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AN ENHANCED ASYMPTOTIC EXPANSION FOR THE STABILITY OF NONLINEAR ELASTIC STRUCTURES

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A new, enhanced asymptotic expansion applicable to stability of structures made of nonlinear elastic materials is established. The method utilizes "hyperbolic" terms instead of the conventional polynomial terms, covers full kinematic nonlinearity and is applied to nonlinear elastic Euler columns with two different types of cross-section. Comparison with numerical results show that our expansion provides more accurate predictions of the behavior than usual expansions.

The method is based on an extended version of the principle of virtual displacements that covers cases with auxiliary conditions, such as inextensibility. Membrane locking and similar problems are also handled by the method.

Part I. Theory

1. Introduction

The asymptotic expansions for elastic postbuckling and imperfection sensitivity originally introduced in [Koiter 1945] may be applied to any linearly elastic structure that experiences bifurcation instability in its geometrically perfect realization. There is, however, an inherent problem with these expansions, in that they employ polynomial terms, which means that the predictions of carrying capacities are inaccurate because the term of highest order approaches plus or minus infinity depending on its sign. This is, of course, not a desirable situation; it may be mended by exploiting some of the ideas introduced in [Christensen and Byskov 2008]. In particular, the concept of enhancing asymptotic expansions by using hyperbolic instead of polynomial terms is central here.

A set of explicit expressions for the coefficients of asymptotic elastic postbuckling and imperfection sensitivity analysis, applicable to linearly elastic structures with moderately large strains, linear loads and linear prebuckling was first proposed in [Budiansky and Hutchinson 1964]. Soon afterward, Fitch [1968] modified this to include nonlinear prebuckling. Later Byskov et al. [1996] extended the previous expansions to include loads which are nonlinear to fourth order in the displacements, and introduced Lagrange multiplier terms to fourth order in the displacements. The Lagrange multiplier terms provide a way to impose constraints on the structure, such as inextensibility, and a way of handling numerical phenomena such as membrane locking.

Here, we concentrate on two main subjects. The first is the development of a general asymptotic method akin to the one in [Byskov et al. 1996], but with strains, constraints and loads that may be arbitrarily nonlinear in the displacements. The constitutive relation is taken to be nonlinear elastic. Our second focus is the application of the enhanced asymptotic expansion developed in [Christensen and

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Byskov 2008], where we utilized the expansion to study postbuckling and imperfection sensitivity of the so-called Shanley–Hutchinson model column; see, e.g., [Hutchinson 1974]. In the present study this expansion is applied to the more realistic case of a nonlinear elastic Euler column.

2. Principle of virtual displacements

Consider a structure which, when it is geometrically perfect, experiences bifurcation at a certain critical load level λ_c. In order to investigate the behavior of the perfect structure and the influence of small geometric imperfections for situations in the vicinity of the critical load, we follow ideas from [Koiter 1945] and [Budiansky 1974], and use the perfect structure as a basis for the group of structures that only differ from the perfect one by an initial geometric imperfection. Regard the initial geometric imperfection as a small stress-free displacement $\hat{\boldsymbol{u}}$, which is not necessarily forced to obey the kinematic boundary conditions and does not depend on the actual stress and deformation state in the structure. A full nonlinear *modified principle of virtual displacements* including *Lagrange multiplier* terms, as previous used in [Byskov et al. 1996], for example, and nonlinear elasticity, may in general be written

$$\boldsymbol{P}(\boldsymbol{u}, \lambda, \hat{\boldsymbol{u}}) = \boldsymbol{\sigma}(\Delta\boldsymbol{\varepsilon}) \cdot \delta\Delta\boldsymbol{\varepsilon}(\boldsymbol{u}, \hat{\boldsymbol{u}}) - \delta\left[\boldsymbol{\eta} \cdot \boldsymbol{C}(\boldsymbol{u}, \hat{\boldsymbol{u}})\right] - \lambda\delta\Delta\boldsymbol{B}(\boldsymbol{u}, \hat{\boldsymbol{u}}) = 0, \tag{1}$$

where $\boldsymbol{u}$ is defined as an extended field of additional displacements that may include both derivatives of the basic displacements with respect to position and Lagrange multipliers, $\boldsymbol{\varepsilon}$ is the strain measured on the perfect structure, $\boldsymbol{\sigma}$ the nonlinear elastic stress, λ is a scalar load parameter, $\boldsymbol{B}$ is a nonlinear loading functional based on the perfect structure, $\boldsymbol{C}$ contains the appropriate constraints with associated Lagrange multiplier fields $\boldsymbol{\eta}$, and Δ indicates the difference between nondeformed and deformed states. Finally, following the notation in [Budiansky and Hutchinson 1964], a dot (·) signifies a generalized inner product over the entire structure.

For later purposes, let $\boldsymbol{u}$ include n components u and define

$$\boldsymbol{u} = \{u^1, u^2, \ldots, u^n\} = u^i. \tag{2}$$

According to these definitions,

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u} + \hat{\boldsymbol{u}}) - \boldsymbol{\varepsilon}(0 + \hat{\boldsymbol{u}}), \qquad \Delta\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{u} + \hat{\boldsymbol{u}}) - \boldsymbol{B}(0 + \hat{\boldsymbol{u}}). \tag{3}$$

The principle of virtual displacements depends linearly on the virtual displacements $\delta\boldsymbol{u}$ and may be written as

$$\boldsymbol{P} = \delta u^l \cdot \boldsymbol{p}_l = 0, \tag{4}$$

where

$$\boldsymbol{p}_l = \boldsymbol{\sigma}\,(\Delta\boldsymbol{\varepsilon}) \cdot \frac{\partial\boldsymbol{\varepsilon}(\boldsymbol{u} + \hat{\boldsymbol{u}})}{\partial u^l} - \frac{\partial\,[\boldsymbol{\eta} \cdot \boldsymbol{C}]}{\partial u^l} - \lambda\frac{\partial\boldsymbol{B}(\boldsymbol{u} + \hat{\boldsymbol{u}})}{\partial u^l}. \tag{5}$$

Note that $\boldsymbol{p}_l$ does not depend on $\delta\boldsymbol{u}$.

3. Perturbation expansion

Let the load be controlled by the scalar load parameter λ. When λ is close to its classical critical value λ_c, the displacement field $\boldsymbol{u}$, the scalar load parameter λ, and the principle of virtual displacements $\boldsymbol{P}(u^i, \lambda, \hat{u}^i)$ may be expanded in perturbation series around the prebuckling solution in the spirit of, for instance, [Budiansky and Hutchinson 1964; Budiansky 1974; Byskov et al. 1996]. Let us choose a characteristic buckling amplitude ξ, which vanishes at the critical point, as our perturbation parameter. Further, let $\bar{\xi}$ denote a characteristic amplitude of the imperfection shape $\tilde{\boldsymbol{u}}$:

$$\hat{\boldsymbol{u}} = \bar{\xi}\tilde{\boldsymbol{u}}. \tag{6}$$

It is our purpose to establish a formula for the maximum value of λ for a given value $\bar{\xi}$ of the imperfection amplitude.

Following [Budiansky 1974] we may imagine that in the space $(\bar{\xi}, \xi, \lambda)$ the values of λ form a surface and assume that the following relation is valid for small values of ξ, $\bar{\xi}$ and $|\lambda - \lambda_c|$:

$$\bar{\xi} = \alpha \xi^{\gamma} \tag{7}$$

where the coefficient α and the exponent γ both are scalars, and the value $\alpha = 0$ implies the traditional postbuckling path.

By choosing values of α and γ appropriately we may reach any point in the $(\bar{\xi}, \xi)$-plane, in particular the point associated with $\lambda_{\max}$, as indicated in Figure 1. In our search for the above mentioned maximum value of λ we select the value $\gamma = 2$ and determine the value of α by inserting (7) into the equations for the boundary value problem for the geometrically imperfect structure after having made an asymptotic expansion in terms of the characteristic buckling amplitude ξ. We shall assume that a perturbation expansion for the equilibria on which (7) holds may be written

$$\frac{\lambda}{\lambda_c} = 1 + \frac{\Delta\lambda}{\lambda_c} = 1 + \bar{a}_1\xi + \bar{a}_2\xi^2 + \bar{a}_3\xi^3 + O(\xi^4). \tag{8}$$

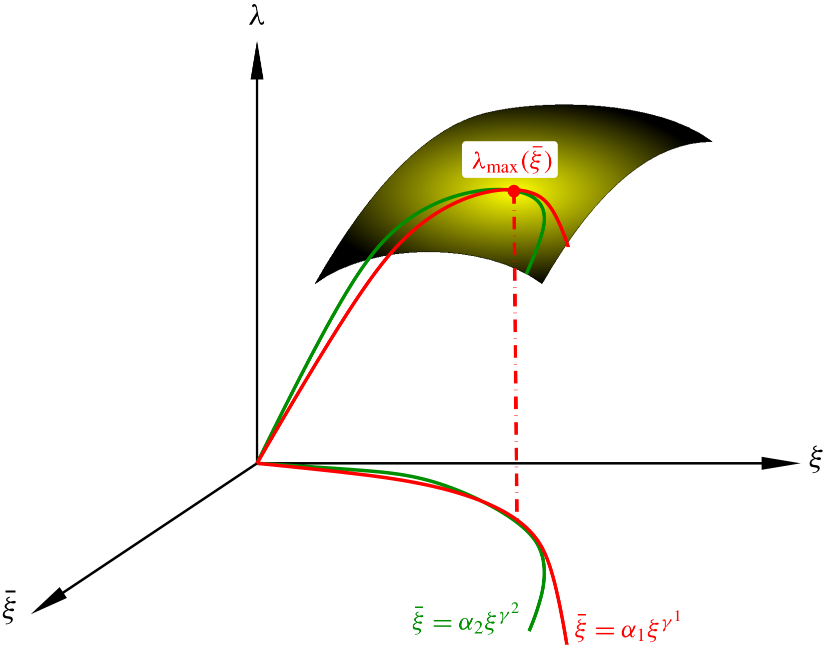

Figure 1. Prebuckling and equilibrium paths on which $\bar{\xi} = \alpha\xi^{\gamma}$ close to bifurcation.

Any such solution to the principle of virtual work includes parts of the prebuckling displacements $\boldsymbol{u}_0$, where subscript $_0$ indicates the prebuckling path. In anticipation of this we write the solution close to prebuckling as

$$\boldsymbol{u} = \boldsymbol{u}_0(\lambda) + \Delta\boldsymbol{u} = \bar{\boldsymbol{u}}_0(\lambda) + \xi\bar{\boldsymbol{u}}_1 + \xi^2\bar{\boldsymbol{u}}_2 + \xi^3\bar{\boldsymbol{u}}_3 + O(\xi^4). \tag{9}$$

The perturbation coefficients $\bar{a}_i$ and $\bar{\boldsymbol{u}}_i$ may be split into a part independent of α and a part which depends on α and vanishes when $\alpha = 0$:

$$\bar{a}_i = a_i + a_i^\alpha, \quad \bar{\boldsymbol{u}}_i = \boldsymbol{u}_i + \boldsymbol{u}_i^\alpha. \tag{10}$$

Now $\boldsymbol{P}$ may be expanded in terms of ξ through its dependence on λ and $\boldsymbol{u}$:

$$\boldsymbol{P} = \boldsymbol{P}_0(\lambda) + \Delta\boldsymbol{P} = \boldsymbol{P}_0(\lambda) + \xi\boldsymbol{P}_1 + \xi^2\boldsymbol{P}_2 + \xi^3\boldsymbol{P}_3 + O(\xi^4) = 0. \tag{11}$$

4. Asymptotic problems

Since the prebuckling path is a solution to the principle of virtual displacements (1), the prebuckling term of (11) is identically zero, i.e., $\boldsymbol{P}_0(\lambda) \equiv 0$. Therefore the matched asymptotic expansion (11) may be rewritten as

$$\Delta\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{u}_0 + \Delta\boldsymbol{u}, \lambda) - \boldsymbol{P}(\boldsymbol{u}_0, \lambda) = \xi\boldsymbol{P}_1 + \xi^2\boldsymbol{P}_2 + \xi^3\boldsymbol{P}_3 + O(\xi^4) = 0, \tag{12}$$

where we demand that (12) is fulfilled exactly for all values of the expansion parameter ξ and obtain the higher-order asymptotic stability problems according to the order in ξ:

first-order problem $\quad \boldsymbol{P}_1 = 0 \quad$ (13)

second-order problem $\quad \boldsymbol{P}_2 = 0 \quad$ (14)

third-order problem $\quad \boldsymbol{P}_3 = 0 \quad$ (15)

4.1. ***First-, second- and third-order problems.*** In order to solve the asymptotic problems (13)–(15), the three high-order operators $\boldsymbol{P}_i$ $(i = 1, 2, 3)$ must be expressed in terms of the expansion coefficients of the basic variables, i.e., $\bar{\boldsymbol{u}}_i$ and $\bar{a}_i$. Prebuckling fields may be considered known at bifurcation and derivatives with respect to the scalar load parameter λ are defined:

$$(\ )' \equiv \frac{d(\ )}{d\lambda}. \tag{16}$$

In the following, subscript $_c$ or superscript ^{c} designates prebuckling values taken at bifurcation. From (12) it is evident that $\boldsymbol{P}_i$ are the coefficient fields of an expansion in ξ of the prebuckling solution given in Appendix B subtracted from the full postbuckling solution. In Appendix A this approach has been utilized to provide the higher-order problems $\boldsymbol{P}_i$ provided that $\boldsymbol{P}$ has continuous derivatives at least up til fourth order with respect to $\bar{u}^i$. This ensures that the indices (i, j, k) in the below expressions for $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ may be swapped freely:

$$\boldsymbol{P}_1 = \boldsymbol{P}^c_{,i} \cdot \bar{u}^i_1, \tag{17}$$

$$\boldsymbol{P}_2 = \boldsymbol{P}^c_{,i} \cdot \bar{u}^i_2 + \tfrac{1}{2}\boldsymbol{P}^c_{,ij} \cdot [\bar{u}^i_1\bar{u}^j_1] + \bar{a}_1\lambda_c(\delta\boldsymbol{B}^c_{,i} + \boldsymbol{P}^c_{,ij}u^{j'}_c) \cdot \bar{u}^i_1 + \alpha\tilde{\boldsymbol{u}}(\boldsymbol{P}^c_{,\hat{\boldsymbol{u}}}), \tag{18}$$

$$\begin{aligned}\boldsymbol{P}_3 = {} & \boldsymbol{P}^c_{,i} \cdot \bar{u}^i_3 + \boldsymbol{P}^c_{,ij} \cdot [\bar{u}^i_2\bar{u}^j_1] + \tfrac{1}{6}\boldsymbol{P}^c_{,ijk} \cdot [\bar{u}^i_1\bar{u}^j_1\bar{u}^k_1] + \bar{a}_2\lambda_c(\delta\boldsymbol{B}^c_{,i} + \boldsymbol{P}^c_{,ij}u^{j'}_c) \cdot \bar{u}^i_1 \\ & + \bar{a}_1\lambda_c\big[(\delta\boldsymbol{B}^c_{,i} + \boldsymbol{P}^c_{,ij}u^{j'}_c) \cdot \bar{u}^i_2 + \tfrac{1}{2}(\delta\boldsymbol{B}^c_{,ij} + \boldsymbol{P}^c_{,ijk}u^{k'}_c) \cdot [\bar{u}^i_1\bar{u}^j_1]\big] \\ & + (\bar{a}_1\lambda_c)^2\big(\delta\boldsymbol{B}^c_{,ij}u^{j'}_c + \tfrac{1}{2}\boldsymbol{P}^c_{,ijk}[u^{j'}_c u^{k'}_c] + \tfrac{1}{2}\boldsymbol{P}^c_{,ij}u^{j''}_c\big) \cdot \bar{u}^i_1 \\ & + \alpha\tilde{\boldsymbol{u}} \cdot \big[\bar{a}_1\lambda_c(\delta\Delta\boldsymbol{B}^c_{,\hat{\boldsymbol{u}}} + \boldsymbol{P}^c_{,i\hat{\boldsymbol{u}}}u^{i'}_c) + \boldsymbol{P}^c_{,i\hat{\boldsymbol{u}}}\bar{u}^i_1\big],\end{aligned} \tag{19}$$

where, according to (4),

$$\boldsymbol{P}_{,i\ldots k} = \delta u^l \cdot \boldsymbol{p}_{l,i\ldots k} = \frac{\partial^n \boldsymbol{P}}{\partial u^i \ldots \partial u^k} \quad \text{and} \quad \boldsymbol{P}_{,\hat{\boldsymbol{u}}} = \delta u^l \cdot \boldsymbol{p}_{l,\hat{\boldsymbol{u}}} = \frac{\partial \boldsymbol{P}}{\partial \hat{u}^i} \tag{20}$$

and it is utilized that

$$\boldsymbol{P}_{,\lambda i\ldots k} = \delta\boldsymbol{B}_{i\ldots k} = \delta u^l \cdot \boldsymbol{B}_{,li\ldots k}. \tag{21}$$

4.2. ***Stability operators.*** In the first- to third-order problems, (17)–(19), a number of scalar operators may be identified, which we call the *stability operators*. These operators are given directly by the loading conditions, the strain measure, the stress-strain relation, the constraints, and the imperfection shape. In the following, the subscript of the operators indicates their degree.

Operators not acting on the imperfection shape. The following stability operators always enter the stability problems. Below the displacement fields, $\boldsymbol{u}_\alpha$, $\boldsymbol{u}_\beta$, $\boldsymbol{u}_\gamma$ and $\boldsymbol{u}_\delta$ may be any displacement field:

$$\begin{aligned}\mathcal{P}^c_1(\boldsymbol{u}_\alpha) &= \boldsymbol{p}^c_l \cdot [u^l_\alpha], \\ \mathcal{P}^c_2(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \boldsymbol{p}^c_{l,i} \cdot [u^l_\alpha u^i_\beta], \\ \mathcal{P}^c_3(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) &= \boldsymbol{p}^c_{l,ij} \cdot [u^l_\alpha u^i_\beta u^j_\gamma], \\ \mathcal{P}^c_4(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma, \boldsymbol{u}_\delta) &= \boldsymbol{p}^c_{l,ijk} \cdot [u^l_\alpha u^i_\beta u^j_\gamma u^k_\delta],\end{aligned} \tag{22}$$

$$\begin{aligned}\mathcal{B}^c_1(\boldsymbol{u}_\alpha) &= \boldsymbol{B}^c_{,l} \cdot [u^l_\alpha], \\ \mathcal{B}^c_2(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \boldsymbol{B}^c_{,li} \cdot [u^l_\alpha u^i_\beta], \\ \mathcal{B}^c_3(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) &= \boldsymbol{B}^c_{,lij} \cdot [u^l_\alpha u^i_\beta u^j_\gamma], \\ \mathcal{B}^c_4(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma, \boldsymbol{u}_\delta) &= \boldsymbol{B}^c_{,lijk} \cdot [u^l_\alpha u^i_\beta u^j_\gamma u^k_\delta].\end{aligned} \tag{23}$$

It is easily shown that $\boldsymbol{p}^c_{l,i\ldots k}$ is not influenced by the imperfection since $\hat{\boldsymbol{u}} = \bar{\xi}\tilde{\boldsymbol{u}} = \alpha\xi^2\tilde{\boldsymbol{u}}$ vanishes at the critical load. Thus $\mathcal{P}^c_i$ and $\mathcal{B}^c_i$ only depend indirectly on the geometric imperfection through the displacements they operate on.

Operators acting on the imperfection shape. These operators enter only when imperfections are present ($\alpha \neq 0$) and operate directly on the imperfection shape, $\tilde{\boldsymbol{u}}$:

$$\bar{\mathcal{P}}^c_2(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) = \boldsymbol{p}^c_{l,\hat{\boldsymbol{u}}} \cdot [u^l_\alpha\tilde{\boldsymbol{u}}] \quad \text{and} \quad \bar{\mathcal{P}}^c_3(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \tilde{\boldsymbol{u}}) = \boldsymbol{p}^c_{l,i\hat{\boldsymbol{u}}} \cdot [u^l_\alpha u^i_\beta\tilde{\boldsymbol{u}}], \tag{24}$$

$$\bar{\mathcal{B}}^c_2(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) = \Delta\boldsymbol{B}^c_{,l\hat{\boldsymbol{u}}} \cdot [u^l_\alpha\tilde{\boldsymbol{u}}] = \boldsymbol{B}^c_{,li} \cdot [u^l_\alpha\tilde{u}^i] \tag{25}$$

5. Bifurcation

The first-order problem (13) is referred to as the eigenvalue problem at bifurcation as it is used to determine the critical load λ_c and its associated bifurcation mode $\bar{\boldsymbol{u}}_1$, which is not necessarily identical to the traditional buckling mode $\boldsymbol{u}_1$ because imperfections may interact. Insert $\boldsymbol{P}_1$ as given by (17) in the first-order problem (13) to furnish

$$0 = \mathcal{P}_2^c(\bar{\boldsymbol{u}}_1, \delta\boldsymbol{u}), \tag{26}$$

where the first postcritical constant $\bar{a}_1$ does not enter. Thus, $\bar{\boldsymbol{u}}_1$ is simply determined as an eigenfield of (26) at the critical load, fixed by the characteristic amplitude ξ. As (26) is independent of the imperfection because the characteristic amplitude is only an expansion parameter, the first postcritical displacement field does not depend on the imperfection, and (10) yields

$$\bar{\boldsymbol{u}}_1 = \boldsymbol{u}_1 \text{ and } \boldsymbol{u}_1^\alpha = 0. \tag{27}$$

6. Higher-order problems

The main purpose of the higher-order problems (14) and (15) which are sometimes called the *postcritical problems*, is to provide a relation between the expansion parameter ξ, which usually is identified as some characteristic buckling mode amplitude, and the load level characterized by the value of the load parameter λ. In order to do this, we need the values of the constants $\bar{a}_1, \bar{a}_2, \dots,$ which determine the initial displacement-load relation after the classical critical load has been reached. Therefore, the higher-order displacement fields $\bar{\boldsymbol{u}}_2, \dots,$ must be found. Interest is, however, usually focused on determining the first nonvanishing postcritical constant. In the case of an unsymmetric structure, or a symmetric structure loaded unsymmetrically, the first postbuckling constant $a_1 = 0$ does not vanish, and we shall not need more than the buckling field $\boldsymbol{u}_1$. On the other hand, when the structure as well as the load is symmetric $\bar{a}_1$ becomes zero, and we need to determine the higher order displacement field $\boldsymbol{u}_2$ and the higher-order postbuckling constant $\bar{a}_2$. When $\bar{a}_1 = 0$ computation of $\hat{a}_2$ is simplified because in this case certain terms vanish from the higher order problems.

6.1. ***First postcritical problem.*** The first postcritical problem (14) determines the second order displacements $\bar{\boldsymbol{u}}_2$ and the first-order postcritical constant, $\bar{a}_1$, when the first-order displacement field has been determined by the buckling problem. Introduce $\boldsymbol{P}_2$ in (14) to obtain

$$0 = \mathcal{P}_2^c(\bar{\boldsymbol{u}}_2, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \delta\boldsymbol{u}) + \bar{a}_1\lambda_c\big(\mathcal{B}_2^c(\boldsymbol{u}_1, \delta\boldsymbol{u}) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \delta\boldsymbol{u})\big) + \alpha\bar{\mathcal{P}}_2^c(\tilde{\boldsymbol{u}}, \delta\boldsymbol{u}). \tag{28}$$

First postcritical constant. To determine the first postcritical constant $\bar{a}_1$ introduce $\delta u^l = \bar{u}_2^l$ in the buckling problem (26) and subtract it from the first postcritical problem (28) with $\delta u^l = u_1^l$ and exploit that, as shown in Appendix C, $\boldsymbol{p}_{l,i}^c = \boldsymbol{p}_{i,l}^c$ and eliminate the unknown postcritical displacement field $\bar{u}_2^i$ from the problem. The solution takes the form

$$\bar{a}_1 = a_1 + a_1^\alpha = a_1 + \alpha\rho_1, \tag{29}$$

where we for later use define a_1 and ρ_1 by

$$a_1 = \frac{a_1^N}{a_1^D}\frac{1}{\lambda_c} \quad \text{and} \quad \rho_1 = \frac{\rho_1^N}{a_1^D}\frac{1}{\lambda_c}, \tag{30}$$

with

$$a_1^N = -\tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1), \quad \rho_1^N = -\bar{\mathcal{P}}_2^c(\tilde{\boldsymbol{u}}, \boldsymbol{u}_1), \tag{31}$$

and

$$a_1^D = \mathcal{B}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1). \tag{32}$$

Postcritical displacement field. When $\bar{a}_1$ has been determined $\bar{\boldsymbol{u}}_2$ is found as a particular solution of the first postcritical problem (28) where $\bar{\boldsymbol{u}}_2$ only enters linearly. It appears from (28) that the complete solution for this problem takes the form

$$\bar{\boldsymbol{u}}_2 = c_1 \boldsymbol{u}_1 + \bar{\boldsymbol{u}}_2^{\text{partic}}. \tag{33}$$

In principle the arbitrary constant c_1 may be chosen freely. Each specific choice will only lead to a different interpretation of the expansion parameter ξ. Conversely, c_1 is fixed when ξ has been chosen. Often some orthogonality condition between the buckling displacement $\boldsymbol{u}_1$ and the postcritical displacement $\bar{\boldsymbol{u}}_2$ is enforced in order to exclude domination of the buckling field on the postcritical field in (33); see, e.g., [Fitch 1968; Budiansky 1974; Byskov et al. 1996]. Since $\bar{\boldsymbol{u}}_1$ does not depend on the imperfection, and because $\bar{a}_1$ depends linearly on α, $\boldsymbol{u}_2$ and $\boldsymbol{u}_2^\alpha$ may be found separately from (28), where we note that $\boldsymbol{u}_2^\alpha$ depends linearly on α. The variational equation for $\boldsymbol{u}_2$ is

$$0 = \mathcal{P}_2^c(\boldsymbol{u}_2, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \delta\boldsymbol{u}) + a_1\lambda_c\left(\mathcal{B}_2^c(\boldsymbol{u}_1, \delta\boldsymbol{u}) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \delta\boldsymbol{u})\right) \tag{34}$$

and the equation for $\boldsymbol{u}_2^\alpha = \alpha\boldsymbol{v}_2$ becomes

$$0 = \mathcal{P}_2^c(\boldsymbol{v}_2, \delta\boldsymbol{u}) + \bar{\mathcal{P}}_2^c(\tilde{\boldsymbol{u}}, \delta\boldsymbol{u}) + \rho_1\lambda_c\left(\mathcal{B}_2^c(\boldsymbol{u}_1, \delta\boldsymbol{u}) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \delta\boldsymbol{u})\right). \tag{35}$$

Note that α does not enter (35), and therefore $\boldsymbol{v}_2$ does not depend on α.

6.2. ***Second postcritical problem.*** The sole purpose of the second postcritical problem (15) is to determine the second postcritical constant $\bar{a}_2$. Utilize $\boldsymbol{P}_3$ in (15) to provide a problem depending on $\bar{a}_2$, $\bar{\boldsymbol{u}}_3$ and lower-order fields alone:

$$\begin{aligned}
0 = {} & \mathcal{P}_2^c(\bar{\boldsymbol{u}}_3, \delta\boldsymbol{u}) + \mathcal{P}_3^c(\bar{\boldsymbol{u}}_2, \boldsymbol{u}_1, \delta\boldsymbol{u}) + \tfrac{1}{6}\mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \delta\boldsymbol{u}) + \bar{a}_2\lambda_c\left(\mathcal{B}_2^c(\boldsymbol{u}_1, \delta\boldsymbol{u}) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \delta\boldsymbol{u})\right) \\
& + \bar{a}_1\lambda_c\left(\mathcal{B}_2^c(\bar{\boldsymbol{u}}_2, \delta\boldsymbol{u}) + \mathcal{P}_3^c(\boldsymbol{u}_c', \bar{\boldsymbol{u}}_2, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{B}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1, \delta\boldsymbol{u})\right) \\
& + (\bar{a}_1\lambda_c)^2\left(\mathcal{B}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_c', \boldsymbol{u}_1, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_c'', \boldsymbol{u}_1, \delta\boldsymbol{u})\right) \\
& + \alpha\left(\bar{\mathcal{P}}_3^c(\boldsymbol{u}_1, \tilde{\boldsymbol{u}}, \delta\boldsymbol{u}) + \bar{a}_1\lambda_c(\bar{\mathcal{B}}_2^c(\tilde{\boldsymbol{u}}, \delta\boldsymbol{u}) + \bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \tilde{\boldsymbol{u}}, \delta\boldsymbol{u}))\right).
\end{aligned} \tag{36}$$

Second postcritical constant. The second postcritical constant $\bar{a}_2$ is determined by a procedure similar to the one used to calculate the first postcritical constant $\bar{a}_1$. Let $\delta u^l = \bar{u}_3^l$ in the buckling problem (26) and subtract it from the second postcritical problem (36) with $\delta u^l = u_1^l$ to eliminate the unknown postcritical displacement field $\bar{u}_3^i$ from the problem:

$$\bar{a}_2\lambda_c = (a_2 + a_2^\alpha)\lambda_c = (a_2 + \rho_2\alpha + \rho_3\alpha^2)\lambda_c = \frac{(a_2^N + \rho_2^N\alpha + \rho_3^N\alpha^2)}{a_1^D}. \tag{37}$$

Here, a_1^D is determined earlier by (32) and a_2^N, a_2, ρ_i^N and ρ_i are parameters that do not depend on α:

$$\begin{aligned} a_2^N = &- \mathcal{P}_3^c(\boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_1) - \tfrac{1}{6}\mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) \\ &- a_1\lambda_c \left(\mathcal{B}_2^c(\boldsymbol{u}_2, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_2, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{B}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1)\right) \\ &- (a_1\lambda_c)^2 \left(\mathcal{B}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_c'', \boldsymbol{u}_1, \boldsymbol{u}_1)\right), \end{aligned} \tag{38}$$

$$\begin{aligned} \rho_2^N = &- \mathcal{P}_3^c(\boldsymbol{v}_2, \boldsymbol{u}_1, \boldsymbol{u}_1) \\ &- a_1\lambda_c \left(\mathcal{B}_2^c(\boldsymbol{v}_2, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{v}_2, \boldsymbol{u}_1)\right) \\ &- \rho_1\lambda_c \left(\mathcal{B}_2^c(\boldsymbol{u}_2, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_2, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{B}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1)\right) \\ &- 2\rho_1 a_1\lambda_c^2 \left(\mathcal{B}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_c'', \boldsymbol{u}_1, \boldsymbol{u}_1)\right) \\ &- \left(\bar{\mathcal{P}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \tilde{\boldsymbol{u}}) + a_1\lambda_c(\bar{\mathcal{B}}_2^c(\boldsymbol{u}_1, \tilde{\boldsymbol{u}}) + \bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \tilde{\boldsymbol{u}}))\right), \end{aligned} \tag{39}$$

$$\begin{aligned} \rho_3^N = &- \rho_1\lambda_c \left(\mathcal{B}_2^c(\boldsymbol{v}_2, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{v}_2, \boldsymbol{u}_1)\right) \\ &- (\rho_1\lambda_c)^2 \left(\mathcal{B}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_c'', \boldsymbol{u}_1, \boldsymbol{u}_1)\right) \\ &- \rho_1\lambda_c(\bar{\mathcal{B}}_2^c(\boldsymbol{u}_1, \tilde{\boldsymbol{u}}) + \bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \tilde{\boldsymbol{u}})). \end{aligned} \tag{40}$$

In the actual computation of a_2, ρ_2 and ρ_3 it may be exploited that several patterns of stability operators appear more than once.

7. Asymptotic problems to lowest order

As mentioned earlier, we are often only interested in determining the lowest-order postcritical effects with the implication that solving the asymptotic problems is simplified a great deal.

7.1. ***Postbuckling of a symmetric structure.*** Here, a_1 equals zero. In order to predict the initial postbuckling behavior for these structures, the second postbuckling constant a_2 must be found. However, when $a_1 = 0$ the formulas (28) and (37) for determining $\boldsymbol{u}_2$ and a_2 simplify considerably, becoming

$$0 = \mathcal{P}_2^c(\boldsymbol{u}_2, \delta\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \delta\boldsymbol{u}) \tag{41}$$

and

$$a_2\lambda_c = \frac{-\mathcal{P}_3^c(\boldsymbol{u}_2, \boldsymbol{u}_1, \boldsymbol{u}_1) - \tfrac{1}{6}\mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1)}{\mathcal{B}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1)}. \tag{42}$$

8. Determination of stability operators

The stress field $\boldsymbol{\sigma}$ may be given by the displacements u^i and $\tilde{u}^i$. However, for simplicity the stress field $\boldsymbol{\sigma}$ is often given as a function of the additional strain field $\Delta\boldsymbol{\varepsilon}$ of (3), which itself is a function of u^i and $\tilde{u}^i$.

The scalar stability operators given in (22) which determine the asymptotic coefficients $\bar{a}_i$ and $\bar{\boldsymbol{u}}_i$ depend on strain terms, Lagrange multiplier terms and load terms through the principle of virtual displacements.

8.1. ***Operators of the principle of virtual displacements.*** Together with the load operators $\mathcal{B}_i^c$ given in (23) the general operators below are used both for calculating the postbuckling equilibrium and the asymptotic effects of initial imperfections. As was the case for the stability operators, $\boldsymbol{u}_\alpha$, $\boldsymbol{u}_\beta$, $\boldsymbol{u}_\gamma$ and $\boldsymbol{u}_\delta$

may be any relevant displacement field:

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha) &= \boldsymbol{\varepsilon}^c_{,l} u^l_\alpha, \\
\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \boldsymbol{\varepsilon}^c_{,li} u^l_\alpha u^i_\beta, \\
\boldsymbol{\mathcal{E}}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) &= \boldsymbol{\varepsilon}^c_{,lij} u^l_\alpha u^i_\beta u^j_\gamma, \\
\boldsymbol{\mathcal{E}}_4^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma, \boldsymbol{u}_\delta) &= \boldsymbol{\varepsilon}^c_{,lijk} u^l_\alpha u^i_\beta u^j_\gamma u^k_\delta,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\mathcal{C}_1^c(\boldsymbol{u}_\alpha) &= (\boldsymbol{\eta} \cdot \boldsymbol{C})^c_{,l} u^l_\alpha, \\
\mathcal{C}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= (\boldsymbol{\eta} \cdot \boldsymbol{C})^c_{,li} u^l_\alpha u^i_\beta, \\
\mathcal{C}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) &= (\boldsymbol{\eta} \cdot \boldsymbol{C})^c_{,lij} u^l_\alpha u^i_\beta u^j_\gamma, \\
\mathcal{C}_4^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma, \boldsymbol{u}_\delta) &= (\boldsymbol{\eta} \cdot \boldsymbol{C})^c_{,lijk} u^l_\alpha u^i_\beta u^j_\gamma u^k_\delta.
\end{aligned} \tag{44}
$$

The derivatives of the stress field with respect to the strain field are

$$
\boldsymbol{\sigma} = \sigma_i, \quad \boldsymbol{D} = \boldsymbol{\sigma}_{,\boldsymbol{\varepsilon}} = D_{ij}, \quad \boldsymbol{D}' = \boldsymbol{\sigma}_{,\boldsymbol{\varepsilon\varepsilon}} = D'_{ijk} \quad \text{and} \quad \boldsymbol{D}'' = \boldsymbol{\sigma}_{,\boldsymbol{\varepsilon\varepsilon\varepsilon}} = D''_{ijkl}, \tag{45}
$$

where i, j, k and l may take on any natural number between 1 and the number of stress components. The operators D_{ij}, D'_{ijk} and D''_{ijkl} are symmetric in their indices for nonlinear elastic materials; thus the indices may be swapped freely. Now use the fact that the initial imperfection $\hat{\boldsymbol{u}}$ does not depend on any displacement component and that $\hat{\boldsymbol{u}} = 0$ at the classical critical load to evaluate the stability operators given in (22). Finally, introduce the operators from (43) to provide the *scalar* stability operators.

Note that in the following the fields in brackets have the same number of dimensions as the corresponding field of stiffnesses, which is one for each $\boldsymbol{\mathcal{E}}^c$. When evaluating the bracket each $\boldsymbol{\mathcal{E}}^c$ is treated as a tensor with a separate index. The order of the $\boldsymbol{\mathcal{E}}^c$-fields in the bracket is unimportant, since the corresponding field of stiffnesses is symmetric.

$$
\begin{aligned}
\mathcal{P}_1^c(\boldsymbol{u}_\alpha) &= \boldsymbol{\sigma}_c \cdot \boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha) - \mathcal{C}_1^c(\boldsymbol{u}_\alpha) - \lambda_c \mathcal{B}_1^c(\boldsymbol{u}_\alpha), \\
\mathcal{P}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \boldsymbol{D}_c \cdot [\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha)] + \boldsymbol{\sigma}_c \cdot \boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) - \mathcal{C}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) - \lambda_c \mathcal{B}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta), \\
\mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) &= \boldsymbol{D}'_c \cdot [\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\gamma)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha)] \\
&\quad + \boldsymbol{D}_c \cdot \big(\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\beta, \boldsymbol{u}_\gamma)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha) + \boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\gamma) + \boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\gamma)\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta)\big) \\
&\quad + \boldsymbol{\sigma}_c \cdot \boldsymbol{\mathcal{E}}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) - \mathcal{C}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma) - \lambda_c \mathcal{B}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma).
\end{aligned} \tag{46}
$$

In the stability problems, the fourth stability operator $\mathcal{P}_4$ operates at least twice on the buckling displacement field, $\boldsymbol{u}_1$. Exploit this to show that

$$
\begin{aligned}
\mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) =& \boldsymbol{D}''_c \cdot [\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_1)^2 \boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta)] \\
&+ \boldsymbol{D}'_c \cdot \big(\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_1)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta) + \boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_1)^2 \\
&\qquad + 2\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_1)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha)\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_\beta) + 2\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_1)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_\alpha)\big) \\
&+ \boldsymbol{D}_c \cdot \big(\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_1)\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) + 2\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_\alpha)\boldsymbol{\mathcal{E}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_\beta) + \boldsymbol{\mathcal{E}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_\alpha)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\beta) \\
&\qquad + \boldsymbol{\mathcal{E}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_\alpha) + 2\boldsymbol{\mathcal{E}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta)\boldsymbol{\mathcal{E}}_1^c(\boldsymbol{u}_1)\big) \\
&+ \boldsymbol{\sigma}_c \cdot \boldsymbol{\mathcal{E}}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) - \mathcal{C}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) - \lambda_c \mathcal{B}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_\alpha, \boldsymbol{u}_\beta).
\end{aligned} \tag{47}
$$

8.2. ***Scalar operators associated with imperfections.*** Let operators evaluated at the *initially* perfect reference before loading be indicated by superscript ^{I}. Then, operators associated with imperfections may be written

$$\mathcal{E}_1^I(\tilde{\boldsymbol{u}}) = \boldsymbol{\varepsilon}_{,l}^I \tilde{u}^l \quad \text{and} \quad \mathcal{D}_2^I(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) = \boldsymbol{\eta}_\alpha \boldsymbol{C}_{,i}^I \tilde{u}^i; \tag{48}$$

thus, the stability operators introducing imperfection given by (24) become

$$\begin{aligned} \bar{\mathcal{P}}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) &= \mathcal{P}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) - \boldsymbol{D}_c \cdot [\mathcal{E}_1^I(\tilde{\boldsymbol{u}})\mathcal{E}_1^c(\boldsymbol{u}_\alpha)] + \mathcal{D}_2^I(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}), \\ \bar{\mathcal{P}}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \tilde{\boldsymbol{u}}) &= \mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \tilde{\boldsymbol{u}}) - \boldsymbol{D}_c' \cdot [\mathcal{E}_1^I(\tilde{\boldsymbol{u}})\mathcal{E}_1^c(\boldsymbol{u}_\alpha)\mathcal{E}_1^c(\boldsymbol{u}_\beta)] - \boldsymbol{D}_c \cdot [\mathcal{E}_1^I(\tilde{\boldsymbol{u}})\mathcal{E}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta)], \end{aligned} \tag{49}$$

and

$$\bar{\mathcal{B}}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) \ = \mathcal{B}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}). \tag{50}$$

The asymptotic coefficients given earlier by the buckling, the first postcritical and the second postcritical problem are simple to compute when the above given stability operators have been computed for the specific structure.

9. Load-carrying capacity of imperfect structures

The asymptotic equilibrium path of both perfect and imperfect structures where $\bar{\xi} = \alpha\xi^2$ may be written

$$\frac{\lambda}{\lambda_c} = 1 + (a_1 + \rho_1\alpha)\xi + (a_2 + \rho_2\alpha + \rho_3\alpha^2)\xi^2 + O(\xi^3). \tag{51}$$

In reality the imperfection amplitude $\bar{\xi}$ is constant for each load case and introduction of $\alpha = \bar{\xi}/\xi^2$ provides an expression which links the load parameter λ to the characteristic buckling amplitude ξ for given $\bar{\xi}$:

$$\frac{\lambda}{\lambda_c} = \big(1 + a_1\xi + a_2\xi^2 + O(\xi^3)\big) + \frac{\bar{\xi}}{\xi}\big(\rho_1 + \rho_2\xi + O(\xi^2)\big) + \left(\frac{\bar{\xi}}{\xi}\right)^2 (\rho_3 + O(\xi)) + O((\bar{\xi}/\xi)^3). \tag{52}$$

9.1. ***Enhanced asymptotic expansion through the origin.*** Structures for which ξ is a characteristic buckling amplitude are subject to the simple boundary condition $\lambda(\xi = 0) = 0$. The asymptotic expansion (52) does not fulfill this condition. An expression which does fulfill $\lambda(\xi = 0) = 0$ *and* matches (52) asymptotically may be constructed as

$$\begin{aligned} \frac{\lambda}{\lambda_c} = \big(1 + a_1\xi + a_2\xi^2 + O(\xi^3)\big) + \left(\frac{\bar{\xi}/\xi}{1 + m_1\bar{\xi}/\xi}\right)\big(\psi_1 + \psi_2\xi + O(\xi^2)\big) \\ + \left(\frac{\bar{\xi}/\xi}{1 + m_1\bar{\xi}/\xi}\right)^2 (\psi_3 + O(\xi)) + O\left(\left(\frac{\bar{\xi}/\xi}{1 + m_1\bar{\xi}/\xi}\right)^3\right). \end{aligned} \tag{53}$$

An asymptotic match of (53) with (52) provides the constants ψ_i as

$$\psi_1 = \rho_1, \quad \psi_2 = \rho_2 \quad \text{and} \quad \psi_3 = \rho_3 + m_1\rho_1, \tag{54}$$

while the condition $\lambda(\xi = 0) = 0$ furnishes for m_1 the expression

$$m_1^2 + 2\rho_1 m_1 + \rho_3 = 0. \tag{55}$$

Remark. Our enhancement is not the only possible one. For instance, Koiter [1945] establishes one in a more physically intuitive way than ours, which is of a more mathematical origin and nature. Koiter's method, which results in modifying the load term in the expression for the potential energy or principle of virtual work by multiplication by the load parameter, has been applied elsewhere, as in [Byskov and Hutchinson 1977] and [Koiter 2009]. After a simple rearrangement of terms, Koiter's expression, valid for linear elasticity and linear prebuckling, is

$$\frac{\lambda}{\lambda_c} = \frac{\xi + b_1\xi + b_2\xi^2}{\xi + \bar{\xi}} \tag{56}$$

which at first appears to be different from our expression. An expansion in terms of $\bar{\xi}$ reveals, however, that the structure of the two formulas is the same for small values of $\bar{\xi}$. At the same time, our choice seems justified by the remarkably accurate results for large values of the rotation θ_0 shown in Figure 11.

9.2. ***Asymptotic maximum load of imperfect structures.*** The maximum load-carrying capacity of the imperfect structure may be determined asymptotically from either the traditional asymptotic expansion (52) or from the enhanced asymptotic expansion (53) with the same *asymptotic* accuracy. The asymptotic procedure for determining maximum load is, however, simpler for the traditional asymptotic expansion (52) and we shall later tie our enhanced expansion to it, but for larger imperfections abandon it in favor of the enhanced expansion.

The asymptotic maximum load of the imperfect structure is found where the derivative of λ/λ_c, given by (51), with respect to the buckling amplitude ξ becomes zero:

$$\frac{d(\lambda/\lambda_c)}{d\xi} = (a_1 - \rho_1\alpha_m) + 2(a_2 - \rho_3\alpha_m^2)\xi_m + O(\xi_m^2) = 0, \tag{57}$$

where subscript $_m$ indicates quantities related to the asymptotic maximum load and where we have used

$$\frac{d\alpha}{d\xi} = -2\frac{\alpha}{\xi}, \tag{58}$$

a consequence of (7).

An asymptotic match in (57) reveals that

$$\alpha_m = c_1 + c_2\xi_m + O(\xi_m^2) \quad \text{or} \quad \bar{\xi} = c_1\xi_m^2 + c_2\xi_m^3 + O(\xi_m^4), \tag{59}$$

where the constants are readily found to be

$$c_1 = \frac{a_1}{\rho_1} \quad \text{and} \quad c_2 = 2\frac{a_2\rho_1 - a_1\rho_3}{\rho_1^2}. \tag{60}$$

Insert the asymptotic expansion (59) for α_m in (51) to obtain a relation between the maximum load, λ_m, and the buckling amplitude at maximum load, ξ_m:

$$\frac{\lambda_m}{\lambda_c} = 1 + 2a_1\xi_m + \big(a_2 + \rho_1 c_2 + \rho_2 c_1 + \rho_3 c_1^2\big)\xi_m^2 + O(\xi_m^3). \tag{61}$$

In order to determine the maximum load for imperfect structures from (61), an asymptotic expression for ξ at maximum load must be found from (59). Since $a_1 = 0$ implies $c_1 = 0$, the asymptotic investigation is split into two cases, one for unsymmetric ($a_1 \neq 0$) and one for symmetric ($a_1 = 0$) structures.

Maximum load of unsymmetric structures. The asymptotic expression for the buckling amplitude ξ_m at maximum load becomes

$$\xi_m = l_1 \bar{\xi}^{\frac{1}{2}} + l_2 \bar{\xi} + O(\bar{\xi}^{\frac{3}{2}}) \quad \text{with} \quad l_1 = \sqrt{\frac{\rho_1}{a_1}} \quad \text{and} \quad l_2 = \frac{a_1\rho_3 - a_2\rho_1}{a_1^2}. \tag{62}$$

The asymptotic expansion for ξ_m, given by (62), may be inserted in (61) to provide the exact initial asymptotic dependency of the maximum load on the imperfection amplitude and shape when $a_1 \neq 0$:

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{1}{2}} \bar{\xi}^{\frac{1}{2}} + c^m_{\frac{2}{2}} \bar{\xi} + O(\bar{\xi}^{\frac{3}{2}}) = 1 - 2\sqrt{a_1\rho_1}\, \bar{\xi}^{\frac{1}{2}} + \left(\frac{a_2\rho_1}{a_1} + \rho_2 + \frac{\rho_3 a_1}{\rho_1} \right) \bar{\xi} + O(\bar{\xi}^{\frac{3}{2}}), \tag{63}$$

with

$$c^m_{\frac{1}{2}} = -2\sqrt{a_1\rho_1} \quad \text{and} \quad c^m_{\frac{2}{2}} = \left(\frac{a_2\rho_1}{a_1} + \rho_2 + \frac{\rho_3 a_1}{\rho_1} \right), \tag{64}$$

where we note that the maximum only exists when $a_1\rho_1 > 0$.

Maximum load of symmetric structures. At maximum load the asymptotic expression for the buckling amplitude ξ_m becomes

$$\xi_m = q_1 \bar{\xi}^{\frac{1}{3}} + O(\bar{\xi}^{\frac{2}{3}}) \quad \text{with} \quad q_1 = \left(\frac{\rho_1}{2a_2} \right)^{\frac{1}{3}}. \tag{65}$$

Insert the asymptotic expansion for ξ_m, given by the imperfection amplitude $\bar{\xi}$ (65) in (61) to provide the exact asymptotic dependency of the maximum load on the imperfection amplitude and shape when $a_1 = 0$ and get

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{2}{3}} \bar{\xi}^{\frac{2}{3}} + O(\bar{\xi}) \quad \text{with} \quad c^m_{\frac{2}{3}} = 3a_2^{\frac{1}{3}} \left(\frac{\rho_1}{2} \right)^{\frac{2}{3}}. \tag{66}$$

Note that the maximum only exists when $a_2 < 0$.

For a symmetric structure the third postbuckling constant a_3 will also vanish and one more term may be added to the asymptotic expansion (66) of the maximum load when $a_1 = 0$ without further expansion of the stability problems.

It is easily shown that the third degree term, t_3, of (51) takes the form

$$t_3 = k_i \alpha^i \xi^3, \quad \text{summed over } i \geq 1. \tag{67}$$

Thus, the expression for the maximum load associated with ξ_m simplifies to the exact asymptotic expansion

$$\frac{\lambda_m}{\lambda_c} = 1 + 3a_2\xi_m^2 + \frac{2a_2\rho_2}{\rho_1}\xi_m^3 + O(\xi_m^4), \tag{68}$$

where it has been exploited that both a_1 and a_3 vanish.

Use ξ_m as given in (65) to obtain the asymptotic expression for the maximum load as a function of the imperfection shape and amplitude:

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{2}{3}} \bar{\xi}^{\frac{2}{3}} + c^m_{\frac{3}{3}} \bar{\xi} + O(\bar{\xi}^{\frac{4}{3}}) \quad \text{with} \quad c^m_{\frac{3}{3}} = \rho_2, \tag{69}$$

where $c^m_{\frac{2}{3}}$ follows from (66)b.

Enhanced maximum load prediction. Because the above asymptotic expansion of the maximum load eventually approaches plus or minus infinity, it often deviates considerably from the real maximum load, even for relatively small imperfection amplitudes; this is the basic reason for our enhanced expansion (53). Due to the fact that the load parameter is zero before loading is applied, the maximum load of any imperfect structure must be greater than or equal to zero. Since we concentrate on structures whose maximum load decreases with the imperfection amplitude $\bar{\xi}$, but is always greater than zero, an enhanced maximum load prediction of the asymptotic method may be obtained by matching the asymptotic maximum load prediction given above with an expression that approaches zero with $\bar{\xi}$.

The traditional polynomial asymptotic expression for maximum load is determined by (63) for $a \neq 0$ and by (66) or (69) when $a = 0$. In the enhanced approach each asymptotic term is chosen to be a hyperbolic function approaching a constant at infinity. Often only the first asymptotic term is determined and therefore the enhanced asymptotic expansion is required to approach zero even if only one asymptotic term is used. The second term in the enhanced asymptotic expansion is required to match the second traditional asymptotic term, if present. A third enhancing term is included to force the expansion to approach zero at infinity. Its presence does, however, not interfere with the asymptotic correctness of the expansion because its dependence on the expansion parameters is beyond the limit of the original expansion. We establish the enhanced asymptotic expansion for the two separate cases:

Enhanced expansion, $a \neq 0$:

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{1}{2}}\left(\frac{\bar{\xi}}{1+(c^m_{\frac{1}{2}})^2\bar{\xi}}\right)^{\frac{1}{2}} + c^m_{\frac{2}{2}}\left(\frac{\bar{\xi}}{1+(c^m_{\frac{1}{2}})^2\bar{\xi}}\right) + c^m_{\frac{2}{2}}c^m_{\frac{1}{2}}\left(\frac{\bar{\xi}}{1+(c^m_{\frac{1}{2}})^2\bar{\xi}}\right)^{\frac{3}{2}} + O(\bar{\xi}^{\frac{3}{2}}) \tag{70}$$

Enhanced expansion, $a = 0$:

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{2}{3}}\left(\frac{\bar{\xi}}{1+(-c^m_{\frac{2}{3}})^{\frac{3}{2}}\bar{\xi}}\right)^{\frac{2}{3}} + c^m_{\frac{3}{3}}\left(\frac{\bar{\xi}}{1+(-c^m_{\frac{2}{3}})^{\frac{3}{2}}\bar{\xi}}\right) - c^m_{\frac{3}{3}}(-c^m_{\frac{2}{3}})^{\frac{1}{2}}\left(\frac{\bar{\xi}}{1+(-c^m_{\frac{2}{3}})^{\frac{3}{2}}\bar{\xi}}\right)^{\frac{4}{3}} + O(\bar{\xi}^{\frac{3}{2}}) \tag{71}$$

Part II. Application: the Euler column

10. Introduction to the Euler column

A vast number of analytical asymptotic and numerical studies have been performed on the postbuckling and imperfection sensitivity of the linear elastic Euler column (see [Kuznetsov and Levyakov 2002], for example), while the effects of nonlinear elasticity on the behavior of Euler columns have been studied less frequently; but see, for instance, [Tvergaard and Needleman 1982].

In the following, the formulas derived above are applied to the pin-ended Euler column (see Figure 2 on the next page), taking into account the effect of nonlinear elasticity as well as full nonlinear kinematics. Two different cross-sections, one symmetric and one asymmetric, of the column are investigated in order to show the ability of our method to handle both kinds of structures. Furthermore, the influence and possible simplifications caused by introduction of inextensibility is examined. In Appendix D the problem of stability of the geometrically perfect column is solved for nonlinear elasticity, considering extensibility as well as inextensibility. It turns out that the extensible and inextensible case yield the same

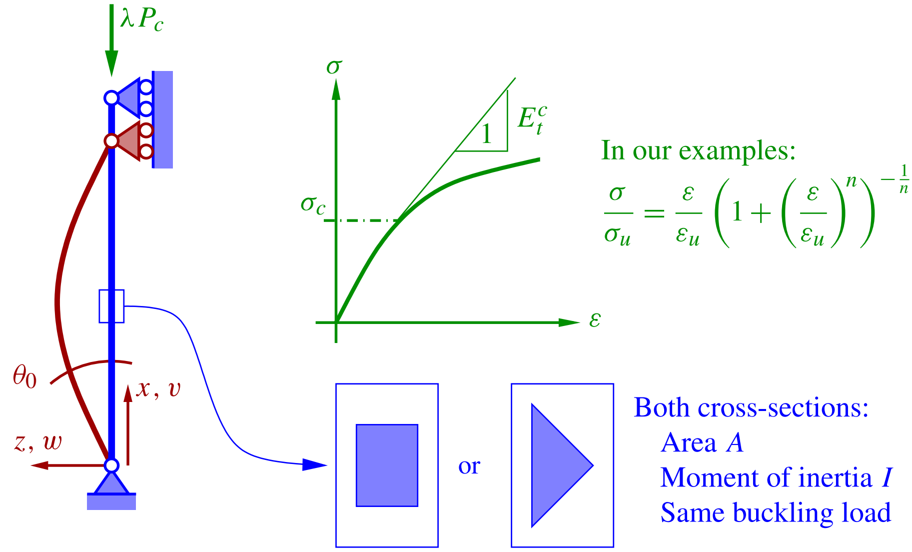

Figure 2. The nonlinear elastic Euler column.

critical load, while the first and second postbuckling constants are only identical or close to identical when the material model is linear. Thus, to solve stability problems for nonlinear elastic structures it is imperative to model extensibility.

11. Geometry and constitutive relation of the Euler column

In solving the stability problems we shall employ *fully nonlinear strain measures* and a *fully nonlinear elastic stress-strain relation*. The column is *not* regarded as inextensible but the usual Bernoulli–Euler beam theory simplifications are implemented, with the following nondimensional cross-sectional constants:

$$
\begin{aligned}
A_0 &\equiv \int_{A_0} dA, & Z_0 &\equiv \int_{A_0} z\, dA = 0 \\
I_j &\equiv \int_{A_0} z^{j+1} dA, & i_j &\equiv \left(\frac{\pi}{L}\right)^{j+1} \frac{I_j}{A_0} \ll 1 \qquad (i = 1, 2, 3),
\end{aligned}
\tag{72}
$$

where A_0 is the initial area of the columns cross-section and it important to note that all i_j vanish compared to unity according to the Bernoulli–Euler beam theory.

For convenience we introduce the nondimensional material stiffnesses

$$
e_c' = i_1 E_{t,\varepsilon}^c / E_t^c \quad \text{and} \quad e_c'' = i_3 E_{t,\varepsilon\varepsilon}^c / E_t^c, \tag{73}
$$

where we note that $E_{t,\varepsilon}^c / E_t^c$ and $E_{t,\varepsilon\varepsilon}^c / E_t^c$ may be very large. Thus, the constants e_c' and e_c'' are not necessarily small compared to unity.

11.1. *Cross-sections.* In order to demonstrate the capability of our method to predict the postbuckling behavior of geometrically perfect columns as well as describing the load-displacement relation of geometrically imperfect columns, either with symmetric or asymmetric cross-sections consider the two types of cross-sections shown in Figure 2. In both cases, which were studied in [Tvergaard and Needleman 1982], we let the area A and the moment of inertia I_1 of the cross-sections be identical for the two columns. In order to obtain the same critical load, given by the value of i_1 and material expense of the two types of column, the height and width of the triangular cross-section must be

$$h_T = \sqrt{\tfrac{3}{2}}h, \quad b_T = \sqrt{\tfrac{2}{3}}b, \tag{74}$$

where h and b are the height and width of the rectangular cross-section, respectively, h_T and b_T denote the equivalent quantities of the triangular cross-section, and

$$i_1 = \frac{1}{12}\left(\frac{h\pi}{L}\right)^2. \tag{75}$$

The higher-order nondimensional moments i_2 and i_3 are, of course, not the same for the nonsymmetric and symmetric cross-sections.

Triangular cross-section. Only the second dimensionless moment of inertia i_2 is needed since $a_1 \neq 0$ when the material is nonlinear:

$$i_2 = \frac{\sqrt{\frac{3}{2}}}{90}\left(\frac{h\pi}{L}\right)^3. \tag{76}$$

Rectangular cross-section. Here, both i_2 and i_3 are necessary because $a_1 = 0$:

$$i_2 = 0, \quad i_3 = \frac{1}{80}\left(\frac{h\pi}{L}\right)^4. \tag{77}$$

11.2. ***Strain-displacement relation.*** According to Bernoulli–Euler beam theory the only nonvanishing strains are in the axis direction and may be given as

$$\varepsilon(x) = \epsilon(x) + z\kappa(x), \tag{78}$$

where ε is the fiber strain, ϵ the strain of the neutral axis, and κ denotes the curvature strain. In the following we consider full kinematic nonlinearity, and thus

$$\epsilon = \sqrt{(1+\dot{v})^2 + \dot{w}^2} - 1, \quad \kappa = \dot{\theta} = \frac{\ddot{w}(1+\dot{v}) - \ddot{v}\dot{w}}{(1+\dot{v})^2 + \dot{w}^2}, \tag{79}$$

where a dot over a quantity denotes differentiation with respect to x, and the coordinates x and z and the displacements v and w are defined in Figure 2.

11.3. ***Stress-strain relation.*** Obviously, the choice of stress-strain relation influences the postbifurcation constants through the nondimensional derivatives with respect to strain at critical load defined in (73), i.e., through e'_c and e''_c, which both equal zero in linear elasticity. Provided that buckling takes place under decreasing stiffness in compression, e'_c may be any positive value and e''_c any value at all, independently of each other and of i_1 and i_3 respectively:

$$e'_c \in \,]0; \infty[, \quad e''_c \in \,]-\infty; \infty[. \tag{80}$$

To investigate the stability behavior of a structure for realistic cases, we shall assume that the column obeys the nonlinear elastic constitutive relation

$$\frac{\sigma}{\sigma_u} = \frac{\varepsilon}{\varepsilon_u}\left(1+\left(\frac{\varepsilon}{\varepsilon_u}\right)^n\right)^{-1/n}, \tag{81}$$

where E is Young's modulus, $\sigma_u < 0$ is the ultimate stress in compression, $\varepsilon_u = \sigma_u/E < 0$ is the strain corresponding to σ_u assuming linear elasticity, and n is a hardening parameter. High values of n imply nearly linear elastic behavior up till σ_u, while low values of n induce nonlinear elastic behavior even at small stresses compared to σ_u.

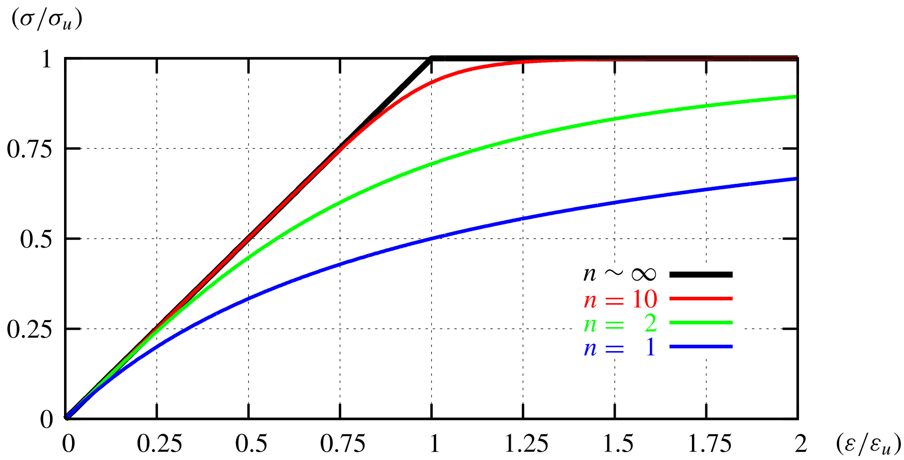

Figure 3. Stress-strain relation for different levels of the strain hardening parameter n. For $n \to \infty$ the constitutive model approaches linear elasticity-perfect plasticity.

11.4. ***Expansion parameter.*** Let the expansion parameter ξ be identified as the rotation of the column at $x = 0$:

$$\xi \equiv \theta(0). \tag{82}$$

12. Geometrically perfect Euler column

Appendix D contains the detailed calculations and derivations which lead to the determination of the asymptotic coefficient fields and critical load. In particular, we note that

$$\lambda_c = 1, \quad \sigma_c = i_1 E_t^c. \tag{83}$$

12.1. ***Unsymmetric elastic triangular cross-section.*** When the cross-section of the column is nonsymmetric *and* the material model is nonlinear elastic at the same time, the first postbuckling constant a_1 does not vanish. Under these conditions it is only necessary to determine the buckling displacement field $\boldsymbol{u}_1$ and a_1. The asymptotic displacement field is

$$\begin{aligned} w &= w_1\xi + O(\xi^2) = (L/\pi)\sin(\pi x/L)\xi + O(\xi^2), \\ v &= v_1\xi + O(\xi^2) = 0 + O(\xi^2), \end{aligned} \tag{84}$$

and the asymptotic load parameter is

$$\frac{\lambda}{\lambda_c} = 1 + a_1\xi + O(\xi^2) = 1 - \frac{4e'_c i_2}{3\pi i_1^2(1+e'_c)}\,\xi + O(\xi^2), \tag{85}$$

which is in exact agreement with the result obtained in [Tvergaard and Needleman 1982] for the non-symmetric cross-section.

12.2. ***Symmetric as well as linear elastic unsymmetric cross-sections.*** When the cross-section of the column is symmetric or when the material is linear elastic, the first postbuckling constant a_1 vanishes. Then, it is necessary to determine the second postbuckling constant a_2 to investigate postbuckling behavior. To the lowest order the asymptotic loads and displacements are

$$\begin{aligned} w &= w_1\xi + w_2\xi^2 + O(\xi^3) = (L/\pi)\sin(\pi x/L)\xi + O(\xi^3), \\ v &= v_1\xi + v_2\xi^2 + O(\xi^3) = -\frac{1}{4}\left((1+e'_c)x + (1-e'_c)\frac{L}{2\pi}\sin(2\pi x/L)\right)\xi^2 + O(\xi^3), \end{aligned} \tag{86}$$

and the asymptotic load parameter is

$$\frac{\lambda}{\lambda_c} = 1 + a_2\xi^2 + O(\xi^3) = 1 + \frac{i_1 - 3(e'_c)^2 + e''_c}{8i_1(1+e'_c)}\,\xi^2 + O(\xi^3). \tag{87}$$

12.3. ***Comparison with known results: the elastica.*** The load-carrying capacity in initial and advanced postbuckling of the linear elastic pin-ended Euler column has received much attention since the original study in [Euler 1744] of the so-called *elastica* and has been investigated in [Britvec 1973; Kuznetsov and Levyakov 2002], among other works, for a full nonlinear strain measure.

Excluding material nonlinearity from our initial postbuckling loads (87) provides the linear elastic postbuckling constants $a_1 = 0$ and $a_2 = \frac{1}{8}$, which agree exactly with the elastica solution in [Britvec 1973]. It may be worth noticing that, according to the kinematically moderately linear theory, a_2 as well as a_1 vanishes which underlines the importance of applying a general full nonlinear stability theory to obtain accurate postbuckling constants. On the other hand, the fact that the results for the second postbuckling constant differ between the two theories should not, in general, discredit the moderately nonlinear theory. The relative difference in predicted load is, after all, only $\xi/8$ (around 9.8% at the very large end rotation 45°).

12.4. ***Postbuckling behavior assuming nonlinear elasticity.*** As is evident from the expressions (85) and (87), introduction of nonlinear elasticity requires, apart from more constitutive parameters, i.e., e'_c and e''_c, a detailed knowledge of the cross-section geometry through i_2 and i_3. Because the expressions are only valid for small i_1 compared to unity the ratio between h and L must be limited. Since the absolute value of ε_u usually is less than one tenth of a per cent, the order of i_1 lies in the same range. In Figures 4–7 results for the Euler column are shown for the constitutive relation given by (81). The critical load, the first and second postbuckling constants are plotted as functions of the ratio between i_1 and the absolute value of ε_u in the range [0; 2] (where $\varepsilon_c/\varepsilon_u \in [0; 1]$ as indicated at the top of Figures 4–6).

Bifurcation load. In Figure 4 the nondimensional critical load, σ_c/σ_u, of the column is plotted for both nonlinear and initially linear elastic behavior. It is not surprising that the difference in critical load between nonlinear and linear elasticity increases with increasing cross-section, given by i_1, relative to the "ultimate" strain ε_u.

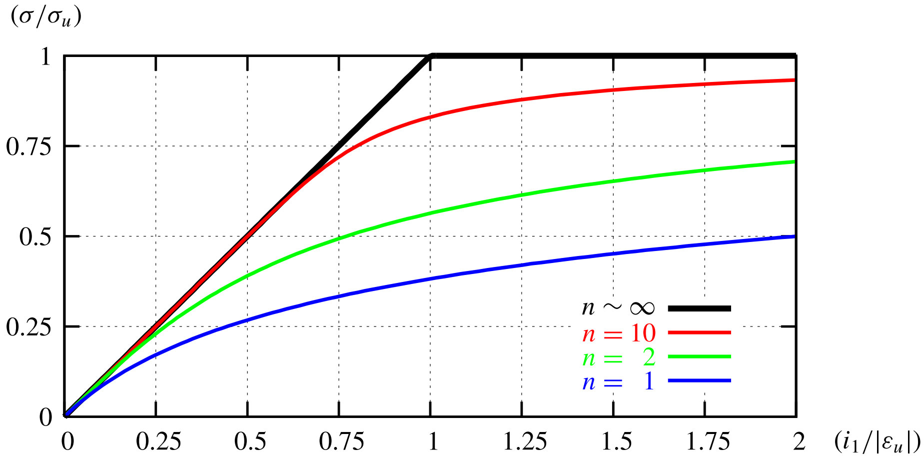

Figure 4. Bifurcation load of nonlinear elastic Euler columns with an ultimate stress.

Triangular cross-section. Figure 5 shows the first postbuckling constant a_1 of the triangular cross-section for nonlinear elasticity. Only at extremely small values of $i_1/|\varepsilon_u|$, i.e., for very slender columns, or extremely high values of n with relatively low values of $i_1/|\varepsilon_u|$ is a_1 according to the nonlinear elastic theory close to vanishing as it does according to linear elastic theory. The fact that nonlinear stress-strain relations deviate faster from initial linearity for lower values of n reflects in that a_1 initially deviates more rapidly from zero with increasing values of $\varepsilon_c/\varepsilon_u$ (and thus $i_1/|\varepsilon_u|$) the lower the value of n. When $\varepsilon_c/\varepsilon_u$ approaches unity ($i_1/|\varepsilon_u|$ approaches 2) high n yields the largest absolute values of a_1 because of the sudden large drop in tangent modulus near ε_u.

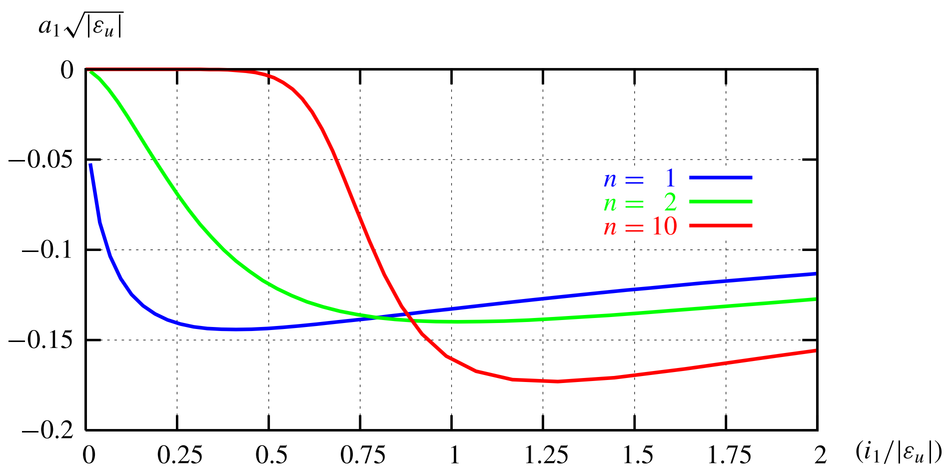

Figure 5. First postbuckling constant a_1 for triangular cross-section.

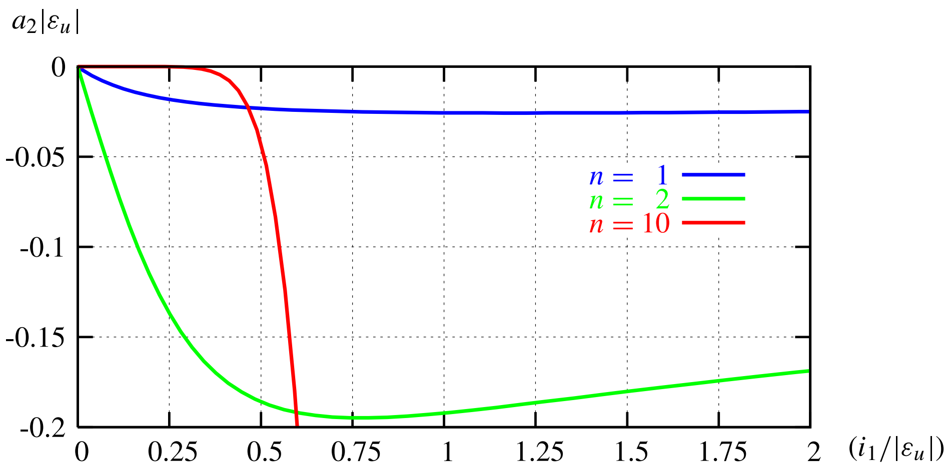

Figure 6. Second postbuckling constant a_2 for rectangular cross-section.

Rectangular cross-section. Results for the second postbuckling constant a_2 of the rectangular cross-section in nonlinear elastic postbuckling are shown in Figure 6. When $i_1/|\varepsilon_u| \to 0$, i.e., for extremely slender columns, the buckling stress is very low, and therefore the second postbuckling constant a_2 of nonlinear elasticity approaches that of linear elasticity. In linear elasticity $a_2|\varepsilon_u|$ is usually of the order 10^{-4}. However, as $i_1/|\varepsilon_u|$ is increased a_2 rapidly grows negative and reaches a global minimum. The higher the value of n the lower the minimum (the minimum for $n = 10$ falls outside the bounds of this plot), and as $n \to \infty$ the minimum value of a_2 becomes $-\infty$ and is reached for $i_1/|\varepsilon_u| = 2$ where $\varepsilon_c = \varepsilon_u$). Thus, assuming material linearity only furnishes reliable values of a_2 for extremely slender columns.

The large deviation of $a_2|\varepsilon_u|$ in Figure 6 from its linear elastic counterpart helps to demonstrate that the linear elastic value of a_2, assuming full nonlinear kinematics differs *very* little from the value obtained under the assumption of moderately nonlinear kinematics compared to the effects of nonlinearity of the stress-strain curve. Therefore, as discussed earlier, the simplifications of moderately nonlinear geometry should not necessarily be discarded.

12.5. ***Comparison with numerical results.*** An immediate and important consequence of including non-linear elastic effects is that both symmetric (except for *extremely* slender columns) and nonsymmetric cross-sections may become imperfection sensitive in contrast to the predictions of linear elasticity. It follows that nonlinear material effects may not be handled safely by assuming that the correct nonlinear elastic bifurcation load shown in Figure 4 predicts the real load-carrying capacity.

The two cross-section types, triangular and rectangular are, as mentioned earlier, constructed to occupy the same amount of material for the same critical load. However, as is clear from Figure 7, the postbuckling paths of the cross-sections are not identical as the column with triangular cross-section experiences asymmetric buckling (in contrast to its linear elastic counterpart), while the column with rectangular cross-section buckles symmetrically. This agrees with the findings in [Tvergaard and Needleman 1982].

Figure 7 shows an example of the load parameter, λ, plotted against the end rotation of the column for both the triangular and the rectangular cross-sections where $|\varepsilon_u| = 0.002$, $n = 2$, $i_1/|\varepsilon_u| = 0.2$ $(h/L = 0.022)$ which yields $a_1\sqrt{|\varepsilon_u|} = -0.0538$ for the triangle and $a_2|\varepsilon_u| = -0.117$ for the rectangle.

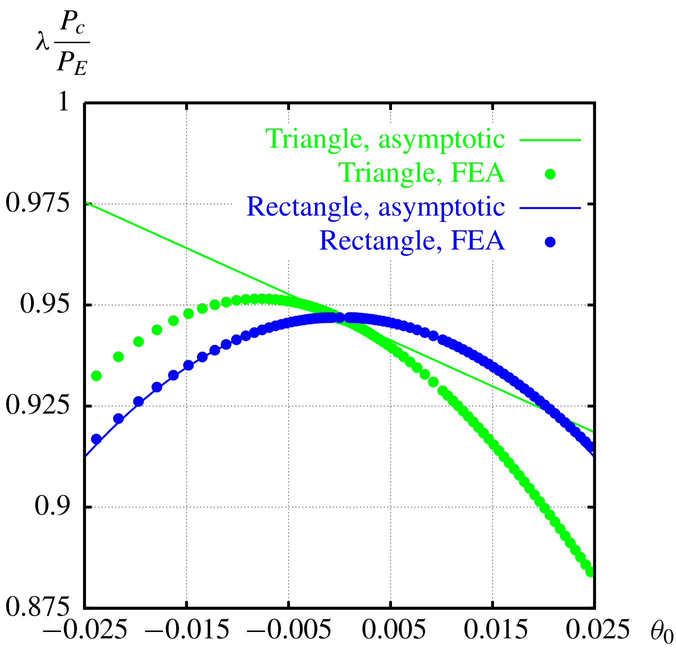

Figure 7. Initial postbuckling behavior for rectangular and triangular cross-sections, $|\varepsilon_u| = 0.002$, $n = 2$, $i_1/|\varepsilon_u| = 0.2$. The classic Euler load of linear elasticity is denoted P_E.

The asymptotic results for initial postbuckling are verified by a finite element analysis. While the symmetric asymptotic analysis including both the postbuckling constants a_1 $(= 0)$ and a_2 matches the finite element analysis results nicely in the range $\theta_0 \in [-0.025; 0.025]$ the precision of the nonsymmetric analysis which is only carried out to the first postbuckling constant a_1 rapidly deteriorates mainly because of the heavy nonlinearities introduced through the material law.

The column with rectangular cross-section always exhibits imperfection sensitivity although not as distinct as with the triangular cross-section because it experiences symmetric postbuckling.

According to the asymptotic analysis the triangular cross-section column is initially stable in postbuckling when forced to bifurcate in the opposite direction of the w-axis (see Figure 2), in the direction of the cross-section axis z while it is imperfection sensitive when buckling in the direction of the w-axis. On the other hand, the finite element analysis shows that the initial postbuckling stability is soon negated by a decrease in load-carrying capacity which is not detected by the asymptotic analysis. Therefore and because the accuracy of the asymptotic approach decreases soon after bifurcation it may be an obvious idea to include the next asymptotic term a_2 for nonsymmetric structures as well as for symmetric ones.

13. Imperfect Euler column

In principle the geometric imperfection may be of any shape. Here, in the spirit of Koiter, we restrict ourselves to columns subject to an initial imperfection in the shape of the buckling displacement field $\boldsymbol{u}_1$ and characterized by the imperfection amplitude $\bar{\xi}$:

$$\widehat{w} = \bar{\xi} w_1, \quad \widehat{v} = \bar{\xi} v_1 = 0. \tag{88}$$

The asymptotic equilibrium path of the imperfect column close to bifurcation may be examined through the stability problems derived earlier by calculating the imperfection shape parameters ρ_1, ρ_2 and ρ_3. However, since the second postbuckling constant a_2 was not determined for the nonsymmetric perfect column, ρ_2 and ρ_3 may not be determined exactly and are therefore excluded for nonsymmetric perfect columns.

13.1. ***Asymptotic imperfection shape parameters.*** In Appendix E the asymptotic stability problems for imperfect structures are solved in details. For small values of i_1 The asymptotic imperfection parameters are given as

$$\rho_1 \lambda_c = \frac{-1}{1+e'_c} \tag{89}$$

for all columns, while ρ_2 and ρ_3 are only determined for symmetric columns:

$$\rho_2 \lambda_c = 0, \qquad \rho_3 \lambda_c = (\rho_1 \lambda_c)^2 - \frac{{e'_c}^2 - \frac{5}{9} e''_c}{2(1+e'_c)^3}. \tag{90}$$

For various values of the hardening parameters n the dependence on the slenderness $i_1/|\varepsilon_u|$ of the two nonvanishing imperfection shape parameters ρ_1 and ρ_3 are shown in Figure 8. For infinitely slender columns, i.e., for $i_1/|\varepsilon_u| \to 0$, the imperfection parameters ρ_1 and ρ_3 approach their linear elastic counterparts $\rho_1 = -1$ and $\rho_3 = -1$, respectively. When $i_1/|\varepsilon_u|$ increases the absolute values of ρ_1 and ρ_3 decrease towards zero, though for ρ_3 the sign changes for large n. The larger the value of the hardening parameter, the faster decrease of $|\rho_1|$ and $|\rho_3|$. In general, this means that the equilibrium of a geometrically imperfect nonlinear elastic column is closer to the equilibrium of its perfect realization

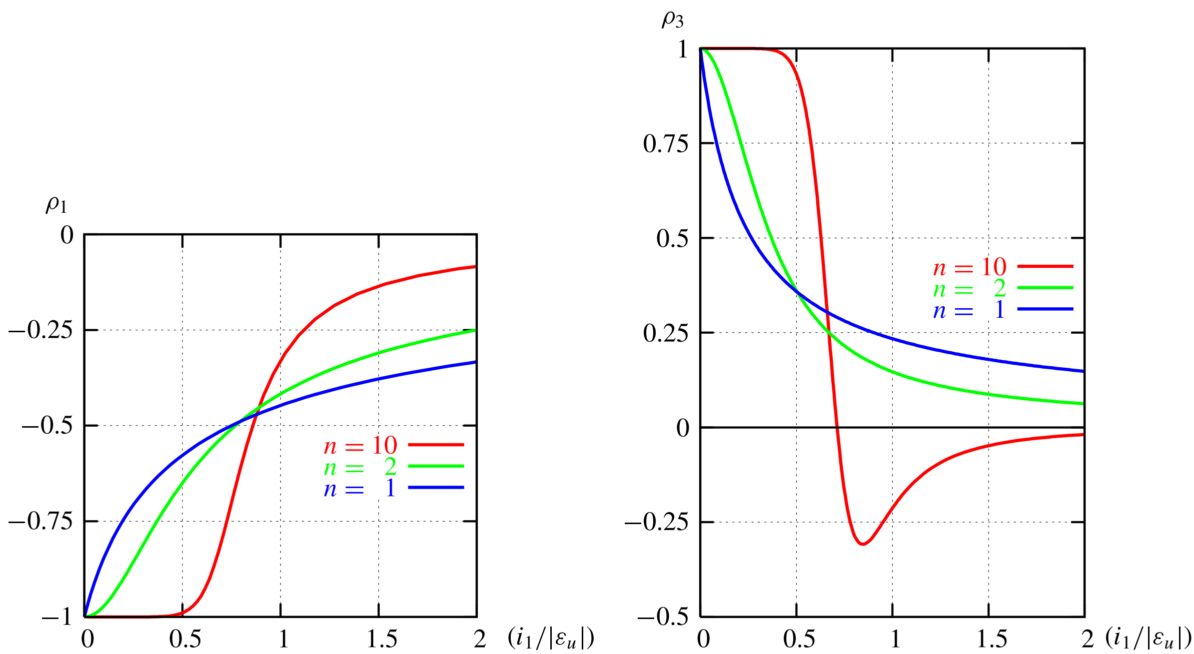

Figure 8. Left: The first imperfection shape parameter ρ_1. Right: The third imperfection shape parameter ρ_3 for symmetric columns.

than it is for a linear elastic column. The initial imperfection sensitivity, described by the maximum load, depends both on the initial postbuckling path and the three imperfection shape parameters. The nonlinear elastic effect of smaller imperfection parameters partly neutralizes the more rapid decrease in postbuckling load capacity shown for the perfect Euler column.

13.2. *Asymptotic maximum load.* As mentioned above, the expression for the asymptotic maximum load depends on whether the perfect structure is symmetric or not.

Nonsymmetric column. The asymptotic maximum load of the nonsymmetric column may be computed from (63):

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{1}{2}} \bar{\xi}^{\frac{1}{2}} + O(\bar{\xi}^1) \tag{91}$$

where the constant $c^m_{\frac{1}{2}}$, shown in Figure 9, is given by

$$c^m_{\frac{1}{2}} = -\frac{4}{1+e'_c}\sqrt{\frac{i_2}{3\pi i_1^2} e'_c}, \tag{92}$$

where λ_c and a_1 are determined for the perfect column and ρ_1 is given by (89).

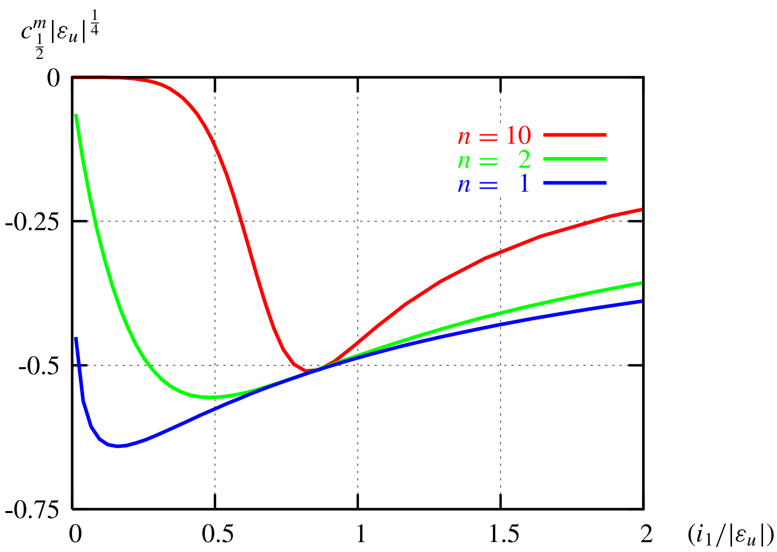

Figure 9. The maximum load constant, $c^m_{\frac{1}{2}}$, for nonsymmetric columns.

Symmetric column. The asymptotic maximum load of the symmetric column is determined from (69):

$$\frac{\lambda_m}{\lambda_c} = 1 + c^m_{\frac{2}{3}} \bar{\xi}^{\frac{2}{3}} + O(\bar{\xi}^{\frac{4}{3}}), \tag{93}$$

where the constant $c^m_{\frac{2}{3}}$, shown in Figure 10, is given by

$$c^m_{\frac{2}{3}} = \frac{3}{2(1+e'_c)}\left(\frac{i_1 - 3(e'_c)^2 + e''_c}{4i_1}\right)^{\frac{1}{3}}. \tag{94}$$

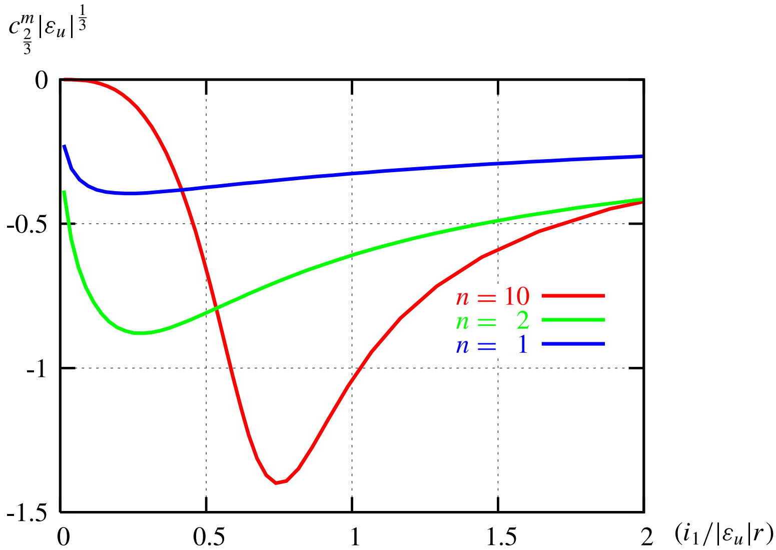

Figure 10. The maximum load constant $c^m_{\frac{2}{3}}$ for symmetric columns.

While λ_c and a_2 are associated with the perfect column, ρ_1 is given by (89). Note that even though ρ_2 and ρ_3 do not affect the maximum load directly in this case, then the fact that ρ_2 vanishes enables us to show that the remainder $O(\bar{\xi}^{\gamma})$ is of order $\bar{\xi}^{\frac{4}{3}}$ and not of order $\bar{\xi}^{1}$.

13.3. ***Comparison with numerical results.*** Here, results of the usual asymptotic and the enhanced asymptotic expansion, both taking imperfections into account, are compared with numerical results obtained by a full nonlinear finite element analysis for the same symmetric column that was used for comparison of perfect column results. Equilibria for the imperfection levels $\bar{\xi} = 0.0025, 0.01, 0.04$ are plotted in Figure 11 for a regular expansion with one and two terms, as well as the enhanced asymptotic expansion of (53), which is forced to obey the condition $\lambda(\xi = 0) = 0$. The one-term asymptotic expansion is the traditional lowest order asymptotic method developed in [Koiter 1945] which only depends on ρ_1 in (52), while ρ_2 is ignored. The two-term asymptotic expansion takes also ρ_2 into account by (52). For the structure in question the relevant constants are

$$\rho_1 = -0.90, \quad \rho_3 = 0.57, \quad m_1 = 1.39, \quad \psi_3 = -0.681. \tag{95}$$

It appears from Figure 11 that, as expected, independent of the imperfection amplitude the enhanced solution through (0, 0) yields the best approximation to the numerical solution, especially for small values of the characteristic buckling amplitude, θ_0. Though both the one- and the two-term solutions diverge close to zero, the two-term solution provides accurate results for much smaller buckling amplitudes than the one-term solution. While the one-term solution provides reliable estimates of the equilibrium path only for very small imperfection levels, the two-term solution approximates loads around the maximum well even for moderate imperfection amplitudes, although the shape of the equilibrium path is badly approximated for smaller amplitudes of the buckling mode (small values of ξ). As seen from the plot, the estimates of the equilibrium paths given by the enhanced asymptotic expansion lie very close to the numerical results for any limited buckling amplitude and even for relatively large imperfection amplitudes.

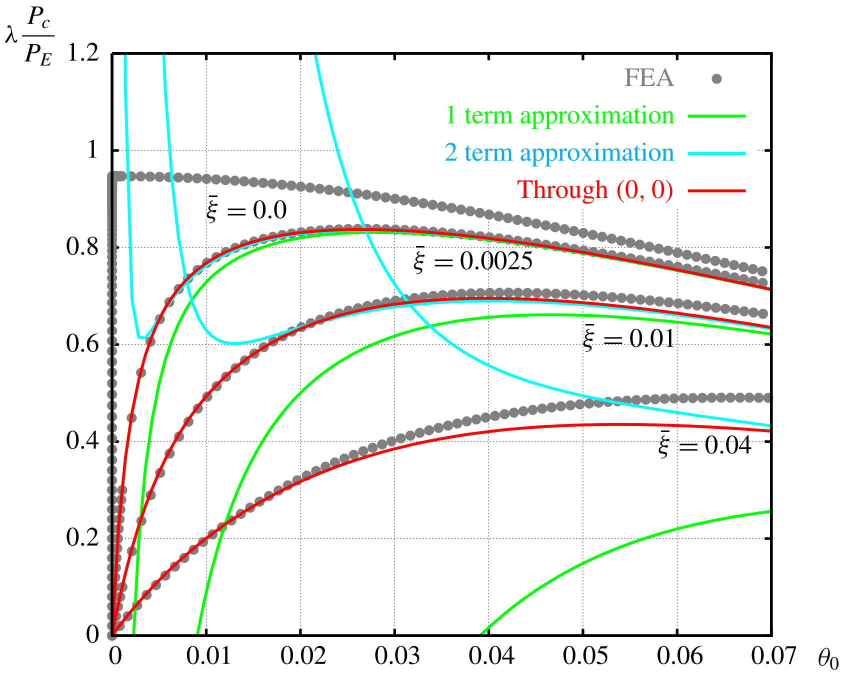

Figure 11. Comparison of equilibrium paths for geometrically imperfect column with rectangular cross-section, $|\varepsilon_u| = 0.002$, $n = 2$, $i_1/|\varepsilon_u| = 0.2$. The classic Euler load of linear elasticity is denoted P_E.

In Figure 12 the dependence of the maximum load on the imperfection amplitude $\bar{\xi}$ is illustrated for the traditional polynomial 1-term asymptotic expansion given by (93), for the enhanced hyperbolic

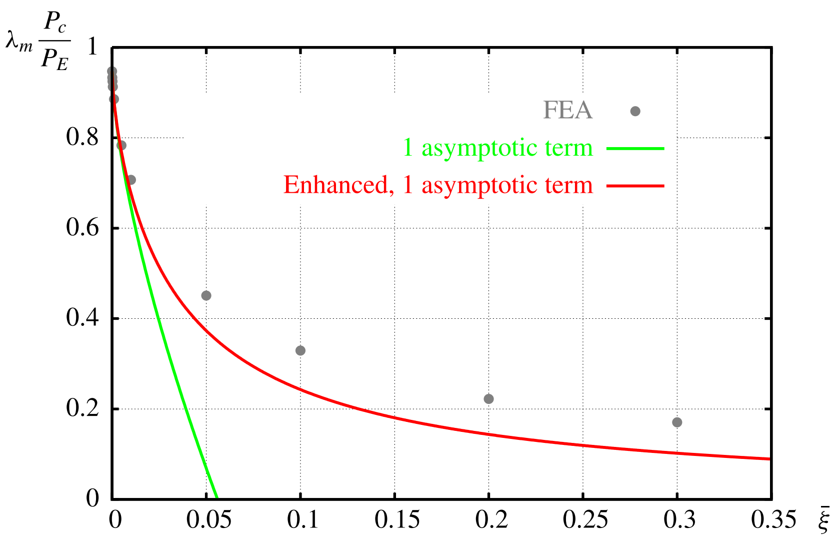

Figure 12. Comparison of maximum load prediction for rectangular cross-section, $|\varepsilon_u| = 0.002$, $n = 2$, $i_1/|\varepsilon_u| = 0.2$. The classic Euler load of linear elasticity is denoted P_E.

asymptotic expansion suggested in (71) and for numerical finite element calculations. For the actual column the relevant constants are

$$c^m_{\frac{2}{3}} = -6.83, \quad c^m_{\frac{3}{3}} = 0. \tag{96}$$

and thus only one nonvanishing asymptotic term exists for both the enhanced and the traditional asymptotic method. Comparison between numerical results and the traditional 1-term polynomial asymptotic expansion shows good agreement only for very small values of $\bar{\xi}$. The enhanced expansion provides relatively accurate approximations of the maximum load even at large values of the imperfection amplitude $\bar{\xi}$. In part, this is due to the fact that the enhanced method utilizes that the maximum load does not drop below zero by letting the maximum load approach zero for large values of $\bar{\xi}$. For this column numerical studies show that the maximum load has a lower limit which is higher than zero, yet the enhanced method provides excellent results.

14. Conclusion

In the body of the text a generally applicable asymptotic expansion valid for determination of postbifurcation behavior and imperfection sensitivity of structures under the assumption of full kinematic and elastic nonlinearity has been established. The asymptotic prediction of equilibria for imperfect structures has been enhanced such that the boundary condition that the buckling amplitude vanishes with the load for any imperfection is fulfilled.

The above comparisons with numerical results indicate that exploitation of additional boundary conditions and limit states imposed on the asymptotic expansion may lead to modified, but still asymptotically correct, expressions for imperfect structures which provide stable and relatively accurate results even for larger values of the imperfection amplitude.

Appendix A. Asymptotic coefficient fields P_i

Consider the function $\boldsymbol{P}(u^i, \lambda)$ of a field of n variables, $\boldsymbol{u}(\xi)$:

$$\boldsymbol{u}(\xi) = \{u^1(\xi), u^2(\xi), \ldots, u^n(\xi)\} = u^i(\xi), \tag{A-1}$$

where $\boldsymbol{P}(u^i, \lambda)$ depends linearly on the scalar load parameter $\lambda(\xi)$, and where we shall assume that the partial derivatives of $\boldsymbol{P}$ with respect to u^i are continuous at least to third order to ensure that the order of differentiation is unimportant.

A.1. ***Expansion of P at bifurcation.*** Suppose $\xi = 0$ at the bifurcation and expand in series in ξ around the singular point:

$$\begin{aligned} \lambda/\lambda_c &= 1 + \bar{a}_1\xi + \bar{a}_2\xi^2 + \bar{a}_3\xi^3 + O(\xi^4), \\ \boldsymbol{u}(\xi) &= \boldsymbol{u}_c + \xi\bar{\boldsymbol{u}}_1^T + \xi^2\bar{\boldsymbol{u}}_2^T + \xi^3\bar{\boldsymbol{u}}_3^T + O(\xi^4), \\ \boldsymbol{P}\,(\boldsymbol{u}(\xi), \lambda(\xi)) &= \boldsymbol{P}_c + \xi\boldsymbol{P}_1^T + \xi^2\boldsymbol{P}_2^T + \xi^3\boldsymbol{P}_3^T + O(\xi^4), \end{aligned} \tag{A-2}$$

where $_c$ designates prebifurcation values taken at the critical point. In the following we exploit that $\boldsymbol{P}$ depends linearly on λ to eliminate higher-order derivatives with respect to λ when the derivatives of $\boldsymbol{P}$ with respect to ξ are obtained.

We define a generalized displacement field consisting of $\boldsymbol{u}$ and the imperfection $\hat{\boldsymbol{u}}$:

$$\boldsymbol{U} \in \{\boldsymbol{u}, \hat{\boldsymbol{u}}\} = \{\boldsymbol{u}, \alpha\tilde{\boldsymbol{u}}\xi^2\}, \tag{A-3}$$

where (6) and (7) are introduced. Then,

$$\begin{aligned}
\frac{\partial \boldsymbol{P}}{\partial \xi} &= \frac{\partial \boldsymbol{P}}{\partial U^i}\frac{\partial U^i}{\partial \xi} + \frac{\partial \boldsymbol{P}}{\partial \lambda}\frac{\partial \lambda}{\partial \xi},\\
\frac{\partial^2 \boldsymbol{P}}{\partial \xi^2} &= \frac{\partial \boldsymbol{P}}{\partial U^i}\frac{\partial^2 U^i}{\partial \xi^2} + \frac{\partial^2 \boldsymbol{P}}{\partial U^i \partial U^j}\frac{\partial U^i}{\partial \xi}\frac{\partial U^j}{\partial \xi} + \frac{\partial \boldsymbol{P}}{\partial \lambda}\frac{\partial^2 \lambda}{\partial \xi^2} + 2\frac{\partial^2 \boldsymbol{P}}{\partial \lambda \partial U^i}\frac{\partial \lambda}{\partial \xi}\frac{\partial U^i}{\partial \xi},\\
\frac{\partial^3 \boldsymbol{P}}{\partial \xi^3} &= \frac{\partial \boldsymbol{P}}{\partial U^i}\frac{\partial^3 U^i}{\partial \xi^3} + 3\frac{\partial^2 \boldsymbol{P}}{\partial U^i \partial U^j}\frac{\partial^2 U^i}{\partial \xi^2}\frac{\partial U^j}{\partial \xi} + \frac{\partial^3 \boldsymbol{P}}{\partial U^i \partial U^j \partial U^k}\frac{\partial U^i}{\partial \xi}\frac{\partial U^j}{\partial \xi}\frac{\partial U^k}{\partial \xi}\\
&\quad + \frac{\partial \boldsymbol{P}}{\partial \lambda}\frac{\partial^3 \lambda}{\partial \xi^3} + \frac{\partial^2 \boldsymbol{P}}{\partial \lambda \partial U^i}\left(\frac{\partial^2 \lambda}{\partial \xi^2}\frac{\partial U^i}{\partial \xi} + \frac{\partial \lambda}{\partial \xi}\frac{\partial^2 U^i}{\partial \xi^2}\right) + 3\frac{\partial^3 \boldsymbol{P}}{\partial \lambda \partial U^i \partial U^j}\frac{\partial \lambda}{\partial \xi}\frac{\partial U^i}{\partial \xi}\frac{\partial U^j}{\partial \xi}.
\end{aligned} \tag{A-4}$$

Now the coefficient fields $\boldsymbol{P}_i^T$ are expressible in terms of $\bar{\boldsymbol{u}}_i^T$ and $\bar{a}_i$ and the imperfection shape $\tilde{\boldsymbol{u}}$ as

$$\begin{aligned}
\boldsymbol{P}_1^T &= \left.\frac{\partial \boldsymbol{P}}{\partial \xi}\right|_c &&= \boldsymbol{P}^c_{,i}\bar{u}_1^{Ti} + \bar{a}_1\lambda_c \boldsymbol{P}^c_{,\lambda},\\
\boldsymbol{P}_2^T &= \frac{1}{2}\left.\frac{\partial^2 \boldsymbol{P}}{\partial \xi^2}\right|_c &&= \boldsymbol{P}^c_{,i}\bar{u}_2^{Ti} + \bar{a}_2\lambda_c \boldsymbol{P}^c_{,\lambda} + \bar{a}_1\lambda_c \boldsymbol{P}^c_{,\lambda i}\bar{u}_1^{Ti} + \tfrac{1}{2}\boldsymbol{P}^c_{,ij}\bar{u}_1^{Ti}\bar{u}_1^{Tj} + \alpha\tilde{\boldsymbol{u}}(\boldsymbol{P}^c_{,\hat{\boldsymbol{u}}}),\\
\boldsymbol{P}_3^T &= \frac{1}{6}\left.\frac{\partial^3 \boldsymbol{P}}{\partial \xi^3}\right|_c &&= \boldsymbol{P}^c_{,i}\bar{u}_3^{Ti} + \bar{a}_3\lambda_c \boldsymbol{P}^c_{,\lambda} + \bar{a}_2\lambda_c \boldsymbol{P}^c_{,\lambda i}\bar{u}_1^{Ti} + \bar{a}_1\lambda_c(\boldsymbol{P}^c_{,\lambda i}\bar{u}_2^{Ti} + \tfrac{1}{2}\boldsymbol{P}^c_{,\lambda ij}\bar{u}_1^{Ti}\bar{u}_1^{Tj})\\
& &&\quad + \boldsymbol{P}^c_{,ij}\bar{u}_2^{Ti}\bar{u}_1^{Tj} + \tfrac{1}{6}\boldsymbol{P}_{,ijk}\bar{u}_1^{Ti}\bar{u}_1^{Tj}\bar{u}_1^{Tk} + \alpha\tilde{\boldsymbol{u}}(\bar{a}_1\lambda_c \boldsymbol{P}^c_{,\lambda\hat{\boldsymbol{u}}} + \boldsymbol{P}^c_{,i\hat{\boldsymbol{u}}}\bar{u}_1^{Ti}),
\end{aligned} \tag{A-5}$$

where

$$(\ )_{,i,\dots,k} = \frac{\partial^n(\ )}{\partial u^i, \dots, \partial u^k}. \tag{A-6}$$

A.2. *Perturbation expansion close to the precritical path.* Let a perturbation expansion of the function $\boldsymbol{P}$ around the precritical path, indicated by subscript $_0$, be given:

$$\begin{aligned}
\boldsymbol{u}(\xi) &= \boldsymbol{u}_0(\xi) + \xi\bar{\boldsymbol{u}}_1 + \xi^2\bar{\boldsymbol{u}}_2 + \xi^3\bar{\boldsymbol{u}}_3 + O(\xi^4),\\
\boldsymbol{P}(\boldsymbol{u}(\xi)) &= \boldsymbol{P}_0(\xi) + \xi\boldsymbol{P}_1 + \xi^2\boldsymbol{P}_2 + \xi^3\boldsymbol{P}_3 + O(\xi^4),
\end{aligned} \tag{A-7}$$

and an expansion of the precritical path in ξ:

$$\begin{aligned}
\boldsymbol{u}_0(\xi) &= \boldsymbol{u}_c + \xi\boldsymbol{u}_1^0 + \xi^2\boldsymbol{u}_2^0 + \xi^3\boldsymbol{u}_3^0 + O(\xi^4),\\
\boldsymbol{P}_0(\boldsymbol{u}(\xi)) &= \boldsymbol{P}_c + \xi\boldsymbol{P}_1^0 + \xi^2\boldsymbol{P}_2^0 + \xi^3\boldsymbol{P}_3^0 + O(\xi^4).
\end{aligned} \tag{A-8}$$

It is now possible to determine the asymptotic coefficient fields $\boldsymbol{P}_i$ when the precritical path is established. Insert the precritical path (A-8) in the perturbation expansion (A-7) and match it with the expansion of $\boldsymbol{P}$ (A-2) to provide

$$\bar{\boldsymbol{u}}_i = \bar{\boldsymbol{u}}_i^T - \boldsymbol{u}_i^0, \tag{A-9}$$

which yields

$$\begin{aligned}
\boldsymbol{P}_1 &= \boldsymbol{P}_1^T - \boldsymbol{P}_1^0 &&= \boldsymbol{P}^c_{,i}\bar{u}_1^i, \\
\boldsymbol{P}_2 &= \boldsymbol{P}_2^T - \boldsymbol{P}_2^0 &&= \boldsymbol{P}^c_{,i}\bar{u}_2^i + \bar{a}_1\lambda_c\boldsymbol{P}^c_{,\lambda i}\bar{u}_1^i + \tfrac{1}{2}\boldsymbol{P}^c_{,ij}(\bar{u}_1^i\bar{u}_1^j + 2\bar{u}_1^i u_1^{0j}) + \alpha\tilde{\boldsymbol{u}}(\boldsymbol{P}^c_{,\hat{\boldsymbol{u}}}), \\
\boldsymbol{P}_3 &= \boldsymbol{P}_3^T - \boldsymbol{P}_3^0 &&= \boldsymbol{P}^c_{,i}\bar{u}_3^i + \bar{a}_2\lambda_c\boldsymbol{P}^c_{,\lambda i}\bar{u}_1^i + \bar{a}_1\lambda_c(\boldsymbol{P}^c_{,\lambda i}\bar{u}_2^i + \tfrac{1}{2}\boldsymbol{P}^c_{,\lambda ij}(\bar{u}_1^i\bar{u}_1^j + 2\bar{u}_1^i u_1^{0j})) \\
&&&\quad + \boldsymbol{P}^c_{,ij}(\bar{u}_2^i\bar{u}_1^j + \bar{u}_2^i u_1^{0j} + u_2^{0j}\bar{u}_1^i) \\
&&&\quad + \tfrac{1}{6}\boldsymbol{P}_{,ijk}(\bar{u}_1^i\bar{u}_1^j\bar{u}_1^k + 3\bar{u}_1^i u_1^{0j}u_1^{0k} + 3\bar{u}_1^i\bar{u}_1^j u_1^{0k}) \\
&&&\quad + \alpha\tilde{\boldsymbol{u}}(\bar{a}_1\lambda_c\boldsymbol{P}^c_{,\lambda\hat{\boldsymbol{u}}} + \boldsymbol{P}^c_{,i\hat{\boldsymbol{u}}}(\bar{u}_1^i + u_1^{0i})),
\end{aligned} \tag{A-10}$$

The prebuckling fields established in Appendix B are introduced to provide the specific expressions for $\boldsymbol{P}_i$ in (17)–(19).

Appendix B. Expansion of $\boldsymbol{u}_0$ in ξ

A traditional expansion of $\boldsymbol{u}_0$ around Bifurcation in λ yields

$$\boldsymbol{u}_0 = \boldsymbol{u}_c + (\lambda - \lambda_c)\boldsymbol{u}_c' + \tfrac{1}{2}(\lambda - \lambda_c)^2\boldsymbol{u}_c'' + O((\lambda - \lambda_c)^3). \tag{B-1}$$

Insertion of λ given by (8) in (B-1) provides the desired expansion in ξ as

$$\boldsymbol{u}_0 = \boldsymbol{u}_c + \xi\boldsymbol{u}_1^0 + \xi^2\boldsymbol{u}_2^0 + O(\xi^3), \tag{B-2}$$

where

$$\boldsymbol{u}_1^0 = a_1\lambda_c\boldsymbol{u}_c' \quad \boldsymbol{u}_2^0 = a_2\lambda_c\boldsymbol{u}_c' + \tfrac{1}{2}(a_1\lambda_c)^2\boldsymbol{u}_c''. \tag{B-3}$$

Appendix C. Symmetry of $\boldsymbol{p}_{l,i}$

Differentiation of $\boldsymbol{p}_l$ given by (5) provides

$$\boldsymbol{p}_{l,i} = \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{,li} + \boldsymbol{\sigma}_{,i}\cdot\boldsymbol{\varepsilon}_{,l} - (\boldsymbol{\eta}\cdot\boldsymbol{C})_{,li} - \lambda\boldsymbol{B}_{,li}. \tag{C-1}$$

The stress field is a function of the strain field alone, i.e., $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$, so differentiation of the stress field with respect to the displacement field yields

$$\boldsymbol{\sigma}_{,i} = \frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_{,i} = \boldsymbol{D}\boldsymbol{\varepsilon}_{,i}, \tag{C-2}$$

where $\boldsymbol{D}$ is the field of tangent stiffnesses.

Because $\boldsymbol{P}$ and therefore also $\boldsymbol{p}_l$ has continuous derivatives of at least to fourth order with respect to $\boldsymbol{u}$, the constituent functions of $\boldsymbol{p}_l$ must be equally differentiable. This ensures that the order of differentiation may be switched without altering the result. Thus

$$\begin{aligned}
\boldsymbol{p}_{l,i} &= \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{,li} + \boldsymbol{D}\boldsymbol{\varepsilon}_{,i}\cdot\boldsymbol{\varepsilon}_{,l} - (\boldsymbol{\eta}\cdot\boldsymbol{C})_{,li} - \lambda\boldsymbol{B}_{,li} \\
&= \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_{,il} + \boldsymbol{D}\boldsymbol{\varepsilon}_{,l}\cdot\boldsymbol{\varepsilon}_{,i} - (\boldsymbol{\eta}\cdot\boldsymbol{C})_{,il} - \lambda\boldsymbol{B}_{,il} = \boldsymbol{p}_{i,l},
\end{aligned} \tag{C-3}$$

which proves that $\boldsymbol{p}_{l,i}$ is symmetric and that the indices may be interchanged freely, i.e., $\boldsymbol{p}_{l,i} = \boldsymbol{p}_{i,l}$.

Appendix D. Perfect Euler column: asymptotic coefficients

D.1. *Prebuckling.* The straightforward prebuckling solution for the Euler column is

$$w_0(x) = 0, \qquad \ddot{v}_0(x) = 0, \tag{D-1}$$

where dots over a quantity denote differentiation with respect to x.

D.2. *Principle of virtual displacements and operators.* The principle of virtual displacements as given in (1) depends on strains, stresses, constraints and load.

The Bernoulli–Euler beam theory assumes the only influential strains to be the strains in the direction of the x-axis, here denoted ε:

$$\varepsilon = \epsilon + z\kappa, \tag{D-2}$$

where

$$\epsilon = \sqrt{(1+\dot{v})^2 + \dot{w}^2} - 1, \quad \kappa = \dot{\theta} = \frac{\ddot{w}(1+\dot{v}) - \ddot{v}\dot{w}}{(1+\dot{v})^2 + \dot{w}^2}, \tag{D-3}$$

and the corresponding stresses $\sigma(\varepsilon)$ may depend on the strains to any degree of nonlinearity.

The load operator $\boldsymbol{B}$ is taken to be linear in the displacements:

$$\boldsymbol{B}(\boldsymbol{u}) = -\boldsymbol{P}_c v(L). \tag{D-4}$$

In the present application no Lagrange constraints are enforced.

The operators $\mathcal{C}_i^c$, $\mathcal{B}_i^c$ and $\boldsymbol{\mathcal{E}}_i^c$. Use (23), (43) and (44) to show that for this column the operators associated with the principle of virtual displacements are as follows:

Constraints:
$$\mathcal{C}_i^c \equiv 0 \tag{D-5}$$

Loads:
$$\mathcal{B}_1^c(\boldsymbol{u}_\alpha) = -\boldsymbol{P}_c v_\alpha(L), \quad \mathcal{B}_i^c = 0,\ i \neq 1 \tag{D-6}$$

Strains: The strain operator may be split up in parts that are independent of the cross-sectional variable z:

$$\boldsymbol{\mathcal{E}}_i^c = \boldsymbol{\mathcal{E}}_i^{c\epsilon} + z\boldsymbol{\mathcal{E}}_i^{c\kappa}. \tag{D-7}$$

The stretch ratio at critical load s_c is

$$\frac{1}{s_c} = \frac{1}{1+\dot{v}_c} = 1 - \dot{v}_c + O(\dot{v}_c^2) = 1 + i_1(1-n_c) + O(i_1^2), \tag{D-8}$$

where $0 \le n_c \le 1$ and $n_c = 0$ for linear elasticity. Thus the strain operators are

$$\boldsymbol{\mathcal{E}}_1^{c\epsilon}(\boldsymbol{u}_\alpha) = \dot{v}_\alpha, \quad \boldsymbol{\mathcal{E}}_2^{c\epsilon}(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) = \dot{w}_\alpha \dot{w}_\beta \frac{1}{s_c} \tag{D-9}$$

and

$$\boldsymbol{\mathcal{E}}_1^{c\kappa}(\boldsymbol{u}_\alpha) = \ddot{w}_\alpha \frac{1}{s_c}, \quad \boldsymbol{\mathcal{E}}_2^{c\kappa}(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) = -\big((\dot{w}_\alpha \dot{v}_\beta)^{\cdot} + (\dot{w}_\beta \dot{v}_\alpha)^{\cdot}\big)\frac{1}{s_c^2}. \tag{D-10}$$

Provided that we only determine the second postbuckling constant a_2 when the column is symmetric $\boldsymbol{\mathcal{E}}_3^c$ enters solely as $\boldsymbol{\mathcal{E}}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1)$ and $\boldsymbol{\mathcal{E}}_4^c$ as $\boldsymbol{\mathcal{E}}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1)$ after $\boldsymbol{u}_1$ has been determined. It is later shown that $v_1 \equiv 0$, and thus we get

$$\begin{aligned}\boldsymbol{\mathcal{E}}_3^{c\epsilon}(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1) &= -(\dot{v}_\alpha \dot{w}_1 \dot{w}_1)\frac{1}{s_c^2},\\ \boldsymbol{\mathcal{E}}_4^{c\epsilon}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= -3(\dot{w}_1 \dot{w}_1 \dot{w}_1 \dot{w}_1)\frac{1}{s_c^3},\end{aligned} \tag{D-11}$$

and

$$\boldsymbol{\mathcal{E}}_3^{c\kappa}(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1) = -2(\dot{w}_\alpha \dot{w}_1 \dot{w}_1)^{\cdot}\frac{1}{s_c^3}, \quad \boldsymbol{\mathcal{E}}_4^{c\kappa}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) = 0. \tag{D-12}$$

Stability operators $\mathcal{P}_i^c$. The relevant stability operators $\mathcal{P}_1^c$–$\boldsymbol{\mathcal{P}}_4^c$ are provided by (46) and (47):

$$\begin{aligned}\mathcal{P}_1^c(\boldsymbol{u}_\alpha) &= \int_0^L (-\lambda_c \boldsymbol{P}_c \dot{v}_\alpha)\,dx + \lambda_c \boldsymbol{P}_c v_\alpha(L),\\ \mathcal{P}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \int_0^L \left(E_t^c A \dot{v}_\alpha \dot{v}_\beta + \frac{E_t^c I_1}{s_c^2}\ddot{w}_\alpha \ddot{w}_\beta - \frac{\sigma_c A}{s_c}\dot{w}_\alpha \dot{w}_\beta\right)dx.\end{aligned} \tag{D-13}$$

Again, we only determine the second postbuckling constant a_2 when the column is symmetric. Thus $\mathcal{P}_3^c$ enters solely as $\mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1)$ and $\boldsymbol{\mathcal{P}}_4^c$ as $\mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1)$ after $\boldsymbol{u}_1$ has been determined. Utilize $v_1 \equiv 0$ in (D-9) and (D-11) to provide

$$\begin{aligned}\mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1) &= \int_0^L \left(\left(\frac{E_{t,\varepsilon}^c}{s_c^2} - \frac{2E_t^c}{s_c^3}\right) I_1 \dot{v}_\alpha \ddot{w}_1^2 + \left(\frac{E_t^c}{s_c} + \frac{\sigma_c}{s_c^2}\right) A \dot{v}_\alpha \dot{w}_1^2 - \frac{2E_t^c}{s_c^3} I_1 \dot{w}_1 \ddot{v}_\alpha \ddot{w}_1 + \frac{E_{t,\varepsilon}^c}{s_c^3} I_2 \ddot{w}_\alpha \ddot{w}_1^2\right)dx,\\ \mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= \int_0^L \left(\frac{E_{t,\varepsilon\varepsilon}^c}{s_c^4} I_3 \ddot{w}_1^4 + \left(\frac{6E_{t,\varepsilon}^c}{s_c^3} - \frac{24E_t^c}{s_c^4}\right) I_1 \dot{w}_1^2 \ddot{w}_1^2 + 3\left(\frac{E_t^c}{s_c^2} + \frac{\sigma_c}{s_c^3}\right) A \dot{w}_1^4\right)dx.\end{aligned} \tag{D-14}$$

D.3. *Expansion parameter and boundary conditions.* Let the expansion parameter ξ be the rotation of the column end, i.e.,

$$\xi = \theta(0), \quad \tan\theta(0) = \frac{\dot{w}(0)}{1 + \dot{v}(0)}. \tag{D-15}$$

The solution to the boundary value problem of the geometrically perfect column must fulfill (D-15)b. Insert (D-15)a in the asymptotic expansion (9) of $\boldsymbol{u}$ and match the right-hand side of (D-15)b with the left-hand side to reveal the rather obvious boundary conditions

$$\dot{w}_1(0) = 1, \quad \dot{w}_2(0) = \dot{v}_1(0). \tag{D-16}$$

D.4. *Buckling.* The buckling equation (26) using the stability operator (D-13) with the operators of (D-9) inserted provides

$$0 = \int_0^L \left(E_t^c A \dot{v}_1 \delta\dot{v} + \frac{E_t^c I_1}{s_c^2}\ddot{w}_1 \delta\ddot{w} - \frac{\lambda_c P_c}{s_c}\dot{w}_1 \delta\dot{w}\right)dx. \tag{D-17}$$

Fulfill (D-17) for all admissible δv and δw to get

$$E_t^c A \ddot{v}_1 = 0, \quad \overset{\dots}{w}_1 + \frac{s_c \lambda_c \boldsymbol{P}_c}{E_t^c I_1}\ddot{w}_1 = 0, \tag{D-18}$$

respectively. Apply the kinematic boundary conditions at the pinned ends to obtain the eigenmode solution of (D-18). The amplitude of the eigenmode is determined by condition (D-16)a:

$$v_1 = 0, \quad w_1 = \frac{L}{\pi}\sin(\pi x/L). \tag{D-19}$$

Normalize the applied force as follows to make $\lambda_c = 1$:

$$P_c/A = \sigma_c = E_t^c\left(i_1 + O(i_1^2)\right) \implies \lambda_c = 1. \tag{D-20}$$

D.5. *First postbuckling problem.*

First postbuckling constant. As mentioned earlier, in the case of a nonsymmetric structure, we shall limit ourselves to determining the first postbuckling constant a_1 and refrain form determining the second, a_2. From (29) with (D-14) inserted we get

$$a_1^N = -\int_0^L \frac{1}{2s_c^3} E_{t,\varepsilon}^c I_2 \ddot{w}_1^3 dx = E_t^c AL\frac{2e_c' i_2}{3\pi i_1 s_c^3}. \tag{D-21}$$

Utilize the necessary coefficient from the expansion of the prebuckling path, namely

$$\dot{v}_c'(x) = -\frac{\sigma_c}{E_t^c} = -i_1, \tag{D-22}$$

to determine

$$\begin{aligned} a_1^D &= -\frac{\sigma_c}{E_t^c}\int_0^L \left(\left(\frac{E_{t,\varepsilon}^c}{s_c^2} - \frac{2E_t^c}{s_c^3}\right) I_1\ddot{w}_1^2 + \left(\frac{E_t^c}{s_c} + \frac{\sigma_c}{s_c^2}\right) A\dot{w}_1^2\right)dx \\ &= -\tfrac{1}{2}E_t^c ALi_1\left(1 + e_c' + O(i_1)\right). \end{aligned} \tag{D-23}$$

As $i_1 \ll 1$, introduction of the nondimensional quantities defined in (72) and (73) yields

$$a_1 = -\frac{4e_c' i_2}{3\pi i_1^2(1+e_c')}. \tag{D-24}$$

From (D-24) it is clear that only when the cross-section is nonsymmetric ($i_2 \neq 0$) *and* the material model is nonlinear will the first postbuckling constant a_1 differ from zero.

Postbuckling displacement field. The postbuckling displacement field, which is only determined when $a_1 = 0$ (implying $e_c' i_2 = 0$), may be determined from the variational equation (41) together with the boundary conditions (D-16). Utilize the stability operators (D-13)–(D-14) and get

$$\begin{aligned} 0 = \int_0^L &\left(E_t^c A\dot{v}_2\delta\dot{v} + \frac{1}{s_c^2}E_t^c I_1\ddot{w}_2\delta\ddot{w} - \frac{1}{s_c}\sigma_c A\dot{w}_2\delta\dot{w}\right)dx \\ &+ \frac{1}{2}\int_0^L\left(\frac{1}{s_c^2}E_{t,\varepsilon}^c I_1\ddot{w}_1^2\delta\dot{v} + \left(\frac{1}{s_c}E_t^c + \frac{1}{s_c^2}\sigma_c\right)A\dot{w}_1^2\delta\dot{v} + \frac{1}{s_c^3}2E_t^c I_1\ddot{w}_1\dot{w}_1\delta\dot{v}\right)dx. \end{aligned} \tag{D-25}$$

Gather terms in (D-25) and introduce $\boldsymbol{u}_1$ from (D-19) and s_c from (D-8) to reach the differential equations

$$\dot{v}_2 = -\tfrac{1}{2}\left((1-n_c i_1)\cos^2\frac{\pi x}{L} + e_c'\left(1+2(1-n_c)i_1\right)\sin^2\frac{\pi x}{L}\right) + O(i_1^2), \quad \ddot{w}_2 + \left(\frac{\pi}{L}\right)^2\dot{w}_2 = 0, \tag{D-26}$$

and the static boundary conditions

$$\ddot{w}_2(0) = 0, \quad \ddot{w}_2(L) = 0. \tag{D-27}$$

Use the kinematic boundary conditions (D-27) at the pinned ends along with the conditions (D-16). When $i_1 \ll 1$ the second postbuckling displacement field becomes

$$v_2 = -\tfrac{1}{4}\Big((1+e_c')x + (1-e_c')\frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big) + O(i_1), \quad w_2 = 0. \tag{D-28}$$

D.6. ***Second postbuckling constant.*** For symmetric cross-sections ($a_1 = 0$, $e_c' i_2 = 0$) the numerator of the second postbuckling constant a_2^N may be found from (42). Utilize the operator expressions (D-13)–(D-14) to obtain

$$\begin{aligned} a_2^N = &-\int_0^L \Big(\Big(\frac{1}{s_c^2}E_{t,\varepsilon}^c - \frac{2}{s_c^3}E_t^c\Big) I_1 \dot{v}_2 \ddot{w}_1^2 + \Big(\frac{1}{s_c}E_t^c + \frac{1}{s_c^2}\sigma_c\Big) A\dot{v}_2\dot{w}_1^2 - \frac{2}{s_c^3}E_t^c I_1 \dot{w}_1 \ddot{v}_2 \ddot{w}_1\Big)dx \\ &- \frac{1}{6}\int_0^L \Big(\frac{1}{s_c^4}E_{t,\varepsilon\varepsilon}^c I_3 \ddot{w}_1^4 + 6\Big(\frac{1}{s_c^3}E_{t,\varepsilon}^c - \frac{4}{s_c^4}E_t^c\Big) I_1 \ddot{w}_1^2\dot{w}_1^2 + 3\Big(\frac{1}{s_c^2}E_t^c + \frac{1}{s_c^3}\sigma_c\Big) A\dot{w}_1^4\Big)dx. \end{aligned} \tag{D-29}$$

Since the terms proportional to $E_t^c A$ vanish, it is essential to include the first-order terms of i_1 in $\boldsymbol{u}_2$ (D-26) and $1/s_c$ (D-8). After some derivations (D-29) yields

$$a_2^N = -\tfrac{1}{16}E_t^c AL\big(i_1 - 3(e_c')^2 + e_c''\big) + O(i_1^2) + e_c' O(i_1) + e_c'' O(i_1). \tag{D-30}$$

The denominator of a_2 is identical to the denominator of a_1, which is given by (D-23). Utilize that $i_1 \ll 1$ to truncate terms of order 1, e_c' and e_c'' respectively to the lowest order of i_1. Then, the second postbuckling constant becomes

$$a_2 = \frac{i_1 - 3(e_c')^2 + e_c''}{8i_1(1+e_c')}. \tag{D-31}$$

Note that the term i_1 in the numerator of a_2 is not necessarily small compared to the other terms e_c' and e_c''. When the column exhibits linear or near linear material behavior at buckling, the absolute value of the nonlinear material constants e_c' and e_c'' decrease and the i_1-term becomes important.

D.7. ***Nonlinear elastic inextensible Euler column.*** When the Euler column is constrained to be inextensible, it is easily shown that

$$C^1 = \boldsymbol{\eta}^1 \cdot \big(\dot{v} + \tfrac{1}{2}\dot{v}^2 + \tfrac{1}{2}\dot{w}^2\big) = 0, \quad C^2 = \boldsymbol{\eta}^2 \cdot (\sin\theta - \dot{w}) = 0, \quad B = -\boldsymbol{P}_c v(L), \tag{D-32}$$

and

$$\epsilon = 0, \quad \kappa = \dot{\theta}. \tag{D-33}$$

Furthermore, when prebuckling is given by

$$v_0 = w_0 = \theta_0 = \boldsymbol{\eta}_0^2 = 0, \quad \boldsymbol{\eta}_0^1 = \lambda \boldsymbol{P}_c, \tag{D-34}$$

the operators associated with the principle of virtual displacements may be found from (23) and (43) and (44) together with (D-32)–(D-33).

Constraints: Inextensibility requires

$$\begin{aligned}\mathcal{C}_1^{1c}(\boldsymbol{u}_\alpha) &= \lambda_c \boldsymbol{P}_c \cdot \dot{v}_\alpha,\\ \mathcal{C}_2^{1c}(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \boldsymbol{\eta}_\beta^1 \cdot \dot{v}_\alpha + \boldsymbol{\eta}_\alpha^1 \cdot \dot{v}_\beta + \lambda_c \boldsymbol{P}_c \cdot (\dot{v}_\alpha \dot{v}_\beta + \dot{w}_\alpha \dot{w}_\beta),\\ \mathcal{C}_3^{1c}(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1) &= \boldsymbol{\eta}_\alpha^1 \cdot \dot{w}_1^2,\\ \mathcal{C}_4^{1c}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= 0\end{aligned} \tag{D-35}$$

and

$$\begin{aligned}\mathcal{C}_1^{2c}(\boldsymbol{u}_\alpha) &= 0,\\ \mathcal{C}_2^{2c}(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \boldsymbol{\eta}_\beta^2 \cdot (\theta_\alpha - \dot{w}_\alpha) + \boldsymbol{\eta}_\alpha^2 \cdot (\theta_\beta - \dot{w}_\beta),\\ \mathcal{C}_3^{2c}(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1) &= 0,\\ \mathcal{C}_4^{2c}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= -4\boldsymbol{\eta}_1^2 \cdot \theta_1^3,\end{aligned} \tag{D-36}$$

where it is utilized that $v_1 = \boldsymbol{\eta}_1^1 \equiv 0$ has been established before $\mathcal{C}_3^i$ and $\mathcal{C}_4^i$ are used.

Loads: $$\mathcal{B}_1^c(\boldsymbol{u}_\alpha) = -\boldsymbol{P}_c v_\alpha(L), \quad \mathcal{B}_i^c = 0,\ i \neq 1. \tag{D-37}$$

Strains: $$\mathcal{E}_1^c(\boldsymbol{u}_\alpha) = z\dot{\theta}_\alpha, \quad \mathcal{E}_i^c = 0. \tag{D-38}$$

Stability operators: The stability operators defined in (46) and (47) become

$$\begin{aligned}\mathcal{P}_1^c(\boldsymbol{u}_\alpha) &= \int_0^L \big(-\lambda_c \boldsymbol{P}_c \dot{v}_\alpha\big) dx + \lambda_c \boldsymbol{P}_c v_\alpha(L),\\ \mathcal{P}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta) &= \int_0^L \Big(E_t^c I_1 \dot{\theta}_\alpha \dot{\theta}_\beta - \big(\boldsymbol{\eta}_\beta^1 \dot{v}_\alpha + \boldsymbol{\eta}_\alpha^1 \dot{v}_\beta + \lambda_c \boldsymbol{P}_c (\dot{v}_\alpha \dot{v}_\beta + \dot{w}_\alpha \dot{w}_\beta)\big)\\ &\qquad - \big(\boldsymbol{\eta}_\beta^2 (\theta_\alpha - \dot{w}_\alpha) + \boldsymbol{\eta}_\alpha^2 (\theta_\beta - \dot{w}_\beta)\big)\Big) dx,\\ \mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1) &= \int_0^L \big(E_{t,\varepsilon}^c I_2 \dot{\theta}_\alpha \dot{\theta}_1^2 - \boldsymbol{\eta}_\alpha^1 \dot{w}_1^2\big)\, dx,\\ \mathcal{P}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= \int_0^L \big(E_{t,\varepsilon\varepsilon}^c I_3 \dot{\theta}_1^4 + 4\boldsymbol{\eta}_1^2 \theta_1^3\big)\, dx.\end{aligned} \tag{D-39}$$

Buckling and postbuckling. Insert the stability operators in the stability problems (26) and (29) together with (41) and (42) to provide the buckling solution

$$\begin{aligned}&\lambda_c = 1, \quad P_c = \frac{\pi^2 E_t^c I_1}{L^2}, \quad \sigma_c = i_1 E_t^c,\\ &v_1 = 0, \quad w_1 = \frac{L}{\pi} \sin \frac{\pi x}{L}, \quad \theta_1 = \cos \frac{\pi x}{L},\\ &\boldsymbol{\eta}_1^1 = 0, \quad \boldsymbol{\eta}_1^2 = P_c \cos \frac{\pi x}{L},\end{aligned} \tag{D-40}$$

the first postbuckling constant a_1 for nonsymmetric cross-sections

$$a_1 = -\frac{4e_c' i_2}{3\pi i_1^2}, \tag{D-41}$$

the postbuckling solution

$$\begin{aligned} v_2 &= -\tfrac{1}{4}\Big(x + \frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big), & w_2 &= 0, \quad \theta_2 = 0, \\ \eta_2^1 &= \tfrac{1}{2}P_c\cos^2\frac{\pi x}{L}, & \eta_2^2 &= 0, \end{aligned} \tag{D-42}$$

and the second postbuckling constant a_2 for symmetric cross-sections

$$a_2 = \frac{i_1 + e_c''}{8i_1}. \tag{D-43}$$

Appendix E. Imperfect Euler column: asymptotic coefficients

The prebuckling, buckling and postbuckling solution may be taken from Appendix D and the strain measure with respect to the perfect reference is given by (D-3).

E.1. ***Scalar stability operators connected with imperfection.*** The effects of initial imperfections are determined by the imperfection shape parameters, ρ_i. The scalar general stability operators $\mathcal{P}_i^c$ determined for the perfect column in (D-13) and (D-14), some of the additional general stability operators defined by (46) and (47) and the scalar operators associated with imperfections defined by (49) are needed in order to compute ρ_i.

Additional general stability operator. Most general stability operators needed to determine ρ_i were determined for the perfect structure in (D-13) and (D-14). Because the second postbuckling constant a_2 only is determined when $a_1 = 0$ some general stability operators enter only the imperfection sensitivity analysis. These stability operators are $\mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_c')$ and $\boldsymbol{\mathcal{E}}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_c')$. First we determine the additional operators associated with the principle of virtual displacements needed to compute $\mathcal{P}_3^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_c')$ and $\boldsymbol{\mathcal{E}}_4^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_c')$ from (43) when it is exploited that $v_1 = 0$ and $w_0 = 0$ and the strains are given by (D-3):

$$\begin{aligned} &\boldsymbol{\mathcal{E}}_3^{c\epsilon}(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_c') = -(\dot{w}_\alpha \dot{w}_1 \dot{v}_c') s_c^{-2}, && \boldsymbol{\mathcal{E}}_4^{c\epsilon}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_c') = 0, \\ &\boldsymbol{\mathcal{E}}_3^{c\kappa}(\boldsymbol{u}_\alpha, \boldsymbol{u}_1, \boldsymbol{u}_c') = 2(\dot{v}_\alpha \dot{w}_1 \dot{v}_c')^{\cdot} s_c^{-3}, && \boldsymbol{\mathcal{E}}_4^{c\epsilon}(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_c', \boldsymbol{u}_c') = 2\dot{w}_1^2(\dot{v}_c')^2 s_c^{-3}. \end{aligned} \tag{E-1}$$

Thus, the additional general stability operators become

$$\begin{aligned} \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_\alpha) &= -\int_0^L \frac{\sigma_c}{E_t^c}\bigg(\Big(\frac{E_{t,\varepsilon}^c}{s_c^2} - 3\frac{E_t^c}{s_c^3}\Big) I_1 \ddot{w}_1 \ddot{w}_\alpha + \frac{E_t^c A}{s_c}\dot{w}_1\dot{w}_\alpha\bigg)dx, \\ \mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= 0, \\ \mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) &= \int_0^L \Big(\frac{\sigma_c}{E_t^c}\Big)^2 \bigg(\Big(\frac{E_{t,\varepsilon\varepsilon}^c}{s_c^2} - 4\frac{E_{t,\varepsilon}^c}{s_c^3} + 6\frac{E_t^c}{s_c^4}\Big) I_1 \ddot{w}_1^2 + \Big(\frac{E_{t,\varepsilon}^c}{s_c} - 2\frac{E_t^c}{s_c^2} + 2\frac{\sigma_c}{s_c^3}\Big) A\dot{w}_1^2\bigg)dx, \end{aligned} \tag{E-2}$$

where we have used that $\dot{v}_c' = -\sigma_c/E_t^c$ and $\ddot{v}_c' = 0$ according to (D-22) and the buckling problem has been utilized to eliminate terms.

Stability operators associated with imperfections. Use (48) to show that the operators associated with imperfections are

$$\boldsymbol{\mathcal{E}}_1^{I\epsilon}(\tilde{\boldsymbol{u}}) = \dot{\tilde{v}}, \quad \boldsymbol{\mathcal{E}}_1^{I\kappa}(\tilde{\boldsymbol{u}}) = \ddot{\tilde{w}}, \quad \mathcal{D}_2^I(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) = 0, \tag{E-3}$$

when no constraints are enforced.

The only stability operators associated with imperfections needed to determine ρ_1 and when $a_1 = 0$, as well as ρ_2 and ρ_3 are $\bar{\mathcal{P}}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}})$, $\bar{\mathcal{P}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \tilde{\boldsymbol{u}})$ and $\bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \tilde{\boldsymbol{u}})$. The third degree operators are only determined for symmetric and/or linear elastic columns, thus I_2 and/or derivatives of E_t with respect to ε vanish. From (49) we have

$$\begin{aligned}
\bar{\mathcal{P}}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) &= \mathcal{P}_2^c(\boldsymbol{u}_\alpha, \tilde{\boldsymbol{u}}) - \int_0^L \left(E_t^c A \dot{v}_\alpha \dot{\tilde{v}} + \frac{E_t^c I_1}{s_c} \ddot{w}_\alpha \ddot{\tilde{w}} \right) dx, \\
\bar{\mathcal{P}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \tilde{\boldsymbol{u}}) &= \mathcal{P}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \tilde{\boldsymbol{u}}) - \int_0^L \left(\frac{E_{t,\varepsilon} I_1}{s_c^2} \dot{\tilde{v}} \ddot{w}_1^2 + \frac{E_t^c A}{s_c} \dot{\tilde{v}} \dot{w}_1^2 \right) dx, \\
\bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \tilde{\boldsymbol{u}}) &= \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \tilde{\boldsymbol{u}}) - \int_0^L \left(\left(\frac{E_{t,\varepsilon}}{s_c} - \frac{E_t^c}{s_c^2} \right) I_1 \ddot{\tilde{w}} \ddot{w}_1 \dot{v}_c' \right) dx.
\end{aligned} \tag{E-4}$$

E.2. *Expansion parameter.* Since the expansion parameter ξ is identified as the rotation of the column end even for the imperfect structure, the condition (D-15) must apply. Similar to the asymptotic match for the perfect structure (D-15) yields

$$\dot{\tilde{w}}_1(0) = 1, \quad \dot{\tilde{w}}_2(0) = \dot{w}_2(0) + \alpha\dot{\omega}_2(0) = \dot{\tilde{v}}_1(0), \tag{E-5}$$

where ω_2 is associated with the shape of the imperfection.

Because $\boldsymbol{u}_1$ is independent of the imperfection and $\dot{w}_2(0)$ already fulfills the condition $\dot{w}_2(0) = \dot{v}_1(0)$, the boundary condition on ω_2 becomes

$$\dot{\omega}_2(0) = 0. \tag{E-6}$$

E.3. *Imperfection in shape of the buckling displacements.* When the shape of the imperfection is given, the imperfect stability operators (E-4) may be evaluated. For simplicity, let the shape of the imperfection be the buckling displacement field $\boldsymbol{u}_1$:

$$\widehat{w} = \bar{\xi} w_1 = \bar{\xi} \left(\frac{L}{\pi} \right) \sin \frac{\pi x}{L}, \quad \widehat{v} = \bar{\xi} v_1 = 0. \tag{E-7}$$

Thus, from the buckling problem $\mathcal{P}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1) = 0$, and the stability operators associated with imperfections become

$$\begin{aligned}
\bar{\mathcal{P}}_2^c(\boldsymbol{u}_\alpha, \boldsymbol{u}_1) &= -\int_0^L \frac{E_t^c I_1}{s_c} \ddot{w}_\alpha \ddot{w}_1 dx, \\
\bar{\mathcal{P}}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) &= \mathcal{P}_3^c(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1) = 0, \\
\bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) &= \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) - \int_0^L \left(\left(\frac{E_{t,\varepsilon}}{s_c} - \frac{E_t^c}{s_c^2} \right) I_1 \ddot{w}_1^2 \dot{v}_c' \right) dx.
\end{aligned} \tag{E-8}$$

First imperfection parameter. The first imperfection shape parameter ρ_1 may be determined from (30) with $\tilde{\boldsymbol{u}} = \boldsymbol{u}_1$:

$$\rho_1 \lambda_c = \frac{\rho_1^N}{a_1^D}, \tag{E-9}$$

where a_1^D has already been determined in (D-23) and

$$\rho_1^N = -\bar{\mathcal{P}}_2^c(\boldsymbol{u}_1, \boldsymbol{u}_1) = \int_0^L \frac{E_t^c I_1}{s_c} \ddot{w}_1^2 dx = \tfrac{1}{2} E_t^c A L i_1\big(1 + O(i_1)\big). \tag{E-10}$$

Thus, as $i_1 \ll 1$,

$$\rho_1 \lambda_c \simeq \frac{-1}{1 + e_c'}. \tag{E-11}$$

Imperfection displacement field for symmetric columns. The lowest degree displacement field which depends on the imperfection is $\boldsymbol{v}_2$. The field $\boldsymbol{v}_2$ may be found as a linear solution of (35). Let $\boldsymbol{v}_2 = (v_2^v, w_2^v)$, where v_2^v is affine with v and w_2^v with w, and insert stability the operators (E-2), (E-8) and (D-13)–(D-14):

$$0 = \int_0^L \left(E_t^c A \dot{v}_2^v \delta \dot{v} + \frac{1}{s_c^2} E_t^c I_1 \ddot{w}_2^v \delta \ddot{w} - \frac{1}{s_c} \sigma_c A \dot{w}_2^v \delta \dot{w} \right) dx - \int_0^L \frac{E_t^c I_1}{s_c} \ddot{w}_1 \delta \ddot{w} dx \\ - \int_0^L \rho_1 \lambda_c \frac{\sigma_c}{E_t^c} \left(\left(\frac{E_{t,\varepsilon}^c}{s_c^2} - 3 \frac{E_t^c}{s_c^3} \right) I_1 \ddot{w}_1 \delta \ddot{w} + \frac{E_t^c A}{s_c} \dot{w}_1 \delta \dot{w} \right) dx. \tag{E-12}$$

Gather terms of the same variational fields and introduce $\boldsymbol{u}_1$ to provide

$$E_t^c A \dot{v}_2^v = 0, \qquad \left(\frac{L}{\pi}\right)^2 \dddot{w}_2^v + \dot{w}_2^v = k_1 \cos \frac{\pi x}{L}, \tag{E-13}$$

where the constant k_1 is

$$k_1 = -s_c - \rho_1 \lambda_c \left(\frac{e_c'}{s_c} - \frac{3 i_1}{s_c^2} + 1 \right) = O(i_1). \tag{E-14}$$

Introducing the geometric boundary conditions and the expansion parameter condition (E-6) yields

$$v_2^v = 0, \quad w_2^v = k_1 \left(\frac{L}{2\pi} \sin(\pi x/L) - \frac{L}{4}(1 - \cos(\pi x/L)) - \frac{1}{2} x \cos(\pi x/L) \right). \tag{E-15}$$

Second imperfection parameter for symmetric columns. The second imperfection parameter ρ_2 may in general be computed from (37). For symmetric columns, and when stability operators which equal zero for the column are excluded, (37) provides

$$\rho_2 \lambda_c = \frac{\rho_2^N}{a_1^D}, \tag{E-16}$$

where a_1^D is given by (D-23), and

$$\rho_2^N = -\mathcal{P}_3^c(\boldsymbol{v}_2, \boldsymbol{u}_1, \boldsymbol{u}_1) - \rho_1 \lambda_c \mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_2, \boldsymbol{u}_1), \tag{E-17}$$

where the operators are given by (D-14) and (E-2), respectively. The first operator depends linearly on $\dot{v}_2^v = 0$ for symmetric postbuckling behavior and the second operator depends linearly on w_2 and derivatives which are all zero. Thus, (E-16) and (E-17) yields

$$\rho_2 \lambda_c = 0 \implies \rho_2 = 0. \tag{E-18}$$

Third imperfection parameter for symmetric columns. The third imperfection parameter ρ_3 may in general be calculated from (37). For symmetric columns, and when stability operators which equal zero for the column are excluded, (37) provides

$$\rho_3 \lambda_c = \frac{\rho_3^N}{a_1^D}, \tag{E-19}$$

where a_1^D is given by (D-23) and the only nonvanishing contributions to ρ_3^N are

$$\rho_3^N = -\rho_1 \lambda_c \left(\mathcal{P}_3^c(\boldsymbol{u}_c', \boldsymbol{v}_2, \boldsymbol{u}_1) + \bar{\mathcal{P}}_3^c(\boldsymbol{u}_c', \boldsymbol{u}_1, \tilde{\boldsymbol{u}})\right) - \tfrac{1}{2}(\rho_1 \lambda_c)^2 \left(\mathcal{P}_4^c(\boldsymbol{u}_c', \boldsymbol{u}_c', \boldsymbol{u}_1, \boldsymbol{u}_1) + \mathcal{P}_3^c(\boldsymbol{u}_c'', \boldsymbol{u}_1, \boldsymbol{u}_1)\right). \tag{E-20}$$

When the relevant displacement fields are introduced in the four operators given by (D-14), (E-2) and (E-4), and $\boldsymbol{u}_c''$ given by

$$\dot{v}_c'' = -e_c' i_1, \tag{E-21}$$

the expression to lowest degree in i_1 for ρ_3^N is

$$\rho_3^N = -\tfrac{1}{2} E_t^c A L i_1 \left(-\rho_1 \lambda_c - \tfrac{1}{2}(\rho_1 \lambda_c)^2 \left({e_c'}^2 - \frac{i_1^2}{i_3} e_c''\right)\right)(1 + O(i_1)). \tag{E-22}$$

Utilize the result of (E-22) with $\rho_1 \lambda_c$ given by (E-11) and a_1^D in (E-19) to provide

$$\rho_3 \lambda_c \simeq (\rho_1 \lambda_c)^2 - \frac{{e_c'}^2 - (i_1^2/i_3) e_c''}{2(1 + e_c')^3} \tag{E-23}$$

when $i_1 \ll 1$. Since i_3 is generally of order i_1^2, both terms in the numerator may become important.

References

[Britvec 1973] S. J. Britvec, *The stability of elastic systems*, Pergamon Unified Engineering Series **12**, Pergamon Press, New York, 1973.

[Budiansky 1974] B. Budiansky, "Theory of buckling and post-buckling behavior of elastic structures", *Adv. Appl. Mech.* **14** (1974), 1–65.

[Budiansky and Hutchinson 1964] B. Budiansky and J. W. Hutchinson, "Dynamic buckling of imperfection-sensitive structures", pp. 636–651 in *Proceedings of the 11th International Congress on Applied Mechanics* (Munich, 1964), edited by H. Görtler, Springer, Berlin, 1964.

[Byskov and Hutchinson 1977] E. Byskov and J. W. Hutchinson, "Mode interaction in axially stiffened cylindrical shells", *AIAA J.* **15**:7 (1977), 941–948.

[Byskov et al. 1996] E. Byskov, C. D. Christensen, and K. Jørgensen, "Elastic postbuckling with nonlinear constraints", *Int. J. Solids Struct.* **33**:17 (1996), 2417–2436.

[Christensen and Byskov 2008] C. D. Christensen and E. Byskov, "Advanced postbuckling and imperfection sensitivity of the elastic-plastic Shanley–Hutchinson model column", *J. Mech. Mater. Struct.* **3**:3 (2008), 459–492.

[Euler 1744] L. Euler, *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Bousquet, Lausanne, 1744.

[Fitch 1968] J. Fitch, "The buckling and post-buckling behavior of spherical caps under concentrated load", *Int. J. Solids Struct.* **4**:4 (1968), 421–466.

[Hutchinson 1974] J. W. Hutchinson, "Plastic buckling", *Adv. Appl. Mech.* **14** (1974), 67–144.

[Koiter 1945] W. T. Koiter, *Over de stabiliteit van het elastisch evenwicht*, Ph.D. thesis, Delft University of Technology, 1945, available at http://tinyurl.com/Koiter-WT-1945-Thesis. In Dutch; translated in *On the stability of elastic equilibrium*, NASA Technical Translation NASA-TT-F-10833 (1967) and *The stability of elastic equilibrium*, Air Force Flight Dynamics Laboratory report AFFDL TR 70-25 (1970), http://contrails.iit.edu/DigitalCollection/1970/AFFDLTR70-025.pdf.

[Koiter 2009] W. T. Koiter, *W. T. Koiter's elastic stability of solids and structures*, edited by A. M. A. van der Heijden, Cambridge University Press, Cambridge, 2009.

[Kuznetsov and Levyakov 2002] V. V. Kuznetsov and S. V. Levyakov, "Complete solution of the stability problem for elastica of Euler's column", *Int. J. Non-Linear Mech.* **37**:6 (2002), 1003–1009.

[Tvergaard and Needleman 1982] V. Tvergaard and A. Needleman, "On the foundations of plastic buckling", pp. 205–233 in *Developments in thin-walled structures*, edited by J. Rhodes and A. C. Walker, Applied Science Publishers, London, 1982.

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