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STRESS AND STRAIN RECOVERY FOR THE IN-PLANE DEFORMATION OF AN ISOTROPIC TAPERED STRIP-BEAM

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The variational-asymptotic method was recently applied to create a beam theory for a thin strip-beam with a width that varies linearly with respect to the axial coordinate. For any arbitrary section, ratios of the cross-sectional stiffness coefficients to their customary values for a uniform beam depend on the rate of taper. This is because for a tapered beam the outward-directed normal to a lateral surface is not perpendicular to the longitudinal axis. This changes the lateral-surface boundary conditions for the cross-sectional analysis, in turn producing different formulae for the cross-sectional elastic constants as well as for recovery of stress, strain and displacement over a cross-section. The beam theory is specialized for the linear case and solutions are compared with those from plane-stress elasticity for stress, strain and displacement. The comparison demonstrates that for beam theory to yield such excellent agreement with elasticity theory, one must not only use cross-sectional elastic constants that are corrected for taper but also the corrected recovery formulae, which are in turn based on cross-sectional in- and out-of-plane warping corrected for taper.

A list of symbols can be found on page 975.

1. Introduction

It is typical in beam theory to assume that taper affects cross-sectional stiffness constants, stress and strain only from the change in section geometry along the beam axis. In other words, if for a homogeneous, isotropic beam, the bending stiffness is EI, then for a homogeneous, isotropic, tapered beam, the bending stiffness is simply written as $EI(x)$, where the area moment of inertia varies with the axial coordinate due to change in the sectional geometry arising from taper. A recent work, [Das et al. 2009], is one among a series of papers on tapered beams by the same authors that follows this methodology. In [Abdel-Jaber et al. 2008] and [de Rosa et al. 2010], the bending energy per unit length is simply written as $EI(x)\kappa^2/2$. Results in [Abdel-Jaber et al. 2008] were compared with those of an older work [Rao and Rao 1988], both of which clearly follow this methodology. These are only a few selected examples out of the many recent works on tapered beams based on cross-sectional stiffnesses that are not corrected for taper.

An asymptotic beam theory for an isotropic strip-beam with linearly tapered width was presented in [Hodges et al. 2008]. Section stiffnesses for this theory depend on taper in ways other than the simplistic approach noted above. The main reason for this is the tilting of the outward-directed normal so that it has a non-zero component along the beam longitudinal axis, and to be accurate the cross-sectional analysis must take this tilt into account. Because of the strip-like geometry, accuracy of the cross-sectional stiffnesses was evaluated using plane-stress elasticity solutions for extension, bending and flexure from [Timoshenko and Goodier 1970] and [Krahula 1975] and were shown to be in excellent agreement. The

Keywords: beam theory, elasticity, asymptotic methods.

plane stress problem of the in-plane deformation of an isotropic tapered strip was chosen because it is a simple example to illustrate the proposed theory. All results are closed-form expressions that can be validated from corresponding elasticity solutions available in the literature.

One purpose of this paper is to show that high-fidelity information is available in beam theories based on asymptotic methods, which are no more complicated than "engineering" theories. This paper focuses on the recovery of the stress, strain, and displacement fields for the linearly tapered isotropic strip-beam. This aspect was not addressed in [Hodges et al. 2008]. The recovery is performed by the variational-asymptotic method (VAM) and is consistent with the derivation of the stiffness constants in that paper. It will be shown that to capture the recovery relations accurately, one needs to evaluate the warping one order higher. The recovery relations are then compared with the corresponding elasticity solutions, a comparison that confirms that a VAM-based beam theory is able to satisfactorily predict all aspects of the behavior of beam-like structures.

Section 2 of this paper revisits the previous work of [Hodges et al. 2008] and reviews the importance of including taper in the stiffness constants. Section 3 provides details of the procedure to determine the recovery relations using the VAM and presents a comparison with the corresponding elasticity solutions. In Section 4, the range of the small parameters used in the VAM is determined for which the VAM solution is in close proximity with the elasticity solutions. Finally conclusions are drawn.

2. Corrected stiffness constants for a tapered beam

For better understanding of the results to be presented, a brief review of the variational-asymptotic method and a summary of the results from [Hodges et al. 2008] is presented here. The VAM is used to perform cross-sectional analysis of beams using the principle of minimum total potential energy, exploiting the presence of small parameters. The total potential energy is developed from a general displacement field subject to a restriction to small strain. The leading terms of the energy can be obtained asymptotically in terms of the small parameters of the analysis, which can be used to obtain the equations governing in- and out-of-plane warping. This procedure can be repeated for successively higher powers of the small parameters until the desired accuracy is achieved. As a result of this analysis, the warping is expressed in terms of one-dimensional (1D) strains and can then be used to calculate the strain energy per unit length. This 1D strain energy per unit length provides the cross-section constants, reducing the 2D plane stress problem to 1D, and formulae that allow for recovery of stress, strain and displacement over the cross-section.

We now proceed to outline a procedure to obtain the cross-sectional constants using the VAM for a tapered-strip beam as in Figure 1. For details, the reader is encouraged to consult [Hodges et al. 2008].

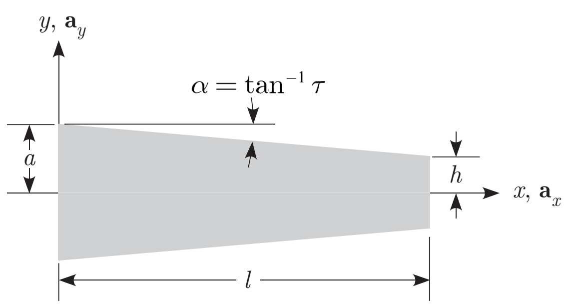

Figure 1. Schematic of the isotropic strip tapered beam.

The two small parameters of the system are a slenderness parameter $\delta = a/l$ and a taper parameter $\tau = -b'(x)$, which are assumed to be of the same asymptotic order. Considering the position vector of an arbitrary point in the undeformed and deformed configurations of the beam, the expressions for strain can be derived as

$$\Gamma_{xx} = \bar{\epsilon} - y\bar{\kappa} + w_{x,x}, \quad \Gamma_{xy} = w_{x,y} + w_{y,x}, \quad \Gamma_{yy} = w_{y,y}, \tag{1}$$

where $\bar{\epsilon}$ and $\bar{\kappa}$ are the classical 1D stretching and bending strain measures, respectively. The strain energy per unit length is then

$$U = \frac{Et}{2(1-\nu^2)} \left\langle \Gamma_{xx}^2 + \Gamma_{yy}^2 + 2\nu\Gamma_{xx}\Gamma_{yy} + \frac{1-\nu}{2}\Gamma_{xy}^2 \right\rangle, \tag{2}$$

where

$$\langle \bullet \rangle = \int_{-b(x)}^{b(x)} \bullet \, dy. \tag{3}$$

The first step is to solve for the zeroth-order warping. For this, we identify and remove all the terms that are first and higher order in the small parameters from the strain energy. The resulting equations obtained using the principle of minimum total potential energy can be used to evaluate the zeroth-order warping, which in turn gives the zeroth-order strain energy per unit length as

$$U_0 = \tfrac{1}{2} EA(x)\bar{\epsilon}^2 + \tfrac{1}{2} EI(x)\bar{\kappa}^2, \tag{4}$$

the expected expression for strain energy per unit length associated with classical Euler–Bernoulli beam theory.

To refine this result it is necessary to solve for the warping corrected to first order in δ and τ. To do so, the solution of warping previously obtained is perturbed to the next higher order. A similar procedure is preformed as described previously, the only difference being that all the terms in the energy correct through second order in the small parameters are retained. The first-order warping thus obtained is used to obtain the strain energy per unit length:

$$\begin{aligned} U_2 = Etb(x)\big(1 - \tfrac{2}{3}(1+\nu)\tau^2\big)\bar{\epsilon}^2 + \tfrac{2}{3}Et\nu\tau b(x)^2\bar{\epsilon}\,\bar{\epsilon}' + \tfrac{1}{9}Etb(x)^3\big(3 + 2(14\nu+15)\tau^2\big)\bar{\kappa}^2 \\ -\tfrac{4}{9}Et\tau(8\nu+9)b(x)^4\bar{\kappa}\,\bar{\kappa}' + \tfrac{4}{15}Et(1+\nu)b(x)^5\bar{\kappa}'^2 + \tfrac{2}{45}Et(11\nu+12)b(x)^5\bar{\kappa}\,\bar{\kappa}'', \end{aligned} \tag{5}$$

which is asymptotically correct through second order.

However, this strain energy per unit length is unsuitable for an engineering beam theory because it contains derivatives of the classical 1D strain measures. Hence, it is transformed into a generalized Timoshenko form as follows: First, the 1D classical strain measures are written in terms of 1D generalized Timoshenko strain measures using simple beam kinematics. A 1D shear strain measure enters into the picture through this transformation. Second, the derivatives of the 1D generalized Timoshenko strain measures are evaluated using equilibrium equations. The equilibrium equations can be simply obtained by the standard textbook approach of considering an element of the beam and writing the force and moment equilibrium.

Thus, the strain energy per unit length of a beam correct through second order, when transformed to the form of a generalized Timoshenko theory, is given by

$$U^* = \tfrac{1}{2} Z\epsilon^2 + \tfrac{1}{2} W\kappa^2 + \tfrac{1}{2} Y\gamma^2 + X\kappa\gamma, \tag{6}$$

where

$$\begin{aligned} Z &= EA(x)\left(1 - \frac{2\tau^2}{3}\right), & W &= EI(x)\left(1 + \frac{(\nu-48)\nu-45}{45(\nu+1)}\tau^2\right), \\ Y &= \tfrac{5}{6} GA(x), & X &= \frac{Et(5\nu+3)b(x)^2\tau}{9(1+\nu)}, \end{aligned} \tag{7}$$

where, for a linearly tapered beam, τ is the tangent of the taper angle α as shown in Figure 1. It should be noted that the stiffness associated with shear is what one obtains from the usual Timoshenko beam theory. There is no taper correction to this term because the shear strain is already one order higher in the small parameter δ than the strains associated with 1D bending and extension measures, so that the overall contribution of the term to the strain energy per unit length is correct through second order. This theory is said to be a generalized Timoshenko theory in that it contains contributions to the strain energy associated with extension, bending and shear. However, it is not subject to any of the usual restrictions on kinematics associated with the original Timoshenko theory. Moreover, it includes a bending-shear coupling term X, which is not found in the original theory.

Validation of these stiffness constants, presented in [Hodges et al. 2008], showed that the theory is only accurate when corrections associated with nonzero τ are included. Unfortunately, a review of the literature shows that there is hardly any awareness among researchers that beam stiffness constants depend on taper, as all references the authors have found to date would provide the above stiffness constants with τ set equal to zero.

An important aspect of the asymptotic theory is that bending and shear are coupled for a tapered beam; hence, the coefficient X is present in the energy. Therefore, if one takes the bending and shear stiffnesses as $EI(x)$ and $5GA(x)/6$ (i.e., only changing the sectional width in the stiffness formulae), the strain energy associated with bending-shear coupling will be missed. This can lead to significant errors in prediction of the beam deflection.

Figure 2 shows the percentage errors in extension and bending stiffnesses (Z and W from (7)) when one neglects the effect of taper and proceeds with the simplistic change in the sectional stiffnesses. It can be concluded that neglecting taper introduces an error in the beam sectional stiffnesses that can be significant, affecting deflections under load as well as natural frequencies.

To assess the importance of the bending-shear coupling term X relative to the pure bending and pure shear term, the coupling stiffness is normalized, such that

$$\overline{X} = \frac{X}{\sqrt{YW}} = \frac{(5\nu+3)\tau}{\sqrt{45(\nu+1) + \left(\nu^2 - 48\nu - 45\right)\tau^2}}. \tag{8}$$

This normalized value can be thought of as a measure of coupling strength to be compared with unity. For a taper (τ) of 0.2, it varies from 0.0215 to 0.1367 as Poisson's ratio varies from -0.5 to 0.5. Moreover, the plot shown in Figure 3 indicates that these values are by no means negligible compared to unity.

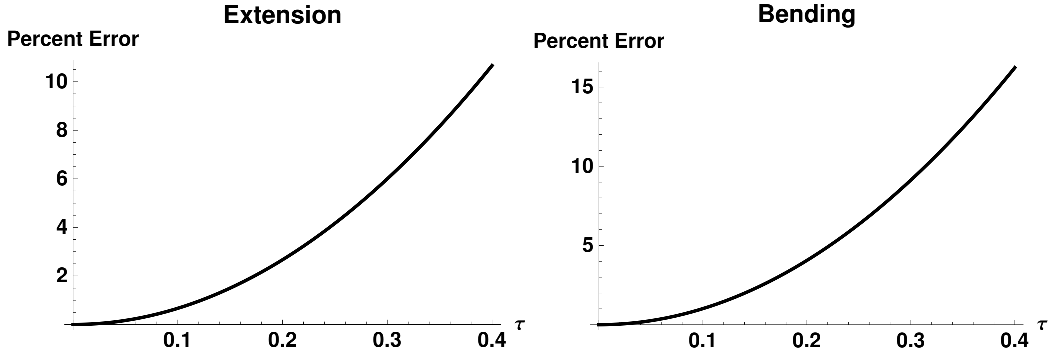

Figure 2. Percentage errors in the stiffnesses for $\nu = 0.3$.

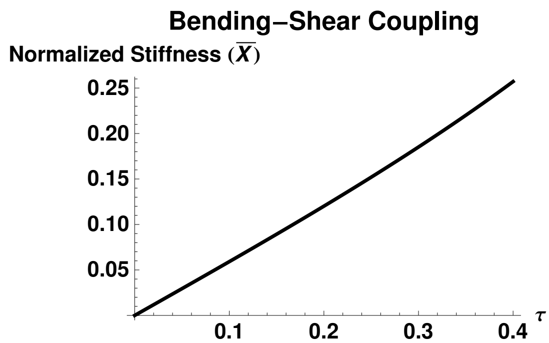

Figure 3. Normalized stiffness for bending-shear coupling, $\nu = 0.3$.

Therefore, its absence may cause significant errors, and it is thus important to include these corrections in the stiffnesses to account correctly for the effects of taper.

3. Recovery relations

This section presents strain, stress and displacement components obtained from the beam theory based on VAM and comparisons with elasticity solutions. Although the baseline elasticity solutions are not restricted to small values of the parameters δ and τ, they are compared to solutions from the beam theory, which are subject to small values of δ and τ. In particular, beam theory based on the VAM is used to analyze the problem of a tapered beam subjected to three different types of loading described as extension, bending and flexure shown in Figure 4. These three cases correspond to constant axial force, constant bending moment and constant shear force, respectively. As in [Hodges et al. 2008], the warping and strain energy are evaluated through first and second orders, respectively.

For greater accuracy than in the earlier paper, the warping is here evaluated to second order. For this, the same procedure outlined in Section 2 is followed. The first-order warping is perturbed and from the perturbed warping, strains are obtained that are, in turn, used to evaluate the strain energy as a function of the unknown warping perturbations. Minimization of the strain energy using calculus of variations

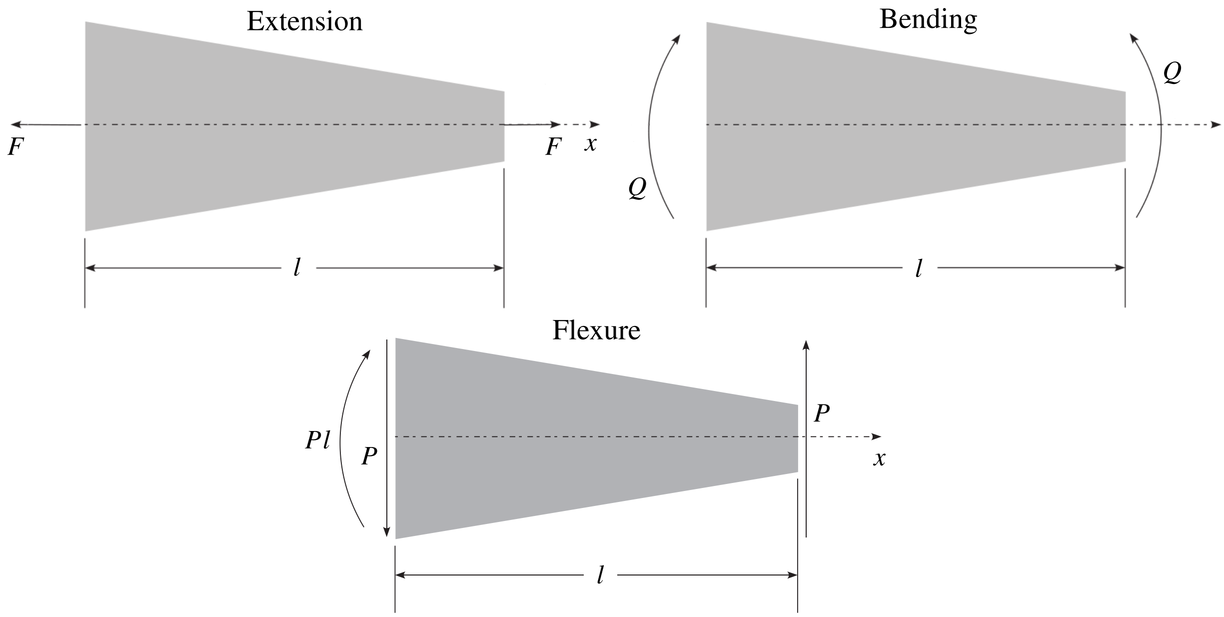

Figure 4. Schematic of beam loaded for extension, bending and flexure.

yields the expression for the second-order terms in warping as

$$w_x^{(2)} = 0, \quad w_y^{(2)} = A_0\bar{\epsilon} + A_1\bar{\epsilon}' + A_2\bar{\epsilon}'' + B_0\bar{\kappa} + B_1\bar{\kappa}' + B_2\bar{\kappa}'', \tag{9}$$

where

$$\begin{aligned}
A_0 &= \tfrac{1}{6}b^{-2}y(\nu+1)\tau^2\big(y^2(\nu+1) - b^2(\nu-3)\big),\\
A_1 &= \tfrac{1}{6}b^{-1}y\tau\big(y^2(\nu+1)^2 - b^2(\nu^2+2\nu+3)\big),\\
A_2 &= \tfrac{1}{6}y\nu^2(b^2-y^2),\\
B_0 &= -\tfrac{1}{18}(8\nu^2+6\nu-3)\tau^2(b^2-3y^2),\\
B_1 &= \tfrac{1}{9}b\nu(5\nu+6)\tau(b^2-3y^2),\\
B_2 &= \tfrac{1}{360}\big(-b^4(40\nu^2+54\nu+7) + 30b^2y^2(4\nu^2+6\nu+1) - 15y^4(2\nu+1)\big).
\end{aligned} \tag{10}$$

Thus, an expression for the warping through second order has been obtained. The derivatives of the 1D classical strain measures make it unsuitable for use in an engineering beam theory. The classical strain measures are transformed into generalized Timoshenko strain measures, whose derivatives are computed using the equilibrium equations. The required sectional stiffnesses for use in the equilibrium equations are given by (7).

Note that the second-order warping functions are not used for obtaining stiffnesses but only for recovery of stress, strain and displacement. The expressions for strain in (1) are restricted only by the assumption of small strain. The 1D strain measures may be used in their geometrically exact form. Herein, however, for the purpose of comparison with linear elasticity theory, we restrict them to small displacement and rotation. Care should be taken to distinguish between the 1D classical strain measures appearing in (1) and 1D generalized Timoshenko strain measures in (6). The relation between the two is detailed in [Hodges 2006] and specialized in [Hodges et al. 2008] and here as

$$\bar{\epsilon} = \epsilon, \qquad \bar{\kappa} = \kappa + \gamma'. \tag{11}$$

Since the problem under consideration is that of plane stress, the stresses are simply obtained from the constitutive law as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \Gamma_{xx} \\ \Gamma_{yy} \\ \Gamma_{xy} \end{Bmatrix}. \tag{12}$$

The 2D displacements from linear beam theory are computed from subtracting the position vector of an arbitrary point on the undeformed cross-sectional plane from the corresponding position vector in the deformed cross-sectional surface, such that

$$u_x = u - y v_{,x} + w_x \quad \text{and} \quad u_y = v + w_y, \tag{13}$$

where u and v are the 1D displacement variables of the beam theory, while u_x and u_y are the 2D displacements of an arbitrary point of the cross-section. These 1D displacement variables can be computed from the 1D strain measures by using the linear 1D strain-displacement equations

$$\epsilon = u', \quad \gamma = v' - \theta, \quad \kappa = \theta'. \tag{14}$$

To completely determine the 1D displacement and rotation variables, i.e., u, v and θ, the boundary condition specified at $x = 0$ sets u, v and θ to zero.

From this the stress, strain and displacement components were obtained from the beam analysis based on VAM. They are compared with the plane-stress elasticity solutions obtained from [Krahula 1975] and [Timoshenko and Goodier 1970]. Results are presented in Figures 5–7 for the three loading cases of extension, bending and flexure, respectively. The two results from the variational-asymptotic method, VAM (I) and VAM (II), correspond to the cases when warping is evaluated through first and second orders, respectively. The elasticity solutions also have been plotted for comparison purposes. For the three loading cases, the recovery relations are plotted at $x = l/2$, versus ζ, a dimensionless variable defined as $y/b(x)$.

It is clear that if the warping is accurate to second order, then the recovery relations of the beam theory agree very well with results from the elasticity solution. On the other hand, if warping is evaluated only to first order as in [Hodges et al. 2008], some results are not in good agreement with the elasticity solutions. Note that for presentation the recovery relations were normalized by certain standard quantities. In the case of strain the normalizing quantities were $F/(ELt)$, $Q/(EL^2t)$ and $P/(ELt)$ for extension, bending and flexure, respectively. For stresses and displacements, they were the strain normalizing factors multiplied by modulus of elasticity and length of the beam, respectively. The results were generated for $\nu = 0.3$, $\tau = 0.2$ and $\delta = 0.25$. It is essential to state here that the VAM solutions are compared with the total elasticity solutions, not with the elasticity solutions expanded to a certain order in small parameters.

Asymptotic expansions of the expressions for recovered strains were carried out, and it was seen that the results are in excellent agreement with the elasticity solutions expanded through the corresponding order. For the extension case, if the warping is correct through second order, i.e., $O(a\delta^2\epsilon)$, then the strains

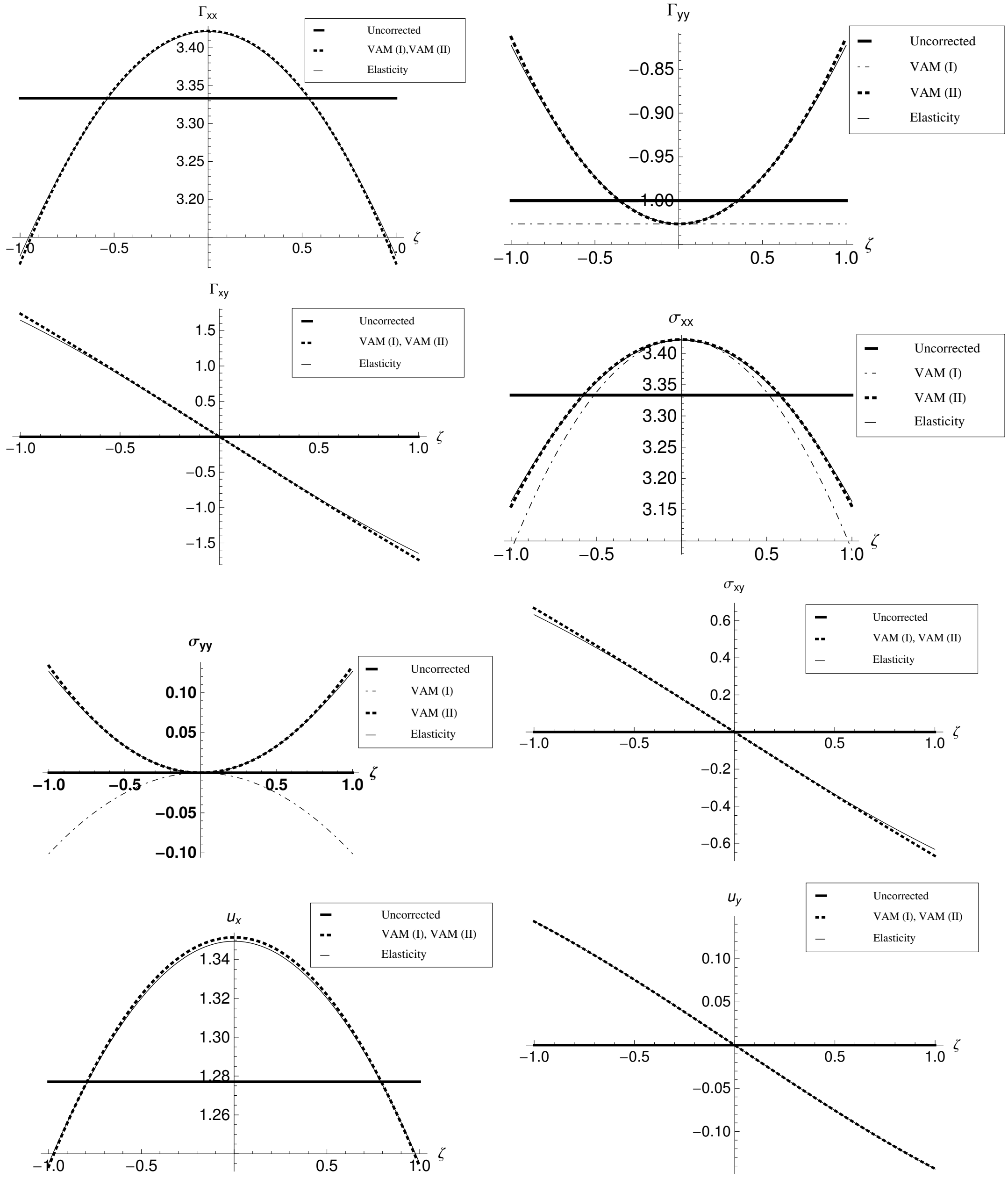

Figure 5. Comparison of the normalized VAM strains, stresses and displacements with the elasticity solutions for extension.

Γ_{xx}, Γ_{xy} and Γ_{yy} are expected to be correct through orders 3, 1 and 2, respectively. However, based on the trends in the evaluation of warping the third-order contribution to the warping, w_y is expected to be zero. Therefore, under these special circumstances, the strains listed in the same order as above are actually correct through orders 3, 2 and 3, respectively, relative to the leading term. Expansions of the

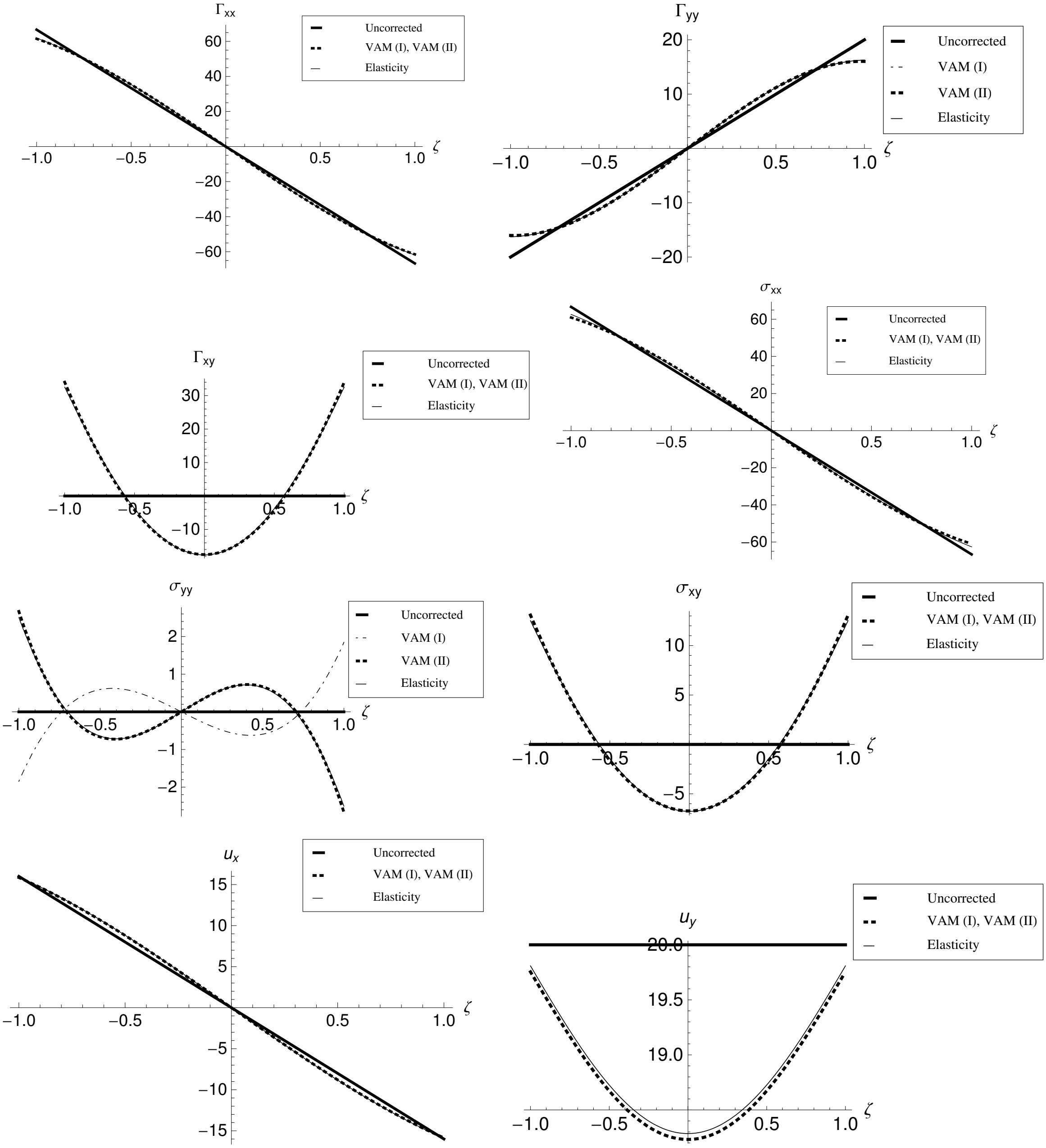

Figure 6. Comparison of the normalized VAM strains, stresses and displacements with the elasticity solutions for bending.

2D strain components for extension are presented in Table 1 on page 973. For brevity, the bending and flexure cases are not included. The third-order terms are zero for Γ_{xx} and Γ_{yy} and hence the expansions are correct through the third order. Also, the second-order terms are zero for Γ_{xy}, and hence it is correct through second order.

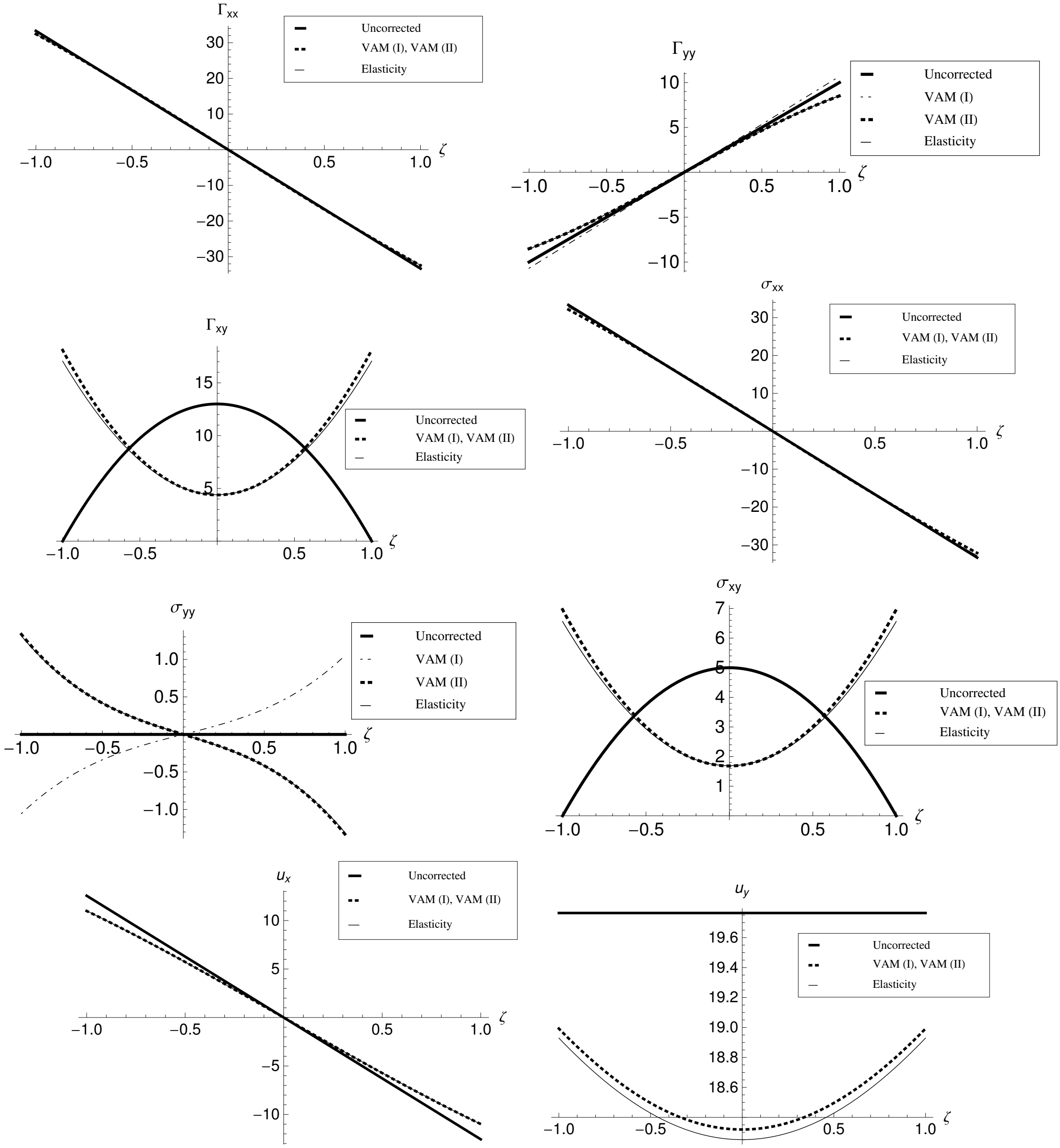

Figure 7. Comparison of the normalized VAM strains, stresses and displacements with the elasticity solutions for flexure.

Recovery relations without taper corrections. When the sectional formulae of an untapered beam are used for a tapered one, with the only effect of taper being a change in the width, it follows that taper does not enter into the expressions for strains and stresses. The stresses, strains and displacements from this type of analysis, which as mentioned in Section 2 is the starting point for most of the research

Strain	Expansion
Γ_{xx}	$\frac{F}{2Etb}\left(1+\frac{1}{3}\tau^2(-3\zeta^2\nu-6\zeta^2+2)\right)$
Γ_{xy}	$-\frac{\tau F\zeta(\nu+1)}{Etb}$
Γ_{yy}	$-\frac{F\nu}{2Etb}\left(\frac{\tau^2(-6\zeta^2\nu-3\zeta^2+2\nu)}{3\nu}+1\right)$

Table 1. Asymptotic expansions of the strains from VAM and elasticity for the extension case.

on tapered beams, has been plotted along with the VAM and elasticity solutions in Figures 5–7. The recovery relations are erroneous and certain trends are incorrect. The bending-shear coupling effect is not captured as expected, and it can be seen from the graphs of Γ_{xy} and σ_{xy} in Figure 6 that the case of bending does not result in any shear stress or strain. Another example is that of σ_{xy} for flexure, where the trend for a tapered beam is exactly opposite that of a prismatic beam, as shown in the graph for σ_{xy} in Figure 7. Using the change in sectional width in the stiffness formulae for prismatic beams leads to erroneous stress, strain and displacement. This implies that the problem is being posed in a fundamentally incorrect way. A prominent error of this type was identified in [Hodges et al. 2008], wherein it was shown that the lateral-surface boundary conditions in the typical tapered beam analysis are incorrect.

4. Validity of the recovery expressions

In the previous sections, the recovery relations obtained from the VAM were compared with the exact elasticity solutions. The VAM analysis was based on considering the parameters δ and τ to be small. This section addresses the definition of the "smallness" of these parameters. In other words, we increase the values of δ and τ till the point at which the VAM solution deviates from the exact elasticity solutions, thus determining the range of applicability of the VAM solution. It is important to note that from their definitions, the value of τ must always be less than or equal to the value of δ. If τ were equal to δ, this is a special case of a tapered beam, i.e., a wedge, for which a singularity exists in the case of flexure and extension, as the force applied at the end in both the cases, acts over a vanishing area. Hence, we will address the cases for which τ is strictly less than δ. The percentage errors for various values of δ at are plotted in Figure 8. By error here we mean the maximum of the percentage errors of the recovery relations for all the three loading cases. The error of a VAM solution is obtained by comparison with the corresponding elasticity solution. Results for those combinations of τ and δ for which the maximum error was below 5% was considered to be satisfactory. It is seen that at the extreme case of $\delta = 0.4$, the results are accurate up to $\tau = 0.26$. Investigations were terminated at $\delta = 0.4$ as for higher values, it is generally expected that an engineering analysis would be done considering the structure as a plate and not a beam.

5. Conclusions

A beam theory has been presented based on the VAM for tapered strip-beam. The strip-beam is sufficiently thin that it can be assumed to be in a state of plane stress. The novel feature of the beam theory is that the effect of the taper parameter τ on the lateral-surface boundary conditions is included. This

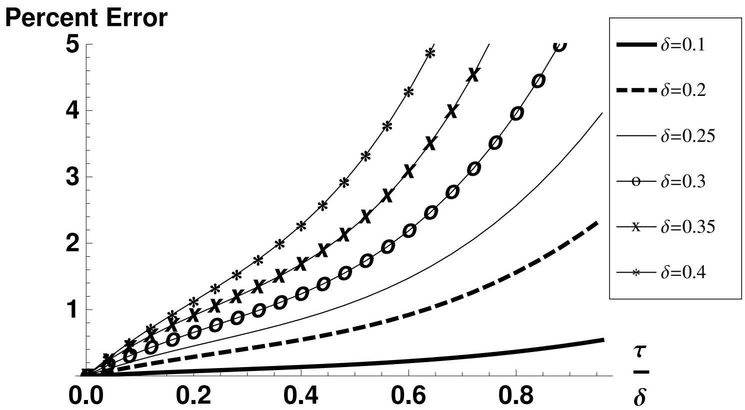

Figure 8. Percentage errors with respect to τ/δ at a given δ for the VAM recovery relations with respect to the elasticity solutions.

effect must be accounted for when performing a cross-sectional analysis, which gives the cross-sectional elastic constants necessary for solving the 1D beam equations, and the recovery relations necessary for accurately capturing stress, strain and displacement. To obtain accurate recovery relations, it is necessary to evaluate warping through second order in the small parameters, while only first-order warping is sufficient for obtaining accurate cross-sectional elastic constants.

When the VAM-based beam theory is linearized and applied to problems for which elasticity solutions exist, such as constant axial force, constant bending moment and constant transverse shear force, the results agree quite well for all values of τ for a beam with δ up to 0.25. Beyond this value of δ, the values of τ for which the solutions are good reduces; and, finally, for $\delta = 0.4$, the maximum value of τ is 0.26, which is satisfactory. Therefore, a VAM-based beam cross-sectional analysis can solve the problem of a tapered beam with sufficient accuracy.

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List of symbols

$A(x)$	cross-sectional area of the strip-beam
E	Young's modulus
F	axial force
G	shear modulus
$I(x)$	area moment of inertia of the strip-beam about the z axis
P	shear force
Q	bending moment
U	strain energy per unit length
U_0	zeroth-order sectional strain energy
U_2	second-order sectional strain energy
U^*	second-order sectional strain energy transformed to generalized Timoshenko form
a	half-width of the strip-beam at $x = 0$, equal to $b(0)$
$b(x)$	half-width of the strip-beam
h	half-width of the strip-beam at $x = l$, equal to $b(l)$
l	length of the strip-beam
t	thickness of the strip-beam
u, v	1D displacements
u_x, u_y	2D displacements
w_x, w_y	warping of the strip-beam section, out-of-plane and in-plane, respectively
$w_x^{(2)}, w_y^{(2)}$	second-order warping of the strip-beam section
x	axial coordinate of the strip-beam
y, z	cross-sectional coordinates of the strip-beam, with z through the thickness
$\Gamma_{xx}, \Gamma_{yy}, \Gamma_{xy}$	2D strain components
α	taper angle of the strip-beam
δ	ratio of the maximum half-width of the strip-beam to its length
$\bar{\epsilon}$	1D classical generalized stretching strain
ϵ	1D generalized Timoshenko stretching strain
$\bar{\kappa}$	1D classical generalized bending strain
κ	1D generalized Timoshenko bending strain
γ	1D generalized Timoshenko shearing strain
ν	Poisson's ratio
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	2D stress components
θ	1D rotation
τ	taper of the strip-beam, equal to $\tan\alpha$
ζ	dimensionless cross-sectional coordinate $y/b(x)$, such that $-1 \leq \zeta \leq 1$

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