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ON THE MYSTERY OF CALDERÓN'S FORMULA FOR THE GEOMETRY OF AN INCLUSION IN ELASTIC MATERIALS

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I write this paper to pay homage to Marie-Louise Steele and in honor of Charles R. Steele. I have had the pleasure and the honor to serve their journals IJSS and JoMMS, with George Herrmann. They have made Solids & Structures and now Material Sciences a subject of nobility to all of us.

We consider the nonlinear inverse problem of determining an inclusion in an elastic body, in antiplane shear loading. The perturbation of the shear modulus due to the inclusion was determined by Calderón (1980) in the case of a small amplitude of perturbation. For the general nonlinear case, the problem is decomposed into two linear problems: a source inverse problem, which determines the geometry of the inclusion, and a Volterra integral equation of the first kind for determining the amplitude. In this paper, we deal only with the determination of the inclusion geometry in the two-dimensional problem. We derive a simple formula for determining the inclusion geometry. This formula enables us to investigate the mystery of Calderón's solution for the linearized perturbation h^0, raised by Isaacson and Isaacson (1986), in the case of axisymmetry. By using a series method for numerical analysis, they found that the supports of the perturbation, in the linearized theory and the nonlinear theory in the axisymmetric case, are practically the same. We elucidate the mystery by discovering that both theories give exactly the same support of the perturbation, $\mathrm{supp}(h^0) \equiv \mathrm{supp}(h)$, for the general case of geometry and loadings. Then, we discuss an application of the geometry method to locate an inclusion and solve the source inverse problem, which gives an indication of the amplitude of the perturbation.

1. Introduction

Inverse problems for defect and crack identification in elasticity have many applications in medicine and the mechanics of materials. In medicine, tomography techniques using mechanical loads such as an antiplane shear loading on animal tissue, are worked out in [Catheline et al. 2004]. Cancer tumors are expected to have higher density and higher stiffness or shear modulus than sound tissues, so the difference in material properties between sound and malicious tissues is detected by mechanical loads and responses. Auscultation by a doctor is nothing but a rudimentary method of endoscopy relying on the same principle.

In the mechanics of materials, damage is known to result from microcracks which lower locally the elastic constants. New topics in mechanical tomography have recently been the subjects of several works. For example, solutions to crack inverse problems in two and three dimensions are known in elasticity [Andrieux and Ben Abda 1992; Andrieux et al. 1999; Bui et al. 2005] and in viscoelasticity [Bui et al. 2009], in statics as well as in dynamics, under the assumption of small frequencies. In elastodynamics, solutions of inverse crack problems are obtained in [Bui et al. 2005] where the solution to an earthquake

Keywords: nonlinear inverse problem, inclusion geometry, antiplane problem.

inverse problem to recover the faulting process was proposed. A review of several exact solutions to inverse problems is found in [Bui 2006].

The case of distributed defects was first studied in [Calderón 1980] for the scalar elastic equation, then in [Bui and Chaillat 2009] for the case of dynamic viscoelasticity in the low frequency domain. Most of the works mentioned used reciprocity functional techniques for solving the following inverse problem. Find functions h and u satisfying the field equation and two superabundant boundary data f and g:

$$\operatorname{div}\big[\big(1+h(\boldsymbol{x})\big)\operatorname{grad} u\big]=0 \quad \text{in } \Omega, \tag{1-1}$$

$$u=f \quad \text{and} \quad \frac{\partial u}{\partial n}=g \quad \text{in } \partial\Omega. \tag{1-2}$$

The pair of data (f, g) is compatible with the perturbation h. We consider normalized constants so that the shear modulus of an homogeneous body is 1 and the inhomogeneity is characterised by the material constant $1+h(\boldsymbol{x})$. We assume nonzero measure of support of h which excludes the hair line inclusion case. For small amplitude of the perturbation h, Calderón [1980] derived explicitly the solution, denoted hereafter by $h^0(\boldsymbol{x})$.

In this paper, we shall reconsider the nonlinear Calderón inverse problem for identifying distributed defects in elasticity of arbitrary geometry. First we give a formula for the geometry inclusion. This nonlinear problem has been solved numerically in [Isaacson and Isaacson 1989] for data corresponding to a circular geometry of the inclusion and body, under axisymmetric radial loads. They used a series method to solve the equation in the axisymmetric case, with data (f, g) corresponding to $h(r)$ not necessary of small amplitude. The data for the numerical test is obtaining first by solving a Dirichlet boundary value with condition $u=f$ on the boundary and with a given h on the inclusion. Then the boundary value of the gradient $g=\partial u/\partial n$ is calculated. They have shown by numerical experiments (with a series method) that the support of the inclusion function $h^0(\boldsymbol{x})$ from Calderón's formula is almost indiscernible from the circle introduced for (f, g). This is the mystery of Calderón's solution, which seems to work for the nonlinear case as well. It is incredible that a formula derived for a small amplitude perturbation still works for the general case. We shall clarify the mystery by comparing the geometry inclusions in both theories. Then we discuss an application to the source inverse problem.

2. The nonlinear Calderón equation for h and the linearized solution

We consider the identification of internal defects in the antiplane problem of elasticity, or stationary heat/mass transfer phenomena, or electricity conduction. In the antiplane problem the shear modulus is of the form $\mu(\boldsymbol{x})=1+h(\boldsymbol{x})$. The shear modulus in the absence of defect ($h=0$) is normalized to unity.

We assume that the defect is characterized by the function $h(\boldsymbol{x})$ with compact support $C\subset\Omega$ and $h=0$ on the boundary $\partial\Omega$, see Figure 1. Defects may not be necessarily unique.

Consider an adjoint problem for the sound solid $h=0$ to determine the function φ:

$$\operatorname{div}\operatorname{grad}\varphi=0 \quad \text{in } \Omega, \tag{2-1a}$$

$$\frac{\partial\varphi}{\partial n}=a(\boldsymbol{x}) \quad \text{on } \partial\Omega, \tag{2-1b}$$

with the equilibrium condition $\int_{\partial\Omega} a\,dS=0$, and let $b(\boldsymbol{x})$ be the boundary value of φ.

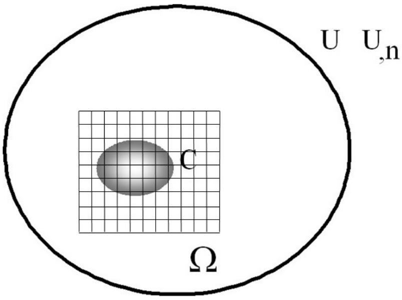

Figure 1. Identification of an inclusion C represented by a perturbation of the material constant. A mesh for numerical analysis with data U and $\partial U/\partial n =: U_{,n}$.

Combining (1-1) and (2-1a), after multiplying (1-1) by φ and (2-1a) by u, integrating by parts and taking account of $h = 0$ on $\partial\Omega$, we obtain the Calderón equation for h for any φ

$$\int_C h(\boldsymbol{x}) \operatorname{grad} u(\boldsymbol{x}; h) \cdot \operatorname{grad} \varphi(\boldsymbol{x}) \, d^2x = \int_{\partial\Omega} (gb - fa) \, dS =: R(f, g; \varphi), \quad \forall\varphi. \tag{2-2}$$

The right-hand side of (2-2) is known from the boundary data (f, g) of the current field u and the data (a, b) of the adjoint function φ. In the following, it will be denoted by $R(f, g; \varphi)$ or simply $R(\varphi)$. We remark that (2-2) is nonlinear in h, firstly because function $u(\boldsymbol{x}; h)$ is yet unknown and secondly because the integration domain C is unknown. Equation (2-2) constitutes the nonlinear variational equation for h. The arbitrariness of φ is the key point to solve the nonlinear problem, as shown in several examples given in [Bui 2006]. An important case is when $u(\boldsymbol{x}; h)$ as well as the geometry of C are known, for which (2-2) becomes linear.

Following [Calderón 1980] we can linearize (2-2) by replacing the unknown function u by the function u^0 which satisfies the harmonic equation

$$\operatorname{div} \operatorname{grad} u^0 = 0 \quad \text{in } \Omega. \tag{2-3}$$

We obtain

$$\int_C h(\boldsymbol{x}) \operatorname{grad} u^0(\boldsymbol{x}) \cdot \operatorname{grad} \varphi(\boldsymbol{x}) \, d^2x = R(f, g; \varphi). \tag{2-4}$$

Whatever the extended function $\operatorname{grad} u$ outside the domain may be, (2-2) can also be written with the integral over the whole plane because the extended function $\tilde{h}$ vanishes outside the domain:

$$\int_{\mathbb{R}^2} \tilde{h}(\boldsymbol{x}) \operatorname{grad} u(\boldsymbol{x}; h) \cdot \operatorname{grad} \varphi(\boldsymbol{x}) \, d^2x = R(f, g; \varphi), \quad \forall\varphi. \tag{2-5}$$

In this case boundary data a, b of the harmonic function φ may not be specified. We consider the adjoint harmonic function, which depends on parameters $\boldsymbol{\xi}$ of the ξ-plane:

$$\varphi(\boldsymbol{x}, \xi) = \exp\bigl(-i(x_1\xi_1 + x_2\xi_2)\bigr) \exp(-x_1\xi_2 + x_2\xi_1). \tag{2-6}$$

Calderón assumed a small amplitude of h and expanded the unknown function in the form $u(\boldsymbol{x}; h) = u(\boldsymbol{x}; 0) + O(h)$. By taking $u(\boldsymbol{x}; 0) = \varphi(\boldsymbol{x}, \xi)$, given in (2-6), he obtained explicitly the solution with

$R(\boldsymbol{\xi}) =: R(f, g; \varphi(\boldsymbol{x}; \boldsymbol{\xi}))$:

$$\tilde{h}^0(\boldsymbol{x}) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{2}{\|\boldsymbol{\xi}\|^2} R(\boldsymbol{\xi}) \exp(i\boldsymbol{x}\cdot\boldsymbol{\xi})\, d^2\xi. \tag{2-7}$$

Strictly speaking, (2-7) is derived for (f, g) compatible with small perturbation. What happens when one applies this formula to the general case of arbitrary h? It seems, from [Isaacson and Isaacson 1989] with their example of a circular inclusion in a circular domain, that the support of the solution given by (2-7) is indiscernible from the circle introduced for obtaining data f, g. To clarify this mystery, we need to derive the solution for the inclusion geometry, in the general nonlinear theory.

3. Reduction to two successive simpler problems

Let us rewrite (1-1) in the following form:

$$\operatorname{div}\operatorname{grad} u + S(\boldsymbol{x}) = 0 \quad \text{in } \Omega, \tag{3-1}$$

where $S(\boldsymbol{x}) = \operatorname{div}(h(\boldsymbol{x}) \operatorname{grad} u(\boldsymbol{x}; h))$; see also [Bui and Chaillat 2009].

Determining S with boundary data $u = f$ and $\partial u/\partial n = g$ is a classical *source* inverse problem; see [Isakov 1990; Alves and Ha-Duong 1997]. Here, we consider the source inverse problem already solved, to obtain the source $S(\boldsymbol{x})$, its support C, and the true displacement field denoted by $U(\boldsymbol{x}) := u(\boldsymbol{x}; h)$, even if $h(\boldsymbol{x})$ is unknown. We obtain also the strain, but not the stress because h is yet unknown. It is interesting to remark that the solution of the source inverse problem (3-1) provides the geometry or support of the unknown $h(\boldsymbol{x})$. As a matter of fact, since the true displacement is known as U, we have

$$S(\boldsymbol{x}) = \operatorname{div}\big(h(\boldsymbol{x}) \operatorname{grad} U(\boldsymbol{x})\big). \tag{3-2}$$

The source term (3-2) is a linear combination of derivatives of h in the distributional sense. Its support is the inclusion itself, $\operatorname{supp}(S) \equiv \operatorname{supp}(h) = C$. Therefore, we have a first method to derive only the geometry by solving a source inverse problem. We will discuss next a simpler method. It is based on the nonlinear Calderón equation rewritten as

$$\int_C h(\boldsymbol{x}) \operatorname{grad} U(\boldsymbol{x}) \cdot \operatorname{grad} \varphi(\boldsymbol{x}; \boldsymbol{\xi})\, d^2x = R\big(\varphi(\boldsymbol{\xi})\big). \tag{3-3}$$

With appropriate choice of φ, for example given by (2-6), we obtain a Volterra equation of the first kind for h, with kernel $K(\boldsymbol{x}, \boldsymbol{\xi}) := \operatorname{grad} U(\boldsymbol{x}) \cdot \operatorname{grad} \varphi(\boldsymbol{x}, \boldsymbol{\xi})$ which determines both geometry and amplitude of the perturbation. However, we do not solve the Volterra integral equation, but show how we derive the support of h directly.

4. The mystery of Calderón's solution

The adjoint function (2-6) as well as its gradient $\operatorname{grad} \varphi(\boldsymbol{x}; \boldsymbol{\xi})$ are analytic in the whole x-space and ξ-space (except at infinity), and thus can be expanded into infinite series of x_r and ξ_h. We expand $\operatorname{grad} \varphi(\boldsymbol{x}; \boldsymbol{\xi}) = \exp(-i(\boldsymbol{x}\cdot\boldsymbol{\xi}))\exp(-x_1\xi_2 + x_2\xi_1)\big((-i\xi_1 - \xi_2)\boldsymbol{e}^1 + (-i\xi_2 + \xi_1)\boldsymbol{e}^2\big)$ as

$$\operatorname{grad} \varphi(\boldsymbol{x}; \boldsymbol{\xi}) = \Bigg(\sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}^{hkrs}_{nmpq} (i\xi_h)^n (i\xi_k)^m x_r^p x_s^q\Bigg) \exp\big(-i(\boldsymbol{x}\cdot\boldsymbol{\xi})\big), \tag{4-1}$$

with constant complex vectors $\boldsymbol{a}_{nmpq}^{hkrs}$. We extend $h(\boldsymbol{x})$ to the infinite plane $\mathbb{R}^2$ by putting $h = 0$ outside C and denote its extension by $\tilde{h}$ and obtain the nonlinear Calderón equation in the form (the dot means a scalar product between vectors)

$$\int_{\mathbb{R}^2} \tilde{h}(\boldsymbol{x}) \operatorname{grad} U(\boldsymbol{x}) \cdot \left(\sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs} (i\xi_h)^n (i\xi_k)^m x_r^p x_s^q \right) \exp(-i\boldsymbol{x} \cdot \boldsymbol{\xi})\, d^2x = R\big(\varphi(\boldsymbol{\xi})\big), \tag{4-2}$$

or equivalently (using the properties of the Fourier transform)

$$\int_{\mathbb{R}^2} \sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs} \cdot \frac{\partial^n}{\partial x_h^n} \frac{\partial^m}{\partial x_k^m} \big(x_r^p x_s^q \tilde{h}(\boldsymbol{x}) \operatorname{grad} U(\boldsymbol{x}) \big) \exp(-i\boldsymbol{x} \cdot \boldsymbol{\xi})\, d^2x = R\big(\varphi(\boldsymbol{\xi})\big). \tag{4-3}$$

Let us define the function appearing in the above series by $F(\boldsymbol{x})$:

$$F(\boldsymbol{x}) = \sum_{h,k,r,s=1}^{2} \sum_{n,m,p,q=0}^{\infty} \boldsymbol{a}_{nmpq}^{hkrs} \cdot \frac{\partial^n}{\partial x_h^n} \frac{\partial^m}{\partial x_k^m} \big(x_r^p x_s^q \tilde{h}(\boldsymbol{x}) \operatorname{grad} U(\boldsymbol{x}) \big), \tag{4-4}$$

$$\int_{\mathbb{R}^2} F(\boldsymbol{x}) \exp(-i\boldsymbol{x} \cdot \boldsymbol{\xi})\, d^2x = R\big(\varphi(\boldsymbol{\xi})\big). \tag{4-5}$$

Because the function $F(\boldsymbol{x})$ is a linear combination of $\tilde{h}(\boldsymbol{x})$ and its partial derivatives, it has the same support $C = \operatorname{supp}(F) = \operatorname{supp}(h)$. Therefore we get the inclusion geometry C by the support of F which satisfies (4-5). It follows that function $F(\boldsymbol{x})$ is the inverse Fourier transform of $R(\varphi(\boldsymbol{\xi}))$. The inclusion geometry for the nonlinear theory is solved by

$$F(\boldsymbol{x}) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} R\big(\varphi(\boldsymbol{\xi})\big) \exp(+i\boldsymbol{x} \cdot \boldsymbol{\xi})\, d^2\xi. \tag{4-6}$$

This exact solution for the geometry of the inclusion enables us to investigate the mystery of Calderón's linearized solution which seems to work well for any perturbation. Let us make first a remark about function F given in terms of $R(\varphi(\boldsymbol{\xi}))$. From the expression of $R(\varphi(\boldsymbol{\xi}))$ we can easily check, for a bounded solid, that R is of the "exponential type", that is, the complex extension $R(z)$ of $R(\boldsymbol{\xi})$, obtained by the substitution $\boldsymbol{\xi} = (\xi_1, \xi_2) \to (\xi_1 + i\eta_1, \xi_2 + i\eta_2) =: z$ in $R(\boldsymbol{\xi})$, has the bound $\|R(z)\| \le C \exp(a\|z\|)$ with $C > 0$ and $a > 0$. According to the Paley–Wiener theorem [Schwartz 1966], the function $R(\boldsymbol{\xi})$ is the Fourier transform of a function with *compact* support. Therefore $F(\boldsymbol{x})$ given by (4-6) is a compactly supported function.

Consider now the linearized solution (2-7) for $\tilde{h}^0(\boldsymbol{x})$ which we recall below:

$$\tilde{h}^0(\boldsymbol{x}) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{2}{\|\boldsymbol{\xi}\|^2} R(\boldsymbol{\xi}) \exp(i\boldsymbol{x} \cdot \boldsymbol{\xi})\, d^2\xi. \tag{4-7}$$

Using the properties of the Fourier transform, we can rewrite (4-7) in the form

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \tilde{h}^0(\boldsymbol{x}) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} R(\boldsymbol{\xi}) \exp(i\boldsymbol{x} \cdot \boldsymbol{\xi})\, d^2\xi \equiv F(\boldsymbol{x}). \tag{4-8}$$

The left-hand side of this equation is a function having the same support as $\tilde{h}^0(\boldsymbol{x})$. Therefore we conclude that both functions $\tilde{h}^0(\boldsymbol{x})$ of the linearized theory and $h(\boldsymbol{x})$ of the nonlinear theory have the same geometry as $\mathrm{supp}(F)$. The mystery revealed by [Isaacson and Isaacson 1989] is finally elucidated. It is amazing that Calderón's linearized theory gives the exact solution for the geometry of the inclusion, whatever the perturbation may be. Of course, the amplitude of function $h(\boldsymbol{x})$ has to be determined by the Volterra integral equation, by a standard numerical approach. In the next section, we shall discuss a numerical procedure to determine not the function $h(\boldsymbol{x})$ itself, which can be found by standard techniques for the Volterra integral equation, but the source term $S(\boldsymbol{x})$, which is directly related to the amplitude of the perturbation.

5. A numerical approach to the source inverse problem by moving windows

It is known that the Volterra integral equation (3-3) is an ill-posed problem. Numerically it is difficult to recover exactly a function h which is strictly equal to zero outside the inclusion, which is assumed here to be unique. Therefore, it is of interest to consider a small *moving window* which is discretized in regular meshes and to solve numerically the source inverse problem for N point sources $S(\boldsymbol{x}) = \sum_{i=1}^{N} \lambda_i \delta(\boldsymbol{x} - \boldsymbol{a}_i)$, with source points at the centers of finite elements, and unknown amplitudes λ_i. With a chosen window, we enforce the condition $h = 0$ outside it. For a large window enclosing the solid, it is shown in [El Badia and Ha-Duong 2000] that the solution for a finite number N of sources approaching the source $S(\boldsymbol{x})$ exists and is unique. By choosing a particular window inside the solid domain, we search a solution which vanishes outside it. If the window does not contain entirely the source, we get a wrong solution and the corresponding image of the numerical solution is then blurred. Only in the case where the window contains the inclusion is a sharp image obtained. This procedure resembles echography imaging of a body. For example, by trial and error, one moves the echography device on the body of an expectant mother in order to search the right location to reveal a sharp image of her fetus. In our examples of the source problem, studying a tumor in live tissue or a damaged zone in materials, the moving window is a 4×5 mesh. For example, in Figure 2, the image on the left corresponds to the wrong solution, while the one on the right is correct.

Therefore it is of great interest to know the right location of the initial mesh. This is precisely provided by the support of the function F given in (4-6).

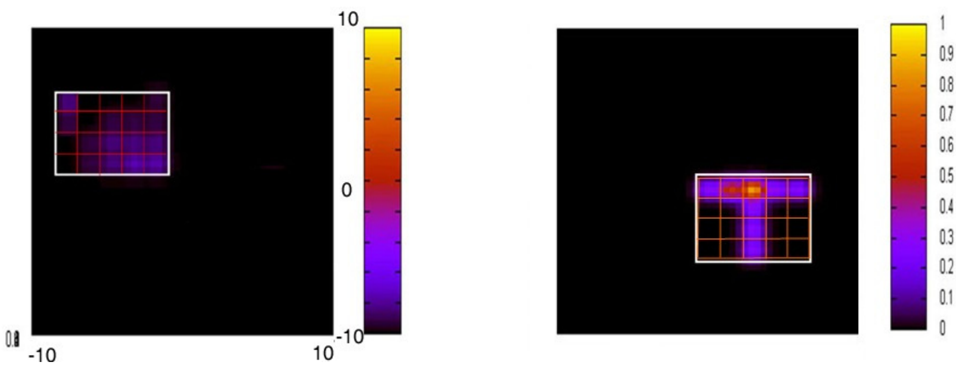

Figure 2. Imaging of a defect. Left: bad window, wrong solution. Right: right window, good solution.

6. Conclusions

In this paper, we consider the problem of finding the perturbation of a material constant in elastic solids which satisfies the nonlinear Calderón equation. The nonlinear problem reduces to two successive ones: a source inverse problem and a Volterra integral equation of the first kind. The first problem provides the inclusion geometry $\mathrm{supp}(h)$ explicitly. The second provides the magnitude of h. We make a comparison between the geometry of an inclusion in the small perturbation case and the geometry in the nonlinear case and find that both inclusion geometries are identical for arbitrary loading and geometry of the solid. Our result elucidates the mystery of the linearized Calderón solution for geometry which works well for the nonlinear case, as revealed numerically in the axisymmetric example given in [Isaacson and Isaacson 1989].

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