Journal of
Mechanics of
Materials and Structures

TIME-HARMONIC ELASTODYNAMIC GREEN'S FUNCTION FOR THE HALF-PLANE MODELED BY A RESTRICTED INHOMOGENEITY OF QUADRATIC TYPE

Tsviatko V. Rangelov and George D. Manolis

Volume 5, No. 6

June 2010

mathematical sciences publishers

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 5, No. 6, 2010

TIME-HARMONIC ELASTODYNAMIC GREEN'S FUNCTION FOR THE HALF-PLANE MODELED BY A RESTRICTED INHOMOGENEITY OF QUADRATIC TYPE

TSVIATKO V. RANGELOV AND GEORGE D. MANOLIS

We derive closed-form solutions for point-force generated motions in a continuously inhomogeneous half-plane, which represent the complete elastic wave-train in the interior domain obeying traction-free boundary conditions at the horizontal surface. More specifically, a special type of material inhomogeneity is studied, where the shear modulus varies quadratically with respect to the depth coordinate. Furthermore, the material density profile varies proportionally to the aforementioned profile, while Poisson's ratio remains fixed at one-quarter. Limit forms for the Green's functions are derived for both zero frequency and for the equivalent homogeneous medium. Next, a series of numerical results serve to validate this mechanical model, and to show the differences in the wave motion patterns developing in media that are inhomogeneous as compared to a reference homogeneous background. These singular solutions are useful within the context of boundary element formulations for the numerical solution of problems involving nonhomogeneous continua, which find applications in fields as diverse as composite materials, geophysical prospecting, petroleum exploration and earthquake engineering.

1. Introduction and problem statement

Detailed knowledge of wave motions produced by point forces in the elastic half-plane [Achenbach 1973] are of paramount importance in mechanics, since they form the backbone of any integral equation formulation whose numerical treatment yields boundary element method solutions to a wide range of boundary-value problems in elastodynamics [Kausel and Manolis 2000].

We examine here a restricted class of inhomogeneous media, where the elastic parameters and the density all vary proportionally with depth, which makes possible a decoupling of the equations of motion for the boundless continuum into pseudopressure and shear wave components in a transformed domain. Although somewhat unlikely, there are situations where this type of inhomogeneity has actually been observed. As an example, we mention the geological profile of the Sofia region in Bulgaria [Bonchev et al. 1982], which was measured from in-situ data. This profile seems to imply both a proportional variation of the shear modulus and of the density, plus a constant Poisson's ratio value of one-quarter, for a nearly thirty kilometer thickness of the local deposits as measured from the surface.

Let (x_1, x_2) be Cartesian coordinates in $\mathbb{R}^2$ and denote the lower half-plane by $\mathbb{R}^2_- = \{(x_1, x_2) : x_2 < 0\}$; see Figure 1. Consider the following boundary-value problem defined in the frequency domain, where

Keywords: inhomogeneous media, elastic waves, Fourier transforms, singular solutions.
The authors wish to acknowledge financial support provided through NATO Collaborative Linkage Grant No. 984136.

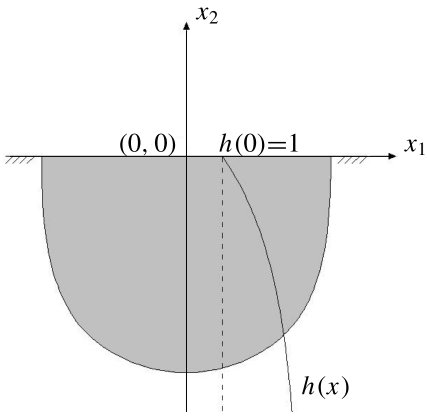

Figure 1. Elastic half-plane with quadratically varying material properties in the depth coordinate as described by the profile function $h(x)$.

all dependent variables have an $e^{i\omega t}$ type dependence on time:

$$L^a(G) \equiv (C_{ikpq}G_{ip,q})_{,j} - \rho\omega^2 G_{ik} = -\delta(x-\xi)\varepsilon_{ik}, \quad \text{where } x, \xi \in \mathbb{R}^2_-, \tag{1}$$

$$T^a(G) \equiv C_{j2pq}G_{ip,q} = 0 \quad \text{on } x_2 = 0, \tag{2}$$

$$G \to 0 \quad \text{for } x_2 \to -\infty. \tag{3}$$

Here Green's tensor G satisfies the Sommerfeld radiation condition along lines parallel to $\{x_2 = 0\}$, i.e., $\{(x_1, x_2), x_1 \to \pm\infty\}$. Furthermore, $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$ are source/receiver points in the continuum; $C_{jkpq} = h(x_2)C^0_{jkpq}$ is the elasticity tensor; $\rho = h(x_2)\rho_0$, with $\rho_0 > 0$, is the material density; and $h(x_2) = (ax_2 + 1)^2$, with $a \le 0$, is the material profile, implying a quadratic variation with depth. In terms of the quantities defined for the corresponding homogeneous background, we have $C^0_{jkpq} = \mu_0(\delta_{jk}\delta_{pq} + \delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp})$, where $\mu_0 > 0$ is the shear modulus, δ_{jk} is Kronecker's delta, and $\omega > 0$ is the frequency. Finally, δ is Dirac's delta function, $\varepsilon = \varepsilon_{ik}$ is the unit tensor, commas denote partial differentiation with respect to the spatial coordinates and summation is implied over repeated indices.

In elastodynamics, the problem defined by (1)–(3) is a model of an isotropic elastic medium in $\mathbb{R}^2_-$ with a point force at ξ and traction-free boundary conditions. Poisson's ratio is fixed at a value of $\nu = 0.25$, while the shear modulus μ and the density ρ depend in the same manner on depth coordinate x_2. A fundamental solution to (1) of this problem in $\mathbb{R}^2_-$ was derived in [Manolis and Shaw 1996] for $a \ne 0$, while a solution of (1)–(3) defining a Green's function for the homogeneous half-plane, i.e., for $a = 0$, has been obtained in M. Kinoshita's M.Sc. thesis, quoted in [Kobayashi 1983]. A corresponding Green's function in the Laplace domain for a homogeneous half-plane can be found in [Guan et al. 1998], while an approximate such function using an image source across the free surface was derived earlier in [Kontoni et al. 1987]. Finally, the transient Green's function due to a suddenly applied load in the homogeneous half-plane, namely Lamb's problem, can be found in [Kausel 2006], a compilation of fundamental solutions in elastodynamics.

2. Solution outline

By following the procedure as outlined in references given above, we will now derive the unique solution to the problem (1)–(3), which corresponds to a Green's function G for the inhomogeneous half-plane with a quadratic variation of the material parameters. Let the matrix-valued function u be a fundamental solution to equation (1):

$$L^a(u) = -\delta(x-\xi)\varepsilon, \quad \text{where} \quad x, \xi \in \mathbb{R}^2_-, \tag{4}$$

while w is a smooth matrix-valued function such that

$$L^a(w) = 0, \qquad \text{where } x, \xi \in \mathbb{R}^2_-, \tag{5}$$

$$T^a(w) = -T^a(u) \quad \text{on } x_2 = 0, \tag{6}$$

where superscript a in the operators corresponds to the degree of inhomogeneity. Then, by using superposition, the complete Green's function is simply $G = u + w$.

The fundamental solution u can be expressed as in [Manolis and Shaw 1996] in the form

$$u(x, \xi, \omega) = h^{-1/2}(\xi_2) U(x, \xi, \omega) h^{-1/2}(x_2), \tag{7}$$

where U is a fundamental solution for the corresponding homogeneous case, i.e.,

$$L^0(U) = -\delta(x-\xi)\varepsilon, \quad \text{with} \quad x, \xi \in \mathbb{R}^2_-. \tag{8}$$

Finally, the traction matrix corresponding to displacements u on free surface $x_2 = 0$ is

$$\begin{aligned} T^a_{1k}(u) &= \mu_0 h^{-1/2}(\xi_2)(-aU_{1k} + U_{1k,2} + U_{2k,1}), \\ T^a_{2k}(u) &= \mu_0 h^{-1/2}(\xi_2)(-3aU_{2k} + U_{1k,1} + 3U_{2k,1}). \end{aligned} \tag{9}$$

The homogeneous matrix-valued function U in $\mathbb{R}^2$ can be found in [Eringen and Şuhubi 1974] as

$$U_{jk} = \frac{i}{4\mu_0}\left[\delta_{jk} H_0^{(1)}(k_2 r) + \frac{1}{k_2^2}\partial^2_{jk}\left(H_0^{(1)}(k_2 r) - H_0^{(1)}(k_1 r)\right)\right]. \tag{10}$$

Here the wave numbers corresponding to pressure and shear body waves are respectively $k_1 = \sqrt{\rho_0/3\mu_0}\,\omega$ and $k_2 = \sqrt{\rho_0/\mu_0}\,\omega$, while the radial distance between source and receiver is $r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}$ and $H_0^{(1)}(z)$ is the Bessel function of third kind (or Hankel function) and zero order.

In order to simplify the calculations, we fix the source point along the vertical axis as $\xi = (0, \xi_2)$, $\xi_2 < 0$. As will be shown later on, Green's function G actually depends on $x_1 - \xi_1$ and separately on x_2 and ξ_2 because the corresponding profile function h is independent of x_1; thus the assumption $\xi_1 = 0$ is not restrictive.

3. Solution methodology

The first step is to recover a general solution w to (5), in the form

$$w(x, \xi, \omega) = h^{-1/2}(x_2) W(x, \xi, \omega). \tag{11}$$

Then, the two corresponding differential operators for the homogeneous and inhomogeneous cases are related as

$$L^a(w) = h^{1/2}(x_2)L^0(W), \tag{12}$$

where

$$\begin{aligned} L^a(w) &= h^{-1/2}C^a_{jkpq}\big(W_{ip,qk} + h^{-1}(h^{1/2}_{,j}W_{ip,q} - h^{1/2}_{,q}W_{ik,j} - h^{1/2}_{,qk}W_{ip})\big) + \rho\omega^2 h^{-1/2}W_{ij} \\ &= h^{1/2}(C_{jkpq}W_{ip,qk} + \rho_0\omega^2 W_{ij}) = h^{1/2}L^0(W). \end{aligned}$$

Thus, if W solves (5) with $a = 0$, then w also solves (5) for $a < 0$ and we seek a solution $W = \{W_{jk}\}$ in the general Rayleigh form [Achenbach 1973; Rajapakse and Wang 1991] as a transformation between distance x_1 and wave number η:

$$W_{jk} = \frac{1}{2\pi}\int_R S_{jk}e^{i\eta x_1}\,d\eta, \tag{13}$$

where the kernel function S_{jk} depends on $e^{\beta x_2}$, η, ω, a, and the parameter β is found as solution of an algebraic system of equations to be developed.

Remark 1. It is not possible to proceed for the inhomogeneous case as in [Kinoshita 1983] for a homogeneous material. The algebraic transformation produces a function

$$\tilde{u}(x, \xi, \omega) = h^{-1/2}(-\xi_2)U(x, \xi, \omega)h^{-1/2}(x_2)$$

that is not well defined for all $\xi_2 < 0$ and is infinite if $h(-\xi_2) = 0$, corresponding to a value $\xi_2 = 1/a$, $a < 0$. Thus, we cannot use superposition as $u(x, \xi, \omega) + \tilde{u}(x, \xi, \omega)$, for which $T_{1k} = 0$, $T_{2k} = 0$ on $x_2 = 0$, but can only use $u(x, \xi, \omega)$ and then add a Rayleigh form to satisfy the boundary conditions.

Thus, in order to find $S = S_{jk}$ we use the Fourier transform $\Im$ with respect to the x_1 coordinate, defined, together with its inverse, by

$$\begin{aligned} \tilde{f}(\eta, x_2) &= \Im_{x_1\to\eta} f = \int_R f(x_1, x_2)e^{-i\eta x_1}\,dx_1, \\ f(x_1, x_2) &= \Im^{-1}_{\eta\to x_1}\tilde{f} = \frac{1}{2\pi}\int_R \tilde{f}(\eta, x_2)e^{i\eta x_1}\,d\eta, \end{aligned} \tag{14}$$

where η is the transform parameter. By applying the Fourier transform to W, we turn (1) with $a = 0$ into

$$L^0(\Im_{x_1\to\eta}(W)) = 0, \tag{15}$$

which in matrix form reads as

$$(M(\eta, \beta) + \rho_0\omega^2 I_2)S = 0, \tag{16}$$

where I_2 is the 2×2 unit matrix and

$$M(\eta, \beta) = \begin{pmatrix} -3\mu_0\eta^2 + \mu_0\beta^2 + \rho_0\omega^2 & 2i\mu_0\eta\beta \\ 2i\mu_0\eta\beta & -\mu_0\eta^2 + 3\mu_0\beta^2 + \rho_0\omega^2 \end{pmatrix}. \tag{17}$$

For every fixed value of η, a nonzero solution to (16) exists if $\det M(\eta, \beta) = 0$, which gives the following biquadratic equation for β:

$$3\mu_0^2\beta^4 - 2\mu_0(3\mu_0\eta^2 - 2\rho_0\omega^2)\beta^2 + \rho_0^2\omega^4 + 3\mu_0\omega^4 - 4\mu_0\eta^2\rho_0\omega^2 = 0. \tag{18}$$

If we set $\gamma_j^2 = \eta^2 - k_j^2$, equation (18) simplifies to

$$\beta^4 - (\gamma_1^2 + \gamma_2^2)\beta^2 + \gamma_1^2\gamma_2^2 = 0, \tag{19}$$

and the solutions are $\beta_j^2 = \pm\gamma_j^2$. In order to satisfy the radiation condition of (3), only the positive root is retained:

$$\beta_j = \gamma_j = \sqrt{\eta^2 - k_j^2}. \tag{20}$$

Since the matrix $M(\eta, \beta_j)$, for $j = 1, 2$, has rank 1, there are two eigenvectors, namely

$$\nu^1 = \begin{pmatrix} \eta \\ -i\beta_1 \end{pmatrix}, \quad \nu^2 = \begin{pmatrix} i\beta_2 \\ \eta \end{pmatrix}, \tag{21}$$

and every solution of (16) has the standard form

$$S = S_{jk} = \sum_{m=1}^{2} C_k^k \nu_j^m e^{\beta_m x_2}. \tag{22}$$

Recapitulating, the matrix form of (11) using indicial notation is

$$w_{jk}(x, \xi, \omega) = h^{-1/2}(x_2) W_{jk}(x, \xi, \omega), \tag{23}$$

and the remaining step is to determine functions $C_m^k(\eta, \xi_2, a)$ such that the boundary condition for zero tractions in (6) is satisfied. The traction field corresponding to displacement field w on $x_2 = 0$ is

$$\begin{aligned} T_{1k}^a &= \frac{1}{2\pi} \int_R \mu_0 \big[\eta(-a + 2\beta_1) C_1^k + i(-a\beta_2 + 2\eta^2 - k_2^2) C_2^k\big] e^{i\eta x_1} dx_1, \\ T_{2k}^a &= \frac{1}{2\pi} \int_R \mu_0 \big[i(3a\beta_1 - 2\eta^2 + k_2^2) C_1^k + \eta(-3a + 2\beta_2) C_2^k\big] e^{i\eta x_1} dx_1. \end{aligned} \tag{24}$$

To determine the traction field corresponding to displacement field u on $x_2 = 0$, we use the representation of $H_0^{(1)}$ based on a Fourier transform with respect to x_1 (see [Gradshteyn and Ryzhik 1980, formulas $6.677_{3,4}$ and Section 8.42]):

$$H_0^{(1)}(rk_j) = \frac{i}{2\pi} \int_R \frac{1}{\beta_j} e^{(\xi_2 - x_2)\beta_j} e^{i\eta x_1} d\eta. \tag{25}$$

Employing (7) and (10) for u and for U, respectively, we obtain

$$T_{jk}^a(u) = \frac{i}{2\pi} \int_R D_{jk} e^{i\eta x_1} d\eta. \tag{26}$$

where the matrix components D_{jk} are

$$
\begin{aligned}
D_{11} &= \frac{h^{-1/2}(\xi_2)}{2k_2^2}\left[(-a\beta_2 - 2\eta^2 + k_2^2)e^{\xi_2\beta_2} + \eta^2(a/\beta_1 + 2)e^{\xi_2\beta_1}\right],\\
D_{21} &= \frac{i\eta h^{-1/2}(\xi_2)}{2k_2^2\beta_1}\left[\beta_1(-3a - 2\beta_2)e^{\xi_2\beta_2} + \eta^2(3a\beta_1 + 2\eta^2 - k_2^2)e^{\xi_2\beta_1}\right],\\
D_{12} &= \frac{i\eta h^{-1/2}(\xi_2)}{2k_2^2\beta_2}\left[(-a\beta_2 - 2\eta^2 + k_2^2)e^{\xi_2\beta_2} + \beta_2(a + 2\beta_1)e^{\xi_2\beta_1}\right],\\
D_{22} &= \frac{i\eta h^{-1/2}(\xi_2)}{2k_2^2}\left[\eta^2(3a/\beta_2 + 2)e^{\xi_2\beta_2} + (-3a\beta_1 - 2\eta^2 + k_2^2)e^{\xi_2\beta_1}\right].
\end{aligned}
\tag{27}
$$

Combining equations (24) and (26), we obtain a system of two linear equations in C_m^1, C_m^2, which appear as kernels of integral equations when substituted in the boundary condition of (6). The determinant of this system is

$$
\Delta^a = \frac{\mu_0^2}{4\pi^2}\begin{vmatrix} \eta(-a + 2\beta_1) & i(-a\beta_2 + 2\eta^2 - k_2^2) \\ i(3a\beta_1 - 2\eta^2 + k_2^2) & \eta(-3a + 2\beta_2) \end{vmatrix},
\tag{28}
$$

which evaluates to

$$
\Delta^a = \frac{\mu_0^2}{4\pi^2}\left[3(\eta^2 - \beta_1\beta_2)a^2 - \big((\beta_1 + \beta_2)k_2^2 + \eta^2\beta_1\big)a - \Delta^0\right],
\tag{29}
$$

where $\Delta^0 = 4\eta^2\beta_1\beta_2 - (2\eta^2 - k_2^2)^2$ is a Rayleigh function [Kobayashi 1983].

The functions C_m^1, C_m^2 are unique solutions of (6), since for every $\eta \in R$, $a < 0$, $\omega > 0$, $\rho_0 > 0$, $\mu_0 > 0$ the condition $\Delta^a \neq 0$ holds true. Possible combinations of values of parameter $|\eta|$ as compared to the two wave numbers k_1, k_2 yield the following cases:

(i) If $|\eta| < k_1$, then $\operatorname{Im}\Delta^a = -\big((|\beta_1| + |\beta_2|)k_2^2 + \eta^2|\beta_1|\big)a > 0.$

(ii) If $|\eta| = k_1$, then $\operatorname{Im}\Delta^a = -\big(|\beta_2|k_2^2 a + (2\eta^2 - k_2^2)^2\big) > 0.$

(iii) If $k_1 < |\eta| \leq k_2$, then $\operatorname{Re}\Delta^a = 3\eta^2 a^2 - \beta_1(k_2^2 + \eta^2)a + (2\eta^2 - k_2^2) > 0.$

(iv) If $k_2 < |\eta|$, then $\Delta^a > \Delta^0 > 0.$

An application of Cramer's rule yields the matrix functions

$$
C_m^k = \Delta_{mk}^a / \Delta^a,
\tag{30}
$$

where the subdeterminants Δ_{mk}^a are given by

$$
\begin{aligned}
\Delta_{11}^a &= \begin{vmatrix} -D_{11} & i\mu_0(-a\beta_2 + 2\eta^2 - k_2^2) \\ -D_{21} & \mu_0\eta(-3a + 2\beta_2) \end{vmatrix}, &\quad \Delta_{21}^a &= \begin{vmatrix} \mu_0\eta(-a + 2\beta_1) & -D_{11} \\ i\mu_0(3a\beta_1 - 2\eta^2 + k_2^2) & -D_{21} \end{vmatrix}\\
\Delta_{12}^a &= \begin{vmatrix} \mu_0\eta(-a + 2\beta_1) & -D_{12} \\ i\mu_0(3a\beta_1 - 2\eta^2 + k_2^2) & -D_{22} \end{vmatrix}, &\quad \Delta_{22}^a &= \begin{vmatrix} -D_{12} & i\mu_0(-a\beta_2 + 2\eta^2 - k_2^2) \\ -D_{22} & \mu_0\eta(-3a + 2\beta_2) \end{vmatrix}.
\end{aligned}
\tag{31}
$$

Finally, the radiation boundary condition in (3) holds true because of the presence of the multiplier $h^{-1/2}(x_2)$ for u and $h^{-1/2}(x_2)e^{x_2\beta}$ under the integral sign on η for w in (13).

Remark 2. This method can be applied for complex wave numbers ($k_j = k_{jR} + ik_{jI}$ with $k_{jR} > 0$, $k_{jI} > 0$) and the structure of Green's function remains the same. This is because the representations for the fundamental solution of (10) and for the Bessel function (25) are valid for complex numbers as well. However, the proof that $\Delta^a \neq 0$ in this case is quite complicated.

Remark 3. The same method can be applied to obtain a transient Green's function in the inhomogeneous half-plane for the equations of motion defined in the time domain as

$$L^a(G) \equiv (C_{jkpq} G_{ip,q})_{,j} - \rho G_{ik,tt} = -f(t)\delta(x-\xi)\varepsilon_{ik}, \tag{32}$$

where $f(t) \in L^1_{\text{loc}}(\mathbb{R}^1)$ and $f = 0$ for $t < 0$. More specifically, $f(t) = H(t)F(t)$, with $H(t)$ the Heaviside function and $|F(t)| \leq Ae^{ct}$ for $t \to \infty$. The transient Green's function is obtained by applying Laplace's transformation to (32) and using a Kelvin function representation of the type $K_0(z) = (i\pi/2)H_0^{(1)}(iz)$. Formally, the Green's function in the Laplace domain is obtained by replacing frequency ω with the Laplace transform parameter written as a purely imaginary number is and then applying the inverse Laplace transform. This path was followed for the homogeneous case, i.e., $a = 0$ and with $F(t) = 1$, in [Guan et al. 1998].

Remark 4. Green's function $G(x, \xi, \omega, a)$ converges in the weak sense to $G(x, \xi, \omega, 0)$ for $a \to 0$, i.e., for every $\varphi(\xi) \in C_0^\infty(\mathbb{R}^2_-)$ we have

$$\int_{\mathbb{R}^2} G(x, \xi, \omega, a)\varphi(\xi)\, d\xi \;\to \int_{\mathbb{R}^2} G(x, \xi, \omega, 0)\varphi(\xi)\, d\xi \quad \text{for } a \to 0.$$

Also, Green's function $G(x, \xi, \omega, a)$ converges in the weak sense to $G(x, \xi, 0, a)$ for $\omega \to 0$, i.e., for every $\varphi(\xi) \in C_0^\infty(\mathbb{R}^2_-)$ we have

$$\int_{\mathbb{R}^2} G(x, \xi, \omega, a)\varphi(\xi)\, d\xi \;\to \int_{\mathbb{R}^2} G(x, \xi, 0, a)\varphi(\xi)\, d\xi \quad \text{for } \omega \to 0.$$

More details for this elastostatic case can be found in the Appendix.

4. Recovery of the homogeneous case

In order to check that it is possible to recover the homogeneous half-plane solution by setting the inhomogeneity parameter a to zero (and, correspondingly, $h(x_2) = h(\xi_2) = 1$ for the profile function) in the solution derived above, we start with the results presented in [Kobayashi 1983]. In that case, (24) reads as

$$\begin{aligned} T_{1k}^0 &= \frac{1}{2\pi}\int_R \mu_0[2\eta\beta_1 C_1^k + i(2\eta^2 - k_2^2)C_2^k]e^{i\eta x_1}dx_1, \\ T_{2k}^a &= \frac{1}{2\pi}\int_R \mu_0[i(-2\eta^2 + k_2^2)C_1^k + 2\eta\beta_2 C_2^k]e^{i\eta x_1}dx_1. \end{aligned} \tag{33}$$

Also, in place of $u(x_1, x_2 - \xi_2)$ we use $u(x_1, x_2 - \xi_2) + \tilde{u}(x_1, x_2 + \xi_2)$, where $\tilde{u}(x_1, x_2 + \xi_2)$ is a smooth matrix-valued function defined in reference to (10) as

$$\tilde{u}_{jk}(x_1, x_2 + \xi_2) = \frac{i}{4\mu_0}\left[\delta_{jk}H_0^{(1)}(k_2\tilde{r}) + \frac{1}{k_2^2}\partial_{jk}^2\left(H_0^{(1)}(k_2\tilde{r}) - H_0^{(1)}(k_1\tilde{r})\right)\right], \tag{34}$$

with $\tilde{r} = \sqrt{x_1^2 + (x_2 + \xi_2)^2}$ the distance between source and receiver. Furthermore, the integral representation for the Hankel function corresponding to (25) is

$$H_0^{(1)}(\tilde{r}k_j) = \frac{i}{2\pi} \int_R \frac{1}{\beta_j} e^{(\xi_2 + x_2)\beta_j} e^{i\eta x_1} d\eta. \tag{35}$$

Thus, the traction vector on the free surface $x_2 = 0$ for the complete displacement field $u + \tilde{u}$ that replaces (26) is

$$T_{jk}^0(u + \tilde{u}) = \frac{i}{2\pi} \int_R \hat{D}_{jk} e^{i\eta x_1} d\eta. \tag{36}$$

with the new definitions

$$\begin{aligned} \hat{D}_{11} &= 0, \quad \hat{D}_{21} = \frac{i\eta}{2k_2^2\beta_1}\left[-2\beta_1\beta_2 e^{\xi_2\beta_2} + (2\eta^2 - k_2^2)e^{\xi_2\beta_1}\right], \\ D_{12} &= \frac{i\eta}{2k_2^2\beta_2}\left[-(2\eta^2 - k_2^2)e^{\xi_2\beta_2} + 2\beta_1\beta_2 e^{\xi_2\beta_1}\right], \quad \hat{D}_{22} = 0. \end{aligned} \tag{37}$$

The new subdeterminants $\bar{\Delta}_{mk}^0$ are now

$$\begin{aligned} \hat{\Delta}_{11}^0 &= \begin{vmatrix} 0 & i\mu_0(2\eta^2 - k_2^2) \\ -\hat{D}_{21} & 2\mu_0\beta_2 \end{vmatrix}, \quad \hat{\Delta}_{21}^0 = \begin{vmatrix} 2\mu_0\beta_1 & 0 \\ -i\mu_0(2\eta^2 - k_2^2) & -\hat{D}_{21} \end{vmatrix}, \\ \hat{\Delta}_{12}^0 &= \begin{vmatrix} -\hat{D}_{12} & i\mu_0(2\eta^2 - k_2^2) \\ 0 & 2\mu_0\beta_2 \end{vmatrix}, \quad \hat{\Delta}_{22}^0 = \begin{vmatrix} 2\mu_0\beta_1 & -\hat{D}_{12} \\ -i\mu_0(2\eta^2 - k_2^2) & 0 \end{vmatrix}, \end{aligned} \tag{38}$$

and the solution for the matrix functions is

$$\hat{C}_m^k = \hat{\Delta}_{mk}^0 / \Delta^0. \tag{39}$$

Finally, the reconstruction of the complete Green's function that replaces (22) is

$$\hat{S} = \hat{S}_{jk} = \sum_{m=1}^{2} \hat{C}_m^k v_j^m e^{\beta_m x_2}, \tag{40}$$

whose components can be explicitly written as

$$\begin{aligned} \hat{S}_{11} &= \frac{i\eta\mu_0}{\Delta_0}\left[(2\eta^2 - k_2^2)e^{x_2\beta_2} - 2\beta_1\beta_2 e^{x_2\beta_1}\right]\hat{D}_{21}, \quad \hat{S}_{21} = \frac{\beta_1\mu_0}{\Delta_0}\left[(2\eta^2 - k_2^2)e^{x_2\beta_2} - 2\eta^2 e^{x_2\beta_1}\right]\hat{D}_{21}, \\ \hat{S}_{12} &= \frac{\beta_2\mu_0}{\Delta_0}\left[-2\eta^2 e^{x_2\beta_2} + (2\eta^2 - k_2^2)e^{x_2\beta_1}\right]\hat{D}_{12}, \quad \hat{S}_{22} = \frac{i\eta\mu_0}{\Delta_0}\left[-2\beta_1\beta_2 e^{x_2\beta_2} + (2\eta^2 - k_2^2)e^{x_2\beta_1}\right]\hat{D}_{21}. \end{aligned} \tag{41}$$

Remark 5. The half-plane Green's function derived above can be used for solving general types of boundary-value problems in the half-plane enclosing singularities such as cracks, holes, cavities, etc. This can be done using boundary element method formulations [Manolis and Beskos 1988], and the advantage here is that a free surface ($x_2 = 0$) discretization is unnecessary.

5. Numerical example

As an example, we consider the lower part of an original full plane, keeping in mind that in the upper half-plane the material function h has a line of degeneracy (see Figure 1). A numerical study will be run for this inhomogeneous half-plane and its equivalent homogeneous limit form ($a = 0 \to h(x_2) = 1.0$) using the Green's functions derived herein. We start with the following source/receiver configuration:

$$(\xi_1, \xi_2) = (0.0, -30.0\,\text{m}), \quad (x_1, x_2) = (30.0\,\text{m}, 0.0). \tag{42}$$

The background homogeneous material corresponds to firm soil and has the following values for the pressure (P) and shear (S) wave speeds and for the density:

$$c_1 = 621.0\,\text{m/sec}, \quad c_2 = 359.0\,\text{m/sec}, \quad \rho = 2100.0\,\text{kg}. \tag{43}$$

The inhomogeneity parameter is assigned a value of $a = -0.001\,\text{m}^{-1}$, which implies that the inhomogeneous profile at the level of the source is stiffer by a factor of 1.07 (i.e., about 7%) compared to the reference value $\mu_0 = 270.0 \times 10^6\,\text{N/m}^2$ at the free surface level. The travel times for the P and S waves to reach the receiver starting from the source are $t_1 = r/c_1 = 42.4/621 = 0.07\,\text{sec}$, $t_2 = 42.4/359 = 0.12\,\text{sec}$, respectively, in the reference homogeneous background material. Choosing a total time $T = 2.0\,\text{sec}$ for the dynamic phenomenon to develop fully yields a frequency value $f = 1.0/T = 0.50\,\text{Hz}$, which is rounded-off to 0.64 Hz so it corresponds to $\Omega = 4.0\,\text{rad/sec}$. This interval is swept in 40 increments of $\Delta\omega = 0.1\,\text{rad/sec}$ starting from zero, where the static solution $G(x, \xi, 0, a)$ is used (see Remark 4).

In reference to the one-sided Fourier transform of (14), this is performed numerically using the fast Fourier transform. More specifically, we use the positive side of the horizontal axis going up to four times the distance of the receiver from the epicenter, i.e., for $X = 120.0\,\text{m}$. For better accuracy, we develop a two-sided transform by projecting symmetric values of the functions to be inverted along the negative X-axis. More specifically, for $N = 1024$ data points, the wave number spectrum $-H \le \eta \le +H$ is set up according to the following formulas:

$$\Delta x = 2X/N = 0.23437\,\text{m}, \quad \Delta\eta = 2\pi/N\Delta x = 0.02618\,\text{m}^{-1}, \quad H = \pi/\Delta x = 13.404\,\text{m}^{-1}. \tag{44}$$

We note in passing that it is possible to introduce viscoelastic material behavior using the Kelvin model with complex values for the material parameters [Flugge 1967], which is compatible with the static solution at zero frequency.

Figures 2 and 3 plot both amplitude and phase angle of the Green's functions $G^{\text{inhom}}(x, \xi, \omega)$ and $G^{\text{hom}}(x, \xi, \omega)$, respectively. The general structure of the Green's functions is

$$G_{IJ}(x, \xi, \omega) = U_{IJ}(x, \xi, \omega) + W_{IJ}(x, \xi, \omega), \quad I, J = 1, 2, \tag{45}$$

where $U_{IJ}(x, \xi, \omega)$ is the full space solution and $W_{IJ}(x, \xi, \omega)$ the Rayleigh-type correction. We note again that in the approach used in [Kinoshita 1983], the component $W_{IJ}(x, \xi, \omega)$ restores traction-free conditions at the free horizontal surface through addition to the full-plane solution $U_{IJ}(x, \xi, \omega)$ plus its image $U_{IJ}(x, -\xi, \omega)$. This latter operation results in a zeroing of the off-diagonal components and a doubling of the diagonal ones for the full-space solution.

We first observe in Figures 2 and 3 that the introduction of inhomogeneity results in a small decrease of a few percentage points in the amplitude of the full-space components $U_{I,J}(x, \xi, \omega)$, since the elastic

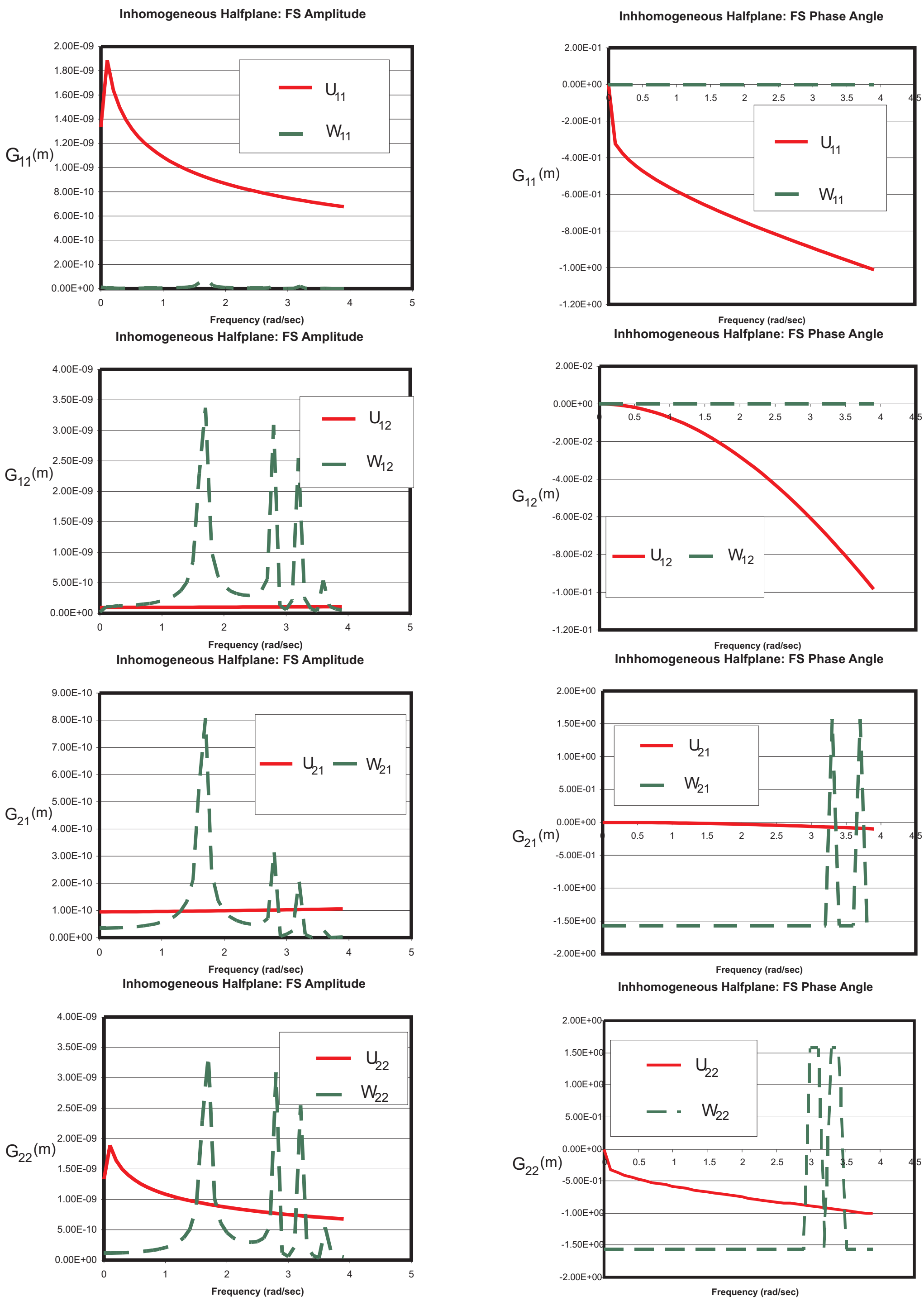

Figure 2. Inhomogenous half-plane ($a = -0.0010$) fundamental solution components: amplitude (left column) and phase angle (right column) versus frequency. From top to bottom, the graphs show G_{11}, G_{12}, G_{21} and G_{22}.

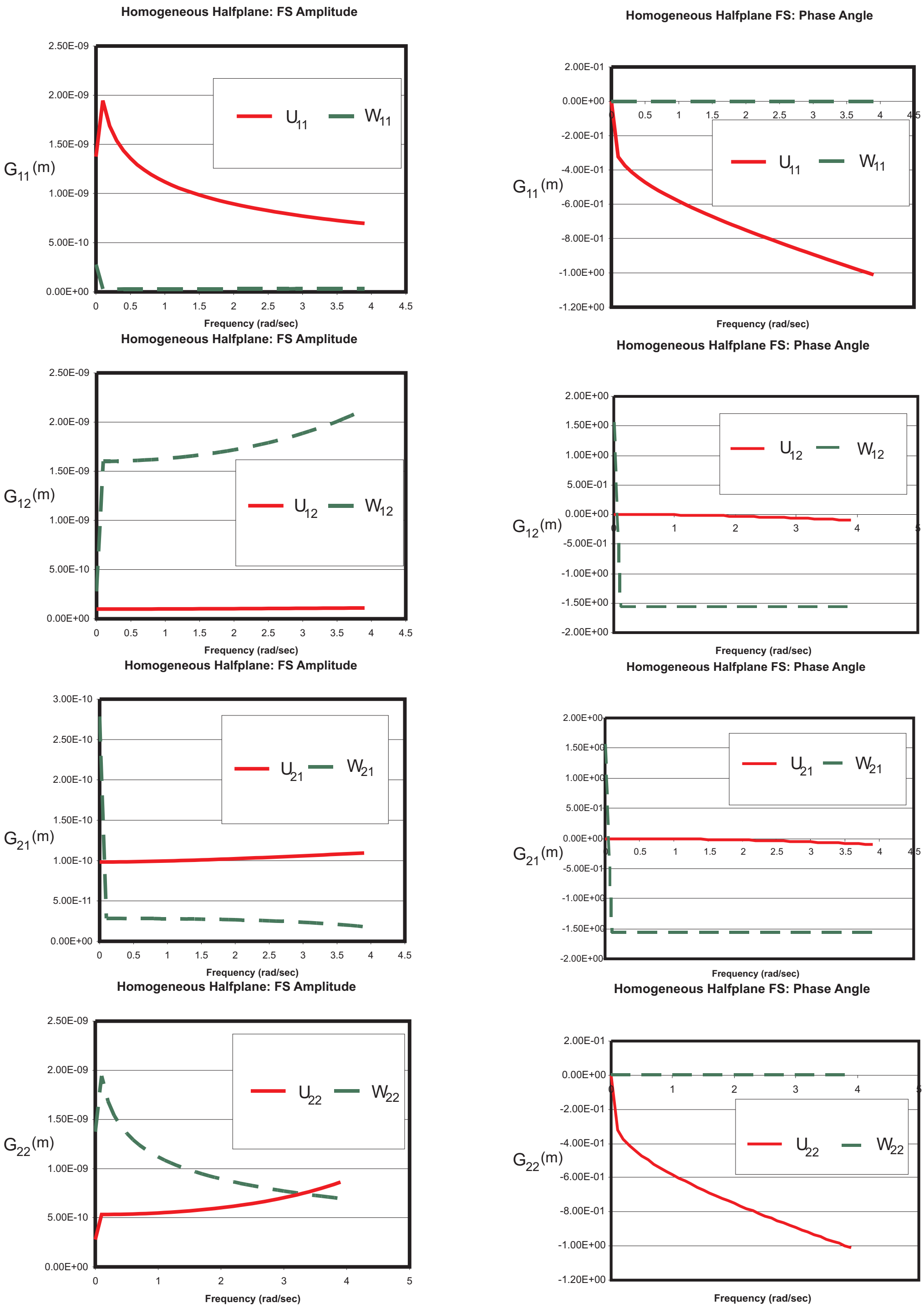

Figure 3. Homogenous half-plane fundamental solution components at surface source S: amplitude (left column) and phase angle (right column) versus frequency. From top to bottom, the graphs show G_{11}, G_{12}, G_{21} and G_{22}.

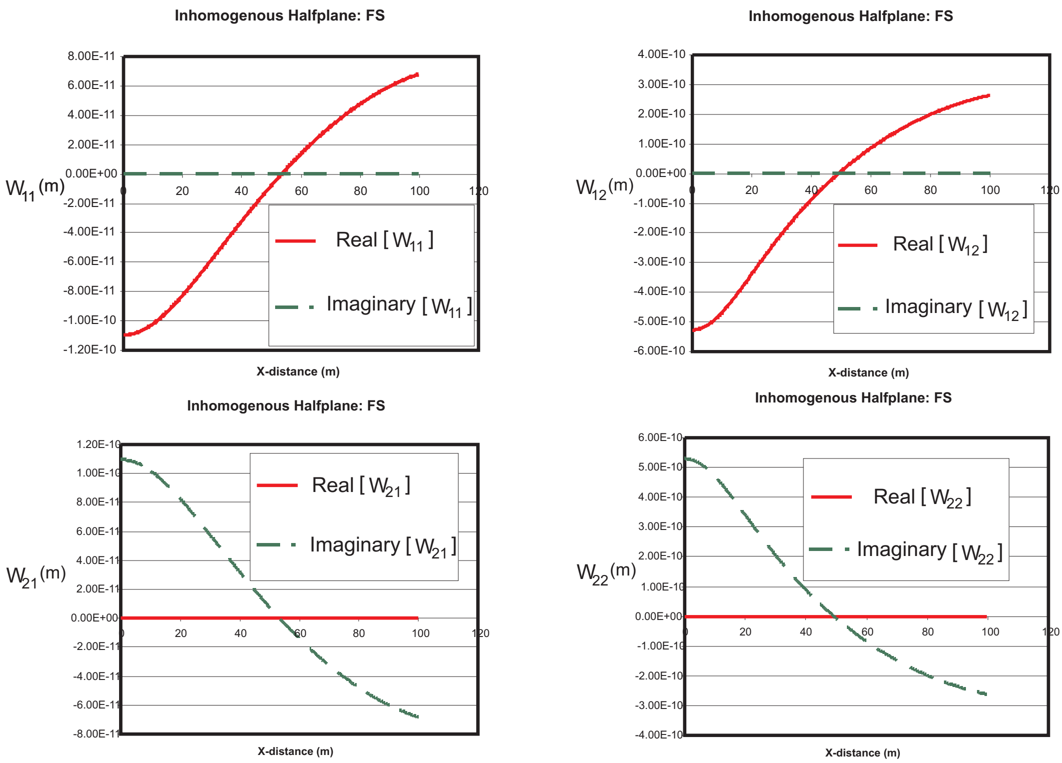

Figure 4. Inhomogeneous half-plane ($a = -0.0010$) Rayleigh-type fundamental solution components W_{11}, W_{12}, W_{21}, and W_{22}, at a frequency $\omega = 1.0$ rad/sec: real and imaginary parts versus distance along the surface.

waves are moving upwards in a medium with decreasing stiffness that is still larger than that of the equivalent homogeneous medium (but becomes equal to it at the surface). The same behavior holds true regarding the phase angle, i.e., there are some small differences between the two cases.. The situation is somewhat different regarding the Rayleigh-type correction. Starting with the homogeneous medium, this correction is substantial for the (1, 2) and (2, 2) components, and less so for the other two. The same trend holds for the inhomogeneous medium, but the correction is not a smoothly decreasing function of frequency as before. Instead, there are peaks at discrete frequency values such as $\omega = 1.5, 2.5, 3.5$ rad/sec. These local peaks are also manifested in the phase angle, with the exception when values that are nearly zero, as is the case with the (1, 1) and (1, 2) components.

In order to further investigate the behavior of the $W_{I,J}(x, \xi, \omega)$ components, Figures 4 and 5 show the variation of the Rayleigh-type solutions (both real and imaginary parts) along the free surface at a fixed value of the external frequency equal to $\omega = 1.0$ rad/sec for both inhomogeneous and equivalent homogeneous media. At first we note that in these solutions, either the real part (for the off-diagonal components) or the imaginary part (for the diagonal components) is zero. Next, these solutions decay slowly with increasing distance from the epicenter and show a sinusoidal type variation. The role of inhomogeneity is primarily seen in the magnitude of the nonzero components, in that they are quite more pronounced compared to the homogeneous medium case.

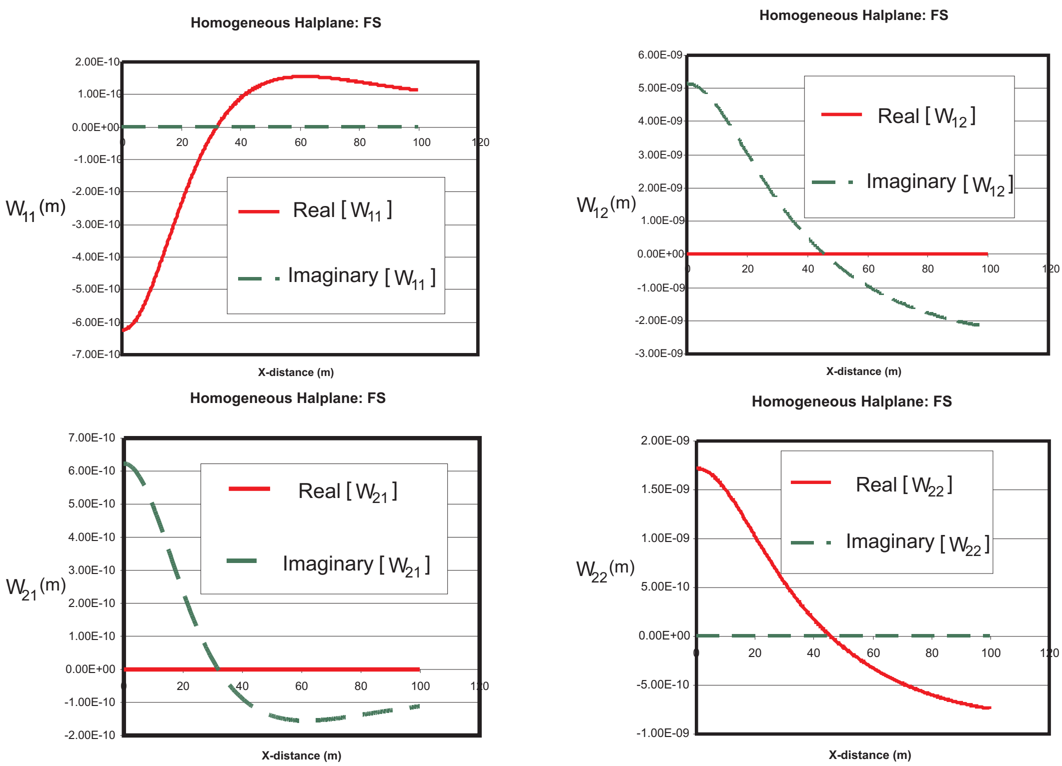

Figure 5. Homogenous half-plane Rayleigh-type fundamental solution components W_{11}, W_{12}, W_{21}, and W_{22}, at a frequency $\omega = 1.0$ rad/sec: real and imaginary parts versus distance along the surface.

We note in closing that approximate solutions using just one positive image source [Kontoni et al. 1987] lead to a doubling of the diagonal components of the displacement field and a zeroing of the off-diagonal ones in order to erase their corresponding traction contributions from the free surface. The use of a negative image source accomplishes the reverse. Thus, it is not possible to reproduce the correct traction conditions at the surface for all four components simultaneously, unless additional sources such as dipoles are added in the form of the Rayleigh integral [Kinoshita 1983].

6. Conclusions

In this work, we derived a new point-force solution in the continuously inhomogeneous half-plane with quadratic-type variation of all material parameters in terms of the depth coordinate. The solution comprises a complete elastic wave-train propagating outwards from the loaded area that satisfies traction-free boundary conditions along the horizontal surface. As such, solutions of this type are useful as kernel functions in boundary element method formulations of problems of engineering importance in elastodynamics and related fields of mechanics, with the added advantage that no free surface discretization is necessary.

Appendix: Elastostatic Green's functions for the half-plane

We derive here the Green's functions for the inhomogeneous half-plane as the external frequency of vibration tends to zero, i.e., the equivalent elastostatic forms. More specifically, based on the continuity of the Fourier transform as $\Im_{\eta\to x_1}(\lim_{\omega\to 0} g(\eta,\omega)) = \lim_{\omega\to 0}\Im_{\eta\to x_1}(g(\eta,\omega))$, it is sufficient to find the limit for $\omega\to 0$ of (22), (30) for the case $a<0$ and of (39), (40) for $a=0$.

To this purpose, we employ L'Hospital rule and re-define the wave numbers $k_j = q_j\omega$ in terms of the two wave slowness $q_1=\sqrt{\rho_0/3\mu_0}$, $q_2=\sqrt{\rho_0/\mu_0}$. Next, we define the following limit forms:

$$\begin{aligned}
(\beta_j)'_\omega &= -\frac{q_j^2}{\sqrt{\eta^2-q_j^2\omega^2}}\omega, \quad \text{with } \lim_{\omega\to 0}\frac{(\beta_j)'_\omega}{\omega} = -\frac{q_j^2}{|\eta|},\\
(\beta_1\beta_2)'_\omega &= -\frac{(q_1^2+q_2^2)\eta^2-q_1^2q_2^2\omega^2}{\sqrt{(\eta^2-q_1^2\omega^2)(\eta^2-q_2^2\omega^2)}}\omega, \quad \text{with } \lim_{\omega\to 0}\frac{(\beta_1\beta_2)'_\omega}{\omega} = -(q_1^2+q_2^2),
\end{aligned}\tag{A1}$$

$$\begin{aligned}
(e^{\xi_2\beta_j})'_\omega &= -\frac{\xi_2 q_j^2 e^{\xi_2\beta_j}}{\sqrt{\eta^2-q_j^2\omega^2}}\omega, \quad \text{with } \lim_{\omega\to 0}\frac{(e^{\xi_2\beta_j})'_\omega}{\omega} = -\frac{\xi_2 q_j^2 e^{\xi_2|\eta|}}{|\eta|},\\
(e^{x_2\beta_j})'_\omega &= -\frac{x_2 q_j^2 e^{x_2\beta_j}}{\sqrt{\eta^2-q_j^2\omega^2}}\omega, \quad \text{with } \lim_{\omega\to 0}\frac{(e^{x_2\beta_j})'_\omega}{\omega} = -\frac{x_2 q_j^2 e^{x_2|\eta|}}{|\eta|},
\end{aligned}\tag{A2}$$

where primes denote derivatives with respect to ω. For the limit of the determinant in (29) we have

$$\Delta^{a,0} = \lim_{\omega\to 0}\Delta^a = -\frac{\eta^2|\eta|\mu_0^2 a}{4\pi^2},\tag{A3}$$

since

$$\lim_{\omega\to 0}\Delta^0 = 0.$$

For the coefficients appearing in (30), we set $D^{a,0}_{jm} = \lim_{\omega\to 0}\Delta^a_{jm}$, and by using (A1), (A2) and (27) we obtain the expressions

$$\begin{aligned}
D^{a,0}_{11} &= \frac{h^{-1/2}(\xi_2)}{4q_2^2}\left[a\left(\frac{q_1^2+q_2^2}{|\eta|}+\xi_2(q_2^2-q_1^2)\right)+2q_2^2+2\xi_2|\eta|(q_2^2-q_1^2)\right]e^{\xi_2|\eta|},\\
D^{a,0}_{21} &= \frac{i\eta h^{-1/2}(\xi_2)}{4q_2^2|\eta|}\left[3a\xi_2(q_1^2-q_2^2)+2q_1^2+2\xi_2|\eta|(q_1^2-q_2^2)\right]e^{\xi_2|\eta|},\\
D^{a,0}_{12} &= -\frac{i\eta h^{-1/2}(\xi_2)}{4q_2^2|\eta|}\left[a\xi_2(q_1^2-q_2^2)+2q_1^2+2\xi_2|\eta|(q_1^2-q_2^2)\right]e^{\xi_2|\eta|},\\
D^{a,0}_{22} &= \frac{h^{-1/2}(\xi_2)}{4q_2^2}\left[3a\left(\frac{q_1^2+q_2^2}{|\eta|}+\xi_2(q_1^2-q_2^2)\right)+2q_2^2+2\xi_2|\eta|(q_1^2-q_2^2)\right]e^{\xi_2|\eta|}.
\end{aligned}\tag{A4}$$

Also, by using (A3) , (A4) and (31) we recover

$$\Delta_{11}^{a,0}=\begin{vmatrix}-D_{11}^{a,0} & i\mu_0(-a|\eta|+2\eta^2)\\ -D_{21}^{a,0} & \mu_0\eta(-3a+2|\eta|)\end{vmatrix},\quad \Delta_{21}^{a,0}=\begin{vmatrix}\mu_0\eta(-a+2|\eta|) & -D_{11}^{a,0}\\ i\mu_0(3a|\eta|-2\eta^2) & -D_{21}^{a,0}\end{vmatrix}$$
$$\Delta_{12}^{a,0}=\begin{vmatrix}\mu_0\eta(-a+2|\eta|) & -D_{12}^{a,0}\\ i\mu_0(3a|\eta|-2\eta^2) & -D_{22}^{a,0}\end{vmatrix},\quad \Delta_{22}^{a,0}=\begin{vmatrix}-D_{12}^{a,0} & i\mu_0(-a|\eta|+2\eta^2)\\ -D_{22}^{a,0} & \mu_0\eta(-3a+2|\eta|)\end{vmatrix}. \tag{A5}$$

Finally, the coefficients for the inhomogeneous case in (30) become

$$C_m^{k,a,0}=\lim_{\omega\to 0}C_m^k=\frac{\Delta_{mk}^{a,0}}{\Delta^{a,0}}. \tag{A6}$$

For completeness, we focus now on the homogeneous half-plane solution. First,

$$(\Delta_0)'_\omega=4\eta^2(\beta_1\beta_2)'_\omega-4(2\eta^2-q_2^2\omega^2)q_2^2\omega,\quad \text{with } \lim_{\omega\to 0}\frac{(\Delta_0)'_\omega}{\omega}=4\eta^2(q_1^2+3q_2^2). \tag{A7}$$

Next we set $\hat{D}_{jm}^0=\lim_{\omega\to 0}\hat{D}_{jm}$, $\hat{\Delta}_{jm}^0=\lim_{\omega\to 0}\hat{\Delta}_{jm}$, and by employing (A1), (A2) and (36) we obtain the following limits for the coefficients:

$$\hat{D}_{11}^0=0,\quad \hat{D}_{12}^0=-\frac{i\eta}{4q_2^2|\eta|}\big[2q_1^2+2|\eta|\xi_2(q_1^2-q_2^2)\big]e^{\xi_2|\eta|},$$
$$\hat{D}_{21}^0=\frac{i\eta}{4q_2^2|\eta|}\big[2q_1^2+2|\eta|\xi_2(q_1^2-q_2^2)\big]e^{\xi_2|\eta|},\qquad \hat{D}_{22}^0=0. \tag{A8}$$

Because of (A7), the limit of (38) does not exist as $\omega\to 0$, but the limit of the coefficients $\hat{S}_{jm}$ does. By setting

$$\bar{S}_{11}=\frac{i\eta\mu_0}{\Delta^0}\big[(2\eta^2-k_2^2)e^{x_2\beta_2}-2\beta_1\beta_2e^{x_2\beta_1}\big],\quad \bar{S}_{21}=\frac{\beta_1\mu_0}{\Delta^0}\big[(2\eta^2-k_2^2)e^{x_2\beta_2}-2\eta^2e^{x_2\beta_1}\big],$$
$$\bar{S}_{12}=\frac{\beta_2\mu_0}{\Delta^0}\big[-2\eta^2e^{x_2\beta_2}+(2\eta^2-k_2^2)e^{x_2\beta_1}\big],\quad \bar{S}_{22}=\frac{i\eta\mu_0}{\Delta^0}\big[-2\beta_1\beta_2e^{x_2\beta_2}+(2\eta^2-k_2^2)e^{x_2\beta_1}\big]. \tag{A9}$$

we obtain

$$\lim_{\omega\to 0}\hat{S}_{11}=(\lim_{\omega\to 0}\bar{S}_{11})\hat{D}_{21}^0,\quad \lim_{\omega\to 0}\hat{S}_{21}=(\lim_{\omega\to 0}\bar{S}_{21})\hat{D}_{12}^0,$$
$$\lim_{\omega\to 0}\hat{S}_{12}=(\lim_{\omega\to 0}\bar{S}_{12})\hat{D}_{12}^0,\quad \lim_{\omega\to 0}\hat{S}_{22}=(\lim_{\omega\to 0}\bar{S}_{22})\hat{D}_{21}^0.$$

Thus, the final expressions completing the elastostatic homogeneous case are

$$\lim_{\omega\to 0}\bar{S}_{11}=\frac{i\mu_0[x_2|\eta|(q_1^2-q_2^2)+q_1^2]}{\eta(q_1^2+3q_2^2)}e^{x_2|\eta|},\quad \lim_{\omega\to 0}\bar{S}_{21}=\frac{\mu_0[x_2|\eta|(q_1^2-q_2^2)-q_2^2]}{\eta(q_1^2+3q_2^2)}e^{x_2|\eta|},$$
$$\lim_{\omega\to 0}\bar{S}_{12}=\frac{i\mu_0[x_2|\eta|(q_2^2-q_1^2)-q_1^2]}{\eta(q_1^2+3q_2^2)}e^{x_2|\eta|},\quad \lim_{\omega\to 0}\bar{S}_{22}=\frac{i\mu_0[x_2|\eta|(q_2^2-q_1^2)+q_1^2]}{\eta(q_1^2+3q_2^2)}e^{x_2|\eta|}. \tag{A10}$$

References

[Achenbach 1973] J. D. Achenbach, *Wave propagation in elastic solids*, North Holland, Amsterdam, 1973.

[Bonchev et al. 1982] E. Bonchev, V. Bune, L. Christoskov, J. Karagyuleva, V. Kostadinov, G. Reisner, S. Rizikova, N. Shebalin, V. Sholpo, and D. Sokerova, "A method for compilation of seismic zoning prognostic maps for the territory of Bulgaria", *Geol. Balkanica* **12**:2 (1982), 3–48.

[Eringen and Şuhubi 1974] A. C. Eringen and E. S. Şuhubi, *Elastodynamics* (2 vol.), Academic Press, New York, 1974.

[Flugge 1967] W. Flugge, *Theory of viscoelasticity*, Blaisdell, Waltham, MA, 1967.

[Gradshteyn and Ryzhik 1980] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, corrected and enlarged ed., Academic Press, New York, 1980.

[Guan et al. 1998] F. Guan, I. D. Moore, and C. C. Spyrakos, "Two dimensional transient fundamental solution due to suddenly applied load in a half-space", *Soil Dyn. Earthquake Eng.* **17** (1998), 269–277.

[Kausel 2006] E. Kausel, *Fundamental solutions in elastodynamics: a compendium*, Cambridge University Press, Cambridge, 2006.

[Kausel and Manolis 2000] E. Kausel and G. D. Manolis (editors), *Wave motion problems in earthquake engineering*, WIT Press, Southampton, 2000.

[Kinoshita 1983] M. Kinoshita, M.Sc. thesis, Department of Civil Engineering, Kyoto University, 1983. in Japanese; quoted in [Kobayashi 1983].

[Kobayashi 1983] S. Kobayashi, "Some problems of the boundary integral equation method in elastodynamics", pp. 775–784 in *Boundary elements V*, edited by C. A. Brebbia et al., Springer, Berlin, 1983.

[Kontoni et al. 1987] D. P. N. Kontoni, D. E. Beskos, and G. D. Manolis, "Uniform half-plane elastodynamic problems by an approximate boundary element method", *Soil Dyn. Earthquake Eng.* **6** (1987), 227–238.

[Manolis and Beskos 1988] G. D. Manolis and D. E. Beskos, *Boundary element methods in elastodynamics*, Unwin and Allen, London, 1988.

[Manolis and Shaw 1996] G. D. Manolis and R. P. Shaw, "Green's function for the vector wave equation in a mildly heterogeneous continuum", *Wave Motion* **24**:1 (1996), 59–83.

[Rajapakse and Wang 1991] R. K. N. D. Rajapakse and Y. Wang, "Elastodynamic Green's functions of orthotropic half plane", *J. Eng. Mech.* (*ASCE*) **117**:3 (1991), 588–604.

Received 4 Feb 2010. Revised 15 Sep 2010. Accepted 29 Sep 2010.

TSVIATKO V. RANGELOV: rangelov@math.bas.bg
Department of Mathematical Physics, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,, acad. G. Bonchev str. bl. 8, 1113 Sofia, Bulgaria

GEORGE D. MANOLIS: gdm@civil.auth.gr
Department of Civil Engineering, Aristotle University, 54124 Thessaloniki, Greece

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

http://www.jomms.org

Founded by Charles R. Steele and Marie-Louise Steele

Cover design: Alex Scorpan Cover photo: Ev Shafrir

See inside back cover or http://www.jomms.org for submission guidelines.

JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2010 is US $500/year for the electronic version, and $660/year (+$60 shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://www.mathscipub.org
A NON-PROFIT CORPORATION
Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 5, No. 6 June 2010