate system. The discrete random functions $\bar{\Lambda}_u$ and $\bar{\boldsymbol{\Lambda}}_\varepsilon$ appear as the transition operators that relate the displacement and strain fields in the remote inclusion to the incident wave. Note that these operators are independent of the locations $\boldsymbol{r}_i$ of the remote inclusion.

Now consider scattering of a set of imaginary inclusions whose internal fields are given by (12) and (13) and whose properties and positions are those of the inclusions (μ_2 and ρ_2, and $\boldsymbol{r}_i$). These inclusions are embedded in the *original* host medium in which a plane incident wave is propagating. The integral in (6) can be written

$$u(\boldsymbol{r}) = u^{\text{in}}(\boldsymbol{r}) + \int_\Omega [\Delta\mu\nabla G(\boldsymbol{r}-\boldsymbol{r}')\cdot\bar{\boldsymbol{\Lambda}}_\varepsilon(\boldsymbol{r}_1) + \Delta\rho\omega^2 G(\boldsymbol{r}-\boldsymbol{r}')\bar{\Lambda}_u(\boldsymbol{r}_1)]\bar{u}^{\text{in}}(\boldsymbol{r}')d\boldsymbol{r}'. \tag{14}$$

Taking an ensemble average on (14) for all possible sets of $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N\}$, one obtains

$$\langle u(\boldsymbol{r})\rangle = u^{\text{in}}(\boldsymbol{r}) + \int_\Omega [\Delta\mu\nabla G(\boldsymbol{r}-\boldsymbol{r}')\cdot\langle\bar{\boldsymbol{\Lambda}}_\varepsilon\rangle + \Delta\rho\omega^2 G(\boldsymbol{r}-\boldsymbol{r}')\langle\bar{\Lambda}_u\rangle]\bar{u}^{\text{in}}(\boldsymbol{r}')d\boldsymbol{r}'. \tag{15}$$

Note that the averages over shape and orientation do not appear since the inclusions are assumed to be identical cylinders with the same volume (Ω_i). Due to the translation invariance the averages of the transition operators are taken over a representative inclusion Ω_i

$$\langle\bar{\Lambda}_u\rangle = \frac{1}{\Omega_t}\int_{\Omega_t}\bar{\Lambda}_u d\boldsymbol{r} = \frac{N}{\Omega_t}\int_{\Omega_i}\hat{\bar{u}}(\boldsymbol{r}_1)e^{-i\bar{k}^i\cdot\boldsymbol{r}_1}d\boldsymbol{r}_1 = v_2\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}, \tag{16}$$

$$\langle\bar{\boldsymbol{\Lambda}}_\varepsilon\rangle = \frac{1}{\Omega_t}\int_{\Omega_t}\bar{\boldsymbol{\Lambda}}_\varepsilon d\boldsymbol{r} = \frac{N}{\Omega_t}\int_{\Omega_i}\nabla\hat{\bar{u}}(\boldsymbol{r}_1)e^{-i\bar{k}^i\cdot\boldsymbol{r}_1}d\boldsymbol{r}_1 = v_2\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i}, \tag{17}$$

where v_2 is the volume fraction of the inclusion phase, Ω_t represents the entire domain covering the matrix and all inclusions, and $\langle\ \rangle_{\Omega_i}$ denotes volume averaging over Ω_i. Substituting (16) and (17) into (15) yields

$$\langle u(\boldsymbol{r})\rangle = u^{\text{in}}(\boldsymbol{r}) + v_2\int_\Omega [\Delta\mu\nabla G(\boldsymbol{r}-\boldsymbol{r}')\cdot\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i} + \Delta\rho\omega^2 G(\boldsymbol{r}-\boldsymbol{r}')\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}]\bar{u}^{\text{in}}(\boldsymbol{r}')d\boldsymbol{r}'. \tag{18}$$

For self-consistency, a second hypothesis is introduced:

Hypothesis 2. The average field (the average displacement in (18)) is equal to the incident plane wave propagating in the effective medium.

That is,

$$\bar{u}^{\text{in}}(\boldsymbol{r}) = \langle u(\boldsymbol{r})\rangle, \tag{19}$$

which leads to

$$\langle u(\boldsymbol{r})\rangle = u^{\text{in}}(\boldsymbol{r}) + v_2\int_\Omega [\Delta\mu\nabla G(\boldsymbol{r}-\boldsymbol{r}')\cdot\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i} + \Delta\rho\omega^2 G(\boldsymbol{r}-\boldsymbol{r}')\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}]\langle u(\boldsymbol{r}')\rangle d\boldsymbol{r}', \tag{20}$$

since (20) is a convolution integral, Fourier transformation of (20) yields

$$\langle u(\boldsymbol{k}_1)\rangle = u^{\text{in}}(\boldsymbol{k}_1) + v_2[\Delta\mu i\boldsymbol{k}_1\cdot\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i} + \Delta\rho\omega^2\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}]G(\boldsymbol{k}_1)\langle u(\boldsymbol{k}_1)\rangle. \tag{21}$$

Applying the operator $A(\boldsymbol{k}_1)$ on (21), it turns out that the average displacement spectrum in (21) is the solution of the operator equation

$$\bar{A}(\boldsymbol{k}_1)\langle u(\boldsymbol{k}_1)\rangle = 0, \quad \bar{A}(\boldsymbol{k}_1) = A(\boldsymbol{k}_1) - v_2\Delta\mu i\boldsymbol{k}_1\cdot\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i} - v_2\Delta\rho\omega^2\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}, \tag{22}$$

the operator for the effective medium. Due to (19), the Fourier transform of the average field is replaced with that of the plane incident wave in the effective medium, $\langle u(\boldsymbol{k}_1)\rangle = \bar{u}^{\text{in}}(\bar{\boldsymbol{k}})$, requiring a necessary condition, $\boldsymbol{k}_1 = \bar{\boldsymbol{k}}$. Then, the expressions for the effective operator and the effective shear modulus and mass density in the EMT-2 formulation are

$$\bar{A}(\bar{\boldsymbol{k}}) = A(\bar{\boldsymbol{k}}) - v_2\Delta\mu i\bar{\boldsymbol{k}}\cdot\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i} - v_2\Delta\rho\omega^2\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}, \tag{23}$$

$$\bar{\mu} = \mu_0 - v_2\Delta\mu i\bar{\boldsymbol{k}}\cdot\frac{\langle\bar{\boldsymbol{\Lambda}}_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_i}}{\bar{k}^2}, \quad \bar{\rho} = \rho_0 + v_2\Delta\rho\langle\bar{\Lambda}_u(\bar{\boldsymbol{k}})\rangle_{\Omega_i}. \tag{24}$$

These are the formulae derived in [Kanaun and Levin 2003]. Note that the signs in the expressions for $\bar{A}$ and $\bar{\mu}$ are different from Equations (3.16) and (3.17) of that reference due to the different time-dependence: $e^{-i\omega t}$ in this paper versus $e^{i\omega t}$ in their paper.

In the EMT-1, an additional hypothesis is introduced:

Hypothesis 3. In the effective medium not only the discrete inclusions but also the surrounding medium acts as a scatterer because its properties differ from those of the effective medium, and these two scattering processes are independent.

The integral equations for the fields in a representative volume for the matrix (Ω_m) and in a representative inclusion (Ω_i) are

$$\bar{u}(\boldsymbol{r}) = \bar{u}^{\text{in}}(\boldsymbol{r}) + \int_{\Omega_m}\Delta\bar{\mu}_1\nabla\bar{G}(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{\varepsilon}_1(\boldsymbol{r}') + \Delta\bar{\rho}_1\omega^2\bar{G}(\boldsymbol{r}-\boldsymbol{r}')u_1(\boldsymbol{r}')\,d\boldsymbol{r}', \tag{25}$$

$$\bar{u}(\boldsymbol{r}) = \bar{u}^{\text{in}}(\boldsymbol{r}) + \int_{\Omega_i}\Delta\bar{\mu}_2\nabla\bar{G}(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{\varepsilon}_2(\boldsymbol{r}') + \Delta\bar{\rho}_2\omega^2\bar{G}(\boldsymbol{r}-\boldsymbol{r}')u_2(\boldsymbol{r}')\,d\boldsymbol{r}', \tag{26}$$

where subscripts 1 and m and 2 and i denote the matrix and the inclusions, respectively, and $\Delta\bar{\mu}_p = (\mu_p - \bar{\mu})$ and $\Delta\bar{\rho}_p = (\rho_p - \bar{\rho})$ for $p = 1, 2$. The displacement and strain fields in the representative volumes (Ω_m and Ω_i) may be obtained in the same way as in (12) and (13).

In the EMT-1, a schizoid medium, a medium that has estimates of the yet-unknown effective properties $\hat{\mu}$ and $\hat{\rho}$, is used for self-consistent embedding, in which the inclusions and the matrix are embedded and insonified by a plane wave. Then, (14) is written

$$\hat{u}(\boldsymbol{r}) = \hat{u}^{\text{in}}(\boldsymbol{r}) + \sum_{p=1,2}\int_{\Omega_p}[\Delta\hat{\mu}_p\nabla\hat{G}(\boldsymbol{r}-\boldsymbol{r}')\cdot\bar{\boldsymbol{\Lambda}}^p_\varepsilon(\boldsymbol{r}_1) + \Delta\hat{\rho}_p\omega^2\hat{G}(\boldsymbol{r}-\boldsymbol{r}')\bar{\Lambda}^p_u(\boldsymbol{r}_1)]\hat{u}(\boldsymbol{r}')\,d\boldsymbol{r}', \tag{27}$$

where variables with a caret are those of the estimate of the effective medium and $\Delta\hat{\mu}_p = (\mu_p - \hat{\mu})$ and $\Delta\hat{\rho}_p = (\rho_p - \hat{\rho})$ for $p = 1, 2$. Following the steps described above, (27) is averaged over the entire domain:

$$\langle\hat{u}(\boldsymbol{r})\rangle=\hat{u}^{\mathrm{in}}(\boldsymbol{r})+\sum_{p=1,2}v_p\int_{\Omega_p}[\Delta\hat{\mu}_p\nabla\hat{G}(\boldsymbol{r}-\boldsymbol{r}')\cdot\langle\bar{\boldsymbol{\Lambda}}^p_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_p}+\Delta\hat{\rho}_p\omega^2\hat{G}(\boldsymbol{r}-\boldsymbol{r}')\langle\bar{\Lambda}^p_u(\bar{\boldsymbol{k}})\rangle_{\Omega_p}]\langle\hat{u}(\boldsymbol{r}')\rangle\,d\boldsymbol{r}'. \tag{28}$$

Fourier transformation of (28) yields

$$\langle\hat{u}(\hat{\boldsymbol{k}})\rangle=\hat{u}^{\mathrm{in}}(\hat{\boldsymbol{k}})+\sum_{p=1,2}v_p[\Delta\hat{\mu}_p i\hat{\boldsymbol{k}}\cdot\langle\bar{\boldsymbol{\Lambda}}^p_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_p}+\Delta\hat{\rho}_p\omega^2\langle\bar{\Lambda}^p_u(\bar{\boldsymbol{k}})\rangle_{\Omega_p}]\hat{G}(\hat{\boldsymbol{k}})\langle\hat{u}(\hat{\boldsymbol{k}})\rangle. \tag{29}$$

Upon applying the operator $\hat{A}(\hat{\boldsymbol{k}})$ on (29), the equation takes the form

$$\Big[\hat{A}(\hat{\boldsymbol{k}})-\sum_{p=1,2}v_p\big(\Delta\hat{\mu}_p i\hat{\boldsymbol{k}}\cdot\langle\bar{\boldsymbol{\Lambda}}^p_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_p}+\Delta\hat{\rho}_p\omega^2\langle\bar{\Lambda}^p_u(\bar{\boldsymbol{k}})\rangle_{\Omega_p}\big)\Big]\langle\hat{u}(\hat{\boldsymbol{k}})\rangle=0, \tag{30}$$

where $\hat{A}(\hat{\boldsymbol{k}})$ is the estimate of the effective operator. By the definition of the effective operator,

$$\bar{A}(\hat{\boldsymbol{k}})\equiv\hat{A}(\hat{\boldsymbol{k}})-\sum_{p=1,2}v_p\big(\Delta\hat{\mu}_p i\hat{\boldsymbol{k}}\cdot\langle\bar{\boldsymbol{\Lambda}}^p_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_p}+\Delta\hat{\rho}_p\omega^2\langle\bar{\Lambda}^p_u(\bar{\boldsymbol{k}})\rangle_{\Omega_p}\big). \tag{31}$$

Then, the formulae for the effective shear modulus and mass density are

$$\bar{\mu}=\hat{\mu}-\sum_{p=1,2}v_p\Delta\hat{\mu}_p i\hat{\boldsymbol{k}}\cdot\frac{\langle\boldsymbol{\Lambda}^p_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_p}}{\hat{k}^2},\qquad \bar{\rho}=\hat{\rho}-\sum_{p=1,2}v_p\Delta\hat{\rho}_p\langle\Lambda^p_u(\bar{\boldsymbol{k}})\rangle_{\Omega_p}. \tag{32}$$

Finally, the self-consistency is that the estimates of the effective shear modulus and density in (32) are the true effective shear modulus and density, that is, $\hat{\mu}=\bar{\mu}$ and $\hat{\rho}=\bar{\rho}$. This statement is, in fact, identical to the hypothesis that the plane wave in the effective medium is coincident with the mean field. Then, two conditions that should be satisfied by the effective medium are

$$\sum_{p=1,2}v_p\overline{\Delta\mu}_p i\bar{\boldsymbol{k}}\cdot\langle\boldsymbol{\Lambda}^p_\varepsilon(\bar{\boldsymbol{k}})\rangle_{\Omega_p}=0,\qquad \sum_{p=1,2}v_p\overline{\Delta\rho}_p\langle\Lambda^p_u(\bar{\boldsymbol{k}})\rangle_{\Omega_p}=0. \tag{33}$$

Kim [1996] obtained the same expressions based more on physical intuition. Adding together the two equations in (33), one gets

$$\sum_{p=1,2}v_p\langle\bar{f}_p(\bar{\boldsymbol{k}}^i)\rangle_{\Omega_p}=0. \tag{34}$$

This implies that the effective medium in the EMT-1 is defined as the one in which the spatial and volume-fraction averaged forward scattering amplitude vanishes. In other words, since the scattered energy is proportional to the forward scattering amplitude (10), the effective medium is the medium in which there is no scattering on the average of the mean field by the constituents. This endows a full physical meaning to the effective medium and its properties in the EMT-1 formulation. One can find the origin of this idea in the solid-state physics problems [Soven 1967; Velicky et al. 1968]; it has also been used in electromagnetic problems [Stroud and Pan 1978; Niklasson et al. 1981].

It is noted that the EMT-1 is a possible dynamic generalization of the theory of Berryman [1979; 1980] and the EMT-2 of [Budiansky 1965; Hill 1965]. It is shown in the Appendix that the EMT-1 yields the same static effective density and shear modulus as the EMT-2 [Kanaun and Levin 2003], and their effective shear moduli are of course identical to those of [Budiansky 1965; Hill 1965; Sabina and Willis

1988]. It is interesting to note that even though these static theories ([Budiansky 1965; Hill 1965] versus [Berryman 1979; 1980]) treat the role of the matrix differently, they yield identical equations for the effective properties.

A parameter which remains unspecified so far in the EMT-1 is the shape and size of the representative volume for the matrix (Ω_m). It is shown also in the Appendix that a natural and meaningful choice is to use the same shape and size of the inclusions, that is, $\Omega_m = \Omega_i$. In the numerical calculations in the next section, Ω_m is taken to be a circular cylinder with the radius a.

4. Results and discussion

Numerical calculations are performed for two-phase composites having different combinations of constituent properties. The mechanical properties of the constituent materials are listed in Table 1 and the composites considered and their characteristics are given in Table 2. As shown in the third column of Table 2, these four composites are all distinctive in their ratios of densities and shear moduli. These distinctive combinations are selected to see how the two model predictions are different for composites with different dynamic characteristics. The effective wave speed and coherent attenuation are calculated for frequencies up to $k_1 a = 10$ for these composites and for different volume concentrations up to 60% of the inclusion phase.

In Figures 1–4, the results from the EMT-1 and EMT-2 are compared. The effective wave speed is normalized with the shear wave speed in the matrix and the coherent attenuation is also presented in a normalized form, $4\pi \mathrm{Im}[\bar{k}]/\mathrm{Re}[\bar{k}]$, which is called the specific attenuation capacity. It is quite surprising that both the wave speed and attenuation predicted by the two models generally agree very well for all composites considered. They are almost identical at frequencies where $k_1 a < 1$ and are also very close to each other at high frequencies. This is quite contrary to the expectation [Kanaun et al. 2004] that these two models would predict substantially different results in the wave speed and coherent attenuation since the matrix phase is treated quite differently in these two models, as a continuous phase in the EMT-2 as opposed to an equivalent inclusion in the EMT-1. Relatively large deviations are seen at the frequencies where the attenuation has a peak due to the rigid-body resonance of the inclusions (see Figures 2 and 3) and at some high frequencies where numerous elastic resonances occur (see Figure 4). Excessive motion will be set up in the inclusions at the resonance frequencies and the motion in the matrix will accordingly be large. Therefore, the relatively large deviations at and near resonance frequencies are due to the amplification effect of the resonance scattering.

To confirm what has been observed in Figures 1–4, these theories are further compared for two composites with extremely different constituent properties, one with hard and heavy inclusions ($\mu_2/\mu_1 = 100$

Material	Density (kg/m^3)	Shear modulus (GPa)	Wave speed (m/s)
Aluminum	2720	38.7	3772
Steel	7800	80.9	3220
Graphite	1310	21.0	4004
Titanium	4510	41.4	3030
SiC (SCS-6)	3200	182.0	7542

Table 1. Elastic properties of constituent materials.

Materials	ρ_2/ρ_1	μ_2/μ_1	Remarks
Steel/aluminum	2.9	2.1	$\rho_2 > \rho_1$, $\mu_2 > \mu_1$
Graphite/aluminum	0.48	0.54	$\rho_2 < \rho_1$, $\mu_2 < \mu_1$
SiC/titanum	0.71	4.4	$\rho_2 < \rho_1$, $\mu_2 > \mu_1$
Steel/SiC	2.44	0.45	$\rho_2 > \rho_1$, $\mu_2 < \mu_1$

Table 2. Ratios of density and shear moduli.

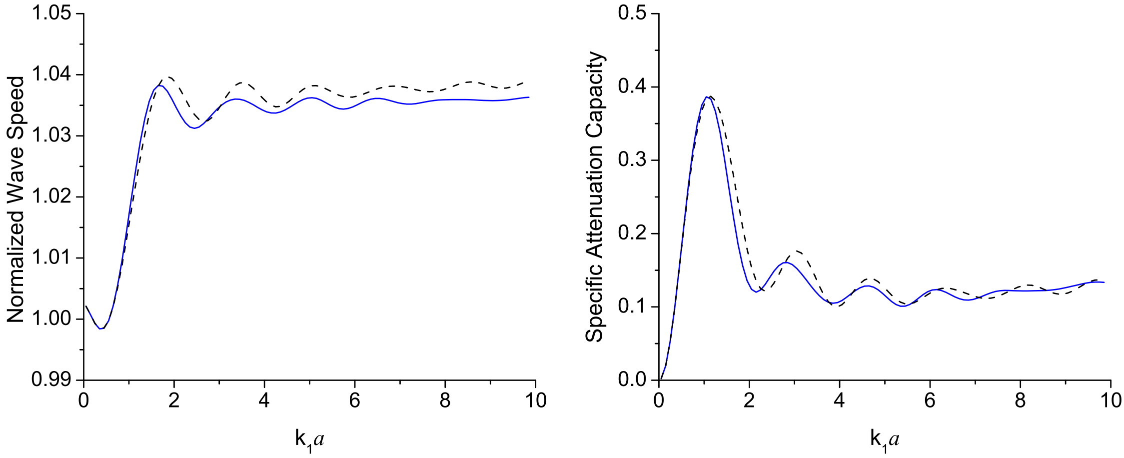

Figure 1. Average wave speed and coherent attenuation in a graphite/aluminum composite with 60% graphite fibers. The continuous line represents the EMT-1, and the dashed line the EMT-2.

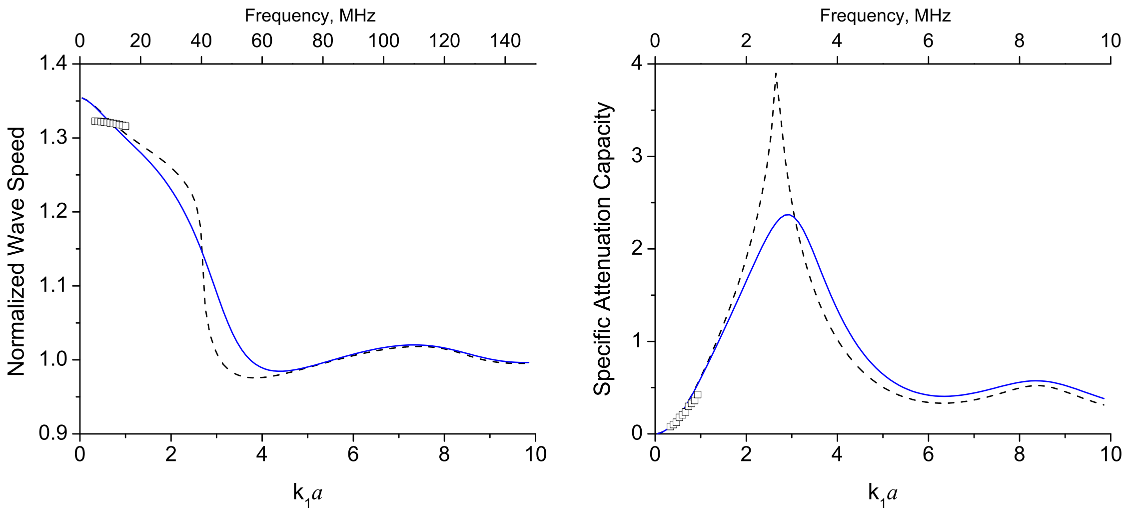

Figure 2. Average wave speed and coherent attenuation in a SiC/Ti composite with 35% SiC fibers. The continuous line represents the EMT-1, the dashed line the EMT-2, and the open squares are from an ultrasonic measurement in the frequency range 5–15 MHz.

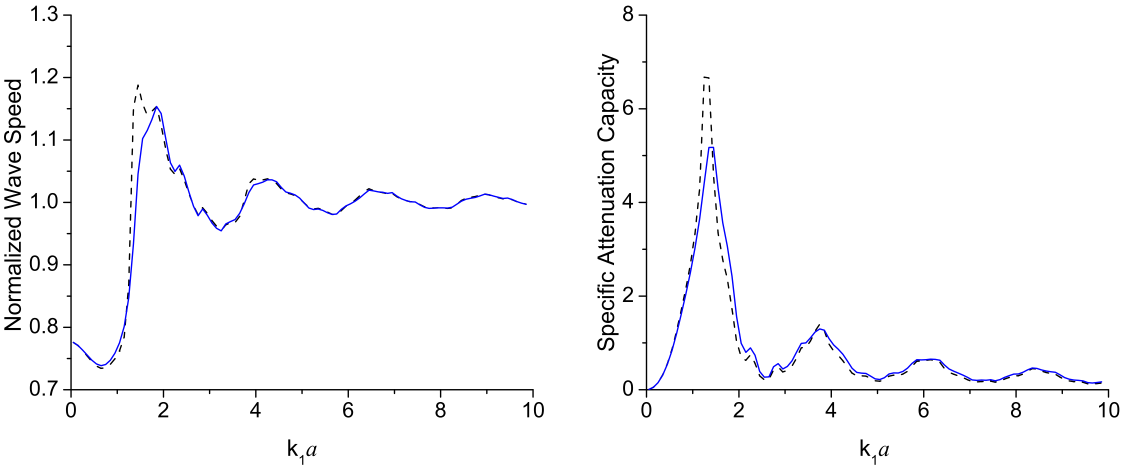

Figure 3. Average wave speed and coherent attenuation in a steel/SiC composite with 35% steel fibers. The continuous line represents the EMT-1, and the dashed line the EMT-2.

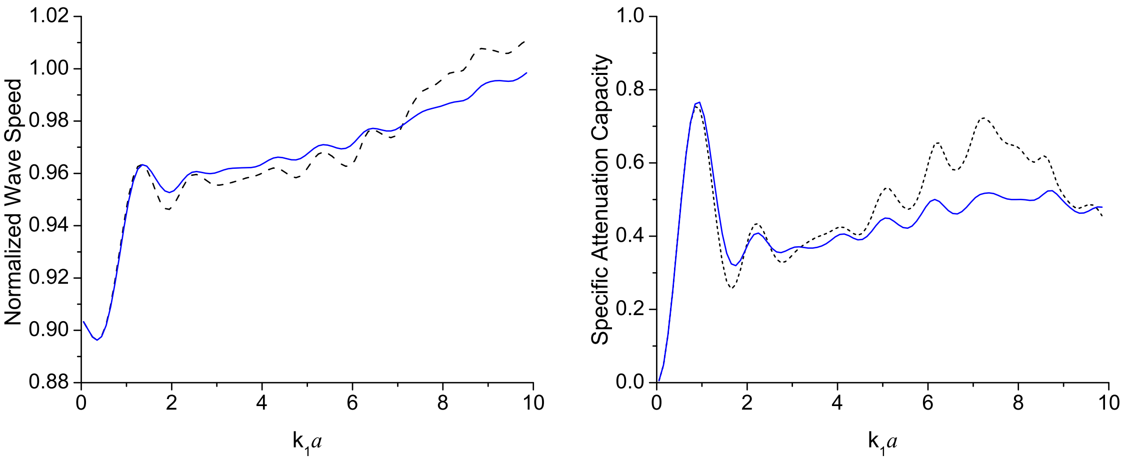

Figure 4. Average wave speed and coherent attenuation in a steel/aluminum composite with 25% steel fibers. The continuous line represents the EMT-1, and the dashed line the EMT-2.

and $\rho_2/\rho_1 = 10$) in Figure 5, and another with soft and light inclusions ($\mu_2/\mu_1 = 0.01$ and $\rho_2/\rho_1 = 0.1$) in Figure 6. Note that these are the numerical examples presented in [Kanaun and Levin 2003]. The frequency range in the numerical simulations is extended to $k_1 a = 50$ to see if any large deviation occurs at frequencies above $k_1 a = 10$. First of all, just as in Figures 1–4, the results from the two theories agree excellently for both cases, which strongly reinforces the conclusion made from the results in Figures 1–4, that the way the EMT-2 treats the matrix phase does not make any substantial difference. Secondly, both the wave speed and attenuation of the EMT-1 converge monotonically to their frequency-independent geometric optic limits: the wave speed of the matrix c_1 and a constant attenuation, respectively. The attenuation factors (of the EMT-1) divided by the inclusion volume fraction ($\mathrm{Im}[\bar{k}a]/v_2$) calculated at $k_1 a = 50$ and for different volume fractions turn out to be nearly a constant — the attenuation predicted

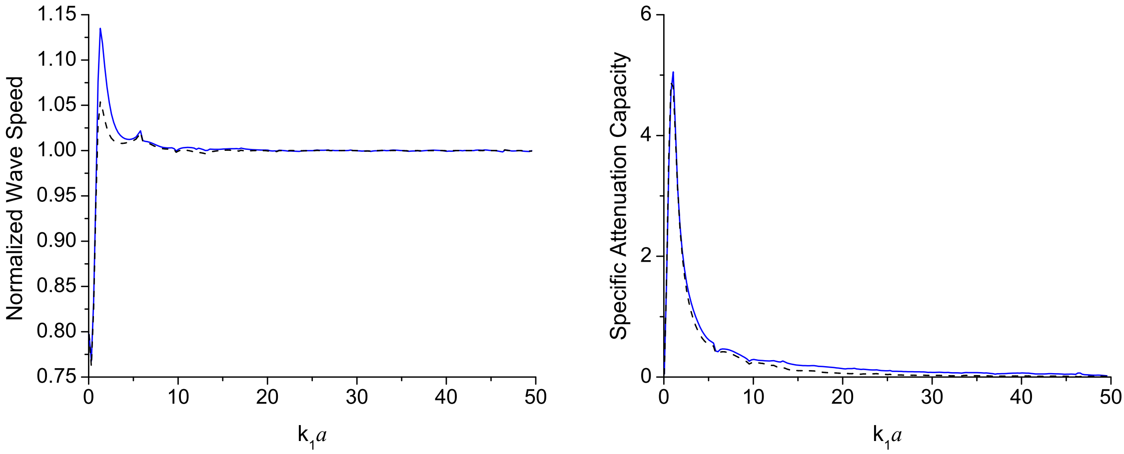

Figure 5. Average wave speed and coherent attenuation in a composite with 30% hard and heavy inclusions ($\mu_2/\mu_1 = 100$ and $\rho_2/\rho_1 = 10$). The continuous line represents the EMT-1, and the dashed line the EMT-2.

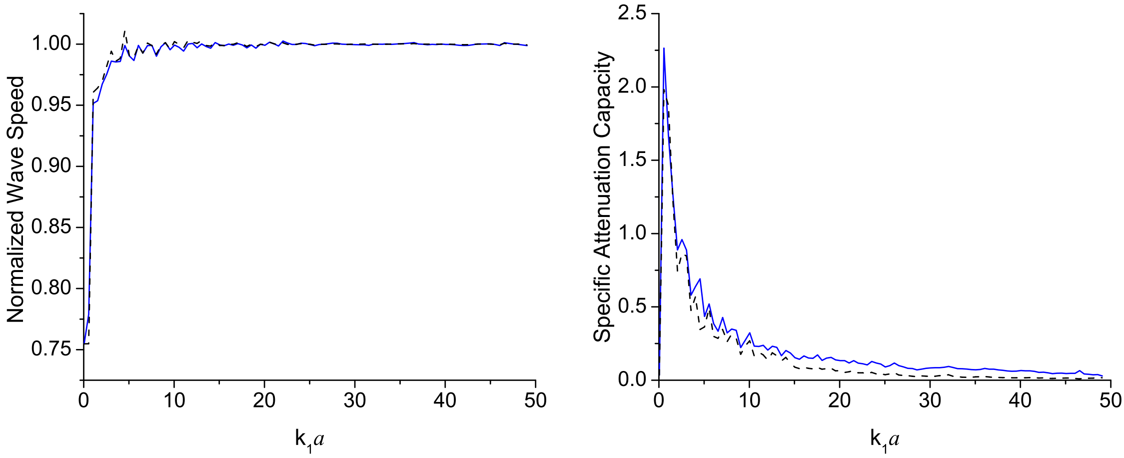

Figure 6. Average wave speed and coherent attenuation in a composite with 30% soft and light inclusions ($\mu_2/\mu_1 = 0.01$ and $\rho_2/\rho_1 = 0.1$). The continuous line represents the EMT-1, and the dashed line the EMT-2.

by the EMT-1 depends only on the volume fraction in the high-frequency limit. These are some expected properties of composites in the high-frequency limit [Kanaun and Levin 2003]. However, it is noted that these high-frequency limit values predicted by the EMT-1 could be inaccurate.

Finally, it should be pointed out that the comparisons given in Figures 5 and 6 are merely to demonstrate the equivalence of the two models. In this high-frequency range, the multiple scattering is so strong that waves quickly lose their mutual coherence, turning to a noise-like random field (called a diffuse field [Sheng 1995]) in a few wavelengths of propagation, and only the mean energy density of this random field is physically meaningful. Therefore, the calculated wave speed and attenuation of coherent waves in this high-frequency range (above $k_1a = 10$) are purely mathematical with no connection to measurable physics. Furthermore, at these high frequencies, the intrinsic material absorption of constituents will be dominant over the coherent attenuation. All these figures show that major changes in the effective properties occur at frequencies below $k_1 a = 10$ and thus it is believed that this is the frequency range where a micromechanical theory is useful, not the high-frequency limit in which the effective properties have no relation to the randomness of the composite. Note also that typical ultrasonic measurements are usually limited to far below $k_1 a = 10$. Again, therefore, the calculation and discussion of the effective properties of coherent waves in this high-frequency region are nonsensical. For all these reasons, regardless whether the EMT-1 is correct or incorrect in the high-frequency limit, it would be more sensible to limit the useful and physically meaningful frequency range of the EMT-1 to below $k_1 a = 10$.

In Figure 2, the wave speed and attenuation calculated for the SiC/Ti composite are compared with the experimental data (the open squares) in the range 5–15 MHz which corresponds to $k_1 a = 0.35$–1.05 [Kim 2010]. In this frequency range, the predictions from the two theories are indistinguishable, and the experimental wave speed shows a nearly constant wave speed ($\bar{c}/c_1 = 1.322$–1.316). This is close to the lower bound static wave speed, $\bar{c}/c_1 = 1.32$, which is the result of the well-isolated fiber arrangement in this composite (see, for example, the micrograph in [Huang and Rokhlin 1995]). Therefore, as mentioned earlier, the EMTs that implicitly assume an aggregated (or granular) microstructure [Yonezawa and Cohen 1983] are not suitable for predicting the effective wave speed of this composite. The low-frequency wave speed in this composite can be better predicted by a model that assumes an isolated arrangement of fibers [Kim 2004]. Every micromechanical model assumes a certain form of the microstructure and as a natural result every model has its utility for the microstructure that it assumes. A blind comparison between theories and experimental data, without considering the microstructure of the sample under examination, will lead to a meaningless conclusion. The attenuation factors predicted by the EMTs are in good agreement with the experimental results.

5. Conclusion

Two effective medium models [Kim 1996; Kanaun and Levin 2003] are formulated in a consistent mathematical procedure. The major difference between these two models lies in how the continuous matrix phase is treated. In spite of the apparently significant difference in these formulations (especially in their averaging schemes), the numerical results show minor discrepancies for all four distinctive composites and in frequencies up to $k_1 a = 10$. This leads to the important conclusion that in the effective medium formulation the self-consistent embedding and the use of the fields in the inclusion obtained by solving the scattering problem in the effective medium are the core operations in which the effective properties are actually determined. Therefore, how the matrix phase is treated must be of minor importance and so could be details as to the representative volume for the matrix. This conclusion is fully supported by two composites with extremely different constituents properties. Deviations appear at the frequencies of inclusion resonances possibly due to their excessive motion. Both models are very efficient computationally compared to other models [Varadan et al. 1985] and take about the same computational cost. In summary, the recent formulation (the EMT-2) of [Kanaun and Levin 2003] does not seem to make an appreciable difference versus the earlier formulation (the EMT-1) of [Kim 1996]. This is analogous to the coincidence between different EMTs in the static limit. The same facts found for electromagnetic waves will be reported elsewhere. As to the question of which model is preferable, it is, of course, up to the reader's discretion to choose one between the two.

Appendix: The EMT-1 in the long wavelength limit

Let us first consider scattering of a plane SH wave ($u^{\text{in}} = \exp(ik_1 x)$) by a single elastic circular inclusion having a radius a embedded in an infinite elastic matrix. The scattered and refracted fields may be expressed by an infinite series of normal modes as [Eringen and Suhubi 1975]

$$u^{\text{sc}} = \sum_{n=0}^{\infty} A_n H_n(k_1 r)\cos n\theta, \tag{A.1}$$

$$u^{\text{re}} = \sum_{n=0}^{\infty} B_n J_n(k_2 r)\cos n\theta, \tag{A.2}$$

where $J_n(z)$ is the Bessel function of order n and $H_n(z)$ the Hankel function of the first kind of order n. Imposing the boundary conditions (continuities of axial displacement and axial shear stress), the scattering coefficients are obtained:

$$A_n = -i^n \varepsilon_n \frac{pJ_n'(\xi_1)J_n(\xi_2) - J_n(\xi_1)J_n'(\xi_2)}{pH_n'(\xi_1)J_n(\xi_2) - H_n(\xi_1)J_n'(\xi_2)}, \tag{A.3}$$

where ε_n is the Neumann factor, $p = \mu_1 k_1/\mu_2 k_2$, $\xi_1 = k_1 a$, and $\xi_2 = k_2 a$. In the long wavelength limit ($\xi_1, \xi_2 \to 0$), the scattering coefficients are

$$A_0 = -\frac{i\pi}{4}\left(\frac{\rho_2}{\rho_1} - 1\right)\xi_1^2 + O(\xi_1^4), \tag{A.4}$$

$$A_n = \frac{i\pi}{4^n n!(n-1)!}\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right)\xi_1^{2n} + O(\xi_1^{2n+1}\ln \xi_1), \qquad n \geq 1. \tag{A.5}$$

Hence, the first two terms in Equation (A.1) with A_0 and A_1 are the leading terms in this limit. The forward scattering amplitude is approximated as

$$f_p(\boldsymbol{k}_1^i) \approx A_0 - iA_1. \tag{A.6}$$

There are two ways to derive the quasistatic effective properties: one using (33) and the other using (34). The second way, which is simpler and is essentially the method of Berryman [1979; 1980], is adopted. Consider scattering by the representative volumes for the matrix (Ω_m) and inclusion (Ω_i). It is first assumed that these volumes are circular and have radii of a_1 and a_2. Using (A.4)–(A.6) together with (34), the following two formulae are obtained:

$$\sum_{p=1,2} v_p\left(\frac{\mu_p - \bar{\mu}}{\mu_p + \bar{\mu}}\right)(\bar{k}a_p)^2 = 0, \qquad \sum_{p=1,2} v_p\left(\frac{\rho_p}{\bar{\rho}} - 1\right)(\bar{k}a_p)^2 = 0. \tag{A.7}$$

Note that the overbar indicates that the scattering occurs in the effective medium. It is obvious that in order for these formulae to be meaningful and consistent, the size of the matrix inclusion should be equal to that of the original inclusions, so $a_1 = a_2$. This means that in the EMT-1 the size of the original inclusion should be taken as the unit volume for all constituents in the composite. Then, one obtains the effective static properties:

$$\sum_{p=1,2} v_p\left(\frac{\mu_p-\bar{\mu}}{\mu_p+\bar{\mu}}\right)=0, \qquad \sum_{p=1,2} v_p\left(\frac{\rho_p}{\bar{\rho}}-1\right)=0. \tag{A.8}$$

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