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ANDREY V. PYATIGORETS, MIHAI O. MARASTEANU,
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The problem of thermal stress development in composite structures containing one linear isotropic viscoelastic phase is considered. The time-temperature superposition principle is assumed to be applicable to the viscoelastic media under consideration. Two methods of solution based on the reduction of the original viscoelastic problem to the corresponding elastic one are discussed. It is argued that the use of a method based on the Laplace transform is impractical for some problems, such as those involving viscoelastic asphalt binders. However, the solution can be obtained by means of the second method considered in the paper, the Volterra correspondence principle, in which the integral operator corresponding to the master relaxation modulus is presented in matrix form. The Volterra principle can be applied to the solution of viscoelastic problems with complex geometry if the analytical solution for the corresponding elastic problem is known. Numerical examples show that the proposed method is simple and accurate. The approach is suitable to the solution of problems involving viscoelastic materials, whose rheological properties strongly depend on temperature. In particular, it can be found useful in the analysis of the low-temperature thermal cracking of viscoelastic asphalt binders.

1. Introduction

Composite structures consisting of elastic and viscoelastic materials play an important role in today's engineering applications. Such structures are used in various environmental conditions, often involving temperature variations. Due to the effects of thermal expansion and contraction of the materials, it is important to be able to predict the behavior of composite structures subjected to varying temperature. For the case of elastic media, temperature effects can be easily incorporated in the analysis since the material's elastic properties are time- and temperature independent. However, temperature changes in viscoelastic media cause changes in the material's rheological properties, which makes the analysis of thermoviscoelastic problems more complex.

Methods of solution of linear viscoelastic problems fall into two major categories: approaches based on the approximate and iterative solutions in the time domain and methods based on the solution of the corresponding elastic problem formulated in the frequency (transformed) domain. Time-stepping algorithms (see [Mesquita et al. 2001], for instance), widely used in the finite element and boundary element methods, belong to the first category. The second category includes approaches based on the application of integral transformations, such as the Laplace or Fourier transforms, to reduce a time-dependent problem to one that depends on the transform parameter only [Lee et al. 1959; Findley et al. 1976]. In the case of nonisothermal conditions, some of these methods can be adopted to the solution of

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thermoviscoelastic problems by using, for example, the time-temperature superposition (TTS) principle, whereas other methods fail.

TTS is widely used to describe the effect of temperature on the mechanical response of a certain class of viscoelastic materials [Ferry 1961; Findley et al. 1976; Wineman and Rajagopal 2000]. Theoretical and experimental results indicate that for such materials the response to a load at high temperature over a short period of time is identical to that at lower temperature but over a longer period, and vice versa. Thus, at any temperature, a material property such as relaxation modulus or creep compliance can be expressed in terms of the material property at the reference temperature and another material property known as the time-shift factor. Following [Schwarzl and Staverman 1952], materials for which the mechanical behavior due to temperature is equivalent to a corresponding shift in the logarithmic time scale are called thermorheologically simple.

A number of studies have been devoted to the solution of thermoviscoelastic problems with the use of TTS. Taylor et al. [1970] proposed an integration scheme that was used together with the Prony series approximation of viscoelastic properties to obtain a recursive expression for the solution of thermoviscoelastic problems at each next time step. Chien and Tzeng [1995] considered the problem of a thick laminated composite cylinder under elevated temperature change. The dependency of creep compliances on time was presented in the form of a simple power function that allowed them to obtain closed form relations in the Laplace domain and use analytical Laplace inversion. Ekel'chik et al. [1994] studied the problem of a thick-walled orthotropic viscoelastic ring; the authors applied a procedure of numerical inversion of the Laplace transform after the solution of the corresponding problem was obtained. A review of some other methods of solving quasistatic problems of linear thermoviscoelasticity is provided in [Bykov et al. 1971].

The aim of this work is to demonstrate, for the first time, that the use of the Volterra correspondence principle can be an effective tool for solving complex thermoviscoelastic problems, such as those encountered in pavement engineering. While the Volterra correspondence principle is well known and has been applied to the solution of a number of viscoelastic problems (in [Rabotnov 1966; Khazanovich 2008], for instance), its application to problems involving thermal effects is, to our knowledge, new. The approach proposed here employs a discrete representation of time-varying functions, and the integral operator corresponding to the master relaxation modulus is presented in matrix form. This technique is a modification of the approach of [Bažant 1972]. Special attention is devoted to thermal stress development in the binders. If tensile stress in asphalt binders or mixtures exceeds their strength upon cooling, cracking occurs. The importance of the study of low-temperature cracking of asphalt pavements stems from the fact that this is one of the dominant distresses occurring in cold climates, where temperature variation can be significant over short periods of time.

This paper is organized as follows. Section 2 summarizes the two approaches used for reducing a viscoelastic problem to an elastic one. Section 3 provides the basic definitions of time-temperature superposition and the description of functional dependencies used for materials characterization. Sections 4 and 5 deal with the relaxation operator and its discrete representation. A simple problem of a restrained bar is considered in Section 6. The solution of this problem can be directly found in the time domain, which allows for comparison results and evaluating the accuracy of the present approach. Section 7 deals with the problem of a composite ring or cylinder undergoing thermal loading, and includes results for thermal stresses and strains in the Asphalt Binder Cracking Device [Kim 2005].

2. Elastic-viscoelastic correspondence principles

One of the most powerful approaches in solving linear viscoelastic problems consists in the reduction to a corresponding elastic problem. This can provide an analytical solution for time-dependent problems and usually is simpler to implement than numerical algorithms for calculating time-varying fields in homogeneous or heterogeneous viscoelastic materials.

Two techniques can be used to obtain a corresponding elastic problem. The first is based on the application of the Laplace transform to all governing equations and boundary conditions describing a viscoelastic body in equilibrium. The Laplace transform of a real function $f(t)$ is defined as

$$f^*(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt \tag{1}$$

where s is the complex transform parameter.

In the case of simple deformations, such as uniaxial tension or shear, the Boltzmann superposition principle for linear viscoelastic materials [Findley et al. 1976] states that the stress σ is related to the mechanical strain ε by

$$\sigma(t) = \int_0^t E(t-\tau)\frac{\partial\varepsilon(\tau)}{\partial\tau}d\tau, \qquad \varepsilon(t) = \varepsilon_{\text{tot}}(t) - \varepsilon_{\text{th}}(t), \tag{2}$$

where E is the relaxation modulus, t is time, ε_{tot} results from a given displacement field, and ε_{th} is related to the effect of thermal expansion. In more general cases, the relaxation modulus E has to be replaced by a tensor of creep functions, and stress and strain tensors should be used. Applying the Laplace transform to the convolution integral (2), an expression formally equivalent to Hooke's law is obtained:

$$\sigma^*(s) = sE^*(s)\varepsilon^*(s). \tag{3}$$

The idea of the approach is to formulate the corresponding elastic problem by replacing time-varying parameters with their transformed counterparts, solve the obtained problem, and apply an inversion of the Laplace transform to arrive at the time domain solution. However, the use of this approach may be limited by the fact that strain or stress histories and/or material properties may be complicated functions of time, whose Laplace transform cannot be found in a closed form. This makes impossible to obtain the solution of the corresponding elastic problem.

The Volterra correspondence principle [1913] can be used in the case when the Laplace transform is not applicable [Rabotnov 1966]. The principle is based on the representation of hereditary integrals of type (2) as integral operators acting on time-varying functions. This technique also leads to the reduction of a time-dependent problem to a corresponding elastic problem.

Consider the case of a simple deformation caused by uniaxial tension of a linear viscoelastic body. Evaluating the integral (2) by parts, it can be found that

$$\sigma(t) = E(0)\varepsilon(t) - \int_0^t \varepsilon(\tau)\,dE(t-\tau) \equiv \tilde{E}\cdot\varepsilon(t) \tag{4}$$

where $\tilde{E}$ is an integral operator acting on a function of time, and $E(0)$ is the value of the relaxation modulus at zero time. It is seen that (4) is formally equivalent to Hooke's law. The operator $\tilde{E}$ is referred to as the relaxation operator. In more general cases, it can be shown [Rabotnov 1966] that other viscoelastic parameters, such as creep compliance, viscoelastic bulk or shear moduli, can be expressed in an operator form in such a way that the constitutive equations written with the use of these operators resemble elastic constitutive equations. This implies that in order to solve a viscoelastic problem, one has to solve a corresponding elastic problem, and substitute the elastic properties with the corresponding integral operators in the obtained elastic solution for each physical variable of interest. Since superposition holds for elasticity, the viscoelastic solution for any physical variable $\gamma(t)$ can be presented as a combination of functions of one or more commutative integral operators[1] acting on known functions of time $f_i(t)$, which can be time-dependent boundary parameters, body forces, temperature, etc. In the case of an isotropic viscoelastic material, such a relation can be presented as

$$\gamma(t) = \sum_i F_i[\tilde{E}, \tilde{\nu}] \cdot f_i(t), \tag{5}$$

where $\tilde{\nu}$ is the integral operator corresponding to the viscoelastic Poisson's ratio and the F_i are functions of operators. Analytical evaluation of the result of the action of integral operators on known functions of time is often a complicated and impractical task, and numerical approximations are used to represent integral operators in matrix form. Thus, the solution of a time-dependent problem is formulated in terms of matrix operators acting on vectors obtained from the discretization of the time functions:

$$\underline{\gamma} = \sum_i F_i\big[\underline{\underline{E}}, \underline{\underline{\nu}}\big] \cdot \underline{f}_i, \tag{6}$$

where $\underline{\underline{E}}$ and $\underline{\underline{\nu}}$ are matrix operators and $\underline{\gamma}$ and $\underline{f}$ are vectors representing the discrete values of functions $\gamma(t)$ and $f(t)$. In the present work it is assumed that the viscoelastic Poisson's ratio ν does not change in time; hence the corresponding matrix operator is represented as ν times the identity matrix.

3. Time-temperature superposition

The problem of deformation of viscoelastic media coupled with temperature variation can be treated with the use of the time-temperature superposition principle [Ferry 1961; Findley et al. 1976; Wineman and Rajagopal 2000]. It follows from the principle that real time t in (2) has to be replaced by reduced time ξ to account for the change in material's properties with temperature. The constitutive equation (2) is now written as

$$\sigma(\xi) = \int_0^{\xi} E(\xi - \xi') \frac{\partial \varepsilon(\xi')}{\partial \xi'} d\xi', \tag{7}$$

where $\varepsilon(\xi) = \varepsilon_{\text{tot}}(\xi) - \varepsilon_{\text{th}}(\xi)$. For transient temperature conditions, the reduced time is connected to the real time by an integral:

$$\xi(t) = \int_0^t \frac{d\tau}{a_T[T(\tau)]} \tag{8}$$

where a_T is shift function (factor) obtained during the process of the construction of the material's master curve. Different functions can be used to fit the plot for the shift factor. The shift factor described by the Williams–Landau–Ferry equation [Williams et al. 1955] is often used to describe the behavior

[1] The term "commutativity" is understood here in the sense used in [Rabotnov 1966; Arutyunyan and Zevin 1997]. Note that the operators are commutative only for the case of nonaging viscoelasticity; see the second of these references.

of polymers. In this work, a linear function is adopted to describe the decimal logarithm of the shift factor a_T:

$$\log a_T = C_1 - C_2 T, \tag{9}$$

where C_1 and C_2 are constants determined through least squares fitting. It was found that (9) provides accurate fitting for the experimental data used in this work.

The thermal strain ε_{th} is given by

$$\varepsilon_{\text{th}} = \alpha \Delta T, \tag{10}$$

where α is the coefficient of thermal expansion (assumed to be constant) and ΔT is the temperature change. For simplicity, the temperature variation is considered to be linear in time:

$$\Delta T(t) = C_0 t, \tag{11}$$

where C_0 is the constant temperature rate. (Nonlinear temperature changes can be treated similarly.) The temperature at time t is thus

$$T = T_0 + \Delta T, \tag{12}$$

where T_0 is the reference temperature.

Combining equations (8)–(11), one obtains the relation between real time t and reduced time ξ, as well as relations between thermal strain ε_{th} and reduced time ξ. These relations are

$$\xi(t) = A_1(1 - 10^{C_0 C_2 t}) \quad \text{with } A_1 = -\big(C_0 C_2 10^{(C_1 - C_2 T_0)} \ln 10\big)^{-1}, \tag{13}$$

$$t(\xi) = \frac{1}{C_0 C_2} \log(1 - \xi/A_1), \quad \varepsilon_{\text{th}}(\xi) = \frac{\alpha}{C_2} \log(1 - \xi/A_1). \tag{14}$$

Because the time shift is zero at the reference temperature, we have $10^{(C_1 - C_2 T_0)} = 1$ in (13).

For asphalt binders, the master relaxation modulus curve is obtained (see Appendix) using the CAM model of [Marasteanu and Anderson 1999], given by the equation

$$E(\xi) = E_g \left(1 + \left(\frac{\xi}{t_c}\right)^v\right)^{-w/v} \tag{15}$$

where E_g is the glassy modulus, t_c is the crossover time, and v and w are the parameters of the model, which, in general, are some rational numbers. The CAM model is considered an effective phenomenological model for characterizing the linear viscoelastic behavior of asphalt binders at low temperatures.

4. Relaxation operator with varying temperature

The approaches presented in Section 2 can be used to solve problems of thermal stress evolution in viscoelastic composite materials. However, when the CAM model (15) is adopted to describe viscoelastic material behavior, its Laplace transform cannot be found in a closed form. Another degree of complexity is introduced by the fact that the Laplace transform of the thermal strain described by (14) is a complex-valued function of s. Because of this, it seemed impractical to use the elastic-viscoelastic correspondence principle based on the Laplace transform for the analysis of the problem. Therefore, we use here the integral operator representation approach.

Because for each value of reduced time ξ one can find a corresponding value of real time t (and vice versa), the stress and strain in the reduced time domain can be replaced by their values found for the corresponding real time: $\sigma(\xi) \equiv \sigma(\xi(t)) \equiv \sigma(t)$, $\varepsilon(\xi) \equiv \varepsilon(\xi(t)) \equiv \varepsilon(t)$. Therefore, it is possible to simplify expression (7):

$$\sigma(t) = \int_0^t E\big(\xi(t) - \xi'(\tau)\big) \cdot \frac{\partial \varepsilon(\tau(\xi'))}{\partial \tau(\xi')} \underbrace{\frac{\partial \tau(\xi')}{\partial \xi'} \cdot \frac{\partial \xi'}{\partial \tau(\xi')}}_{\equiv 1} d\tau(\xi') = \int_0^t E\big(\xi(t) - \xi'(\tau)\big) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau. \tag{16}$$

The integral (16) is now similar to (2), with the only difference that the master relaxation modulus is still given in the reduced time domain. This allows us to use the stress-strain relation in the form

$$\sigma(t) = \tilde{E} \cdot \varepsilon(t), \tag{17}$$

where the integral operator (relaxation operator) $\tilde{E}$ is now defined as

$$\tilde{E} \cdot f(t) \equiv E(0) \cdot \left(f(t) - \frac{1}{E(0)} \int_0^t f(\tau)\, dE\left[\xi(t) - \xi'(\tau)\right] \right). \tag{18}$$

5. Matrix representation of the relaxation operator

For the analysis of viscoelastic problems considered in the present work, a procedure of numerical solution based on the representation of relaxation operator $\tilde{E}$ given by (18) as a matrix operator is adopted.

Consider a discrete set of time moments $t_0, t_1, \ldots, t_n$, with $t_0 = 0$ and $t_n \equiv t$. Strain ε, as well as temperature variation $\Delta T(t)$ are presented as column vectors

$$\underset{\sim}{\varepsilon} \equiv \varepsilon_k = \varepsilon(t_k) \quad \text{and} \quad \underset{\sim}{\Delta T} \equiv \Delta T_k = T(t_k) - T(t_0). \tag{19}$$

To obtain a matrix representation for the integral operator $\tilde{E}$, we integrate numerically using the trapezoid rule [Bažant 1972]. Thus, the expression $\sigma = \tilde{E} \cdot \varepsilon$ for each moment of time k can be written as

$$\begin{aligned} \sigma_k \equiv (\tilde{E} \cdot \varepsilon)_k &= E(0) \cdot \left(\varepsilon_k - \frac{1}{E(0)} \int_0^{t_k} \varepsilon(\tau)\, dE\big(\xi(t) - \xi'(\tau)\big) \right) \\ &= E_{k,k} \varepsilon_k - \frac{1}{2} \sum_{i=1}^{k} (\varepsilon_i + \varepsilon_{i-1})\, (E_{k,i} - E_{k,i-1}), \end{aligned} \tag{20}$$

where $E_{k,i} \equiv E\big(\xi(t_k) - \xi(t_i)\big)$ and $E_{k,k} \equiv E(0)$. Recalling that $\varepsilon_0 \equiv 0$ and rearranging terms in (20), the result is a column vector $\underset{\sim}{\sigma}$ consisting of terms

$$\sigma_k = \frac{1}{2} \left(\sum_{i=1}^{k-1} \big(E_{k,i-1} - E_{k,i+1}\big)\, \varepsilon_i + (E_{k,k-1} + E_{k,k}) \varepsilon_k \right), \qquad k = 1, \ldots, n, \tag{21}$$

or, in matrix form,

$$\underset{\sim}{\sigma} = \underset{\approx}{E} \cdot \underset{\sim}{\varepsilon}. \tag{22}$$

The matrix $\underline{\underline{E}}$ is lower triangular:

$$\underline{\underline{E}} = \begin{vmatrix} E_{1,0}+E_{1,1} & 0 & 0 & \dots & 0 \\ E_{2,0}-E_{2,2} & E_{2,1}+E_{2,2} & 0 & \dots & 0 \\ E_{3,0}-E_{3,2} & E_{3,1}-E_{3,3} & E_{3,2}+E_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{n,0}-E_{n,2} & E_{n,1}-E_{n,3} & E_{n,2}-E_{n,4} & \dots & E_{n,n-1}+E_{n,n} \end{vmatrix}, \tag{23}$$

and

$$\underline{\varepsilon} = \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{vmatrix} = \begin{vmatrix} \varepsilon_{\text{tot}\,1} \\ \varepsilon_{\text{tot}\,2} \\ \vdots \\ \varepsilon_{\text{tot}\,n} \end{vmatrix} - \alpha \begin{vmatrix} \Delta T_1 \\ \Delta T_2 \\ \vdots \\ \Delta T_n \end{vmatrix}. \tag{24}$$

The lower triangular structure of $\underline{\underline{E}}$ represents the history-dependent nature of viscoelastic problems. Thus, it is obvious that for longer time histories the size of the matrix $\underline{\underline{E}}$ grows, which may reduce the numerical efficiency. However, it has been found that for the problems considered in the present study, a relatively small number of subdivisions in the trapezoidal rule is sufficient to obtain accurate solutions over a large interval of temperatures (for instance, ranging from −50°C to +20°C).

In general case of a boundary value problem describing an isotropic viscoelastic material, two integral operators are present in the solution. The problem can be simplified greatly, if one assumes that Poisson's ratio is independent of time and temperature. This allows using only relaxation operator $\tilde{E}$ in the analysis of the problem. The assumption of constant Poisson's ratio is employed in the present work due to the following facts. Experimental data reveal that the Poisson's ratio of viscoelastic asphalt mixtures varies very little at low temperatures, and therefore it can be considered constant [Huang 1993; Levenberg and Uzan 2007]. Furthermore, the study of asphalt binders in [Marasteanu and Anderson 2000] indicates that binders' Poisson's ratio can be reasonably assumed constant at low temperatures. In addition, the results of numerical simulations conducted in the present work also indicate that using constant but different values of Poisson's ratio within a realistic range has very small effect on the final results. Therefore, the solution for viscoelastic problems can be represented in discrete form as a function of the only matrix operator $\underline{\underline{E}}$ multiplied by a vector of a known time function, e.g.,

$$\underline{\sigma} = F\left[\underline{\underline{E}}\right] \cdot \underline{\varepsilon}_{\text{tot}} + G\left[\underline{\underline{E}}\right] \cdot \underline{\Delta T} \tag{25}$$

where F and G are some functions of matrices. In many cases it is much easier to calculate rational functions of matrices than functions of integral operators.

6. Analysis of a viscoelastic restrained bar

Consider the problem of thermal stress evolution in a restrained isotropic homogeneous viscoelastic bar (Figure 1). The total strain in this case is zero, and $\varepsilon = -\alpha \Delta T$. The problem is described by (16), which simplifies to the form

$$\sigma(t) = -\alpha C_0 \int_0^t E\big(\xi(t) - \xi'(\tau)\big)\, d\tau. \tag{26}$$

Figure 1. Geometry for the problem of a restrained bar.

Due to the simple structure of the integrand in (26), one can use various numerical integration techniques to obtain the thermal stress in the bar. Comparison of the results of integration with those obtained from (22)–(24) allows us to evaluate the accuracy of the present approach for the considered class of functions describing material properties (CAM model).

In the present analysis, we used the parameters obtained by the procedure in the Appendix for a modified asphalt binder tested at temperatures −24°C, −30°C, and −36°C.[2] These parameters are as follows:

(CAM model) $E_g = 3\,\text{GPa}$, $t_c = 1.04655 \cdot 10^{-6}\,\text{sec}$, $v = 0.177585$, $w = 0.359506$,

(Other) $\alpha = 2 \times 10^{-4}\,°\text{C}^{-1}$, $C_2 = 0.18562\,°\text{C}^{-1}$, $C_0 = -10°\text{C/hour}$, $T_0 = 20°\text{C}$.

The tensile stress in the bar caused by the temperature drop was determined by the present approach and by numerical integration using 24-point Gaussian quadrature, a technique that provides highly accurate results and whose application to this problem was originally proposed by Voller [Marasteanu et al. 2004].

The comparative plots are presented in Figure 2. It is apparent from the figure that both approaches produce virtually identical results. By testing the accuracy of the present method on various asphalt

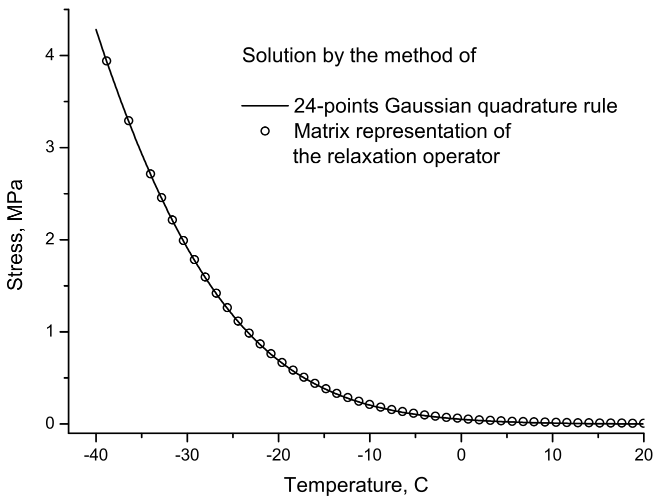

Figure 2. Comparison of stresses obtained from (26) using 24-point Gaussian quadrature and by the method of matrix representation of the relaxation operator $\tilde{E}$.

[2]The binder was modified with styrene-butadiene diblock copolymer (Black Max™) and produced by Husky Energy (Canada). The binder's performance grade is PG 64-34 [AASHTO 2005]. The binder was tested in the Pavement Laboratory of the Department of Civil Engineering at the University of Minnesota. More information about its properties can be found in [Marasteanu et al. 2007]

binders described by the CAM model (15), it was found that quite accurate results can be obtained for the number of integration points $n \gtrsim 1000$ used in the trapezoid rule. The results presented in Figure 2 are obtained with $n = 2000$. Note that for longer time histories more segments in the trapezoidal rule have to be used to obtain accurate results. For the case when the properties of viscoelastic materials cannot be described by the CAM model and/or a linear shift function, the use of a more accurate integration rule such as Simpson's rule may be required.

7. Analysis of viscoelastic composite cylinder/ring

Consider the problem of a composite cylinder subjected to uniform temperature variation. The geometry of the problem is given in Figure 3. The hollow inner cylinder (inclusion) is assumed to be elastic and the outer (binder) to be linearly viscoelastic. The plane strain condition is assumed, but the plane stress solution can also be easily obtained by replacing the elastic Young's moduli and Poisson's ratios [Barber 1992] in the corresponding elastic problem. A perfect bond between the binder and inclusion is assumed; the distribution of circumferential stresses and strains in each cylinder is of primary interest.

To obtain the solution of the corresponding elastic problem, we assume the binder is an elastic material with Young's modulus E_b, Poisson's ratio ν_b, and coefficient of thermal expansion α_b. The material properties of the inner cylinder (inclusion) are denoted by E_i, ν_i and α_i. The elastic solution for the circumferential stresses in the binder can be obtained in the form

$$\sigma_{\theta\theta}^{\text{bind}}(r) = \frac{a(r)E_i + b(r)E_b}{cE_i + E_b} E_b \Delta T, \tag{27}$$

where c is a constant and the parameters a and b depend on the radius. Expressions for these parameters are given in the Appendix as relations (A3) and (A4), following an outline of the derivation of (27).

Assuming that the Poisson's ratio of the binder does not vary with time and temperature, the solution for time-dependent circumferential stresses corresponding to the viscoelastic problem is

$$\underline{\sigma}_{\theta\theta}^{\text{bind}}(r) = \left[aE_i\underline{\underline{I}} + b\underline{\underline{E}}_b\right] \cdot \left[cE_i\underline{\underline{I}} + \underline{\underline{E}}_b\right]^{-1} \cdot \underline{\underline{E}}_b \underline{\Delta T}, \tag{28}$$

where $\underline{\sigma}_{\theta\theta}^{\text{bind}}$ and $\underline{\Delta T}$ are vectors of size n, $\underline{\underline{I}}$ is identity $n \times n$ matrix, and $\underline{\underline{E}}_b$ is $n \times n$ matrix of relaxation functions given by (23). It is seen from (28) that the calculation of the inverse of matrix $\underline{\underline{M}} = \left(cE_i\underline{\underline{I}} + \underline{\underline{E}}_b\right)$ is required. This operation can be easily performed, since $\underline{\underline{M}}$ is a lower triangular matrix.

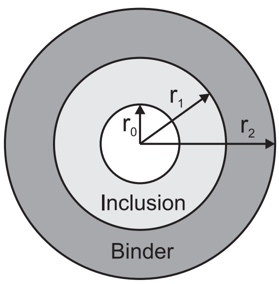

Figure 3. Geometry of an axisymmetric problem with a hollow inner part.

The solutions for circumferential strains in the binder and in the inclusion can be obtained in a form similar to (28). In particular, the total circumferential strain $\underline{\varepsilon}_{\theta\theta}^{\text{inc}}(r)$ in the inclusion is

$$\underline{\varepsilon}_{\theta\theta}^{\text{inc}}(r) = \left[\alpha_i F_1\left(r, \underline{\underline{E}}_b\right) + \alpha_b F_2\left(r, \underline{\underline{E}}_b\right)\right] \cdot \underline{\Delta T} \tag{29}$$

where we omit the subscript "tot", and the matrix functions $F_1\big(r, \underline{\underline{E}}_b\big)$ and $F_2\big(r, \underline{\underline{E}}_b\big)$ are given by

$$\begin{aligned} F_1\big(r, \underline{\underline{E}}_b\big) &= (1+\nu_i)\underline{\underline{I}} - 2\frac{r_1^2}{r^2}(1-\nu_i^2)\big[d\underline{\underline{I}} + gE_i\big(\underline{\underline{E}}_b\big)^{-1}\big]^{-1}, \\ F_2\big(r, \underline{\underline{E}}_b\big) &= 2\frac{r_1^2}{r^2}(1-\nu_i)(1+\nu_b)\big[d\underline{\underline{I}} + gE_i\big(\underline{\underline{E}}_b\big)^{-1}\big]^{-1}, \end{aligned} \tag{30}$$

with constants d and g given as (A5) in the Appendix.

Comparison with the analytical solution. The calculations presented in the analysis of the bar problem (Section 6) were simple and did not involve evaluating the inverse of $\tilde{E}$. The present problem of a composite cylinder (Figure 3) is a more realistic test of accuracy. The plane strain elastic solution for this problem is given by (28)–(30). The results are compared with an analytical solution obtained by the application of the Laplace transform. To be able to use the Laplace transform, the expression for the shift factor a_T given by (9) is replaced with linear function

$$\hat{a}_T(T) = \hat{C}_1 - \hat{C}_2 T, \tag{31}$$

(The hat over the quantities means that they have a different interpretation from before — say in (9).) Using (31) and (10), the reduced time becomes

$$\hat{\xi} = -\frac{1}{\hat{C}_0\hat{C}_2} \ln\left|1 + \frac{\hat{C}_2(T_0 - T)}{\hat{C}_1 - \hat{C}_2 T_0}\right| \tag{32}$$

and we have

$$\hat{t}(\xi) = \frac{\hat{C}_1 - \hat{C}_2 T_0}{\hat{C}_0\hat{C}_2}\big(1 - \exp(-\hat{C}_0\hat{C}_2\xi)\big), \quad \Delta\hat{T}(\xi) = \left(\frac{\hat{C}_1}{\hat{C}_2} - T_0\right)\big(1 - \exp(-\hat{C}_0\hat{C}_2\xi)\big). \tag{33}$$

Despite the fact that the expressions (31)–(33) do not have a physical interpretation (e.g., $\hat{C}_1 - \hat{C}_2 T_0 \neq 0$, which follows from (32)), the choice of a linear form for $\hat{a}_T$ allows the determination of the Laplace transform of $\Delta\hat{T}(\xi)$:

$$\Delta\hat{T}^*(s) = \left(\frac{\hat{C}_1}{\hat{C}_2} - T_0\right)\frac{\hat{C}_0\hat{C}_2}{s\big(s + \hat{C}_0\hat{C}_2\big)}. \tag{34}$$

Similarly, the CAM model used for the description of the master relaxation modulus is replaced by a function for which the Laplace transform can be easily found. A series of several exponents (Prony series) is used instead of (15):

$$\hat{E}(\xi) = \sum_i \hat{E}_i \exp(\hat{\gamma}_i\xi), \quad \hat{E}^*(s) = \sum_i \frac{\hat{E}_i}{s - \hat{\gamma}_i}. \tag{35}$$

For illustration purposes it is enough to use only a few terms in the series; we preserve them up to $i = 3$.

Even though, for comparison purposes, the parameters $\hat{C}_1$, $\hat{C}_2$, $\hat{E}_i$, and $\hat{\gamma}_i$ could be chosen arbitrarily, we chose them via a least squares fitting of the functions $a_T(T)$ and $E(\xi)$ experimentally obtained for a modified asphalt binder (PG58-40). (It is not essential to have perfect fitting of experimental data with the new functions $\hat{a}_T(T)$ and $\hat{E}(\xi)$; we just need to use the same functions in solving the problem using the two approaches being compared.)

The values of the parameters involved in $\hat{a}_T(T)$ are

$$\hat{C}_1 = 41.156640, \quad \hat{C}_2 = 2.276346°\text{C}^{-1}.$$

Those appearing in $\hat{E}(\xi)$ are

$$\begin{array}{lllll} \text{(in MPa)} & \hat{E}_0 = 38.598647, & \hat{E}_1 = 349.348243, & \hat{E}_2 = 195.939985, & \hat{E}_3 = 126.336051, \\ (\text{in sec}^{-1}) & \hat{\gamma}_0 = 0, & \hat{\gamma}_1 = -0.517286 \cdot 10^9, & \hat{\gamma}_2 = -0.265315 \cdot 10^8, & \hat{\gamma}_3 = -0.137463 \cdot 10^7. \end{array}$$

We next present the parameters of the composite cylinder, chosen according to an ABCD specimen described in detail in the next subsection. The specimen consisted of an elastic ring with thickness $r_1 - r_0$, surrounded by a viscoelastic binder with thickness $r_2 - r_1$ (compare Figure 3 on page 845). These geometric parameters are

$$r_0 = 23.75 \cdot 10^{-3}\,\text{m}, \quad r_1 = 25.4 \cdot 10^{-3}\,\text{m}, \quad r_2 = 31.75 \cdot 10^{-3}\,\text{m}, \tag{36}$$

while the remaining parameters are

$$T_0 = 18°\text{C}, \quad C_0 = -1°\text{C/hour}, \quad \begin{array}{l} \alpha_1 = 1.4 \cdot 10^{-6}\,°\text{C}^{-1}, \\ \alpha_2 = 2.0 \cdot 10^{-4}\,°\text{C}^{-1}, \end{array} \quad E_1 = 141\,\text{GPa}, \quad \begin{array}{l} \nu_1 = 0.3, \\ \nu_2 = 0.33. \end{array} \tag{37}$$

Figure 4 compares the analytical plane strain solution for $\sigma_{\theta\theta}^{\text{bind}}(r_1)$ obtained with the use of the Laplace

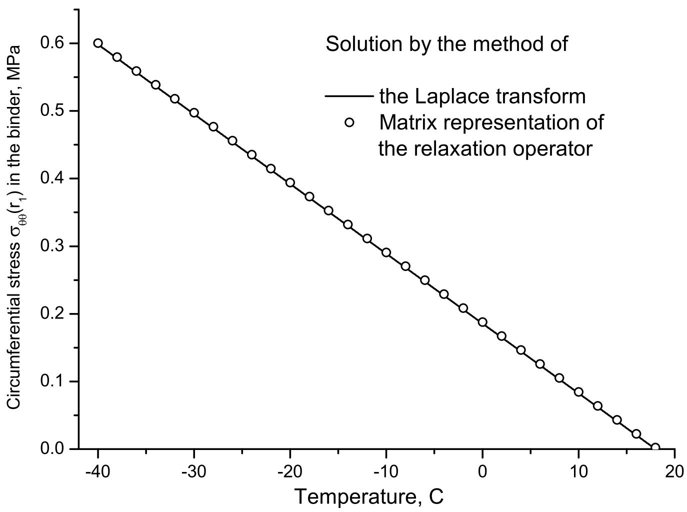

Figure 4. Comparison of circumferential stress $\sigma_{\theta\theta}^{\text{bind}}(r_1)$ in the composite cylinder problem obtained analytically (Laplace transform method) and numerically by the matrix representation of the relaxation operator $\tilde{E}$ of (22)–(24).

transform with the numerical solution found by the present approach for $n = 2500$. As in the previous problem, the results from the both methods match very well. The stress rises linearly in this case due to the use of modified shift factor $\hat{a}_T(T)$, which depends linearly on temperature.

Simulation of thermal stresses in Asphalt Binder Cracking Device. This section is devoted to the study of viscoelastic fields evolution in specimens undergoing thermal loading in the Asphalt Binder Cracking Device (ABCD). ABCD was developed in [Kim 2005; Kim et al. 2006] for direct measurements of cracking temperature of asphalt binders. Based on the present approach, it is possible to relate measured strains to time-dependent thermal stresses that cause the cracking of the binder and determine its strength at the cracking temperature.

The ABCD, schematically represented in Figure 5, consists of a hollow invar (Ni-Fe alloy) ring with uniform thickness surrounded by a layer of asphalt binder. A silicone mold (not shown in the figure) is used to form asphalt binder into a ring. The mold contains a cylindrical protrusion (shown as a hole in Figure 5) that extends through the thickness of the binder and touches the invar ring. An electrical strain gage is glued on the inner surface of the invar ring across the protrusion. To determine the cracking temperature, the whole structure is placed into a temperature chamber, where it is cooled down. Because the coefficient of thermal expansion of asphalt binders ($\alpha \sim 2 \cdot 10^{-4}\,{}^\circ\mathrm{C}^{-1}$) is much larger than invar's ($\alpha = 1.4 \cdot 10^{-6}\,{}^\circ\mathrm{C}^{-1}$), the contraction of the binder is constrained, which causes tensile circumferential stress development in the binder. As a result of cooling, the binder cracks around the hole, where the thickness (in the axial direction) is least. At the moment of cracking the accumulated stress is relieved, and this is expressed as a sudden drop in the strain readings by the strain gage. Thus, the cracking temperature of the asphalt binder can be determined as the temperature at which the sudden drop in measured strain occurs.

In contrast with the cracking temperature, the magnitude of tensile stress in the binder that causes cracking cannot be directly determined from the strain gage readings. Using the present approach, it is possible to simulate both circumferential strain on the inner surface of the invar ring and circumferential stress in the binder. The knowledge of such strain-stress relations can provide a simple tool for the measurement of the tensile circumferential stress causing the crack. This stress can be referred to as the binder's strength at the cracking temperature.

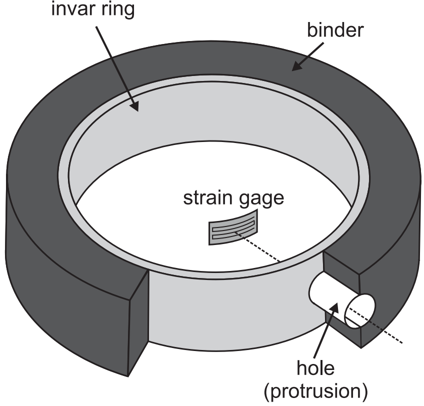

Figure 5. An ABCD specimen consisting of an elastic ring surrounded by a viscoelastic binder.

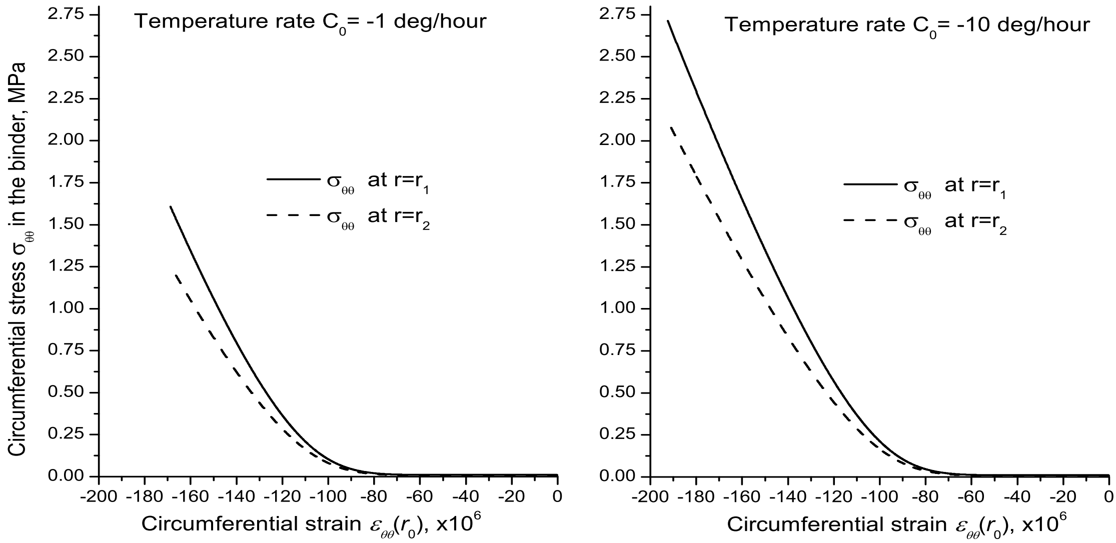

Figure 6. Stress-strain curves for strains on the inner surface of the invar ring and stresses in asphalt binder.

In order to simulate thermal stresses and strains in an ABCD specimen, the model of a composite cylinder (Figure 3) under the condition of plane stress is used. Circumferential strain at the inner side of the invar ring is given by (29), and the expression (28) is used to calculate circumferential stress in the binder. The material properties corresponding to a modified asphalt binder [3] are used, as follows:

(CAM model) $E_g = 3\,\text{GPa}$, $t_c = 148.381583\,\text{sec}$, $v = 0.11346$, $w = 3.73643$,

(Other) $C_1 = 3.03155$, $C_2 = 0.16842°\text{C}^{-1}$, $T_0 = 18°\text{C}$.

The rest of the parameters is given in (36) and (37) on page 847.

The stress-strain relations obtained for the temperature rates $C_0 = -1°\text{C/hour}$ and $C_0 = -10°\text{C/hour}$ are shown in Figure 6. Circumferential stress is found at $r = r_1$ (inner surface of the binder ring) and $r = r_2$ (outer surface of the binder ring). The plots in Figure 6 reveal that the magnitude of circumferential stress is higher at the inner surface of the binder ring. Based on this, it is reasonable to conclude that cracking starts from the inner side of the binder ring. Simulation results also show that at the same strain value, stresses in the binder are larger when the magnitude of temperature rate is larger. This is related to the fact that stress in the binder is not able to fully relax when the temperature drops faster. Plots similar to those given in Figure 6 can be obtained for other binders.

The results just presented are for a model that does not take into account the presence of the hole in the asphalt binder. To estimate the influence of the hole on the stress/strain fields around it, an elastic finite element (FE) model replicating the geometry of an ABCD specimen was built. The effect of the influence of the hole in viscoelastic case would be qualitatively similar to that in the elastic problem. A

[3] This binder was modified with styrene-butadiene-styrene copolymer and produced by Flint Hills (USA). Binder's performance grade (PG) is PG 58-40 [AASHTO 2005]. This binder was tested in the Pavement Laboratory of the Department of Civil Engineering at the University of Minnesota. For more on the properties of this binder, see [Marasteanu et al. 2007].

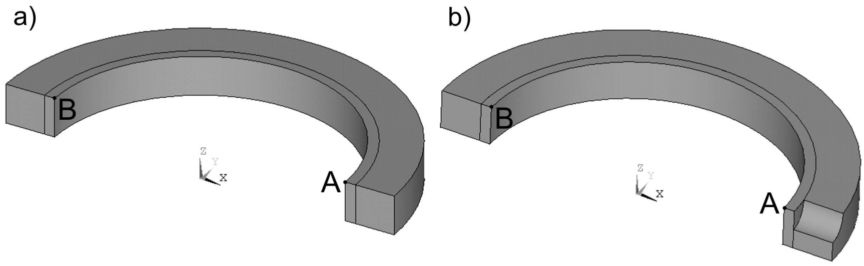

Figure 7. Finite element models used for simulations.

full viscoelastic finite element analysis would be beyond the scope of this paper, though it is possible to conduct such an analysis.

For simplicity, the Young's modulus of the binder in FE model was taken to be constant: $E = 3\,\mathrm{GPa}$, the initial value of the binder's relaxation modulus. The properties of the inner ring and geometric parameters corresponded to those of an actual ABCD specimen, and are listed on page 847. A full bond between the binder and the invar ring was assumed. Due to the symmetry of the problem, only a quarter of the ABCD rings had to be modeled. Two models, one without a hole (Figure 7a) and another with a hole in the external ring (Figure 7b), were considered. Both were subjected to a temperature change of $\Delta T = -50°\mathrm{C}$, and the circumferential strain along the arc AB in Figure 7 was determined.

The results of the FE simulations, shown in Figure 8, reveal that the magnitude of circumferential strain in the model with a hole (Figure 7b) increases significantly at point A. At this location the difference reaches ∼55% in comparison with the case when no hole is present in the outer ring. Indeed, the presence of the hole causes stress concentration around it, and the additional stress causes local bending of the invar ring. This suggests that experimental strain measurements conducted across the hole cannot be directly used for the calculation of thermal stresses in the binder based on the results given in Figure 6. As follows from Figure 8, thermal stress may be estimated more accurately if strain measurements are

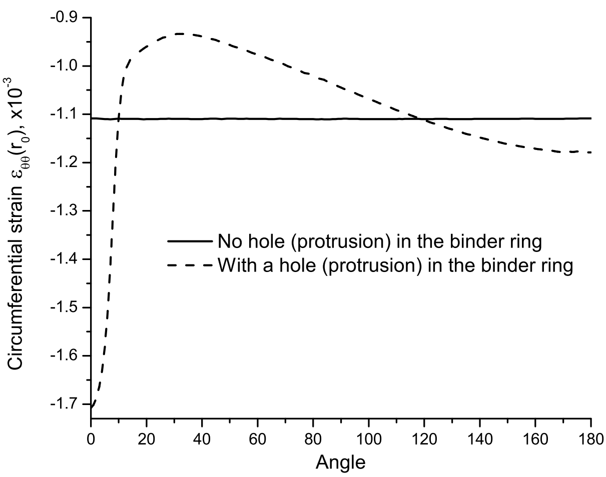

Figure 8. Circumferential strain $\varepsilon_{\theta\theta}(r_0)$ along the arc AB of the finite element models (Figure 7), with A corresponding to 0° and B to 180°.

conducted at an angle different from 0°. Measurements taken at angles around 120° are likely to provide more accurate results.

In order to be able to fully incorporate the advantages of the present approach into the calculations of thermal stresses in ABCD specimens, a binder ring should be tested without a hole in it. In this case strain measurements can be conducted at any location. Based on this suggestion, it may be concluded that using ABCD (without a hole) together with the present approach may substitute two separate tests, the bending beam rheometer and indirect tensile tests, which are currently used for determining thermal stresses in binders, their strength and cracking temperature [AASHTO 2007; 2008].

8. Conclusion

A problem of thermal stress development in composite structures containing one linear isotropic viscoelastic phase is considered. The study is particularly devoted to problems involving viscoelastic asphalt binders, whose properties strongly depend on temperature. The ability to determine thermal stresses in asphalt binders and/or mixtures and the knowledge of their strength is critical for the proper design of road pavements. The approach presented in the paper can be equally useful for the calculation of thermal stresses in other viscoelastic composite structures made of materials — including many polymers and resins — for which the time-temperature superposition principle is applicable.

Two methods of solution of viscoelastic problems are discussed. Both are based on the reduction of viscoelastic problem to an elastic one. It is argued that the use of the method based on the application of the Laplace transform to the analysis of the problems involving viscoelastic asphalt binders is impractical. Therefore, an approach based on the use of the Volterra correspondence principle is adopted.

It is assumed that the Poisson's ratios of the materials do not depend on time and temperature, which allows reducing the number of integral operators to one for the case of an isotropic linear viscoelastic material. This assumption is valid for asphalt binders at low temperatures. Incorporating a numerical technique of solution, the integral operator corresponding to the master relaxation modulus is presented in matrix form. The solution of a problem of viscoelastic composite structures is then reduced to the calculation of functions of matrices, the form of which is determined by the corresponding elastic solution. The application of this technique to the analysis of composite structures (including viscoelastic and elastic constituents) under varying temperature is novel.

Several examples are discussed, in an attempt to estimate the accuracy of the method. It is shown that for the case of viscoelastic asphalt binders, the simple integration technique adopted in the method provides accurate results. For the case of other viscoelastic materials, it may be necessary to use a more accurate integration scheme in the derivation of the matrix representation of the relaxation operator. The last example involves an approach that can potentially be used for the calculation of thermal stresses and binder strength in the Asphalt Binder Cracking Device.

The method can be applied to isotropic viscoelastic problems in which both material parameters change with time and temperature. This is because the integral operators for the case of nonaging viscoelasticity commute with each other.

Composite materials with complex internal structure, such as asphalt mixtures, can be treated with the present method if effective mechanical and thermal properties are used. The method can also be extended to problems involving temperature gradients and heat transfer. However, the application may be limited in the following cases: the solution of the corresponding elastic problem is a transcendental function of elastic properties; the viscoelastic medium exhibits strong aging effects; the time-temperature superposition principle is not applicable. These are topics for future research.

Appendix

Master relaxation modulus curve. The master relaxation modulus $E(\xi)$ of asphalt binders is determined from experimental data for creep compliances, which are obtained with the use of the bending-beam rheometer (BBR) test. In this test [AASHTO 2008], a small beam made of an asphalt binder or asphalt mix is subjected to three-point bending at different low temperatures (for example, −18°C, −24°C, −30°C). The deflection and the applied load are measured in real time. According to Euler–Bernoulli beam theory, the maximum elastic deflection at the mid-span of the beam is

$$\delta_{\max} = \frac{PL^3}{48EI}, \tag{A1}$$

where P is the concentrated load, L the beam span, I the moment of inertia, and E the Young's modulus of the material. Using (A1), the creep compliance of the viscoelastic beam $D(t)$ can be determined to be

$$D(t) = \frac{48I}{PL^3}\delta(t),$$

where $\delta(t)$ is the mid-span deflection history of the beam caused by a constant load P applied at zero time.

The creep compliance $D(t)$ is converted to the relaxation modulus $E(t)$ using the Hopkins–Hamming method [1957] of numerical integration of a convolution integral:

$$\int_0^t E(t)D(t-\tau)\,d\tau = t.$$

After $E(t)$ is obtained for each testing temperature, the shift function $a_T(T)$ can be found by shifting each relaxation curve along time axis in such a way that they create a single smooth curve, called the master relaxation curve. The direction of shifting is chosen in correspondence with the reference and current temperatures. The master curve is fitted using least squares by the CAM model (15), and the shift function is fitted using expression (9). This finally yields the constants C_1 and C_2 and reduced time $\xi(t)$.

Outline of the solution of the composite elastic cylinder problem. (Refer to Figure 3.) The constitutive equations for total strains in each cylinder are written in polar coordinates (r, θ) as

$$\varepsilon_{rr} = \frac{1+\nu}{E}\big((1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta} + \alpha E\Delta T\big), \qquad \varepsilon_{\theta\theta} = \frac{1+\nu}{E}\big((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr} + \alpha E\Delta T\big). \tag{A2}$$

It is assumed that only radial displacement u exists, and the strain components are expressed (see, e.g., [Ugural and Fenster 2003]) as

$$\varepsilon_{rr} = u_{,r}, \qquad \varepsilon_{\theta\theta} = \frac{u}{r}, \qquad \varepsilon_{r\theta} = 0.$$

where the subscript comma means derivative. Substituting these expressions into (A2) and using the equilibrium equation

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0,$$

it is straightforward to obtain the solution for the unknown radial displacement. After cancellation of the terms containing temperature, the result is

$$u = c_1 r + \frac{c_2}{r},$$

where c_1 and c_2 are constants. The general solution for the stresses in each elastic cylinder is

$$\sigma_{rr} = \frac{E}{1+\nu}\left(\frac{c_1}{1-2\nu} - \frac{c_2}{r^2}\right) - \frac{E}{1-2\nu}\alpha\Delta T, \qquad \sigma_{\theta\theta} = \sigma_{rr} + r\sigma_{r,r}.$$

Using the boundary conditions for the problem of composite cylinder (Figure 3), namely

$$\sigma_{rr}^{\text{inc}} = 0 \quad \text{at } r = r_0, \qquad \sigma_{rr}^{\text{inc}} = \sigma_{rr}^{\text{bind}} \quad \text{at } r = r_1, \qquad u_r^{\text{inc}} = u_r^{\text{bind}} \quad \text{at } r = r_1, \qquad \sigma_{rr}^{\text{bind}} = 0 \quad \text{at } r = r_2,$$

the constants c_1 and c_2 are found for each cylindrical layer. Then the solution for the circumferential stresses in the binder is given by (27)–(28), in which

$$a(r) = -\frac{r_1^2}{r^2} \cdot \frac{r_0^2 - r_1^2}{r_1^2 - r_2^2} \cdot \frac{r^2 + r_2^2}{r^2 - 2\nu_i r_1^2 + r_0^2} \cdot \frac{\alpha_b(\nu_b+1) - (\nu_i+1)}{\nu_i+1}, \tag{A3}$$

$$b(r) = -\frac{r_1^2}{r^2} \cdot \frac{r^2 + r_2^2}{r_1^2 - 2\nu_b r_1^2 + r_2^2} \cdot \frac{(\alpha_i - 1)(\nu_i+1)}{\nu_b+1}, \qquad c = \frac{r_0^2 - r_1^2}{r_1^2 - r_2^2} \cdot \frac{r_1^2 - 2\nu_b r_1^2 + r_2^2}{r^2 - 2\nu_i r_1^2 + r_0^2} \cdot \frac{\nu_b+1}{\nu_i+1}. \tag{A4}$$

The solution for the total circumferential strains in the inclusion is given by (29)–(30), in which

$$d = 1 + (1-2\nu_i)\,\frac{r_1^2}{r_0^2}, \qquad g = \frac{r_1^2}{r_0^2} \cdot \frac{r_0^2 - r_1^2}{r_1^2 - r_2^2}\left(1 - 2\nu_b + \frac{r_2^2}{r_1^2}\right)\frac{1+\nu_b}{1+\nu_i}. \tag{A5}$$

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ANDREY V. PYATIGORETS: pyati002@umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States

MIHAI O. MARASTEANU: maras002@umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States

LEV KHAZANOVICH: khaza001@umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States

HENRYK K. STOLARSKI: stola001@umn.edu
Department of Civil Engineering, University of Minnesota, 500 Pillsbury Drive SE, Minneapolis, MN 55455, United States

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