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PROPAGATION OF WAVES IN AN INCOMPRESSIBLE TRANSVERSELY ISOTROPIC ELASTIC SOLID WITH INITIAL STRESS: BIOT REVISITED

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In this paper, the general constitutive equation for a transversely isotropic hyperelastic solid in the presence of initial stress is derived, based on the theory of invariants. In the general finite deformation case for a compressible material this requires 18 invariants (17 for an incompressible material). The equations governing infinitesimal motions superimposed on a finite deformation are then used in conjunction with the constitutive law to examine the propagation of both homogeneous plane waves and, with the restriction to two dimensions, Rayleigh surface waves. For this purpose we consider incompressible materials and a restricted set of invariants that is sufficient to capture both the effects of initial stress and transverse isotropy. Moreover, the equations are specialized to the undeformed configuration in order to compare with the classical formulation of Biot. One feature of the general theory is that the speeds of homogeneous plane waves and surface waves depend nonlinearly on the initial stress, in contrast to the situation of the more specialized isotropic and orthotropic theories of Biot. The speeds of (homogeneous plane) shear waves and Rayleigh waves in an incompressible material are obtained and the significant differences from Biot's results for both isotropic and transversely isotropic materials are highlighted with calculations based on a specific form of strain-energy function.

1. Introduction

Initial stresses in solids have an important influence on the mechanical response of the material from an initially stressed configuration. Applications range from geophysics to the components of engineering structures and the behavior of soft biological tissues. The term *initial stresses* embraces situations in which the stress is accompanied by finite deformation from an unstressed configuration, in which case the term *pre-stresses* is commonly used, as, for example, in [Ieşan 1989], and situations in which the initial stress arises from some other process, such as manufacturing or growth, and is present in the absence of applied loads. In this latter case the initial stress is referred to as *residual stress* according to the definition of [Hoger 1985].

In the present paper we are concerned with the effect of initial stress on the propagation of small amplitude (linearized) elastic waves. A static theory of initial stress was developed long ago by Biot [1939]. He then extended it to wave propagation problems [1940]; this work is summarized in [Biot 1965]. In [Biot 1940] he states: "No assumption is made on how the initial state of stress is produced"— he requires only that it satisfy the equilibrium equations. Biot's theory has since formed the basis of many contributions to the literature, particularly in the geophysical context; see, for example, [Tolstoy

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1982] and the more recent [Dey and De 1999; Sharma and Garg 2006]. The latter was concerned with an initially stressed anisotropic material and further references can be found therein. In the context of modern continuum theory, however, Biot's formulation of the equations is not straightforward, and part of the purpose of the present work is to show how Biot's formulation fits into a more general and more transparent framework.

Surface waves in elastic solids were first studied by Lord Rayleigh [1885] for an isotropic elastic solid. The extension of surface wave analysis and other wave propagation problems to anisotropic elastic materials has been the subject of many studies; see, for example, [Musgrave 1959; Anderson 1961; Stoneley 1963; Chadwick and Smith 1977; Royer and Dieulesaint 1984; Barnett and Lothe 1985; Mozhaev 1995; Nair and Sotiropoulos 1997; 1999; Destrade 2001a; 2001b; Destrade et al. 2002; Ting 2002a; 2002c; 2002b; Destrade 2003; Ogden and Vinh 2004]. For problems involving surface waves in a finitely deformed pre-stressed elastic solid (strain-induced anisotropy) we refer to [Hayes and Rivlin 1961; Flavin 1963; Chadwick and Jarvis 1979; Dowaikh and Ogden 1990; 1991; Norris and Sinha 1995 (concerning a solid/fluid interface); Chadwick 1997; Prikazchikov and Rogerson 2004 (concerning pre-stressed transversely isotropic solids); Destrade et al. 2005; Edmondson and Fu 2009]; see also [Song and Fu 2007]. As representatives of other works concerning waves in initially stressed elastic solids we cite [Norris 1983] on plane waves, the review [Guz 2002] and the analysis [Akbarov and Guz 2004] of waves in circular cylinders. Here we shall study the effect of initial stress on the propagation of surface waves based on a general formulation of the constitutive law of an elastic material that would be transversely isotropic in the absence of initial stress.

In Section 2, the equations governing small amplitude waves in a deformed transversely isotropic elastic solid with initial stress are derived, both for compressible and incompressible materials, the transverse isotropy being associated with a preferred direction in the initially stressed reference configuration. The constitutive law of the material is based on a strain-energy function (defined per unit reference volume) which depends on the combined invariants of the right Cauchy–Green deformation tensor, the initial stress tensor and the preferred direction. For a compressible material there are 18 such independent invariants in the general three-dimensional case, a number which reduces to 17 for an incompressible material. Expressions for the Cauchy stress and nominal stress tensors and the elasticity tensor are given in general forms but, because the large number of invariants makes the theory unwieldy in general, their forms are made explicit only for a restricted number of invariants, and attention is then confined largely to incompressible materials with seven invariants.

In Section 3, the equations of motion are specialized in order to study the effect of initial stress on the wave speed of homogeneous plane waves. It is noted, in particular, that the wave speed depends in a nonlinear fashion on the initial stress. In order to make contact with Biot's theory and to see how it sits within the general framework considered here, we give, in Appendix B, a derivation of Biot's equations, their connection with the equations herein, and a formula for the relation between the components of the elasticity tensor used here (which depend nonlinearly on the initial stress) and the components of Biot's elasticity tensor. In Section 4, the theory is specialized to two-dimensional motions (for incompressible materials) and then applied, in Section 5, to the study of Rayleigh waves in a half-space subject to initial stress parallel to its boundary with the preferred direction of transverse isotropy either parallel or normal to the boundary. The secular equation is derived and then specialized to give corresponding results for Biot's isotropic and orthotropic theories. The final section, Section 6, provides numerical results that show the significant differences between the predictions of the general theory and of Biot's theory, and some concluding discussion is contained therein.

2. Equations of motion

We consider an elastic body whose initial geometry defines a *reference configuration*, which we denote by $\mathcal{B}_r$. In this configuration the body is in equilibrium and may in general be subject to a stress distribution, and we denote the Cauchy stress in $\mathcal{B}_r$ by $\boldsymbol{T}$. If there is a body force $\boldsymbol{b}_r$ per unit mass acting then the equation of equilibrium is

$$\operatorname{Div} \boldsymbol{T} + \rho_r \boldsymbol{b}_r = \boldsymbol{0}, \tag{1}$$

where ρ_r is the mass density of the material in $\mathcal{B}_r$ and Div is the divergence operator in $\mathcal{B}_r$, i.e., with respect to position vector $\boldsymbol{X}$ in $\mathcal{B}_r$. If the traction on the boundary $\partial\mathcal{B}_r$ of $\mathcal{B}_r$ vanishes pointwise then $\boldsymbol{T}$ is referred to as a *residual stress*, and it is necessarily non-uniform [Hoger 1985; Ogden 2003]. If the traction is not zero then we refer to $\boldsymbol{T}$ as an *initial stress* or *pre-stress*, and in general this may be accompanied by some prior deformation required to reach the configuration $\mathcal{B}_r$ from an unstressed state. Here we shall not be concerned with how the initial stress is produced.

A motion of the body from $\mathcal{B}_r$ may be described by a function $\boldsymbol{\chi}$ so that the current position $\boldsymbol{x}$ of the material point initially at $\boldsymbol{X}$ is given by $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, where t is time. The deformation gradient tensor, denoted $\boldsymbol{F}$, is given by $\boldsymbol{F} = \operatorname{Grad} \boldsymbol{\chi}(\boldsymbol{X}, t)$, where Grad is the gradient operator in $\mathcal{B}_r$. Let $\mathcal{B}$ be the configuration occupied by the body at time t.

The constitutive law of an elastic material may be described in terms of a strain-energy function, which is a function of $\boldsymbol{F}$ and defined per unit volume in $\mathcal{B}_r$. We denote this by $W(\boldsymbol{F})$, but note that in general W depends also on the initial stress $\boldsymbol{T}$ and on implicit material symmetries, which are suppressed for the present. Let $\boldsymbol{S}$ denote the nominal stress tensor in the configuration $\mathcal{B}$. Then, for a material not subject to any internal mechanical constraints $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}. \tag{2}$$

For an incompressible material the constraint

$$\det \boldsymbol{F} = 1 \tag{3}$$

is enforced and (2) is modified to

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1}, \tag{4}$$

where p is a Lagrange multiplier associated with the constraint.

The motion $\boldsymbol{\chi}$ is governed by the equation

$$\operatorname{Div} \boldsymbol{S} + \rho_r \boldsymbol{b} = \rho_r \boldsymbol{x}_{,tt}, \tag{5}$$

where $\boldsymbol{b}$ is the body force acting in the current configuration, which may in general be different from $\boldsymbol{b}_r$, and a subscript t following a comma signifies the material time derivative, i.e., the time derivative at fixed $\boldsymbol{X}$, so that $\boldsymbol{x}_{,t}$ is the particle velocity and $\boldsymbol{x}_{,tt}$ the acceleration.

Although this is not strictly necessary, we now assume, for simplicity, that the body force is uniform and independent of the deformation. Then, $\boldsymbol{b} = \boldsymbol{b}_r$ and we get from (1) and (5)

$$\mathrm{Div}(\boldsymbol{S} - \boldsymbol{T}) = \rho_r \boldsymbol{x}_{,tt}. \tag{6}$$

Now consider a finitely deformed equilibrium configuration $\mathcal{B}_0$ defined by $\boldsymbol{\chi}_0(\boldsymbol{X})$ and let $\boldsymbol{F}_0$ be the deformation gradient in this configuration and $\boldsymbol{S}_0$ the corresponding nominal stress. Then, $\mathrm{Div}(\boldsymbol{S}_0 - \boldsymbol{T}) = \boldsymbol{0}$. Next, we consider an incremental motion from this latter configuration with displacement $\dot{\boldsymbol{x}} = \boldsymbol{\chi}(\boldsymbol{X}, t) - \boldsymbol{\chi}_0(\boldsymbol{X})$, and corresponding increment $\dot{\boldsymbol{F}} = \mathrm{Grad}\, \dot{\boldsymbol{x}} = \boldsymbol{F} - \boldsymbol{F}_0$ in the deformation gradient. Let $\dot{\boldsymbol{S}} = \boldsymbol{S} - \boldsymbol{S}_0$ denote the increment in the nominal stress. Then,

$$\mathrm{Div}\, \dot{\boldsymbol{S}} = \rho_r \dot{\boldsymbol{x}}_{,tt}. \tag{7}$$

For a compressible material the linearized form of $\dot{\boldsymbol{S}}$ is

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{A}} \dot{\boldsymbol{F}}, \quad \dot{S}_{\alpha i} = \mathcal{A}_{\alpha i \beta j} \dot{F}_{j\beta}, \tag{8}$$

while for an incompressible material

$$\dot{\boldsymbol{S}} = \boldsymbol{\mathcal{A}} \dot{\boldsymbol{F}} + p \boldsymbol{F}^{-1} \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} - \dot{p} \boldsymbol{F}^{-1}, \tag{9}$$

where $\dot{p}$ is the increment in p and $\boldsymbol{\mathcal{A}}$ is the elasticity tensor, which, for either a compressible or an incompressible material, is defined by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \boldsymbol{F} \partial \boldsymbol{F}}, \quad \mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} = \mathcal{A}_{\beta j \alpha i}, \tag{10}$$

evaluated in the configuration $\mathcal{B}_0$. The convention that Greek indices refer to the configuration $\mathcal{B}_r$ and Roman indices to $\mathcal{B}_0$ is adopted here. The linearized incompressibility condition is

$$\mathrm{tr}(\dot{\boldsymbol{F}} \boldsymbol{F}_0^{-1}) = 0. \tag{11}$$

For details of the background on the theory of incremental deformations superimposed on a finite deformation we refer to [Ogden 1984; 2007], for example.

It is convenient to work with $\mathcal{B}_0$ as the reference configuration, which requires that all quantities are updated, i.e., pushed forward, from $\mathcal{B}_r$ to $\mathcal{B}_0$ and incremental quantities are treated as functions of $\boldsymbol{x}$ and t. In particular, we define the Eulerian form of the displacement vector by $\boldsymbol{u}(\boldsymbol{x}, t) = \dot{\boldsymbol{x}}(\boldsymbol{X}, t)$ via the connection $\boldsymbol{x} = \boldsymbol{\chi}_0(\boldsymbol{X})$. The updated forms of the incremental constitutive laws (8) and (9) are, respectively,

$$\dot{\boldsymbol{S}}_0 = \boldsymbol{\mathcal{A}}_0 \boldsymbol{L}, \quad \dot{S}_{0pi} = \mathcal{A}_{0piqj} L_{jq}, \tag{12}$$

and

$$\dot{\boldsymbol{S}}_0 = \boldsymbol{\mathcal{A}}_0 \boldsymbol{L} + p \boldsymbol{L} - \dot{p} \boldsymbol{I}, \tag{13}$$

where $\boldsymbol{L} = \mathrm{grad}\, \boldsymbol{u}$ is the displacement gradient, $\boldsymbol{I}$ is the identity tensor and a subscript 0 indicates a pushed forward quantity. In particular,

$$J_0 \mathcal{A}_{0piqj} = F_{0p\alpha} F_{0q\beta} \mathcal{A}_{\alpha i \beta j}, \tag{14}$$

where $J_0 = \det \boldsymbol{F}_0 = \rho_r/\rho_0$ and ρ_0 is the density in $\mathcal{B}_0$. For an incompressible material $J_0 = 1$ and $\rho_0 = \rho_r$. The (linearized) incremental form of the incompressibility constraint (11) is then expressed as

$$\operatorname{tr} \boldsymbol{L} \equiv \operatorname{div} \boldsymbol{u} = 0. \tag{15}$$

For a compressible material, the (Cartesian) component form of the equation of motion is

$$(\mathcal{A}_{0piqj} u_{j,q})_{,p} = \rho_0 u_{i,tt}, \tag{16}$$

and for an incompressible material

$$(\mathcal{A}_{0piqj} u_{j,q})_{,p} - \dot{p}_{,i} + p_{,j} u_{j,i} = \rho_r u_{i,tt}, \quad \text{with } u_{i,i} = 0. \tag{17}$$

Note that as well as possessing the major symmetry $\mathcal{A}_{0piqj} = \mathcal{A}_{0qjpi}$ induced by (10), $\boldsymbol{\mathcal{A}}$ has the property

$$\mathcal{A}_{0piqj} + \delta_{jp}\sigma_{0iq} = \mathcal{A}_{0ipqj} + \delta_{ij}\sigma_{0pq} \tag{18}$$

for a compressible material, and

$$\mathcal{A}_{0piqj} + \delta_{jp}(\sigma_{0iq} + p\delta_{iq}) = \mathcal{A}_{0ipqj} + \delta_{ij}(\sigma_{0pq} + p\delta_{pq}) \tag{19}$$

for an incompressible material, where σ_{0ij} are the components of the Cauchy stress tensor $\boldsymbol{\sigma}_0$ in $\mathcal{B}_0$. These are easily established by considering the symmetry of Cauchy stress expressed in the form $\boldsymbol{F}\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}$, taking the increment of this and then updating the reference configuration to $\mathcal{B}_0$ to obtain

$$\dot{\boldsymbol{S}}_0 + \boldsymbol{L}\boldsymbol{\sigma}_0 = \dot{\boldsymbol{S}}_0^{\mathrm{T}} + \boldsymbol{\sigma}_0\boldsymbol{L}^{\mathrm{T}}. \tag{20}$$

Suppose now that the material is homogeneous. This requires, in particular, that the initial stress $\boldsymbol{T}$ is uniform. We recall, however, that a residual stress cannot be uniform [Hoger 1985; Ogden 2003], so the following analysis does not apply if the initial stress is a residual stress. Suppose further that $\boldsymbol{\chi}_0(\boldsymbol{X})$ is a homogeneous deformation. Then, the configuration $\mathcal{B}_0$ is uniform and hence $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{A}}_0$ and p are constant, and the equations of motion (16) and (17) reduce to

$$\mathcal{A}_{0piqj} u_{j,pq} = \rho_0 u_{i,tt} \tag{21}$$

and

$$\mathcal{A}_{0piqj} u_{j,pq} - \dot{p}_{,i} = \rho_r u_{i,tt}, \quad u_{i,i} = 0, \tag{22}$$

respectively.

In general $\boldsymbol{\mathcal{A}}$, and hence $\boldsymbol{\mathcal{A}}_0$, depends on the deformation through $\boldsymbol{F}$, on the initial stress $\boldsymbol{T}$ and on any material symmetry present in the configuration $\mathcal{B}_r$. Here we consider a transversely isotropic material with preferred direction $\boldsymbol{M}$ in $\mathcal{B}_r$, where $\boldsymbol{M}$ is a unit vector. To make the dependence of $\boldsymbol{\mathcal{A}}$ (and $\boldsymbol{\mathcal{A}}_0$) on these quantities explicit we consider the scalar invariants of the tensors involved.

2.1. ***Invariant formulation.*** By objectivity, the dependence of the strain-energy function W on $\boldsymbol{F}$ is through the right Cauchy–Green tensor $\boldsymbol{C}$, which is defined by $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$, and we therefore consider W to depend on the invariants of the three tensors $\boldsymbol{C}$, $\boldsymbol{T}$ and $\boldsymbol{M} \otimes \boldsymbol{M}$ since the material properties are assumed to be independent of the sense of $\boldsymbol{M}$.

The invariants of $\boldsymbol{C}$ most commonly used are the *principal* invariants, defined by

$$I_1 = \operatorname{tr}\boldsymbol{C}, \quad I_2 = \tfrac{1}{2}[(\operatorname{tr}\boldsymbol{C})^2 - \operatorname{tr}(\boldsymbol{C}^2)], \quad I_3 = \det\boldsymbol{C}. \tag{23}$$

The (anisotropic) invariants associated with $\boldsymbol{M}$ and $\boldsymbol{C}$ are usually taken as

$$I_4 = \boldsymbol{M}\cdot(\boldsymbol{C}\boldsymbol{M}), \quad I_5 = \boldsymbol{M}\cdot(\boldsymbol{C}^2\boldsymbol{M}). \tag{24}$$

The notation $I_1, \ldots, I_5$ is fairly standard for these invariants; see, for example, [Merodio and Ogden 2002; 2003]. In the reference configuration $\mathcal{B}_r$ these reduce to $I_1 = I_2 = 3$, $I_3 = I_4 = I_5 = 1$. A set of independent invariants of $\boldsymbol{T}$ that do not involve $\boldsymbol{M}$ may be taken as

$$\operatorname{tr}\boldsymbol{T}, \quad \operatorname{tr}(\boldsymbol{T}^2), \quad \operatorname{tr}(\boldsymbol{T}^3), \quad \operatorname{tr}(\boldsymbol{T}\boldsymbol{C}), \quad \operatorname{tr}(\boldsymbol{T}\boldsymbol{C}^2), \quad \operatorname{tr}(\boldsymbol{T}^2\boldsymbol{C}), \quad \operatorname{tr}(\boldsymbol{T}^2\boldsymbol{C}^2), \tag{25}$$

invariants of $\boldsymbol{T}$ independent of $\boldsymbol{C}$ as

$$\boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{M}), \quad \boldsymbol{M}\cdot(\boldsymbol{T}^2\boldsymbol{M}), \tag{26}$$

and invariants depending on $\boldsymbol{C}, \boldsymbol{M}$ and $\boldsymbol{T}$ as

$$\boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{C}\boldsymbol{M}), \quad \boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{C}^2\boldsymbol{M}), \quad \boldsymbol{M}\cdot(\boldsymbol{T}^2\boldsymbol{C}\boldsymbol{M}), \quad \boldsymbol{M}\cdot(\boldsymbol{T}^2\boldsymbol{C}^2\boldsymbol{M}). \tag{27}$$

These are the only independent invariants, 18 in total. For an incompressible material we have $I_3 = 1$ and hence there are 17 independent invariants in this case. In the reference configuration $\mathcal{B}_r$ the fourth, fifth, sixth and seventh invariants in (25) and the invariants (27) reduce to the first two in (25) and the two in (26). For full discussion of the relevant background on invariants of tensors we refer to [Adkins 1960; Spencer 1971; Zheng 1994]. A set of invariants equivalent to the above has been used by [Hoger 1993a; 1996]. For related work concerned with the constitutive equations and material symmetry for a residually stressed elastic material we refer to [Coleman and Noll 1964; Hoger 1986; 1993b; Man and Lu 1987; Johnson and Hoger 1993; Man 1998; Saravanan 2008; Tanuma and Man 2008].

We have not for the moment defined particular notation for the invariants (25)–(27). In the general case suppose there are N invariants, which we denote by I_i, $i = 1, 2, \ldots, N$. Then, the expressions for the stress and elasticity tensors require the calculation of

$$\frac{\partial W}{\partial \boldsymbol{F}} = \sum_{i=1}^{N} W_i \frac{\partial I_i}{\partial \boldsymbol{F}} \tag{28}$$

and

$$\frac{\partial^2 W}{\partial \boldsymbol{F}\,\partial \boldsymbol{F}} = \sum_{i=1}^{N} W_i \frac{\partial^2 I_i}{\partial \boldsymbol{F}\,\partial \boldsymbol{F}} + \sum_{i=1}^{N}\sum_{j=1}^{N} W_{ij} \frac{\partial I_i}{\partial \boldsymbol{F}} \otimes \frac{\partial I_j}{\partial \boldsymbol{F}}, \tag{29}$$

where we have used the shorthand notations $W_i = \partial W/\partial I_i$, $W_{ij} = \partial^2 W/\partial I_i \partial I_j$, $i, j = 1, 2, \ldots, N$.

Such a large number of invariants is impractical for applications, so for simplicity we restrict attention to incompressible materials and to the following invariants, which capture the main features of the combined anisotropy and initial stress. In particular, we omit invariants that are nonlinear in $\boldsymbol{T}$. Thus, we consider

$$I_1 = \operatorname{tr} \boldsymbol{C}, \qquad I_4 = \boldsymbol{M}\cdot(\boldsymbol{C}\boldsymbol{M}), \qquad I_5 = \boldsymbol{M}\cdot(\boldsymbol{C}^2\boldsymbol{M}), \tag{30}$$

$$I_6 = \operatorname{tr}(\boldsymbol{T}\boldsymbol{C}), \qquad I_7 = \operatorname{tr}(\boldsymbol{T}\boldsymbol{C}^2), \qquad I_8 = \boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{C}\boldsymbol{M}), \qquad I_9 = \boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{C}^2\boldsymbol{M}), \tag{31}$$

which identify the invariants I_6, I_7, I_8, I_9, while the standard notation I_1, I_4, I_5 is retained and, since we are considering incompressible materials, $I_3 \equiv 1$. We may also include $\operatorname{tr} \boldsymbol{T}$ and $\boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{M})$ in the functional dependence of W since they do not depend on $\boldsymbol{C}$ and hence their derivatives with respect to $\boldsymbol{F}$ do not contribute to the stress or elasticity tensors. In the following we give explicit expressions for the stress tensors and the elasticity tensor based on this restricted set of invariants.

2.2. ***Stress tensors.*** The strain-energy function W is now taken to depend on the seven deformation-dependent invariants $I_1, I_4, \dots, I_9$ together (possibly) with $\operatorname{tr} \boldsymbol{T}$ and $\boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{M})$. For the considered incompressible material we have

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1} = \sum_{\substack{1\le i\le 9\\ i\ne 2,3}} W_i \frac{\partial I_i}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1}, \tag{32}$$

and the corresponding Cauchy stress is

$$\boldsymbol{\sigma} = \boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{F}} - p\boldsymbol{I} = \sum_{\substack{1\le i\le 9\\ i\ne 2,3}} W_i \boldsymbol{F}\frac{\partial I_i}{\partial \boldsymbol{F}} - p\boldsymbol{I}, \tag{33}$$

where $W_i = \partial W/\partial I_i$, $i = 1, 4, \dots, 9$. The required expressions for $\partial I_i/\partial \boldsymbol{F}$ are listed for convenience in Appendix A in component form. These enable the Cauchy stress to be expanded as

$$\begin{aligned}\boldsymbol{\sigma} = {} & -p\boldsymbol{I} + 2W_1\boldsymbol{B} + 2W_4\boldsymbol{m}\otimes\boldsymbol{m} + 2W_5(\boldsymbol{m}\otimes\boldsymbol{B}\boldsymbol{m} + \boldsymbol{B}\boldsymbol{m}\otimes\boldsymbol{m})\\ & + 2W_6\boldsymbol{\Sigma} + 2W_7(\boldsymbol{B}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{B}) + W_8(\boldsymbol{F}\boldsymbol{T}\boldsymbol{M}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{F}\boldsymbol{T}\boldsymbol{M})\\ & + W_9(\boldsymbol{F}\boldsymbol{T}\boldsymbol{M}\otimes\boldsymbol{B}\boldsymbol{m} + \boldsymbol{B}\boldsymbol{m}\otimes\boldsymbol{F}\boldsymbol{T}\boldsymbol{M} + \boldsymbol{F}\boldsymbol{C}\boldsymbol{T}\boldsymbol{M}\otimes\boldsymbol{m} + \boldsymbol{m}\otimes\boldsymbol{F}\boldsymbol{C}\boldsymbol{T}\boldsymbol{M}),\end{aligned} \tag{34}$$

where $\boldsymbol{m} = \boldsymbol{F}\boldsymbol{M}$, $\boldsymbol{\Sigma} = \boldsymbol{F}\boldsymbol{T}\boldsymbol{F}^{\mathrm{T}}$ and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the left Cauchy–Green deformation tensor.

In the reference configuration $\mathcal{B}_r$ the Cauchy stress must be equal to the initial stress. Thus, when (34) is evaluated in $\mathcal{B}_r$, we obtain

$$\begin{aligned}\boldsymbol{T} = (2W_1 - p)\boldsymbol{I} + 2(W_4 + 2W_5)\boldsymbol{M}\otimes\boldsymbol{M} + 2(W_6 + 2W_7)\boldsymbol{T}\\ + (W_8 + 2W_9)(\boldsymbol{T}\boldsymbol{M}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\boldsymbol{T}\boldsymbol{M}),\end{aligned} \tag{35}$$

where $W_1, W_4, \dots, W_9$ are evaluated in the reference configuration, where $\boldsymbol{F} = \boldsymbol{I}$, and may in general depend on $\operatorname{tr} \boldsymbol{T}$ and $\boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{M})$. For consistency it is therefore appropriate to set

$$p = 2W_1, \quad W_4 + 2W_5 = 0, \quad W_6 + 2W_7 = \tfrac{1}{2}, \quad W_8 + 2W_9 = 0, \tag{36}$$

in the reference configuration. Indeed, if (35) holds for all possible $\boldsymbol{T}$ then these conditions necessarily follow.

2.3. ***The elasticity tensor.*** Next, we note that the elasticity tensor $\boldsymbol{\mathcal{A}}$ is given by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \boldsymbol{F} \partial \boldsymbol{F}} = \sum_{\substack{1 \le i \le 9 \\ i \ne 2,3}} W_i \frac{\partial^2 I_i}{\partial \boldsymbol{F} \partial \boldsymbol{F}} + \sum_{\substack{1 \le i \le 9 \\ i \ne 2,3}} \sum_{\substack{1 \le j \le 9 \\ j \ne 2,3}} W_{ij} \frac{\partial I_i}{\partial \boldsymbol{F}} \otimes \frac{\partial I_j}{\partial \boldsymbol{F}}. \tag{37}$$

This requires expressions for the second derivatives of the invariants. In component form these are given in Appendix A. Since the resulting formula for the components of $\boldsymbol{\mathcal{A}}$ is quite long we do not give it here. Instead, we give its specialization to the situation in which there is no finite deformation and $\mathcal{B}_0$ coincides with $\mathcal{B}_r$. The subscript 0 on $\boldsymbol{\mathcal{A}}_0$ may now be omitted, and taking account of the conditions (36) prevailing in the reference configuration the components of $\boldsymbol{\mathcal{A}}$ in the reference configuration can be arranged in the (still fairly lengthy) form

$$\begin{aligned}
\mathcal{A}_{piqj} = {} & 2W_1\delta_{ij}\delta_{pq} + 2W_5(\delta_{ij}M_pM_q + \delta_{pq}M_iM_j + \delta_{iq}M_jM_p + \delta_{jp}M_iM_q) + \delta_{ij}T_{pq} \\
& + 2W_7(\delta_{ij}T_{pq} + \delta_{pq}T_{ij} + \delta_{iq}T_{jp} + \delta_{jp}T_{iq}) + W_9[\delta_{ij}(M_pT_{qr} + M_qT_{pr}) \\
& + \delta_{pq}(M_iT_{jr} + M_jT_{ir}) + \delta_{iq}(M_jT_{pr} + M_pT_{jr}) + \delta_{jp}(M_iT_{qr} + M_qT_{ir})]M_r \\
& + 4W_{11}\delta_{ip}\delta_{jq} + 4(W_{44} + 4W_{45} + 4W_{55})M_iM_jM_pM_q \\
& + 4(W_{14} + 2W_{15})(\delta_{ip}M_jM_q + \delta_{jq}M_iM_p) + 4(W_{16} + 2W_{17})(\delta_{ip}T_{jq} + \delta_{jq}T_{ip}) \\
& + 2(W_{18} + 2W_{19})[\delta_{ip}(M_jT_{qr} + M_qT_{jr}) + \delta_{jq}(M_iT_{pr} + M_pT_{ir})]M_r \\
& + 4(W_{46} + 2W_{47} + 2W_{56} + 4W_{57})(M_iM_pT_{jq} + M_jM_qT_{ip}) \\
& + 2(W_{48} + 2W_{49} + 2W_{58} + 4W_{59})(T_{ir}M_jM_pM_q + T_{jr}M_iM_pM_q \\
& + T_{pr}M_iM_jM_q + T_{qr}M_iM_jM_p)M_r + 4(W_{66} + 4W_{67} + 4W_{77})T_{ip}T_{jq} \\
& + 2(W_{68} + 2W_{69} + 2W_{78} + 4W_{79})[T_{ip}(T_{qr}M_j + T_{jr}M_q) + T_{jq}(T_{pr}M_i + T_{ir}M_p)]M_r \\
& + (W_{88} + 4W_{89} + 4W_{99})(M_iT_{pr} + M_pT_{ir})(M_jT_{qs} + M_qT_{js})M_rM_s,
\end{aligned} \tag{38}$$

all the derivatives of W being evaluated in $\mathcal{B}_r$. With the restricted set of invariants adopted these depend in general on $\operatorname{tr}\boldsymbol{T}$ and $\boldsymbol{M}\cdot(\boldsymbol{TM})$. Note that when (38) is substituted into the equation of motion all the terms involving δ_{jp} or δ_{jq} disappear by virtue of the incompressibility condition.

We now consider three special cases that will be used subsequently: (1) the underlying material is isotropic; (2) $\boldsymbol{TM} = \mathbf{0}$; (3) $\boldsymbol{T} = T\boldsymbol{M} \otimes \boldsymbol{M}$.

Case 1: Isotropy. If there is no preferred direction in $\mathcal{B}_r$ and in the absence of initial stress the material is isotropic then in the presence of initial stress (38) reduces simply to

$$\begin{aligned}
\mathcal{A}_{piqj} = {} & 2W_1\delta_{ij}\delta_{pq} + \delta_{ij}T_{pq} + 2W_7(\delta_{ij}T_{pq} + \delta_{pq}T_{ij} + \delta_{iq}T_{jp} + \delta_{jp}T_{iq}) + 4W_{11}\delta_{ip}\delta_{jq} \\
& + 4(W_{16} + 2W_{17})(\delta_{ip}T_{jq} + \delta_{jq}T_{ip}) + 4(W_{66} + 4W_{67} + 4W_{77})T_{ip}T_{jq};
\end{aligned} \tag{39}$$

that is, all terms in which W has a subscript 4, 5, 8 or 9 are omitted.

Case 2: $\boldsymbol{TM} = \mathbf{0}$. In this case all the terms in (38) in which W has a subscript 8 or 9 vanish and (38) reduces to

$$
\begin{aligned}
\mathcal{A}_{piqj} = {} & 2W_1\delta_{ij}\delta_{pq} + 2W_5(\delta_{ij}M_pM_q + \delta_{pq}M_iM_j + \delta_{iq}M_jM_p + \delta_{jp}M_iM_q) + \delta_{ij}T_{pq} \\
& + 2W_7(\delta_{ij}T_{pq} + \delta_{pq}T_{ij} + \delta_{iq}T_{jp} + \delta_{jp}T_{iq}) + 4W_{11}\delta_{ip}\delta_{jq} \\
& + 4(W_{44} + 4W_{45} + 4W_{55})M_iM_jM_pM_q + 4(W_{14} + 2W_{15})(\delta_{ip}M_jM_q + \delta_{jq}M_iM_p) \\
& + 4(W_{16} + 2W_{17})(\delta_{ip}T_{jq} + \delta_{jq}T_{ip}) + 4(W_{46} + 2W_{47} + 2W_{56} \\
& + 4W_{57})(M_iM_pT_{jq} + M_jM_qT_{ip}) + 4(W_{66} + 4W_{67} + 4W_{77})T_{ip}T_{jq}.
\end{aligned}
$$

We note, in particular, the connections

$$\mathcal{A}_{ijji} = \mathcal{A}_{jiij} = \mathcal{A}_{ijij} - (2W_1 + T_{ii}) = \mathcal{A}_{jiji} - (2W_1 + T_{jj}), \quad i \neq j, \tag{40}$$

$$\mathcal{A}_{jijj} = \mathcal{A}_{ijjj} - T_{ij}, \quad i \neq j, \quad \mathcal{A}_{ijjk} = \mathcal{A}_{ijkj} - T_{ik}, \tag{41}$$

where i, j, k are distinct.

Case 3: $\boldsymbol{T} = T\boldsymbol{M} \otimes \boldsymbol{M}$. In this case the components of $\boldsymbol{\mathcal{A}}$ have a relatively simple structure and can be written compactly as

$$
\begin{aligned}
\mathcal{A}_{piqj} = {} & 2W_1\delta_{ij}\delta_{pq} + 4W_{11}\delta_{ip}\delta_{jq} + A(\delta_{ij}M_pM_q + \delta_{pq}M_iM_j + \delta_{iq}M_jM_p + \delta_{jp}M_iM_q) \\
& + B(\delta_{ip}M_jM_q + \delta_{jq}M_iM_p) + CM_iM_jM_pM_q + T\delta_{ij}M_pM_q,
\end{aligned} \tag{42}
$$

where

$$A = 2[W_5 + T(W_7 + W_9)], \tag{43}$$

$$B = 4[W_{14} + 2W_{15} + T(W_{16} + 2W_{17}) + T^2(W_{18} + 2W_{19})], \tag{44}$$

$$
\begin{aligned}
C = {} & 4[W_{44} + 4W_{45} + 4W_{55} + T(W_{46} + 2W_{47} + W_{48} + 2W_{49} + 2W_{56} + 4W_{57} + 2W_{58} + 4W_{59}) \\
& + T^2(W_{66} + 4W_{67} + 4W_{77} + W_{68} + 2W_{69} + 2W_{78} + 4W_{79} + W_{88} + 4W_{89} + 4W_{99})].
\end{aligned} \tag{45}
$$

It is noteworthy that in each of the three cases the components $\mathcal{A}_{piqj}$ are quadratic in the components of the initial stress even though we have not included in the model invariants that are nonlinear in the initial stress.

3. Homogeneous plane waves

With the focus on incompressible materials we now apply the equation of motion and the incompressibility condition in (22) to the analysis of homogeneous plane waves. In particular, we consider the incremental displacement $\boldsymbol{u}$ and Lagrange multiplier $\dot{p}$ to have the forms

$$\boldsymbol{u} = f(\boldsymbol{n} \cdot \boldsymbol{x} - vt)\boldsymbol{d}, \quad \dot{p} = g(\boldsymbol{n} \cdot \boldsymbol{x} - vt), \tag{46}$$

where $\boldsymbol{d}$ is a constant unit (polarization) vector, the unit vector $\boldsymbol{n}$ is the direction of propagation of the plane wave, v is the wave speed, f is a function that need not be made explicit, but is subject to the restriction $f'' \neq 0$, and g is a function related to f. A prime on f or g indicates differentiation with respect to its argument.

Substitution of (46) into (22) then yields

$$[\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{d} - \rho v^2\boldsymbol{d}]f'' - g'\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{d} \cdot \boldsymbol{n} = 0, \tag{47}$$

where the (symmetric) *acoustic tensor* $\boldsymbol{Q}(\boldsymbol{n})$ is defined by

$$Q_{ij}(\boldsymbol{n}) = \mathcal{A}_{piqj} n_p n_q. \tag{48}$$

On taking the dot product of $(47)_1$ with $\boldsymbol{n}$ we obtain $g' = [\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{d}]\cdot\boldsymbol{n} f''$, and on substituting back into $(47)_1$ and eliminating the factor f'' we may arrange the resulting *propagation condition* in the form

$$[\boldsymbol{Q}(\boldsymbol{n}) - \boldsymbol{n}\otimes\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{n}]\boldsymbol{d} = \rho v^2 \boldsymbol{d}, \tag{49}$$

or equivalently as

$$\bar{\boldsymbol{Q}}(\boldsymbol{n})\boldsymbol{d} = \rho v^2 \boldsymbol{d}, \tag{50}$$

where $\bar{\boldsymbol{Q}}(\boldsymbol{n})$ is the projection of $\boldsymbol{Q}(\boldsymbol{n})$ on to the plane normal to $\boldsymbol{n}$ defined by

$$\bar{\boldsymbol{Q}}(\boldsymbol{n}) = \bar{\boldsymbol{I}}\boldsymbol{Q}(\boldsymbol{n})\bar{\boldsymbol{I}}, \tag{51}$$

where $\bar{\boldsymbol{I}} = \boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n}$. The symmetrization (51) was originally derived in [Scott and Hayes 1985]. Thus, $\bar{\boldsymbol{Q}}(\boldsymbol{n})$ is symmetric and the two-dimensional eigenvalue problem (50) (in the plane normal to $\boldsymbol{n}$) therefore has two real solutions for ρv^2. The wave speeds are real if the two eigenvalues are positive, and this is guaranteed if the strong ellipticity condition holds. This is expressed as

$$[\bar{\boldsymbol{Q}}(\boldsymbol{n})\boldsymbol{d}]\cdot\boldsymbol{d} > 0 \quad \text{for all nonzero vectors } \boldsymbol{d} \text{ and } \boldsymbol{n} \text{ such that } \boldsymbol{d}\cdot\boldsymbol{n} = 0. \tag{52}$$

For given $\boldsymbol{n}$ and $\boldsymbol{d}$ the wave speed v is obtained from

$$\rho v^2 = [\bar{\boldsymbol{Q}}(\boldsymbol{n})\boldsymbol{d}]\cdot\boldsymbol{d} = [\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{d}]\cdot\boldsymbol{d}. \tag{53}$$

As a first application we consider Case 1 in Section 2.3. Then we obtain

$$\bar{\boldsymbol{Q}}(\boldsymbol{n}) = [2W_1 + (2W_7+1)(\boldsymbol{T}\boldsymbol{n})\cdot\boldsymbol{n}]\bar{\boldsymbol{I}} + 2W_7\bar{\boldsymbol{T}} + 2(W_{66} + 4W_{67} + 4W_{77})\bar{\boldsymbol{I}}\boldsymbol{T}\boldsymbol{n}\otimes\bar{\boldsymbol{I}}\boldsymbol{T}\boldsymbol{n}, \tag{54}$$

where $\bar{\boldsymbol{T}} = \bar{\boldsymbol{I}}\boldsymbol{T}\bar{\boldsymbol{I}}$ is the projection of $\boldsymbol{T}$ on to the plane normal to $\boldsymbol{n}$. The wave speed is then easily calculated from (53). For illustration we consider the initial stress to be uniaxial and along the x_1 axis with $T_{11} = T$. Then, if we consider motion in the (x_1, x_2) plane with $n_1 = \cos\theta$, $n_2 = \sin\theta$, $d_1 = -\sin\theta$, $d_2 = \cos\theta$, we obtain

$$\rho v^2 = 2W_1 + 2W_7 T + T\cos^2\theta + (W_{66} + 4W_{67} + 4W_{77})T^2\sin^2 2\theta. \tag{55}$$

Clearly, except for the angles $\theta = 0$ and $\theta = \pi/2$ the squared wave speed depends quadratically (at least) on the initial stress T, bearing in mind that in general the coefficients $W_1, W_7, W_{66}, \ldots$ could depend on T. If all the invariants had been included in W then the nonlinearity would be quartic (at least). This is in contrast to the predictions of Biot's theory, and we now make contact with that for comparison. For this purpose we have provided in Appendix B a derivation of the connection between the general theory used here and Biot's theory. If we denote by $\mathcal{B}_{ijkl}$ the material constants used in Biot's theory then, from Appendix B, we have

$$\mathcal{A}_{ijkl} = \mathcal{B}_{ijkl} - \tfrac{1}{2}\delta_{il}T_{jk} - \tfrac{1}{2}\delta_{ik}T_{jl} - \tfrac{1}{2}\delta_{jk}T_{il} + \tfrac{1}{2}\delta_{jl}T_{ik} + \delta_{kl}T_{ij}. \tag{56}$$

For the present special problem the relevant $\mathcal{B}_{ijkl}$ components are given by

$$\mathcal{B}_{1111} = \mu - T, \quad \mathcal{B}_{2222} = \mu, \quad \mathcal{B}_{1122} = -\mu - T, \quad \mathcal{B}_{2211} = -\mu, \quad \mathcal{B}_{1212} = \mu, \tag{57}$$

where $\mu > 0$ is the shear modulus. It follows that

$$\mathcal{A}_{1111} = \mu - T,\ \ \mathcal{A}_{2222} = \mu = -\mathcal{A}_{1122},\ \ \mathcal{A}_{1212} = \mu + \tfrac{1}{2}T,\ \ \mathcal{A}_{2121} = \mu - \tfrac{1}{2}T = \mathcal{A}_{1221}. \tag{58}$$

The wave speed in this case is then given by

$$\rho v^2 = \mu - \tfrac{1}{2}T + T\cos^2\theta. \tag{59}$$

Clearly, this result can be recovered from (55) if we take $2W_1 = \mu$, $W_7 = -\frac{1}{4}$ (and hence, by $(36)_3$, $W_6 = 1$), and $W_{66} + 4W_{67} + 4W_{77} = 0$. The individual components in (58) are recovered from the general expression (39) by also setting $W_{11} = 0$ and $W_{16} + 2W_{17} = 0$ in (39), except that we obtain $\mathcal{A}_{1122} = 0$ instead of $-\mu$ and $\mathcal{A}_{1221} = -T/2$ instead of $\mu - T/2$. However, this difference is of no consequence since the sum of these two terms is the same in each case and it is only their sum, as the coefficient of $2d_1d_2n_1n_2$, that contributes to the expression for ρv^2 in (53). In fact, because of the incompressibility constraint, there is an intrinsic non-uniqueness in the components $\mathcal{A}_{ijkl}$ (and hence in $\mathcal{B}_{ijkl}$) since a term of the form $p^*\delta_{il}\delta_{jk} + q^*\delta_{ij}\delta_{kl}$ may be added to $\mathcal{A}_{ijkl}$, where p^* and q^* are arbitrary scalars, possibly dependent on tr $\boldsymbol{T}$ and $\boldsymbol{M}\cdot(\boldsymbol{T}\boldsymbol{M})$ in the general case. The term in q^* disappears from the incremental constitutive relation by incompressibility and the term in p^* vanishes identically in the incremental equation of motion, again by incompressibility.

As a second example we consider Case 3 in Section 2.3, for which we obtain

$$\bar{\boldsymbol{Q}}(\boldsymbol{n}) = [2W_1 + (A+T)(\boldsymbol{M}\cdot\boldsymbol{n})^2]\bar{\boldsymbol{I}} + [A + C(\boldsymbol{M}\cdot\boldsymbol{n})^2]\bar{\boldsymbol{M}}\otimes\bar{\boldsymbol{M}}, \tag{60}$$

where $\bar{\boldsymbol{M}} = \bar{\boldsymbol{I}}\boldsymbol{M}$, and hence from (53) we obtain

$$\rho v^2 = 2W_1 + (A+T)(\boldsymbol{M}\cdot\boldsymbol{n})^2 + [A + C(\boldsymbol{M}\cdot\boldsymbol{n})^2](\boldsymbol{M}\cdot\boldsymbol{d})^2. \tag{61}$$

Let us take, for illustration, $\boldsymbol{M}$ to be along the x_1 axis with the same $\boldsymbol{d}$ and $\boldsymbol{n}$ as in the previous example, so the motion is confined to the (x_1, x_2) plane. Then

$$\rho v^2 = 2W_1 + A + T\cos^2\theta + C\sin^2\theta\cos^2\theta, \tag{62}$$

where A and C are given by (43) and (45), respectively. We note, in particular, that C is quadratic in T.

Again we compare with the corresponding formula from Biot's theory, which involves two material constants N and Q for this (two-dimensional orthotropic) situation, for which we obtain

$$\mathcal{A}_{1111} = N - T,\ \mathcal{A}_{2222} = N = -\mathcal{A}_{1122},\ \mathcal{A}_{1212} = Q + \tfrac{1}{2}T,\ \mathcal{A}_{2121} = Q - \tfrac{1}{2}T = \mathcal{A}_{1221}, \tag{63}$$

and hence

$$\rho v^2 = Q - \tfrac{1}{2}T + T\cos^2\theta + 4(N-Q)\sin^2\theta\cos^2\theta. \tag{64}$$

This can be obtained from (62) by setting

$$2W_1 = N, \quad 2W_1 + 2W_5 = Q, \quad W_7 = -\tfrac{1}{4}, \tag{65}$$

$$C \equiv 4(W_{44} + 4W_{45} + 4W_{55}) = 4(N-Q), \tag{66}$$

and all the other derivatives of W to zero. The case of isotropy is recovered by taking $N = Q = \mu$.

Case 2 produces very similar results when the motion is again restricted to the (x_1, x_2) plane. If we take $\boldsymbol{M}$ now to be aligned along the x_2 axis and $\boldsymbol{T}$ as before then we obtain

$$\rho v^2 = 2W_1 + A' + T\cos^2\theta + C'\sin^2\theta\cos^2\theta. \tag{67}$$

Compared with (62) A' is obtained from A in (43) by omitting the term in W_9, while C' is given by

$$C' = 4[W_{44} + 4W_{45} + 4W_{55} - 2T(W_{46} + 2W_{47} + 2W_{56} + 4W_{57}) + T^2(W_{66} + 4W_{67} + 4W_{77})], \quad (68)$$

which differs from C in (45).

For each of the cases considered above, the appropriate specialization of the strong ellipticity condition (52) requires $\rho v^2 > 0$, and conditions on the material parameters and the initial stress for this to be satisfied for all angles θ may be inferred. For example, from (67), if $T = 0$ then necessary and sufficient conditions for $\rho v^2 > 0$ are $W_1 + W_5 > 0$ and $2W_1 + 2W_5 + W_{44} + 4W_{45} + 4W_{55} > 0$. On the other hand, if $T \neq 0$ then restrictions are imposed on the permissible range of values of T. For example, by considering $\theta = \pi/2$ and $\theta = 0$, we deduce that $2W_1 + A' > 0$ and $2W_1 + A' + T > 0$, respectively, are necessary conditions for $\rho v^2 > 0$, but to obtain necessary and sufficient conditions for $\rho v^2 > 0$ for all θ requires consideration of several possible cases, depending on the signs of the material coefficients W_7, W_{46}, $W_{66}, \ldots$, and since this is algebraically lengthy the details are omitted.

4. Equations governing two-dimensional motions

In this section we restrict attention to two-dimensional motions of incompressible materials. Specifically, we consider motions in the (x_1, x_2) plane with displacement components $\boldsymbol{u} = (u_1, u_2, 0)$, where u_1 and u_2 depend only on x_1, x_2 and t. The two in-plane components of the equations of motion (14) are

$$\mathcal{A}_{p1qj} u_{j,pq} - \dot{p}_{,1} = \rho u_{1,tt}, \quad \mathcal{A}_{p2qj} u_{j,pq} - \dot{p}_{,2} = \rho u_{2,tt}, \quad (69)$$

where the summation is over indices 1 and 2. In general, the third component of the equations of motion should also be considered; here this has the form $\mathcal{A}_{p3qj} u_{j,pq} - \dot{p}_{,3} = 0$. However, henceforth we specialize the form of $\boldsymbol{T}$ and the orientation of $\boldsymbol{M}$ so that this reduces to $\dot{p}_{,3} = 0$, i.e., $\dot{p}$ depends only on x_1, x_2 and t. In particular, we take the (x_1, x_2) plane to be a principal plane of $\boldsymbol{T}$, with $\boldsymbol{M}$ either lying parallel to the (x_1, x_2) plane or aligned with the x_3 direction. Then, it is easy to show that $\mathcal{A}_{p3qj} u_{j,pq}$ vanishes identically.

Elimination of $\dot{p}$ by cross differentiation in (69) yields

$$\mathcal{A}_{p1qj} u_{j,pq2} - \mathcal{A}_{p2qj} u_{j,pq1} = \rho(u_{1,2tt} - u_{2,1tt}). \quad (70)$$

By incompressibility we can introduce a scalar function ψ such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}, \quad (71)$$

and substitution into (70) then yields an equation for ψ, namely

$$\mathcal{A}_{1212}\psi_{,1111} + 2(\mathcal{A}_{1222} - \mathcal{A}_{1211})\psi_{,1112} + (\mathcal{A}_{1111} + \mathcal{A}_{2222} - 2\mathcal{A}_{1122} - 2\mathcal{A}_{1221})\psi_{,1122} + 2(\mathcal{A}_{1121} - \mathcal{A}_{2221})\psi_{,1222} + \mathcal{A}_{2121}\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}). \quad (72)$$

We now apply these equations to Cases 2 and 3 specialized to the (x_1, x_2) plane for $\boldsymbol{T} = T\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, with $\boldsymbol{M} = \boldsymbol{e}_2$ (Case 2) and $\boldsymbol{M} = \boldsymbol{e}_1$ (Case 3).

Case 2: $\boldsymbol{T} = T\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ and $\boldsymbol{M} = \boldsymbol{e}_2$. In this case (72) reduces to

$$(a + T)\psi_{,1111} + (2a + T + b)\psi_{,1122} + a\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}), \quad (73)$$

where

$$a = \mathcal{A}_{1212} - T = \mathcal{A}_{2121} = 2W_1 + A' = 2W_1 + 2W_5 + 2TW_7, \tag{74}$$

$$\begin{aligned} b &= \mathcal{A}_{1111} + \mathcal{A}_{2222} - 2\mathcal{A}_{1122} - 2\mathcal{A}_{1221} - \mathcal{A}_{1212} - \mathcal{A}_{2121} = C' \\ &= 4(W_{44}+4W_{45}+4W_{55}) - 8T(W_{46}+2W_{47}+2W_{56}+4W_{57}) + 4T^2(W_{66}+4W_{67}+4W_{77}). \end{aligned} \tag{75}$$

Case 3: $\boldsymbol{T} = T\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$ and $\boldsymbol{M} = \boldsymbol{e}_1$. Similarly to the previous case, (72) reduces to

$$(\alpha + T)\psi_{,1111} + (2\alpha + T + \beta)\psi_{,1122} + \alpha\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}), \tag{76}$$

where

$$\alpha = \mathcal{A}_{1212} - T = \mathcal{A}_{2121} = 2W_1 + A = 2W_1 + 2W_5 + 2TW_7 + 2TW_9, \tag{77}$$

$$\begin{aligned} \beta &= \mathcal{A}_{1111} + \mathcal{A}_{2222} - 2\mathcal{A}_{1122} - 2\mathcal{A}_{1221} - \mathcal{A}_{1212} - \mathcal{A}_{2121} = C \\ &= 4[W_{44} + 4W_{45} + 4W_{55} + T(W_{46} + 2W_{47} + 2W_{56} + 4W_{57} + W_{48} + 4W_{49} + 2W_{58} + 4W_{59}) \\ &\qquad + T^2(W_{66} + 4W_{67} + 4W_{77} + W_{68} + 2W_{69} + 2W_{78} + 4W_{79} + W_{88} + 4W_{89} + 4W_{99})]. \end{aligned} \tag{78}$$

If we compare the coefficients of Case 2 and Case 3, it may be remarked that $\alpha = a + 2TW_9$, while β and b differ significantly, although their structures and the equations are very similar. If there is no initial stress then $T = 0$ and the two sets of equations are the same except that I_4 and I_5 will be different for the two cases because the directions of $\boldsymbol{M}$ are different. If there is initial stress but no preferred direction (underlying isotropy) then all the derivatives of W with a subscript 4, 5, 8 or 9 vanish and the two equations are identical, with coefficients reducing to $a = \alpha = 2W_1 + 2TW_7$ and $b = \beta = 4T^2(W_{66} + 4W_{67} + 4W_{77})$.

5. Rayleigh surface waves

We now consider a half-space occupying the region $x_2 < 0$ in the reference configuration with boundary $x_2 = 0$ and surface waves propagating along the direction x_1. The initial stress and preferred direction are confined to the (x_1, x_2) plane so that the equations of motion (69) are applicable and the third equation is again satisfied trivially. Furthermore, we take the initial stress to have the form $\boldsymbol{T} = T\boldsymbol{e}_1 \otimes \boldsymbol{e}_1$. Then the components $\mathcal{A}_{1112}$, $\mathcal{A}_{1121}$, $\mathcal{A}_{2212}$, $\mathcal{A}_{2221}$ all vanish, as for Cases 2 and 3 in Section 4, and the equations simplify accordingly. On the surface we take the incremental surface traction to vanish, so that $\dot{S}_{021} = 0$, $\dot{S}_{022} = 0$ on $x_2 = 0$. For the considered incompressible material this yields

$$\mathcal{A}_{2121}u_{1,2} + (\mathcal{A}_{2112} + p)u_{2,1} = 0, \quad \mathcal{A}_{2211}u_{1,1} + (\mathcal{A}_{2222} + p)u_{2,2} - \dot{p} = 0 \quad \text{on } x_2 = 0, \tag{79}$$

and with $u_1 = \psi_{,2}$ and $u_2 = -\psi_{,1}$ these boundary conditions become

$$\mathcal{A}_{2121}\psi_{,22} - (\mathcal{A}_{2112} + p)\psi_{,11} = 0, \quad \mathcal{A}_{2211}\psi_{,12} - (\mathcal{A}_{2222} + p)\psi_{,12} - \dot{p} = 0 \quad \text{on } x_2 = 0. \tag{80}$$

The second of these equations may be expressed in terms of ψ by differentiating with respect to x_1 and then using $(69)_1$, appropriately specialized, to eliminate $\dot{p}_{,1}$. The result is

$$(\mathcal{A}_{1111} + \mathcal{A}_{2222} - 2\mathcal{A}_{1122} - 2\mathcal{A}_{2112} + \mathcal{A}_{2121})\psi_{,112} + \mathcal{A}_{2121}\psi_{,222} = \rho\psi_{,2tt}. \tag{81}$$

The two boundary conditions can now be written compactly as

$$a(\psi_{,22} - \psi_{,11}) = 0, \quad (3a + b + T)\psi_{,112} + a\psi_{,222} = \rho\psi_{,2tt} \quad \text{on } x_2 = 0. \tag{82}$$

These apply for Case 2. The corresponding equations for Case 3 are obtained by replacing a and b by α and β, respectively. Thus, in the following we work with the parameters a and b.

We consider harmonic waves propagating in the x_1 direction and we write ψ in the form

$$\psi(x_1, x_2, t) = \phi(z)\exp[ik(x_1 - vt)], \tag{83}$$

where k is the wave number, v is the wave speed, $z = kx_2$, and the function ϕ is to be determined. Substituting this into the equation of motion (73) we obtain

$$a\phi'''' - (2a + b + T - \rho v^2)\phi'' + (a + T - \rho v^2)\phi = 0, \tag{84}$$

wherein and in the following equations a prime on ϕ denotes differentiation with respect to z. In terms of ϕ the boundary conditions (82) become

$$\phi''(0) + \phi(0) = 0, \quad a\phi'''(0) - (3a + b + T - \rho v^2)\phi'(0) = 0. \tag{85}$$

The factor a has been omitted from the first of these equations on the assumption that $a \neq 0$. For the solution of (84) we require the decay condition $\phi(x_2) \to 0$ as $x_2 \to -\infty$ to hold, and the general solution satisfying this condition is

$$\phi(z) = c_1 \exp(s_1 z) + c_2 \exp(s_2 z), \tag{86}$$

where c_1 and c_2 are constants and s_1, s_2 are the solutions of

$$as^4 - (2a + b + T - \rho v^2)s^2 + (a + T - \rho v^2) = 0 \tag{87}$$

with positive real parts. From the latter it follows that

$$s_1^2 + s_2^2 = \frac{2a + b + T - \rho v^2}{a}, \quad s_1^2 s_2^2 = \frac{a + T - \rho v^2}{a}. \tag{88}$$

If s_1^2 and s_2^2 are real, then they must be positive to ensure that s_1 and s_2 have positive real part. If they are complex then they are conjugate. In either case the product $s_1^2 s_2^2$ must be positive. Assuming that $a > 0$, a real (surface) wave speed v satisfies the inequality

$$0 < \rho v^2 < a + T. \tag{89}$$

Note that if $T \leq 0$ then this cannot be satisfied unless $a > 0$, which requires $W_1 + W_5 + TW_7 > 0$. In Biot's theory, for example, this, in conjunction with (89), yields $-2Q < T < 2Q$.

Substitution of the solution (86) into the boundary conditions (85) yields the equations

$$(1 + s_1^2)c_1 + (1 + s_2^2)c_2 = 0, \tag{90}$$

$$[as_1^3 - (3a + b + T - \rho v^2)s_1]c_1 + [as_2^3 - (3a + b + T - \rho v^2)s_2]c_2 = 0. \tag{91}$$

In order to construct a nontrivial solution of this system, corresponding to vanishing of the determinant of coefficients, it is convenient to introduce the notations

$$\bar{T} = \frac{T}{a}, \quad \xi = \frac{\rho v^2}{a}, \quad \bar{b} = \frac{b}{a}, \quad \eta = \sqrt{1 + \bar{T} - \xi}, \tag{92}$$

and then we obtain the cubic equation

$$\eta^3 + \eta^2 + (3 + \bar{b})\eta - 1 = 0 \tag{93}$$

for η. Once this is solved for (positive) η the wave speed is obtained from the formula

$$\rho v^2 = a\xi = a + T - a\eta^2. \tag{94}$$

The inequalities satisfied by η corresponding to (89) are $0 < \eta < \sqrt{1 + \bar{T}}$.

Note that (93) can be written as

$$(4 + \bar{b} + \bar{T} - \xi)\eta + \bar{T} - \xi = 0. \tag{95}$$

If $T = 0$ then this can only be satisfied for $\xi > 0$ if $\xi < 4 + \bar{b}$, and hence we must have $4 + \bar{b} > 0$ in this case. If $T \neq 0$ then since, by definition, $\eta \geq 0$, (95) can only be satisfied if either

$$\bar{T} < \xi < 4 + \bar{b} + \bar{T}, \tag{96}$$

which requires $4 + \bar{b} > 0$, or

$$4 + \bar{b} + \bar{T} < \xi < \bar{T}, \tag{97}$$

which requires $4 + \bar{b} < 0$, or the transitional case

$$\xi = \bar{T}, \tag{98}$$

which, since then $\eta = 1$, corresponds to $4 + \bar{b} = 0$. Note that (96) can hold for either positive or negative T provided $4 + \bar{b} + \bar{T} > 0$, but (97) and (98) are only possible if $T > 0$.

If we denote the function on the left-hand side of (93) by $f(\eta)$ and note that $f(0) = -1$ and $f''(\eta) > 0$ for $\eta \geq 0$ we see that (93) has a unique positive solution for $\eta \in (0, \sqrt{1 + \bar{T}}]$ if $f(\sqrt{1 + \bar{T}}) \geq 0$. If $T = 0$ then this is guaranteed if $4 + \bar{b} > 0$, and if this inequality holds a unique positive solution also exists for $T > 0$ (subject to $a > 0$). On the other hand, if $T < 0$ then there is a value $\bar{T}_0$ of $\bar{T}$ with $-1 < \bar{T}_0 < 0$ such that a unique positive solution exists only for $\bar{T} > \bar{T}_0$. The value of $\bar{T}_0$ is determined as the value of $\bar{T}$ satisfying $f(\sqrt{1 + \bar{T}}) = 0$, bearing in mind that in general $\bar{b}$ depends on T. The exact details depend in a fairly complicated way on the form of the strain-energy function W and are not discussed here. Instead we illustrate the results using a specific form of W in the following section.

6. Numerical results and discussion

We remain focused on Case 2 in order to compare the results of the general theory with those of the Biot theory for both anisotropic and isotropic models. In particular, we compare the wave speeds for both plane (shear) waves and Rayleigh waves. In each case this involves the initial stress T and material parameters a and b, which also depend on T. For the general constitutive law considered in Case 2 we record here the expressions for a and b from Section 4 for ease of reference:

$$a = 2W_1 + 2W_5 + 2TW_7, \tag{99}$$

$$b = 4(W_{44} + 4W_{45} + 4W_{55}) - 8T(W_{46} + 2W_{47} + 2W_{56} + 4W_{57}) + 4T^2(W_{66} + 4W_{67} + 4W_{77}). \tag{100}$$

If the second derivative terms in W involving a subscript 6 or 7 are identically zero then these reduce to

$$a = 2W_1 + 2W_5 + 2TW_7, \quad b = 4(W_{44} + 4W_{45} + 4W_{55}), \tag{101}$$

which can be specialized to Biot's anisotropic case by setting $W_7 = -\frac{1}{4}$, $2W_1 + 2W_5 = Q$ and $W_{44} + 4W_{45} + 4W_{55} = N - Q$ in terms of Biot's constants N and Q. Finally, the case of isotropy is then recovered by also dropping terms with a subscript 4 or 5, yielding

$$a = 2W_1 + 2TW_7, \quad b = 0, \tag{102}$$

and Biot's case is obtained by setting $2W_1 = \mu$ and $W_7 = -\frac{1}{4}$, where μ is the shear modulus.

For plane waves in the (x_1, x_2) plane the wave speed v is given by

$$\rho v^2 = a + T\cos^2\theta + b\sin^2\theta\cos^2\theta, \tag{103}$$

while the corresponding formula for the Rayleigh wave speed is

$$\rho v^2 = a + T - a\eta^2, \tag{104}$$

where η is the unique positive solution of the equation

$$\eta^3 + \eta^2 + (3 + b/a)\eta - 1 = 0, \quad 0 < \eta < \sqrt{1 + T/a}. \tag{105}$$

In order to capture the effect of the T^2 term in b, which provides the main distinction from the classical theory, it suffices for purposes of illustration to consider a specific form of the strain-energy function W. In the absence of initial stress an incompressible transversely isotropic linear elastic solid is characterized in terms of three elastic constants. Thus, for the transversely isotropic part of W three material constants are needed. The connections between the derivatives of W with respect to the invariants I_1, I_2, I_4 and I_5 evaluated in the undeformed configuration and the three classical constants were provided in [Merodio and Ogden 2005] and can be specialized to the present situation in which I_2 is omitted. We denote the transversely isotropic constants here by μ, ν and κ and consider a transversely isotropic strain-energy function of the form

$$W = \tfrac{1}{2}\mu(I_1 - 3) + \tfrac{1}{2}\nu(I_4 - I_5)^2 - 2\kappa(I_4 - 1) + \kappa(I_5 - 1), \tag{106}$$

which consists of an isotropic neo-Hookean term with constant μ and two anisotropic terms associated with the preferred direction and involving two anisotropic constants ν and κ. The condition $W_4 + 2W_5 = 0$ identified in (36) is then satisfied in the reference configuration $\mathcal{B}_r$, $W_{45} + W_{55} = 0$ and $W_{44} = \nu$.

The initial stress is next incorporated in the model by introducing two additional material constants λ and γ and the invariants I_6 and I_7, and ensuring that the condition $W_6 + 2W_7 = \frac{1}{2}$ in (36) is satisfied in the configuration $\mathcal{B}_r$. A simple example of this inclusion, which we adopt here, extends (106) to the form

$$\begin{aligned} W = \tfrac{1}{2}\mu(I_1 - 3) + \tfrac{1}{2}\nu(I_4 - I_5)^2 - 2\kappa(I_4 - 1) + \kappa(I_5 - 1) + \tfrac{1}{2}\gamma I_6^2 - \gamma(\operatorname{tr}\boldsymbol{T})I_6 \\ + (\tfrac{1}{2} - 2\lambda)I_6 + \lambda I_7 + \tfrac{1}{2}\gamma(\operatorname{tr}\boldsymbol{T})^2 + (\lambda - \tfrac{1}{2})\operatorname{tr}\boldsymbol{T}, \end{aligned} \tag{107}$$

the final two terms being included merely to ensure that W vanishes in $\mathcal{B}_r$. They do not contribute to the stress. The only nonzero derivatives of W when evaluated in $\mathcal{B}_r$ are the first derivatives

$$W_1 = \tfrac{1}{2}\mu, \quad W_4 = -2\kappa, \quad W_5 = \kappa, \quad W_6 = \tfrac{1}{2} - 2\lambda, \quad W_7 = \lambda, \tag{108}$$

and the second derivatives

$$W_{44} = \nu, \quad W_{45} = -\nu, \quad W_{55} = \nu, \quad W_{66} = \gamma. \tag{109}$$

The expressions for a and b become

$$a = \mu + 2\kappa + 2\lambda T, \quad b = 4(\nu + \gamma T^2). \tag{110}$$

Clearly, if $\lambda = -\frac{1}{4}$ then the term in T within a has the same form as in the Biot theory. Inclusion of the dimensionless constant λ allows more flexibility in the model.

In what follows we present results in dimensionless form by defining the dimensionless quantities

$$\xi = \rho v^2/\mu, \;\; a^* = a/\mu, \;\; b^* = b/\mu, \;\; \kappa^* = \kappa/\mu, \;\; \nu^* = \nu/\mu, \;\; \gamma^* = \gamma\mu, \;\; T^* = T/\mu, \tag{111}$$

and for consistency of notation for the constants we set $\lambda^* = \lambda$.

Thus, for plane waves,

$$\xi = 1 + 2\kappa^* + 2\lambda^* T^* + T^* \cos^2\theta + 4(\nu^* + \gamma^* T^{*2}) \sin^2\theta \cos^2\theta. \tag{112}$$

In Figure 1 we plot ξ as a function of T^* for the values $0, \pi/6, \pi/4, \pi/3, \pi/2$ of θ and specifically we take $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0.5$. The results are qualitatively very similar for other values of these parameters. The parameter λ^* has been set at $-\frac{1}{4}$. By considering the values $\theta = 0$ and $\theta = \pi/2$ it can be deduced from (112) that T^* must be restricted to the range of values $-2(1+2\kappa^*) < T^* < 2(1+2\kappa^*)$. Also in Figure 1 we show, for comparison, the results corresponding to $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0$ (the thick dashed lines — essentially the specialization to Biot's anisotropic theory) and to $\kappa^* = 0$, $\nu^* = 0$,

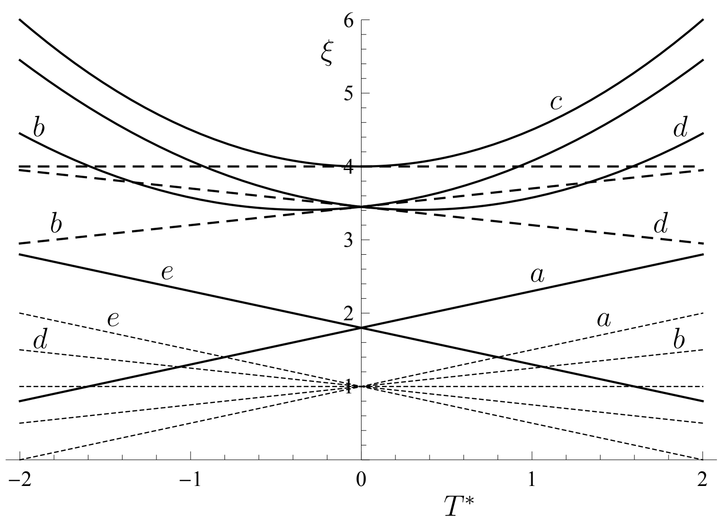

Figure 1. Plot of dimensionless squared wave speed $\xi = \rho v^2/\mu$ against dimensionless initial stress T^* for $\theta = 0$ (a), $\pi/6$ (b), $\pi/4$ (c), $\pi/3$ (d), $\pi/2$ (e), based on (112): continuous curves for parameter values $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0.5$; thick dashed lines for $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0$; thin dashed lines for $\kappa^* = 0$, $\nu^* = 0$, $\gamma^* = 0$ (isotropy). In each case $\lambda^* = -\frac{1}{4}$. The horizontal dashed lines correspond to $\theta = \pi/4$.

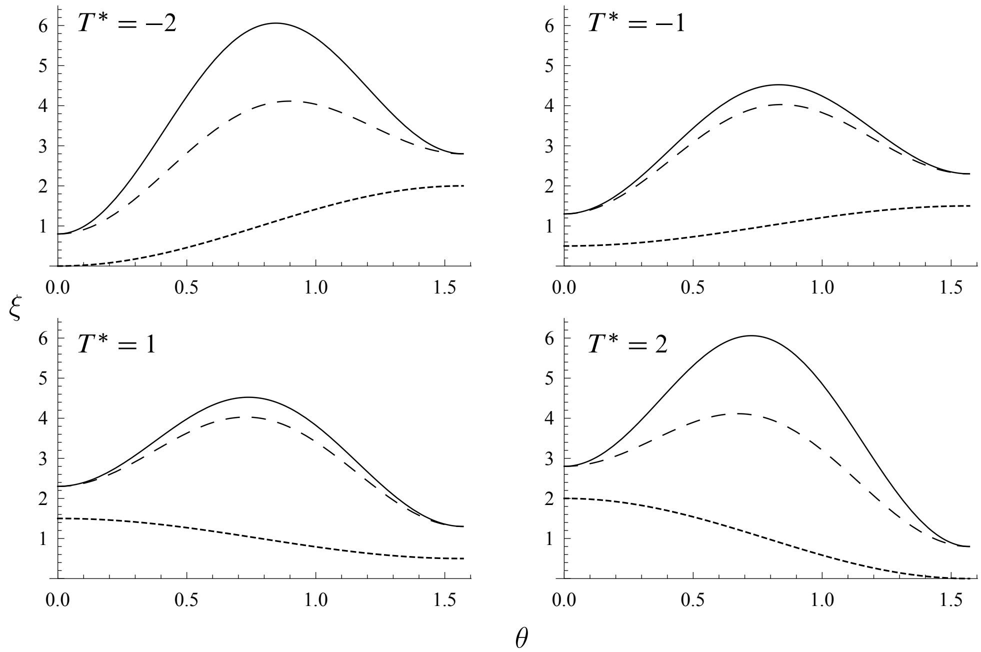

Figure 2. Dimensionless squared wave speed $\xi = \rho v^2/\mu$ versus the angle $\theta \in [0, \pi/2]$, based on (112), for $\lambda^* = -\frac{1}{4}$ and various values of the dimensionless initial stress T^*. Remaining parameter values are $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0.5$ (solid curves); $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0$ (thick dashed curves); $\kappa^* = 0$, $\nu^* = 0$, $\gamma^* = 0$ (thin dashed curves).

$\gamma^* = 0$ (the thin dashed lines — Biot's isotropic theory). For $\theta = 0$ and $\theta = \pi/2$ there is no difference between the two anisotropic models. For intermediate values of θ there is a very significant difference between the results for the general model and Biot's anisotropic model due to the term in T^{*2}. Note that in the isotropic case the wave speed vanishes for $T^* = \pm 2$, which correspond to the extreme values of T^* identified previously.

An alternative view of the results is shown in Figure 2, in which ξ is plotted against the angle θ separately for four values of T^*. For each value of T^* the curves corresponding to the three models are shown. Again the significant difference between the general model and Biot's anisotropic model should be noted, which is particularly strong for the larger values of T^*.

The parameter λ^* also has a significant effect and we illustrate this in Figure 3 in which ξ is plotted against T^* for three different values of λ^* and for $\kappa^* = 0.4, \nu^* = 2.2, \gamma^* = 0.2$ and $\theta = \pi/3$. Even though the (nonlinear) strain-energy function (107) is a considerable specialization of the most general such model it nevertheless demonstrates that the effect of initial stress on the speed of plane waves can be much stronger than is the case with the classical Biot theory.

Next we illustrate the effect of the model (107) on the Rayleigh wave speed based on the solution of (105), with $\xi = \rho v^2/\mu$ then given by (104). In Figure 4, ξ is plotted against T^* for representative values of the parameters ($\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0.5$) and two values of λ^*: -0.15 and -0.3. In each case the dependence on T^* is effectively linear, and this is found also to be the case even when the coefficient γ^* of T^{*2} is quite large. Also shown for comparison are the specializations corresponding to Biot's

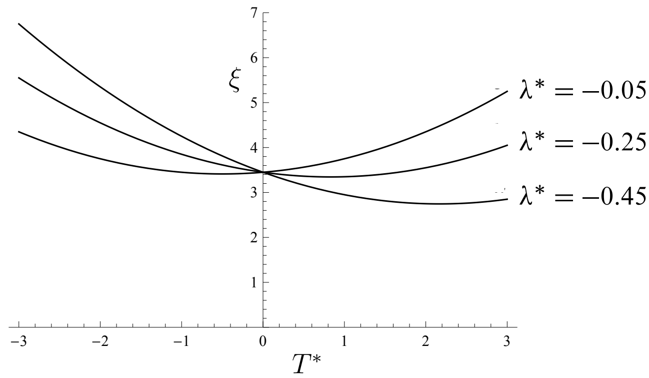

Figure 3. Dimensionless squared wave speed $\xi = \rho v^2/\mu$ versus T^*, based on (112), for $\theta = \pi/3$, $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0.2$ and different values of λ^*.

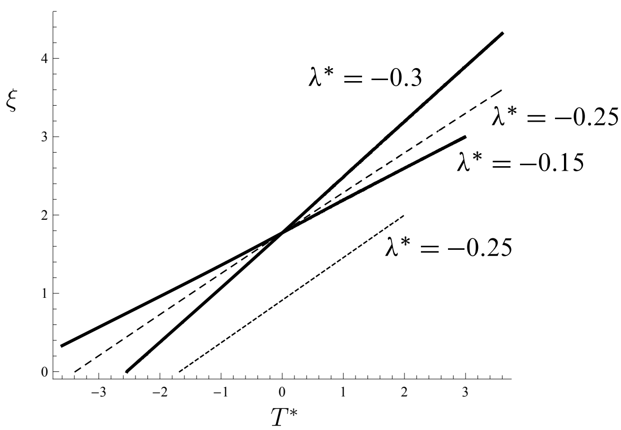

Figure 4. Dimensionless squared Rayleigh wave speed $\xi = \rho v^2/\mu$ versus T^*, based on (104) and (105), for different values of λ^*. Remaining parameter values are $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0.5$ (solid lines); $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0$ (dashed line); $\kappa^* = 0$, $\nu^* = 0$, $\gamma^* = 0$ (dotted line).

anisotropic and isotropic results, for which, respectively, the parameter values are taken as $\kappa^* = 0.4$, $\nu^* = 2.2$, $\gamma^* = 0$, $\lambda^* = -0.25$ and $\kappa^* = 0$, $\nu^* = 0$, $\gamma^* = 0$, $\lambda^* = -0.25$.

In the isotropic case we have

$$\xi = 1 + \tfrac{1}{2}T^* - (1 - \tfrac{1}{2}T^*)\eta_0^2, \tag{113}$$

where η_0 is the unique positive root of (105) when $b = 0$ and is given approximately as $\eta_0 = 0.2956$. As noted earlier, a positive real upper limit for η requires that $1 + T^*/a^* > 0$, and for isotropy this gives $-2 < T^* < 2$. The upper limit corresponds to $\xi = 2$, as can be seen in the figure, and this is a cut-off value beyond which a Rayleigh surface wave cannot propagate. At the lower limit $\xi < 0$ and there is also a cut-off value of T^*, approximately -1.679. These limiting values of T^* are associated with the onset of instability of the underlying initially stressed configuration. For a related discussion of

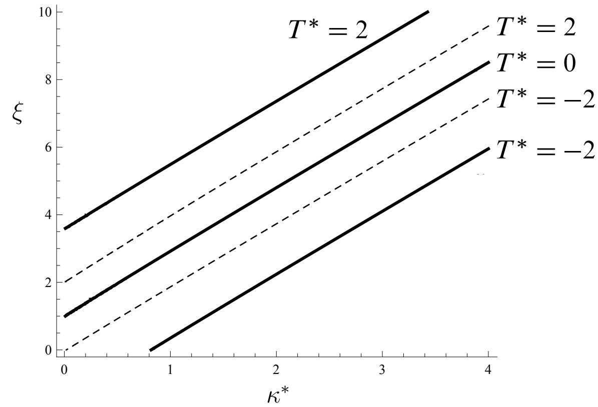

Figure 5. Dimensionless squared Rayleigh wave speed $\xi = \rho v^2/\mu$ versus the parameter κ^* for various values of T^*, with $\nu^* = 2.2$, $\gamma^* = 0$, and $\lambda^* = 0.15$ (solid lines) or $\lambda^* = -0.25$ (dashed lines).

such instability in the case of a pre-stressed (and deformed) half-space we refer to [Dowaikh and Ogden 1990]. Analogous cut-off values of T^* are evident also for the anisotropic models.

As a further illustration of the influence of the parameter λ^*, Figure 5 shows ξ plotted as a function of the parameter κ^* for each of the initial stress values $T^* = -2, 0, 2$ and for the set of parameters $\nu^* = 2.2$, $\gamma^* = 0$, $\lambda^* = 0.15$ compared with results for the set $\nu^* = 2.2$, $\gamma^* = 0$, $\lambda^* = -0.25$. Clearly, λ^* has a significant effect on the value of the wave speed (except, of course, for $T^* = 0$). Thus, as is the case with plane waves, the initial stress dependence of the nonlinear model (107) can have a strong influence on the value of the Rayleigh wave speed compared with its classical linear specialization. Moreover, other specific choices of the nonlinear model within the general framework outlined here can equally and even more substantially affect the wave speed, when, for example, invariants that are nonlinear in the initial stress are included in the form of W.

In this paper we have developed a general theory of transversely isotropic hyperelasticity incorporating initial stress and used a particular specialization of the theory to calculate the elasticity tensor for an undeformed initially stressed configuration. This was then used to examine the propagation of homogeneous plane waves and Rayleigh surface waves with particular reference to the effect of the initial stress. The results, which involve nonlinear terms in the initial stress, are significantly different from those based on the classical theory of Biot, which can be recovered as a special case of the present formulation.

Appendix A. Derivatives of the invariants

First derivatives. The first derivatives of the invariants $I_1, I_4, \ldots, I_9$ with respect to $\boldsymbol{F}$ are obtained, in component form, as

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_4}{\partial F_{i\alpha}} = 2M_\alpha F_{i\beta} M_\beta, \quad \frac{\partial I_5}{\partial F_{i\alpha}} = 2(M_\alpha F_{i\beta} C_{\beta\gamma} + C_{\alpha\beta} M_\beta F_{i\gamma}) M_\gamma,$$

$$\frac{\partial I_6}{\partial F_{i\alpha}} = 2T_{\alpha\beta} F_{i\beta}, \quad \frac{\partial I_7}{\partial F_{i\alpha}} = 2(T_{\alpha\beta} C_{\beta\gamma} + C_{\alpha\beta} T_{\beta\gamma}) F_{i\gamma},$$

$$\frac{\partial I_8}{\partial F_{i\alpha}} = (T_{\alpha\beta} M_\beta F_{i\gamma} + M_\alpha F_{i\beta} T_{\beta\gamma}) M_\gamma,$$
$$\frac{\partial I_9}{\partial F_{i\alpha}} = (T_{\alpha\beta} M_\beta F_{i\gamma} C_{\gamma\delta} + C_{\alpha\beta} M_\beta F_{i\gamma} T_{\gamma\delta} + M_\alpha F_{i\beta} C_{\beta\gamma} T_{\gamma\delta} + F_{i\beta} M_\beta C_{\alpha\gamma} T_{\gamma\delta}) M_\delta.$$

When evaluated in the reference configuration these reduce to

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2\delta_{i\alpha}, \quad 2\frac{\partial I_4}{\partial F_{i\alpha}} = 4M_i M_\alpha = \frac{\partial I_5}{\partial F_{i\alpha}}, \quad 2\frac{\partial I_6}{\partial F_{i\alpha}} = 4T_{i\alpha} = \frac{\partial I_7}{\partial F_{i\alpha}},$$
$$2\frac{\partial I_8}{\partial F_{i\alpha}} = 2(M_i T_{\alpha\beta} + M_\alpha T_{i\beta}) M_\beta = \frac{\partial I_9}{\partial F_{i\alpha}}.$$

Since the latter apply in the reference configuration there is strictly no distinction between Greek and Roman indices in this case.

Second derivatives. The second derivatives of the invariants $I_1, I_4, \ldots, I_9$ with respect to $\boldsymbol{F}$ are, in component form,

$$\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta}, \quad \frac{\partial^2 I_4}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij} M_\alpha M_\beta, \quad \frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij} T_{\alpha\beta},$$
$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}(M_\alpha C_{\beta\gamma} + M_\beta C_{\alpha\gamma}) M_\gamma + 2B_{ij} M_\alpha M_\beta + 2\delta_{\alpha\beta} F_{i\gamma} M_\gamma F_{j\delta} M_\delta + 2(F_{i\beta} F_{j\gamma} M_\alpha + F_{j\alpha} F_{i\gamma} M_\beta) M_\gamma,$$
$$\frac{\partial^2 I_7}{\partial F_{i\alpha}\partial F_{j\beta}} = 2[\delta_{ij}(T_{\alpha\gamma} C_{\beta\gamma} + T_{\beta\gamma} C_{\alpha\gamma}) + B_{ij} T_{\alpha\beta} + \delta_{\alpha\beta}\Sigma_{ij} + F_{i\beta} F_{j\gamma} T_{\alpha\gamma} + F_{j\alpha} F_{i\gamma} T_{\beta\gamma}],$$
$$\frac{\partial^2 I_8}{\partial F_{i\alpha}\partial F_{j\beta}} = \delta_{ij}(M_\alpha T_{\beta\gamma} + M_\beta T_{\alpha\gamma}) M_\gamma,$$
$$\begin{aligned}\frac{\partial^2 I_9}{\partial F_{i\alpha}\partial F_{j\beta}} = {}& \delta_{ij}(T_{\alpha\gamma} M_\gamma C_{\beta\delta} + T_{\beta\gamma} M_\gamma C_{\alpha\delta} + C_{\alpha\gamma} T_{\gamma\delta} M_\beta + C_{\beta\gamma} T_{\gamma\delta} M_\alpha) M_\delta \\ &+ \delta_{\alpha\beta}(F_{i\gamma} T_{\gamma\delta} F_{j\varepsilon} + F_{j\gamma} T_{\gamma\delta} F_{i\varepsilon}) M_\delta M_\varepsilon + B_{ij}(M_\alpha T_{\beta\gamma} + T_{\alpha\gamma} M_\beta) M_\gamma \\ &+ (M_\alpha F_{i\beta} F_{j\gamma} + M_\beta F_{j\alpha} F_{i\gamma}) T_{\gamma\delta} M_\delta + (T_{\alpha\gamma} F_{i\beta} F_{j\delta} + T_{\beta\gamma} F_{j\alpha} F_{i\delta}) M_\gamma M_\delta.\end{aligned}$$

When evaluated in the reference configuration these specialize to

$$\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta}, \quad \frac{\partial^2 I_4}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij} M_\alpha M_\beta, \quad \frac{\partial^2 I_6}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij} T_{\alpha\beta},$$
$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 6\delta_{ij} M_\alpha M_\beta + 2(\delta_{\alpha\beta} M_i M_j + \delta_{i\beta} M_j M_\alpha + \delta_{j\alpha} M_i M_\beta),$$
$$\frac{\partial^2 I_7}{\partial F_{i\alpha}\partial F_{j\beta}} = 6\delta_{ij} T_{\alpha\beta} + 2(\delta_{\alpha\beta} T_{ij} + \delta_{i\beta} T_{j\alpha} + \delta_{j\alpha} T_{i\beta}),$$
$$\frac{\partial^2 I_8}{\partial F_{i\alpha}\partial F_{j\beta}} = \delta_{ij}(M_\alpha T_{\beta\gamma} + M_\beta T_{\alpha\gamma}) M_\gamma,$$

$$\frac{\partial^2 I_9}{\partial F_{i\alpha}\partial F_{j\beta}} = 3\delta_{ij}(M_\alpha T_{\beta\gamma} + M_\beta T_{\alpha\gamma})M_\gamma + \delta_{\alpha\beta}(M_i T_{j\gamma} + M_j T_{i\gamma})M_\gamma + \delta_{i\beta}(M_\alpha T_{j\gamma} + M_j T_{\alpha\gamma})M_\gamma + \delta_{j\alpha}(M_i T_{\beta\gamma} + M_\beta T_{i\gamma})M_\gamma.$$

Appendix B. Connections with Biot's equations

It is instructive to relate the formulation of the incremental equations adopted here to that of Biot (see, for example, [Biot 1965] for details). For a compressible material, which we deal with here, the nominal stress tensor $\boldsymbol{S}$ and Cauchy stress tensor $\boldsymbol{\sigma}$ are related by $\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}$, where $\boldsymbol{F}$ is deformation gradient and $J = \det \boldsymbol{F}$. On taking the increment of this equation we obtain

$$\dot{\boldsymbol{S}} = J\boldsymbol{F}^{-1}\dot{\boldsymbol{\sigma}} + \dot{J}\boldsymbol{F}^{-1}\boldsymbol{\sigma} + J\dot{\overline{\boldsymbol{F}^{-1}}}\boldsymbol{\sigma} = J\boldsymbol{F}^{-1}[\dot{\boldsymbol{\sigma}} + (\operatorname{tr}\boldsymbol{L})\boldsymbol{\sigma} - \boldsymbol{L}\boldsymbol{\sigma}],$$

where $\boldsymbol{L} = \operatorname{grad}\boldsymbol{u}$ and $\boldsymbol{u}$ is the displacement vector. On updating the reference configuration from $\mathcal{B}_r$ to $\mathcal{B}_0$ we obtain

$$\dot{\boldsymbol{S}}_0 = \dot{\boldsymbol{\sigma}} + (\operatorname{tr}\boldsymbol{L})\boldsymbol{\sigma} - \boldsymbol{L}\boldsymbol{\sigma}.$$

Taking the divergence of both sides leads to

$$\operatorname{div}\dot{\boldsymbol{S}}_0 = \operatorname{div}\dot{\boldsymbol{\sigma}} + \operatorname{div}[(\operatorname{tr}\boldsymbol{L})\boldsymbol{\sigma} - \boldsymbol{L}\boldsymbol{\sigma}].$$

If the configuration $\mathcal{B}_0$ is uniform then we have

$$\operatorname{div}[(\operatorname{tr}\boldsymbol{L})\boldsymbol{\sigma} - \boldsymbol{L}\boldsymbol{\sigma}] = L_{kk,j}\sigma_{ji} - L_{jk,j}\sigma_{ki} = L_{kk,j}\sigma_{ji} - L_{jj,k}\sigma_{ki} = 0,$$

and the incremental equation of motion becomes

$$\operatorname{div}\dot{\boldsymbol{S}}_0 \equiv \operatorname{div}\dot{\boldsymbol{\sigma}} = \rho\boldsymbol{u}_{,tt}.$$

Now, Biot works in terms of components referred to different sets of axes, in particular a set of Cartesian axes and a set of axes obtained by rotation therefrom, with the rotation associated with the incremental deformation. Let $\boldsymbol{e}_i$, $i = 1, 2, 3$, be a Cartesian coordinate basis in $\mathcal{B}_0$, and let $\boldsymbol{e}'_i$, $i = 1, 2, 3$, be the axes obtained by the rotation. Then, $\boldsymbol{e}'_i \equiv \boldsymbol{e}_i + \boldsymbol{W}\boldsymbol{e}_i$, where the tensor $\boldsymbol{W}$ is the antisymmetric part of the displacement gradient $\boldsymbol{L}$. After the increment the total Cauchy stress is $\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}$, which has components on the rotated axes given by, to the first order in incremental quantities,

$$[(\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}})\boldsymbol{e}'_i]\cdot\boldsymbol{e}'_j = [(\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}})(\boldsymbol{e}_i + \boldsymbol{W}\boldsymbol{e}_i)]\cdot(\boldsymbol{e}_j + \boldsymbol{W}\boldsymbol{e}_j) = \sigma_{ij} + \dot{\sigma}_{ij} + \sigma_{ik}W_{kj} - W_{ik}\sigma_{kj},$$

where the symmetry of $\boldsymbol{\sigma}$ and the antisymmetry of $\boldsymbol{W}$ have been used, and σ_{ij} and W_{ij} are their components referred to the original axes (note that W_{ij} here are not the same as the second derivatives of the strain-energy function with respect to the invariants introduced in (29)). The component form of the incremental equation of motion with uniform $\boldsymbol{\sigma}$ therefore becomes

$$\dot{\sigma}_{ji,j} + \sigma_{ik}W_{kj,j} - \sigma_{jk}W_{ik,j} = \rho u_{i,tt}.$$

This is translated into Biot's notation [1965] by setting $\sigma_{ij} = S_{ij}$, $\dot{\sigma}_{ij} = s_{ij}$ and $W_{ij} = -\omega_{ij}$, and then yields Biot's equation

$$s_{ij,j} + S_{jk}\omega_{ik,j} + S_{ik}\omega_{jk,j} = \rho u_{i,tt}$$

appropriate for homogeneous S_{ij} in the absence of body forces [Biot 1965, p. 264]. Biot's constitutive equation for incremental deformations may be written in the form $\dot{\sigma}_{ij} = \mathcal{B}_{ijkl} L_{kl}$, where $L_{kl} = u_{k,l}$ and

$$\mathcal{B}_{ijkl} = \mathcal{B}_{jikl} = \mathcal{B}_{ijlk}.$$

Since $W_{ij} = (L_{ij} - L_{ji})/2$, it follows that

$$\mathcal{A}_{ijkl} L_{lk} = \mathcal{B}_{ijkl} L_{lk} - \tfrac{1}{2}(L_{ik} + L_{ki})\sigma_{kj} - \tfrac{1}{2}\sigma_{ik}(L_{kj} - L_{jk}) + L_{kk}\sigma_{ij},$$

and since this holds for arbitrary L_{ij} we deduce that

$$\mathcal{A}_{ijkl} = \mathcal{B}_{ijkl} - \tfrac{1}{2}\delta_{il}\sigma_{jk} - \tfrac{1}{2}\delta_{ik}\sigma_{jl} - \tfrac{1}{2}\delta_{jk}\sigma_{il} + \tfrac{1}{2}\delta_{jl}\sigma_{ik} + \delta_{kl}\sigma_{ij}.$$

This identity holds in the configuration $\mathcal{B}_0$ and when $\mathcal{B}_0$ is taken to coincide with $\mathcal{B}_r$ the Cauchy stress $\boldsymbol{\sigma}$ becomes the initial stress $\boldsymbol{T}$. Using the symmetry $\mathcal{A}_{ijkl} = \mathcal{A}_{klij}$, it follows that

$$\mathcal{B}_{ijkl} - \mathcal{B}_{klij} = \delta_{ij}\sigma_{kl} - \delta_{kl}\sigma_{ij},$$

which recovers a formula of Biot [1965, p. 71], albeit in different notation.

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