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The well-known closed-form solution given by Hult and McClintock for an antiplane crack in an elastic-perfectly plastic material is reconsidered using the crack line analysis method. A precise elastic-plastic solution near the crack line region, different from Hult and McClintock's, is deduced by matching the general solution of the plastic field with that of the exact elastic field. It is verified from the deduction that the Hult–McClintock elastic-plastic solution is inadequate for many purposes.

Introduction

The Hult–McClintock closed-form solution [1957] for an antiplane crack in an elastic-perfectly plastic material was a significant achievement in the development of fracture mechanics. Its importance lies not only in being the first closed-from elastic-plastic solution in fracture mechanics, but also in that the two usual assumptions of small-scale yielding originate from it: (1) the plastic zone ahead of the crack tip is so small that the elastic field out of the plastic zone is the K-dominant elastic singular field for the crack; and (2) the crack tip of the K-dominant elastic singular field effectively behaves as if it lies a distance x_e ahead of the actual crack tip, along the crack line, giving rise to the notion of an "imaginary crack" (see Figure 1). These assumptions were preserved in many subsequent works on the problem, such as [Koshinen 1963; Rice 1966; 1967; Edmunds and Willis 1976; Hutchinson 1979].

Because of the authority of the Hult–McClintock elastic-plastic solution, its limitations have largely gone unaddressed. These shortcomings are difficult to verify directly by conventional crack tip asymptotic

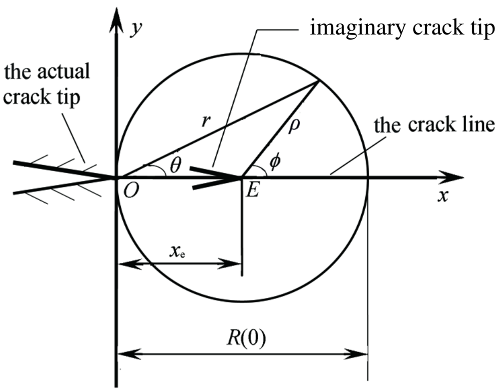

Figure 1. The crack-tip plastic zone [Hult and McClintock 1957].

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analysis under small-scale yielding. However, if the crack line analysis method is used, the method's inadequacy becomes apparent, and already in [Yi 1993; 1994] we were compelled to find more reasonable solutions. In this paper we continue these investigations by deducing an elastic-plastic solution near the crack line region, different from Hult and McClintock's. We do this by matching the general solution of the plastic field to that of the exact elastic field, to discuss the validity of the Hult–McClintock solution.

1. Review of the Hult–McClintock solution

General assumptions. For an antiplane crack in an elastic-perfectly plastic solid, the displacement w and stress components τ_{xz} and τ_{yz} are assumed to depend only on x and y. The crack tip region is shown in Figure 1. The equilibrium equation is

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{1}$$

and the Huber–von Mises yield criterion is

$$\tau_{xz}^2 + \tau_{yz}^2 = k^2, \tag{2}$$

where k is the yield stress in pure shear. The strain is given by $\gamma_{xz} = \dfrac{\partial w}{\partial x}$ and $\gamma_{yz} = \dfrac{\partial w}{\partial y}$. For a stationary crack, the Hencky deformation constitutive relations are

$$\frac{\partial w}{\partial x} = \frac{1}{G}\tau_{xz} + \lambda \tau_{xz}, \qquad \frac{\partial w}{\partial y} = \frac{1}{G}\tau_{yz} + \lambda \tau_{yz}, \tag{3}$$

where G is the elastic shear modulus and λ a nonnegative factor.

Statement of the Hult–McClintock solution. In an (r, θ) polar coordinate system, Hult and McClintock suggested that the stresses in the plastic zone are

$$\tau_{xz} = -k \sin\theta, \qquad \tau_{yz} = k \cos\theta. \tag{4}$$

They then obtained the displacement in the plastic zone, the strains in the plastic zone, and the elastic-plastic boundary by matching the plastic stress field (4) and its corresponding displacement field for the actual crack with the usual crack tip K-dominant elastic singular field of an "imaginary crack" (see Figure 1) at the elastic-plastic boundary. The Hult–McClintock solutions are

$$w = \frac{K_{\mathrm{III}}^2}{G\pi k}\sin\theta, \quad \gamma_{xz} = -\frac{K_{\mathrm{III}}^2}{G\pi k}\frac{\sin\theta\cos\theta}{r}, \quad \gamma_{yz} = \frac{K_{\mathrm{III}}^2}{G\pi k}\frac{\cos^2\theta}{r}, \quad R(\theta) = \frac{1}{\pi}\Big(\frac{K_{\mathrm{III}}}{k}\Big)^2\cos\theta, \tag{5}$$

where K_{III} is the stress intensity factor. The strain $(5)_{2,3}$ in the plastic zone has a $1/r$ singularity.

According to $(5)_4$, the plastic zone is a circle tangent to the actual crack tip, having diameter $d = R(0) = K_{\mathrm{III}}^2/(\pi k^2)$. The imaginary crack tip of the K-dominant elastic singular field lies at the center of the plastic zone, at a distance $x_e = d/2$ ahead of the actual tip.

Beyond the Hult–McClintock plastic field. Solutions satisfying the equilibrium equations (1) and the Huber–von Mises yield criterion (2) are countless: the Hult–McClintock plastic stress field (4) is just a particular solution, not a general one. What are its limitations? A fruitful approach to this question is to use the crack line analysis method and expansion in power series of θ, reducing the solution of the partial differential equations to that of ordinary differential equations, through which a description of the general solution for the plastic field near the crack line region can be obtained.

The crack line analysis method only focuses on the field near the crack line. The method has been used for an antiplane crack before [Achenbach and Li 1984; Guo and Li 1987; Yi 1992]. The elastic-plastic solutions obtained in those references are the same as those given by Hult and McClintock near the crack line, but any such solution is still inadequate and is still confined by the small-scale yielding assumptions.

The crack line analysis method was improved in [Yi 1993; 1994; Yi et al. 1996; 1997; Yi and Yan 2001], and has been used to solve other crack problems [Wu and Wang 1996; Wang and Zhang 1998; Wang and Wu 2003; Wang and Zhou 2004; Zhou and Wang 2005; Zhou and Ling 2006]. The method also applies well to linear elastic fracture mechanics and has been developed into an effective way of solving the stress intensity factors of cracks [Yi 1991; 1992; Wang 1996; 2002]. The significance of the improved crack line method is that the general solution in power series of the plastic field near the crack line can be obtained exactly. By matching the general power series form solution of the plastic field with the precise elastic field outside the plastic zone, the assumptions of the usual small-scale yielding condition can be completely given up. Thus, a more reasonable elastic-plastic solution can be obtained.

2. Introduction to the crack line analysis method: General power series solution for the plastic field near the crack line

To proceed, consider the region near the crack line, shown in Figure 2 and corresponding to $\theta = 0$. The stress components and the displacement are continuous across the crack line. So in the plastic zone near the crack line region, τ_{xz}, τ_{yz}, and w can be expressed in polar coordinates as power series up to second order in θ as

$$\begin{aligned} \tau_{xz} &= \tau_1(r)\theta + O(\theta^3), & w &= w_1(r)\theta + O(\theta^3), \\ \tau_{yz} &= \tau_0(r) + \tau_2(r)\theta^2 + O(\theta^4), & \lambda &= \lambda_0(r) + \lambda_2(r)\theta^2 + O(\theta^4), \end{aligned} \tag{6}$$

where λ is the nonnegative factor in (3). Here we have taken into account that τ_{xz} and w are antisymmetric, while τ_{yz} and λ are symmetric with respect to $\theta = 0$. (In [Guo and Li 1987; Yi 1992] the corresponding expressions in rectangular coordinates are considered.)

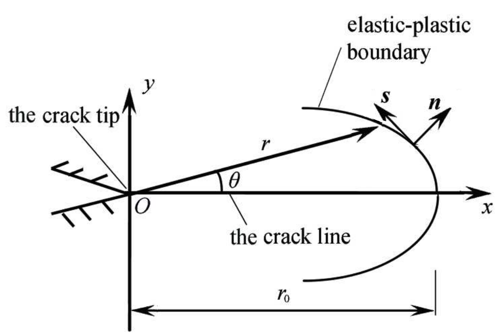

Figure 2. The region near the crack line.

Substituting (6) into (1)–(3) by using the relationships $r^2 = x^2 + y^2$ and $\theta = \arctan \dfrac{y}{x}$ and collecting terms of the same order θ yields

$$\begin{gathered} \frac{d\tau_1}{dr} + \frac{d\tau_0}{dr} - \frac{\tau_1}{r} + \frac{2\tau_2}{r} = 0, \qquad \tau_0^2 = k^2, \qquad \tau_1^2 + 2\tau_0\tau_2 = 0, \\ \frac{dw_1}{dr} - \frac{w_1}{r} = \Big(\frac{1}{G} + \lambda_0\Big)\tau_1, \qquad \frac{w_1}{r} = \Big(\frac{1}{G} + \lambda_0\Big)\tau_0. \end{gathered} \tag{7}$$

Thus the system of partial differential equations (1)–(3) has been transformed into a system of ordinary differential equations. We can solve the equations (7) to obtain *in closed form* the coefficients τ_0, τ_1, τ_2, and w_1 appearing in (6). We find

$$\tau_0 = k, \quad \tau_1 = -\frac{kr}{r+L}, \quad \tau_2 = -\frac{kr^2}{2(r+L)^2}, \quad w_1 = \frac{Cr}{r+L}, \tag{8}$$

where L and C are constants of integration. Thus we have

$$\tau_{xz} = -\frac{kr}{r+L}\theta + O(\theta^3), \quad \tau_{yz} = k - \frac{kr^2}{2(r+L)^2}\theta^2 + O(\theta^4), \quad w = \frac{Cr}{r+L}\theta + O(\theta^3). \tag{9}$$

When converted to rectangular coordinates, the solutions (9) are the same as those given in [Yi 1994].

The strains corresponding to (9) are

$$\gamma_{xz} = -\frac{Cr}{(r+L)^2}\theta + O(\theta^3), \quad \gamma_{yz} = \frac{C}{r+L} + O(\theta^2), \tag{10}$$

which have no singularities as $r \to 0$ if $L > 0$.

Remark. Although the preceding discussion only considered terms up to θ^2, it can be extended to a higher-order analysis using the same idea. Suppose, for example, that we wish to go up to θ^4, writing

$$\begin{aligned} \tau_{xz} &= \tau_1(r)\theta + \tau_3(r)\theta^3 + O(\theta^5), \\ \tau_{yz} &= \tau_0(r) + \tau_2(r)\theta^2 + \tau_4(r)\theta^4 + O(\theta^6). \end{aligned} \tag{11}$$

Substituting these equalities into (1) and (2) and collecting terms of the same order in θ yields two new equations besides the ones appearing on the first line of (7):

$$-\frac{1}{6}\frac{d\tau_0}{dr} - \frac{1}{2}\frac{d\tau_1}{dr} + \frac{1}{6}\frac{\tau_1}{r} + \frac{d\tau_2}{dr} - \frac{\tau_2}{r} + \frac{d\tau_3}{dr} - 3\frac{\tau_3}{r} + 4\frac{\tau_4}{r} = 0, \quad 2\tau_1\tau_3 + \tau_2^2 + 2\tau_0\tau_4 = 0.$$

The number of equations and unknowns has increased from three to five, but we can still use the same method to solve the system of five equations, obtaining, besides the first three equations in (8), the expressions

$$\begin{aligned} \tau_3 &= \frac{k}{6}\frac{r}{r+L} - \frac{k}{2}\Big(\frac{r}{r+L}\Big)^2 + \Big(\frac{k}{2} + \frac{D}{r+L}\Big)\Big(\frac{r}{r+L}\Big)^3, \\ \tau_4 &= \frac{k}{6}\Big(\frac{r}{r+L}\Big)^2 - \frac{k}{2}\Big(\frac{r}{r+L}\Big)^3 + \Big(\frac{3k}{8} + \frac{D}{r+L}\Big)\Big(\frac{r}{r+L}\Big)^4, \end{aligned}$$

which are then plugged into (11) to give explicit expressions for the stress components to order four. The displacement is handled similarly.

Further, the elastic-plastic boundary is assumed to be continuous across the crack line and to have equation $r = r_p(\theta)$ (Figure 2). Again by symmetry, the function r_p is even, and we have, to second order,

$$r_p(\theta) = r_0 + r_2\theta^2 + O(\theta^4), \tag{12}$$

where r_0 is the length of the plastic zone along the crack line [Yi 1994]. The values of r_0 and r_2 can be determined by matching the plastic field with the elastic field at the elastic-plastic boundary.

It follows from (12) that the unit normal vector $\boldsymbol{n} = (n_x, n_y)$ of the elastic-plastic boundary is

$$n_x = 1 - \tfrac{1}{2} B_1^2\theta^2 + O(\theta^4), \quad n_y = B_1\theta + (\theta^3), \quad \text{where } B_1 = 1 - 2\frac{r_2}{r_0}. \tag{13}$$

Returning to the analysis to second order, the idea now is to match (9) and (10) with a sufficiently precise elastic field near the crack line. Before doing this, we recall the derivation of the Hult–McClintock equations in the context of our analysis, to understand its limitations and set the scenario for our solution.

3. Further discussion of the Hult–McClintock matching result

Rederivation of the Hult–McClintock elastic-plastic boundary. In polar coordinates (ρ, ϕ) centered at the point $x = x_e$, $y = 0$ (the "imaginary crack tip"), let the elastic-plastic boundary be written as

$$\rho_p(\phi) = \rho_0 + \rho_2\phi^2 + O(\phi^4), \tag{14}$$

(see Figure 3). The unit normal vector $\boldsymbol{n} = (n_x, n_y)$ of the boundary is then

$$n_x = 1 - \tfrac{1}{2}\beta_1^2\phi^2 + O(\phi^4), \quad n_y = \beta_1\phi + O(\phi^3), \quad \text{where } \beta_1 = 1 - 2\frac{\rho_2}{\rho_0} \tag{15}$$

(compare (13)), and the equations relating the two polar coordinate systems are

$$\theta = \arctan\frac{\rho\sin\phi}{x_e + \rho\cos\phi}, \quad r^2 = x_e^2 + \rho^2 - 2\rho x_e\cos(\pi - \phi). \tag{16}$$

In the Hult–McClintock solution, the usual K-dominant elastic stress field for the imaginary crack in polar coordinates (ρ, ϕ) (see Figure 1 or Figure 3) is

$$\tau_{xz} = -\frac{K_{\text{III}}}{\sqrt{2\pi\rho}}\sin\frac{\phi}{2}, \quad \tau_{yz} = \frac{K_{\text{III}}}{\sqrt{2\pi\rho}}\cos\frac{\phi}{2}, \tag{17}$$

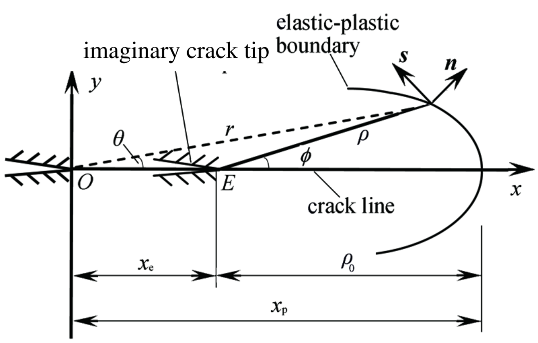

Figure 3. The region near the crack according to [Hult and McClintock 1957].

where the second assumption of small scale yielding is adopted.

Expanding (17) to second order gives

$$\tau_{xz} = -\frac{1}{2}\frac{K_{\rm III}}{\sqrt{2\pi\rho}}\phi + O(\phi^3), \quad \tau_{yz} = \frac{K_{\rm III}}{\sqrt{2\pi\rho}}\Big(1-\tfrac{1}{8}\phi^2\Big) + O(\phi^4). \tag{18}$$

The corresponding displacement is

$$w = \sqrt{\frac{\rho}{2\pi}}\cdot\frac{K_{\rm III}}{G}\phi + O(\phi^3). \tag{19}$$

Substituting (14) into (18) and (19), we get for the elastic stresses and displacement at the elastic-plastic boundary the expressions

$$\tau^e_{xz} = -\frac{1}{2}\frac{K_{\rm III}}{\sqrt{2\pi\rho_0}}\phi + O(\phi^3), \quad \tau^e_{yz} = \frac{K_{\rm III}}{\sqrt{2\pi\rho_0}}\left(1-\Big(\frac{1}{2}\frac{\rho_2}{\rho_0}+\frac{1}{8}\Big)\phi^2\right) + O(\phi^4), \tag{20}$$

$$w = \sqrt{\frac{\rho_0}{2\pi}}\cdot\frac{K_{\rm III}}{G}\phi + O(\phi^3). \tag{21}$$

The Hult–McClintock plastic stresses (4) have the expansion

$$\tau_{xz} = -k\theta + O(\theta^3), \quad \tau_{yz} = k - \tfrac{1}{2}k\theta^2 + O(\theta^4). \tag{22}$$

The expansion of the Hult–McClintock displacement in the plastic zone is

$$w = C\theta + O(\theta^3), \tag{23}$$

where C is a constant, and the expansion of the strain is

$$\gamma_{xz} = -\frac{C}{r}\theta + O(\theta^3), \quad \gamma_{yz} = \frac{C}{r} + O(\theta^2), \tag{24}$$

Combining (14) with (22), (23) and (16), we obtain the plastic stresses and displacement of the real crack at the elastic-plastic boundary:

$$\tau^p_{xz} = -k\frac{\rho_0}{x_e+\rho_0}\phi + O(\phi^3), \quad \tau^p_{yz} = k - \frac{k}{2}\Big(\frac{\rho_0}{x_e+\rho_0}\Big)^2\phi^2 + O(\phi^4), \tag{25}$$

$$w^p = C\frac{\rho_0}{x_e+\rho_0}\phi + O(\phi^3). \tag{26}$$

Now the plastic stresses (25) are made to match the crack tip K-dominant elastic stresses (20) at the elastic-plastic boundary (14), and likewise for the displacements (26) and (21). In the normal local coordinate frame $(\boldsymbol{n}, \boldsymbol{s})$ along the elastic-plastic boundary (see Figure 1), let σ_{nz} and σ_{sz} denote the stress components, so

$$\sigma_{nz} = \tau_{xz}n_x + \tau_{yz}n_y, \quad \sigma_{sz} = \tau_{xz}n_y - \tau_{yz}n_x. \tag{27}$$

Then the matching conditions for the stresses are

$$\sigma^e_{nz} = \sigma^p_{nz}, \quad \sigma^e_{sz} = \sigma^p_{sz} \qquad \text{along the elastic-plastic boundary.} \tag{28}$$

where superscript e and p represent the elastic and plastic sides of the boundary. The right-hand sides of $(28)_{1,2}$ can be obtained by substituting (25) and (15) into (27), and the left-hand sides by substituting (20) and (15) into (27). In this way we obtain three matching equations:

$$\frac{K_{\mathrm{III}}}{\sqrt{2\pi\rho_0}} = k, \quad \frac{1}{2}\frac{K_{\mathrm{III}}}{\sqrt{2\pi\rho_0}} = k\frac{\rho_0}{x_e+\rho_0}, \quad \frac{K_{\mathrm{III}}}{\sqrt{2\pi\rho_0}}\Big(\frac{1}{2}\frac{\rho_2}{\rho_0}+\frac{1}{8}\Big) = \frac{1}{2}k\Big(\frac{\rho_0}{x_e+\rho_0}\Big)^2. \tag{29}$$

Solving the system (29) yields

$$\rho_0 = \frac{K_{\mathrm{III}}^2}{2\pi k^2}, \quad \rho_2 = 0, \quad x_e = \frac{K_{\mathrm{III}}^2}{2\pi k^2}. \tag{30}$$

Thus $\rho_0 = x_e$, expressing that the imaginary crack tip moves to the center of the plastic zone along the crack line. The length of the plastic zone is $x_p = x_e + \rho_0 = K_{\mathrm{III}}^2/(\pi k^2)$, in agreement with the case $\theta = 0$ of $(5)_4$. The result $\rho_2 = 0$ in $(30)_2$ agrees with Hult–McClintock's elastic-plastic solution, in which the elastic-plastic boundary is a circle. Similarly, the constant C in (23) and (24) can be obtained by comparing (21) with (26) and using (29):

$$C = \frac{k}{G}(\rho_0 + x_e), \tag{31}$$

leading to the following expression for the strain (24) in the plastic zone:

$$\gamma_{xz} = -\frac{k}{G}\frac{\rho_0+x_e}{r}\theta + O(\theta^3), \quad \gamma_{yz} = \frac{k}{G}\frac{\rho_0+x_e}{r} + O(\theta^2); \tag{32}$$

this again agrees with the Hult–McClintock solution in $(5)_{2,3}$.

It follows that, to second order in θ, the Hult–McClintock elastic-plastic solution is a necessary consequence of the underlying assumptions.

Critique of the assumptions underlying the Hult–McClintock elastic-plastic solution. Nevertheless, we must inquire whether the assumptions are reasonable. If the plastic zone is small enough, the first assumption — that the dominant elastic field matches the plastic field at the elastic-plastic boundary — is acceptable. However, the second assumption, concerning the "imaginary crack tip", is questionable on several grounds. First, it has no clear physical meaning. Second, it is arbitrary; it is introduced essentially in order to gain one free parameter, the distance x_e. Finally, according to the Hult–McClintock elastic-plastic solution, the strain in the plastic zone, given by (32) or (24), has a singularity. Such a result is incorrect even under small-scale yielding, as will be explained later.

In the alternative formulation we started to develop in Section 2, there is no singularity in the corresponding expression for the strain, (10), unless the parameter L is taken equal to 0 (which corresponds to the Hult–McClintock situation).

4. Abandoning the second assumption

We saw in Section 2 that the three matching equations in (29) involve three constants, ρ_0, ρ_2 and x_e. If we let $x_e = 0$, thus giving up the second assumption of small scale yielding (under which condition the coordinates (ρ, ϕ) coincide with (r, θ)), a conflict inevitably occurs in that the three independent matching equations (29) involve only two unknowns ρ_0 and ρ_2. But if the general solution (9) is used in the matching, the constant L can replace the artificial x_e in providing the additional degree of freedom necessary for a natural match at the actual elastic-plastic boundary. Thus, the second assumption of small-scale yielding becomes unnecessary, and a more natural matching solution can be obtained even under small-scale yielding.

The crack tip K-dominant elastic singular field for the actual crack (see Figure 2) is

$$\tau_{xz} = -\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}, \quad \tau_{yz} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}, \tag{33}$$

(compare (17)). The power series expansion of (33) to second order is

$$\tau_{xz} = -\frac{1}{2}\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\theta + O(\theta^3), \quad \tau_{yz} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}\Big(1 - \frac{1}{8}\theta^2\Big) + O(\theta^4) \tag{34}$$

(compare with (18), which expresses the same relationship but for the imaginary crack tip). The corresponding displacement is

$$w = \sqrt{\frac{r}{2\pi}} \cdot \frac{K_{\mathrm{III}}}{G}\theta + O(\theta^3). \tag{35}$$

The expression (9) of the plastic field is now required to match that of the crack-tip elastic dominant field, (34). In the same vein as in Section 3, we can do this by taking the expressions for σ_{nz}^p and σ_{sz}^p obtained by combining (9), (12), (13) and (27), and equating it, coefficient-wise, to the expressions for σ_{nz}^e and σ_{sz}^e obtained by combining (34), (12), (13) and (27) (see (28)). Solving the resulting equations yields

$$r_0 = \frac{1}{2\pi}\Big(\frac{K_{\mathrm{III}}}{k}\Big)^2, \quad r_2 = 0, \quad L = \frac{1}{2\pi}\Big(\frac{K_{\mathrm{III}}}{k}\Big)^2. \tag{36}$$

According to $(36)_1$ and Figure 2, the length of the plastic zone along the crack line is r_0, or half the length of the plastic zone

$$x_p = x_e + \rho_0 = 2\rho_0 = \frac{K_{\mathrm{III}}^2}{\pi k^2}$$

obtained in the Hult–McClintock solution.

From the continuity condition $w^e = w^p$ for the displacement at the elastic-plastic boundary, the constant C in (9) can be obtained from (35), (36), and (12) as

$$C = \frac{k}{G}(r_0 + L). \tag{37}$$

The strain near the crack line can be deduced as

$$\gamma_{xz} = -\frac{k}{G}\frac{(r_0+L)r}{(r+L)^2}\theta + O(\theta^3), \quad \gamma_{yz} = \frac{k}{G}\frac{r_0+L}{r+L} + O(\theta^2). \tag{38}$$

Thus no singularity is present in the plastic zone, in contrast with the Hult–McClintock strain, (32).

5. Abandoning the first assumption

In Section 4, we kept the first assumption usually made for small-scale yielding. Although the results are more natural than the Hult–McClintock solution (Section 3), they are still confined by small-scale yielding.

The crack line analysis method allows us to abandon also the first assumption. In this section we go over an example of how this can be done in special cases. The general idea is this: the general solution

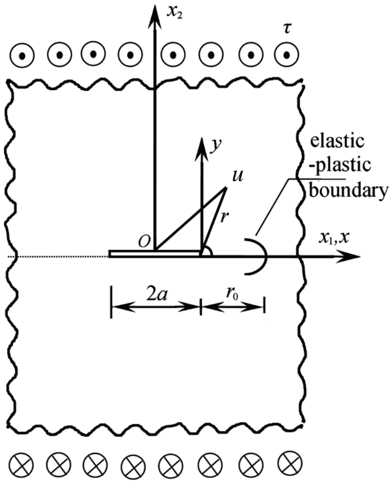

Figure 4. An antiplane crack in an infinite plate.

(9)–(10) for the plastic stress field near the crack line is correct to second order in the plastic zone, whether it be small or large. Hence, if a sufficiently accurate elastic field is known outside the plastic zone, the small-scale yielding assumptions can be relaxed by matching the elastic field with the general plastic field. The key, therefore, is to obtain the precise elastic field, and this can be done with sufficient accuracy near the crack line for some problems.

Our example involves an antiplane crack in an infinite body; see Figure 4. As discussed in [Yi 1994], the exact elastic stresses satisfying the far field boundary condition and the boundary condition that the crack surface is traction-free can be shown to be (see [Paris and Sih 1965; Gdoutos 2005, pp. 25–27])

$$\tau_{xz} = \operatorname{Im} Z_{\mathrm{III}}(u), \qquad \tau_{yz} = \operatorname{Re} Z_{\mathrm{III}}(u), \tag{39}$$

where $u = x + iy$ and $Z_{\mathrm{III}}(u) = \tau u/\sqrt{u^2 - a^2}$ is the Westergaard complex stress function. The corresponding displacement is $w = \operatorname{Im} \tilde{Z}_{\mathrm{III}}(u)/G$, where $\tilde{Z}_{\mathrm{III}}(u) = \int Z_{\mathrm{III}}(u)\,du$, the integral being over the contour $u = a + re^{i\theta}$. In the polar coordinate system centered at the crack tip, when $r \to 0$, the elastic K-dominant field can be obtained from (39) as in (33). But in the following analysis we will focus not on the elastic dominant term near the crack tip where $r \to 0$, but on the terms that are sufficiently precise near the crack line region when $\theta \to 0$.

Equation (39) is a classical analytical solution satisfying the basic equations and boundary conditions for the problem, which contains nonsingular terms besides the K-dominant field. Expanding (39) as a power series in θ, we get for the elastic stresses near the crack line

$$\tau_{xz} = -\frac{\tau}{\sqrt{r(2a+r)}}\,\frac{a^2}{2a+r}\theta + O(\theta^3), \qquad \tau_{yz} = \frac{\tau}{\sqrt{r(2a+r)}}\left((a+r) - \frac{2a^2r + a^3}{2(2a+r)^2}\theta^2\right) + O(\theta^4). \tag{40}$$

As $r \to 0$, this degenerates into (34); but note that (40) is sufficiently precise near the crack line, while (34) is valid only within a tiny area around the crack tip near the crack line.

Matching the elastic stress (40) and its corresponding displacement with the plastic stress and displacement, given in (9), one can obtain an accurate matching solution without the small-scale yielding assumptions. For details, see [Yi 1994].

Comparison between the general matching results and Hult and McClintock's results. In the example just given, elastic stresses precise enough near the crack line are obtained. By matching the general solution of the plastic stress (9) (and the corresponding plastic displacement) with the precise elastic stress (40) (and the corresponding elastic displacement) at the elastic-plastic boundary (12), the small-scale yielding assumptions can be completely abandoned and the matching results are correct, whether the plastic zone is small or large.

The matching results of the example for the infinite cracked plate are

$$r_0 = a\left(\sqrt{\frac{k^2}{k^2-\tau^2}} - 1\right), \quad \frac{r_2}{r_0} = \frac{1}{2}\sqrt{\frac{k^2-\tau^2}{k^2}} - \frac{k}{k+\sqrt{k^2-\tau^2}}, \quad L = a\left(1+\frac{2\tau^2-k^2}{k^2-\tau^2}\sqrt{\frac{k^2}{k^2-\tau^2}}\right), \tag{41}$$

$$C = \frac{k}{G}(r_0+L), \qquad \gamma_{xz} = -\frac{k}{G}\frac{(r_0+L)r}{(r+L)^2}\theta + O(\theta^3), \qquad \gamma_{yz} = \frac{k}{G}\frac{r_0+L}{r+L} + O(\theta^2). \tag{42}$$

Expanding the first of these equations in a power series of τ/k, when $\tau/k \ll 1$, we have

$$r_0 = a\left(\sqrt{\frac{1}{1-\tau^2/k^2}} - 1\right) = \frac{a}{2}\left(\frac{\tau}{k}\right)^2 + O\left(\frac{\tau}{k}\right)^4 = \frac{1}{2\pi}\left(\frac{\tau^2 a\pi}{k^2}\right) = \frac{1}{2\pi}\left(\frac{K_{\mathrm{III}}}{k}\right)^2. \tag{43}$$

Similarly, when $\tau/k \ll 1$, we have to second order

$$\frac{r_2}{r_0} = 0, \quad L = \frac{1}{2\pi}\left(\frac{K_{\mathrm{III}}}{k}\right)^2. \tag{44}$$

Equations (43) and (44) say the same as (36), where $K_{\mathrm{III}} = \tau(\pi a)^{1/2}$.

It is obvious that when the first assumption of small-scale yielding is introduced, the matching results (41) reduce to (43) and (44).

Figure 5 compares the plastic zone lengths for three solutions: the present solution (41), obtained after abandoning the two small-scale yielding assumptions; the solution (43) or $(36)_1$, obtained by maintaining first assumption of small-scale yielding reserved; and Hult and McClintock's solution $(5)_4$, which relies on both the small-scale yielding assumptions.

We see that the result (41) is not confined by the yielding scale: when $\tau \to k$ the length of the plastic zone in (41) approaches ∞, which is reasonable for a plate with infinite width. By contrast, (43) and

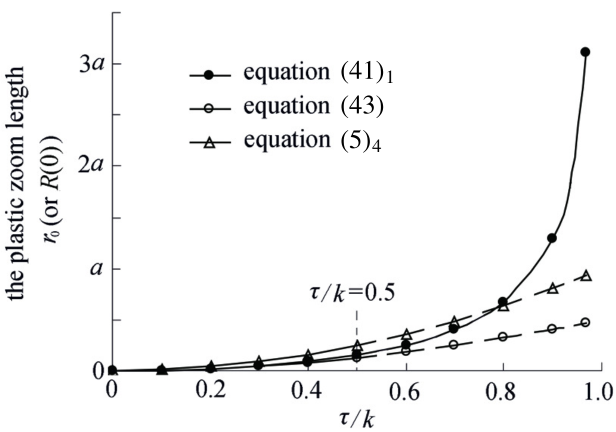

Figure 5. Comparison of the plastic zone lengths.

$(36)_1$ behave correctly only when τ/k is relatively small ($\tau/k \leq 0.5$), while Hult and McClintock's result $(5)_4$, also meant for τ/k small, lies some distance from either of the above.

However, when focusing on the plastic strain, neither our solution $(42)_{2,3}$ nor the small-scale yielding solution (38) have any singularities. In contrast, Hult and McClintock's solution (32) shows a physically unreasonable singularity $1/r$.

6. Conclusions

Three elastic-plastic matching solutions are given in the present paper.

The first matching solution is exhibited by the crack line analysis method to demonstrate the crack-tip elastic-plastic solution given by Hult and McClintock, where the particular plastic stresses (4) and the corresponding plastic strains suggested by Hult and McClintock are expanded in power series forms, (22) and (24), to match with the crack tip K-dominant elastic singular fields. The two small-scale yielding assumptions have to be used during deduction and the resultant plastic strains (32) have singularities. Although the matching solution is obtained around the crack line, it is in fact the same crack tip asymptotic solution as that of Hult and McClintock because the crack tip K-dominant elastic singular fields are introduced in matching.

The second matching solution (Section 4) takes the general power series form (but not the above particular form) plastic stresses (9) and corresponding plastic strains (10) near the crack line, to match with the crack tip K-dominant elastic singular fields. A more rational elastic-plastic solution is obtained with only the first small-scale yielding assumption adopted, in which the plastic strains do not have singularities. Since the crack tip K-dominant elastic singular fields are still applied in matching, the resultant solution can be considered as a new crack tip asymptotic solution, distinct from Hult and McClintock's. This new solution has no singularity in the plastic strain, unlike Hult and McClintock's.

The third matching solution (Section 5) shows how to obtain the precise elastic-plastic solution near the crack line with the usual small-scale yielding assumptions completely removed; the general power series form plastic fields are used to match the precise elastic fields near the crack line. The matching results in the case of an infinite plate will degenerate to those of the second matching solution, the validity of which is strictly justified by the matching conditions and the boundary conditions of the real problem. The resultant plastic strains also have no singularities.

The following observations can be made:

- Hult and McClintock's solution is inappropriate. Firstly, the plastic stress field (4) suggested by Hult and McClintock is just one particular solution of countless solutions to the system of partial differential equations, (1) and (2), instead of a general solution. Near the crack line, the plastic stress field of the Hult–McClintock particular solution is shown in (22) while that of the author's general solution is shown in (9). A comparison between (9) and (22) indicates that the general solution (9) has one constant L but the particular solution (22) does not. Secondly, owing to the adoption of particular plastic solution (4), the assumption that the crack-tip elastic field moves a distance x_e along the crack line has to be expediently introduced in Hult and McClintock's matching solution. Thus the elastic field used to match with the real plastic field is just an imaginarily offset elastic field, not the real one. Thirdly, the plastic strain, (24) or (32), corresponding to Hult and McClintock's matching solution obtained from the particular plastic stress (22) has a singularity, which is incorrect even under small scale yielding.

• The second assumption of the usual small-scale yielding is removable. The usual small-scale yielding involves two assumptions. If the plastic zone is small enough, the first assumption can be adopted, meaning, to match the K-dominant elastic field with the plastic field is acceptable. However, to adopt the second assumption, that is, to assume that the crack tip of the elastic field moves a distance along the crack line as Hult and McClintock have done, is inappropriate and should be rationally abandoned.

• The crack line analysis method offers a precise Taylor series form general solution (that is, a general power series form solution) of the plastic field (9), and provides the possibility of completely abandoning the two small-scale yielding assumptions. Then the elastic-plastic matching solution is decided by how the two small-scale yielding assumptions are treated. If both assumptions are still embraced, as Hult and McClintock have done, the same matching solution as Hult and McClintock's will be reached, as shown in Section 3; if only the first assumption is adopted a mathematically approximate matching solution will be gained which is more appropriate than Hult and McClintock's, as shown in Section 4; finally, if both assumptions are given up, a rigorously precise matching solution will be attained, as shown in Section 5, which goes far beyond Hult and McClintock's.

• There exists an obvious difference between the crack line analysis method and the crack tip asymptotic analysis method. The crack tip area is really a crucial position in the analysis of a crack problem, but the same is true of the crack line area: the stresses, strains, and plastic length near the crack line are all crucial parameters in analysis. When the dominant order terms are used to characterize the stress or strain field, the crack tip asymptotic analysis method only gives solutions appropriate for all points infinitely approaching the crack tip, but which are inappropriate for those points beyond a certain distance from the crack tip. In the polar coordinate system with the crack tip as its origin, the crack tip analysis method is usually restricted by the condition of $r \to 0$ but not by any range of θ $(-\pi \le \theta \le \pi)$, so the solution is restricted by the small-scale yielding conditions and remains valid only when the plastic zone is small enough. While the crack line analysis method only gives solutions appropriate for all points infinitely approaching the crack line, restricted by the condition of $\theta \to 0$ but not by any range of r $(0 \le r \le \infty)$, then the results are sufficiently precise within an area close enough to the crack line and not restricted by the small-scale yielding conditions. The crack line analysis method has the following merits: it can give the precise power-series-form plastic field solution near the crack line, the precise power-series-form plastic field can match with the precise elastic field near the crack line to give sufficiently precise results with the small scale yielding conditions completely abandoned, and it bears a physical clarity of related concepts and mathematical simplicity in deduction.

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