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VARIABLE-ORDER FINITE ELEMENTS FOR NONLINEAR, FULLY INTRINSIC BEAM EQUATIONS

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Fully intrinsic equations and boundary conditions involve only force, moment, velocity, and angular velocity variables, but no displacement or rotation variables. This paper presents variable-order finite elements for the geometrically exact, nonlinear, fully intrinsic equations for both nonrotating and rotating beams. The finite element technique allows for *hp*-adaptivity. Results show that these finite elements lead to very accurate solutions for the static equilibrium state as well as for modes and frequencies for infinitesimal motions about that state. For the same number of variables, the accuracy of the finite elements increases with the order of the finite element. The results based on the Galerkin approximation (which is a special case of the present approach) are the most accurate but require evaluation of complex integrals. Cubic elements are shown to provide a near optimal combination of accuracy and complexity.

1. Introduction

Beam-like structures are used in a wide range of applications, from rotor blades and high-aspect-ratio wings to nanosensors. The displacement formulation of the geometrically exact equations of motion for beams was presented in [Borri and Mantegazza 1985; Simo and Vu-Quoc 1988; Bauchau and Kang 1993]. A mixed formulation was presented in [Hodges 1990], which gave the partial differential equations of motion in so-called intrinsic form, following the terminology of [Green and Laws 1966; Reissner 1973] for beams and [Danielson 1970] for shells. Intrinsic equations are independent of the way displacement and rotation are parametrized. In the mixed formulation of [Hodges 1990], the kinematical partial differential equations relate the chosen displacement and rotation variables to the generalized strain and velocity measures. It was later discovered that one can have the complete beam formulation of [Hodges 1990] without carrying displacement or rotation variables as unknowns. This approach may be inferred from [Hegemier and Nair 1977] though exclusion of rotation variables was not mentioned therein; see [Hodges 2009] for additional details of the pertinent history. As a result, the fully intrinsic, geometrically exact partial differential equations of motion and kinematical partial differential equations were presented in [Hodges 2003]. These fully intrinsic equations and boundary conditions involve only force, moment, velocity and, angular velocity variables, with no displacement or rotation variables; see (1)–(6) below.

The goal of this paper is to present a variable-order finite element solution to the fully intrinsic equations. The research assumes that a suitable cross-sectional analysis is available for beams of arbitrary geometry and material distribution. These cross-sectional properties can be calculated using an analytical thin-walled theory [Volovoi and Hodges 2000; Patil and Johnson 2005] or computational FEM analysis

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[Giavotto et al. 1983; Cesnik and Hodges 1997; Yu et al. 2002] for general configurations. Although the work of [Volovoi and Hodges 2000] addresses both closed- and open-section beams, the present work pertains only to closed-section beams. The beam formulation of [Hodges 1990; 2003] is not appropriate for beams with open cross-section, which require a separate warping displacement variable and associated boundary conditions [Simo and Vu-Quoc 1991; Yu et al. 2005]. The book [Hodges 2006] gives a more comprehensive treatment for beams of all types.

Beam dynamics analysis can be broken up into five parts:

- Cross-sectional analysis to determine elastic constants and stress/strain recovery relations;
- Beam partial differential equations;
- Techniques for solving those equations (including discretization);
- Nonlinear model-order reduction schemes;
- Application to coupled-field problems such as aeroelasticity, active blades, and control design.

The focus of the present work is on the third item, that is, discretization and solution of the geometrically exact, intrinsic beam equations.

Beam equations. There are a number of geometrically exact formulations for the nonlinear dynamics of beams that can be used for analysis and design [Borri and Mantegazza 1985; Simo and Vu-Quoc 1988; Bauchau and Kang 1993]. The present work is based on the "intrinsic" formulation developed in [Hodges 1990], which can be written in a simple matrix form with only second-degree nonlinearities. To say that these equations are intrinsic is to say that they are independent of displacement and rotation measures. They were used along with generalized strain- and velocity-displacement measures to solve for beam dynamics, yielding excellent agreement with experimental results as given in [Hodges and Patil 2004]. Recently, a set of generalized strain-velocity compatibility relations were derived in [Hodges 2003] which, along with the equations of motion, make up a complete set of intrinsic equations that can be solved without using displacement/rotation measures for certain loading and boundary conditions. Although they incorporate all the nonlinearities and anisotropic couplings, these equations are very simple.

Discrete equations of motion. The equations described above are partial differential equations in space and time for 12 variables (force, moment, velocity, and angular velocity vectors). The solution of these equations requires discretization in space to convert the equations to ordinary differential equations in time. For example, one may use finite elements or a series of assumed functions. It is also possible to use a combination of the two, where the order of the polynomials within the elements as well as the number of elements both vary, leading to variable-order (or *hp*) finite elements [Gui and Babuška 1986].

A simple nonlinear finite element representation of the beam equations was presented in [Hodges 2003]. The FEM equations were used successfully to conduct nonlinear dynamic analysis and control design of integrally actuated helicopter blades [Traugott et al. 2006] as well as the nonlinear aeroelastic analysis of high-aspect-ratio flying wing configurations [Patil and Hodges 2006]. A new nonlinear, energy-consistent, Galerkin approach has been developed recently and can be found in [Patil and Althoff 2006]. The Galerkin approach leads to a highly accurate solution with a low computational cost. The application of the nonlinear Galerkin approach is feasible because of the simplicity of the intrinsic equations. Since the highest degree nonlinearity is quadratic, the Galerkin integrals can be evaluated exactly without resorting to numerical quadrature when the properties are uniform along the span. This contributes to the method's accuracy and efficiency.

The Galerkin approach leads to the exact nonlinear solution (to five decimal places) with as few as 10 assumed functions per variable. Also it is clear [Patil and Althoff 2006] from the slow convergence of the low-order FEM approach, that one would require very large number of finite element nodes to generate results with the same order of accuracy as those generated by the Galerkin approach using 10 functions per variable.

The nonlinear Galerkin approach leads to an approximate solution that is more accurate for a given number of unknowns (or that has fewer unknowns for a given level of accuracy). This is especially important if one needs to conduct numerous optimizations, simulations, and control scenarios. The limitation of the Galerkin approach is the handling of discontinuities. Thus, when one has a complex beam, such as a helicopter rotor blade with multiple changes in properties along the span, then it will be more accurate to break the beam into multiple finite elements, to each of which one can apply the Galerkin approach. This would lead to variable-order finite elements. It is the focus of this research to develop a variable-order finite element scheme to optimally model future structures, including helicopter blades and aircraft wings.

2. Nonlinear, intrinsic beam equations

The nonlinear, fully intrinsic governing equations for the dynamics of a nonuniform, initially curved and twisted, anisotropic beam undergoing large deflections and rotations are given as

$$F' + (\tilde{\mathrm{k}} + \tilde{\kappa})F + \mathrm{f} = \dot{P} + \widetilde{\Omega} P, \tag{1}$$

$$M' + (\tilde{\mathrm{k}} + \tilde{\kappa})M + (\tilde{\mathrm{e}}_1 + \tilde{\gamma})F + \mathrm{m} = \dot{H} + \widetilde{\Omega} H + \widetilde{V} P, \tag{2}$$

$$V' + (\tilde{\mathrm{k}} + \tilde{\kappa})V + (\tilde{\mathrm{e}}_1 + \tilde{\gamma})\Omega = \dot{\gamma}, \tag{3}$$

$$\Omega' + (\tilde{\mathrm{k}} + \tilde{\kappa})\Omega = \dot{\kappa}, \tag{4}$$

where $(\ )'$ denotes the partial derivative with respect to the axial coordinate of the undeformed beam, $\dot{(\ )}$ denotes the partial derivative with respect to time, $\tilde{z}$ is the skew-symmetric cross-product operator matrix corresponding to the column matrix z, $F(x,t)$ and $M(x,t)$ are the measure numbers of the internal force and moment vectors (cross-section stress resultants), $P(x,t)$ and $H(x,t)$ are the measure numbers of the linear and angular momentum vector (generalized momenta), $\gamma(x,t)$ and $\kappa(x,t)$ are the beam strains and curvatures (generalized strains), $V(x,t)$ and $\Omega(x,t)$ are velocity and angular velocity measures (generalized velocities), and $\mathrm{f}(x,t)$ and $\mathrm{m}(x,t)$ are the external force and moment measures. Measure numbers of all variables except for k are calculated in the deformed beam cross-sectional frame. The initial twist and curvature of the beam are represented by $\mathrm{k}(x) = \lfloor \mathrm{k}_1(x) \;\; \mathrm{k}_2(x) \;\; \mathrm{k}_3(x) \rfloor$, the measure numbers of which are in the undeformed beam cross-sectional frame. Finally, $\mathrm{e}_1 = \lfloor 1 \;\; 0 \;\; 0 \rfloor^T$. The first two equations in the above set are the equations of motion [Hodges 1990] while the last two are the intrinsic kinematical equations [Hodges 2003] derived from the generalized strain- and velocity-displacement equations.

The cross-sectional stress resultants of the beam are related to the generalized strains via the cross-sectional stiffnesses or flexibilities. These cross-sectional properties can be calculated using an analytical thin-walled theory [Volovoi and Hodges 2000; Patil and Johnson 2005] or computational FEM analysis

[Cesnik and Hodges 1997; Yu et al. 2002] for a general configuration. Such an analysis gives the linear constitutive law

$$\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{bmatrix} \mathrm{R} & \mathrm{S} \\ \mathrm{S}^T & \mathrm{T} \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix}, \tag{5}$$

where $\mathrm{R}(x)$, $\mathrm{S}(x)$, and $\mathrm{T}(x)$, are the cross-sectional flexibilities of the beam. This linear constitutive law is valid only for small strain, but the beam deflections and rotations due to deformation still may be large.

The generalized momenta are related to the generalized velocities via the cross-sectional inertia matrix

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu\Delta & -\mu\tilde{\xi} \\ \mu\tilde{\xi} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} = \begin{bmatrix} \mathrm{G} & \mathrm{K} \\ \mathrm{K}^T & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix}, \tag{6}$$

where $\mu(x)$, $\xi(x)$, $I(x)$ are, respectively, the mass per unit length, the mass center offset (a vector in the cross-section from the beam reference axis to the cross-sectional mass center), and the cross-sectional inertia matrix consisting of mass moments of inertia per unit length on the diagonals, with $I_{11} = I_{22} + I_{33}$, plus the mass product of inertia per unit length I_{23}.

Usually, the constitutive laws are used to replace some variables in terms of others. Here it was decided to express the generalized strains in terms of the cross-section stress resultants, allowing easy specification of zero flexibility, and the generalized momenta in terms of generalized velocities, allowing easy specification of zero inertia. Thus, the primary variables of interest are F, M, V, and Ω.

Finally the boundary conditions need to be specified. For the given beam of length L, there will be two boundary conditions at each end. In this paper we consider only primitive boundary conditions of the form

$$\begin{aligned} &V(0,t) = \mathrm{V}^0 \quad \text{or} \quad F(0,t) = \mathrm{F}^0, &&\Omega(0,t) = \Omega^0 \quad \text{or} \quad M(0,t) = \mathrm{M}^0, \\ &V(\mathrm{L},t) = \mathrm{V}^\mathrm{L} \quad \text{or} \quad F(\mathrm{L},t) = \mathrm{F}^\mathrm{L}, &&\Omega(\mathrm{L},t) = \Omega^\mathrm{L} \quad \text{or} \quad M(\mathrm{L},t) = \mathrm{M}^\mathrm{L}, \end{aligned} \tag{7}$$

Boundary conditions involving attached springs or bodies at the ends may be formulated in terms of fully intrinsic variables as well.

For ease of presentation, we consider a beam clamped at its root. It should be noted that the formulation as well as the conclusions presented are general enough to be applicable to all possible boundary conditions [Sotoudeh and Hodges 2011]. Thus, the assumed boundary conditions are

$$V(0,t) = \mathrm{V}^0, \quad \Omega(0,t) = \Omega^0, \quad F(\mathrm{L},t) = \mathrm{F}^\mathrm{L}, \quad M(\mathrm{L},t) = \mathrm{M}^\mathrm{L}. \tag{8}$$

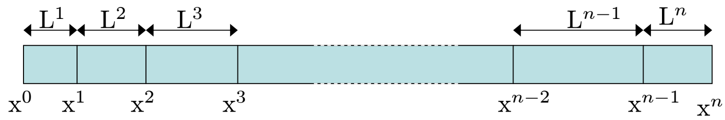
These equations are intrinsic and do not contain displacement and rotation variables. To calculate the displacements and rotations, the following equations, which relate the strains and curvatures to displacements and rotations, are used:

$$(r+u)' = C^{\mathrm{def}}(\gamma + e_1), \quad C^{\mathrm{def}\prime} = -(\tilde{\kappa} + \tilde{k})C^{\mathrm{def}}, \tag{9}$$

where r is the position vector of the beam axis from the origin of the reference frame, u is the displacement, and C^{def} is the rotation matrix of the deformed beam cross-sectional frame relative to the undeformed beam cross-sectional frame.

3. Energy-consistent weighting

Let us assume that the beam is discretized into n elements as shown in the figure. To create a finite element model we need to choose trial functions as well as weighting functions.



Let the solution in the i-th element be given by V^i, Ω^i, F^i, and M^i. We require that it satisfy (approximately) the equations of motion, the kinematic equations, and the boundary conditions given above. We also require that continuity equations be satisfied (approximately) between adjacent elements:

$$V^i(\mathrm{L}^i,t)=V^{i+1}(0,t),\quad \Omega^i(\mathrm{L}^i,t)=\Omega^{i+1}(0,t),\quad F^i(\mathrm{L}^i,t)=F^{i+1}(0,t),\quad M^i(\mathrm{L}^i,t)=M^{i+1}(0,t).$$

These continuity conditions will be modified for any node at which there is a concentrated mass, a rigid body, a nodal force or moment, or a kink in the axis.

Now consider the following weighting of the equations of motion, kinematical equations, continuity conditions, and boundary conditions:

$$\begin{aligned}
&\sum_{i=1}^{n}\Big\langle\int_0^{\mathrm{L}^i}\big\{V^{i^T}[\dot P^i+\widetilde{\Omega}^i P^i-F^{i\prime}-(\tilde{\mathrm{k}}^i+\tilde\kappa^i)F^i-\mathrm{f}^i]+\Omega^{i^T}[\dot H^i+\widetilde{\Omega}^i H^i+\widetilde{V}^i P^i-M^{i\prime}-(\tilde{\mathrm{k}}^i+\tilde\kappa^i)M^i\\
&\qquad-(\tilde{\mathrm{e}}_1+\tilde\gamma^i)F^i-\mathrm{m}^i]+F^{i^T}[\dot\gamma^i-V^{i\prime}-(\tilde{\mathrm{k}}^i+\tilde\kappa^i)V^i-(\tilde{\mathrm{e}}_1+\tilde\gamma^i)\Omega^i]+M^{i^T}[\dot\kappa^i-\Omega^{i\prime}-(\tilde{\mathrm{k}}^i+\tilde\kappa^i)\Omega^i]\big\}\,dx^i\Big\rangle\\
&\quad+\sum_{i=1}^{n-1}\big\langle F^{i+1}(0,t)[V^i(\mathrm{L}^i,t)-V^{i+1}(0,t)]+M^{i+1}(0,t)[\Omega^i(\mathrm{L}^i,t)-\Omega^{i+1}(0,t)]\\
&\quad+V^i(\mathrm{L}^i,t)[F^i(\mathrm{L}^i,t)-F^{i+1}(0,t)]+\Omega^i(\mathrm{L}^i,t)[M^i(\mathrm{L}^i,t)-M^{i+1}(0,t)]\big\rangle-F^1(0,t)^T[V^1(0,t)-\mathrm{V}^0]\\
&\qquad-M^1(0,t)^T[\Omega^1(0,t)-\Omega^0]+V^n(\mathrm{L}^n,t)^T[F^n(\mathrm{L}^n,t)-\mathrm{F}^{\mathrm{L}}]+\Omega^n(\mathrm{L}^n,t)^T[M^n(\mathrm{L}^n,t)-\mathrm{M}^{\mathrm{L}}]=0.
\end{aligned}$$

Note that the constitutive equations are not included, as these equations are satisfied exactly. Since the continuity conditions and the boundary conditions are satisfied weakly, we will not get exact satisfaction of the boundary conditions and the variables will not be continuous at the nodes.

We are conducting a weighted residual above and thus there are no guarantees in terms of energy conservation. Furthermore, we are adding the weighted residuals of various types of equations including equations describing kinetics, kinematics, and continuity. But, if we appropriately choose the weighting function (for each equation over space and relative to the other equations), energy conservation can result. We know that the exact solution satisfies energy conservation, but by the correct choice of the weighting functions we can maintain energy conservation in the reduced space. To prove energy conservation, we integrate these weighted residual equation by parts and simplify to give

$$\begin{aligned}
&\sum_{i=1}^{n}\int_0^{\mathrm{L}^i}(V^{i^T}\dot P^i+\Omega^{i^T}\dot H^i)\,dx+\sum_{i=1}^{n}\int_0^{\mathrm{L}^i}(F^{i^T}\dot\gamma^i+M^{i^T}\dot\kappa^i)\,dx\\
&=\sum_{i=1}^{n}\int_0^{\mathrm{L}^i}(V^{i^T}\mathrm{f}^i+\Omega^{i^T}\mathrm{m}^i)\,dx+[V^n(\mathrm{L}^n,t)^T\mathrm{F}^{\mathrm{L}}+\Omega^n(\mathrm{L}^n,t)^T\mathrm{M}^{\mathrm{L}}-F^1(0,t)^T\mathrm{V}^0-M^1(0,t)^T\Omega^0]. \quad (10)
\end{aligned}$$

The first term above is the rate of change of kinetic energy, and the second is the rate of change of potential energy. The third is the rate of work done (power) due to applied forces in the interior of the beam, and the fourth is the power due to applied forces at the boundaries. The equation states that the rate of change of the energy of the beam is equal to the rate of work done on the beam. Thus, the above weighting of all the equations leads to an energy balance, on the basis of which we derive the FEM equations. It should be noted that energy conservation is not an approximation but is satisfied exactly.

4. Variable-order FEM

The independent trial functions used are the shifted Legendre polynomials [Abramowitz and Stegun 1964], denoted by $\mathcal{P}^j(\bar{x})$, which constitute a complete set of polynomials that are orthogonal over the shifted interval $0 \le \bar{x} \le 1$, so that

$$\int_0^1 \mathcal{P}^j(\bar{x})\mathcal{P}^k(\bar{x})\,d\bar{x} = \frac{\delta_{jk}}{2i+1}. \tag{11}$$

These polynomials can be obtained from the recursive relations

$$\mathcal{P}^0(\bar{x}) = 1, \quad \mathcal{P}^1(\bar{x}) = 2\bar{x}-1, \quad \mathcal{P}^{i+1}(\bar{x}) = \frac{(2i+1)(2\bar{x}-1)\mathcal{P}^i(\bar{x}) - i\,\mathcal{P}^{i-1}(\bar{x})}{i+1}. \tag{12}$$

The use of orthogonality makes certain of the linear coefficient matrices diagonal.

Expanding all twelve variables in terms of these polynomials, one finds that the unknowns can be written as

$$\begin{aligned} V^i(x^i,t) &= \sum_{j=0}^{m} \mathcal{P}^j(\bar{x}^i) v^{j,i}(t), & \Omega^i(x^i,t) &= \sum_{j=0}^{m} \mathcal{P}^j(\bar{x}^i)\omega^{j,i}(t), \\ F^i(x^i,t) &= \sum_{j=0}^{m} \mathcal{P}^j(\bar{x}^i) f^{j,i}(t), & M^i(x^i,t) &= \sum_{j=0}^{m} \mathcal{P}^j(\bar{x}^i) m^{j,i}(t), \end{aligned} \tag{13}$$

where $\bar{x}^i = x^i/\mathrm{L}^i$, and $v^{j,i}$, $\omega^{j,i}$, $f^{j,i}$, and $f^{j,i}$ are column matrices of the unknowns of the formulation, corresponding to the i-th element and j-th order. With $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m$, we have a total of $12(m+1)n$ variables.

The FEM equations for the i-th element can be derived based on the energy-conserving integral equation (10) as

$$\begin{aligned} \int_0^{\mathrm{L}^i} \mathcal{P}^k\big[&(\mathrm{G}^i\mathcal{P}^j\dot{v}^{j,i} + \mathrm{K}^i\mathcal{P}^j\dot{\omega}^{j,i}) + \widetilde{\mathcal{P}^l\omega^{l,i}}(\mathrm{G}^i\mathcal{P}^j v^{j,i} + \mathrm{K}^i\mathcal{P}^j\omega^{j,i}) - \mathcal{P}^{j\prime} f^{j,i} \\ &- (\tilde{\mathrm{k}}^i + \widetilde{\mathrm{S}^{i^T}\mathcal{P}^l f^{l,i}} + \widetilde{\mathrm{T}^i\mathcal{P}^l m^{l,i}})\mathcal{P}^j f^{j,i} - \mathrm{f}^i\big]dx^i + \mathcal{P}^k(1)[\mathcal{P}^j(1) f^{j,i} - \mathcal{P}^j(0) f^{j,i+1}] = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} \int_0^{\mathrm{L}^i} \mathcal{P}^k\big[&(\mathrm{K}^{i^T}\mathcal{P}^j\dot{v}^{j,i} + I^i\mathcal{P}^j\dot{\omega}^{j,i}) + \widetilde{\mathcal{P}^l\omega^{l,i}}(\mathrm{K}^{i^T}\mathcal{P}^j v^{j,i} + I^i\mathcal{P}^j\omega^{j,i}) + \widetilde{\mathcal{P}^l v^{l,i}}(\mathrm{G}^i\mathcal{P}^j v^{j,i} + \mathrm{K}^i\mathcal{P}^j\omega^{j,i}) - \mathcal{P}^{j\prime} m^{j,i} \\ &- (\tilde{\mathrm{k}}^i + \widetilde{\mathrm{S}^{i^T}\mathcal{P}^l f^{l,i}} + \widetilde{\mathrm{T}^i\mathcal{P}^l m^{l,i}})\mathcal{P}^j m^{j,i} - (\tilde{\mathrm{e}}_1 + \widetilde{\mathrm{R}^i\mathcal{P}^l f^{l,i}} + \widetilde{\mathrm{S}^i\mathcal{P}^l m^{l,i}})\mathcal{P}^j f^{j,i} - \mathrm{m}^i\big]dx^i \\ &+ \mathcal{P}^k(1)[\mathcal{P}^j(1) m^{j,i} - \mathcal{P}^j(0) m^{j,i+1}] = 0, \end{aligned} \tag{15}$$

$$\int_0^{\mathrm{L}^i} \mathcal{P}^k[(\mathrm{R}^i\mathcal{P}^j\dot{f}^{j,i}+\mathrm{S}^i\mathcal{P}^j\dot{m}^{j,i})-\mathcal{P}^{j\prime}v^{j,i}-(\tilde{\mathrm{k}}^i+\widetilde{\mathrm{S}^{i^T}\mathcal{P}^l f^{l,i}}+\widetilde{\mathrm{T}^i\mathcal{P}^l m^{l,i}})\mathcal{P}^j v^{j,i} \\ -(\tilde{\mathrm{e}}_1+\widetilde{\mathrm{R}^i\mathcal{P}^l f^{l,i}}+\widetilde{\mathrm{S}^i\mathcal{P}^l m^{l,i}})\mathcal{P}^j\omega^{j,i}]dx^i+\mathcal{P}^k(0)[\mathcal{P}^j(1)v^{j,i-1}-\mathcal{P}^j(0)v^{j,i}]=0, \quad (16)$$

$$\int_0^{\mathrm{L}^i} \mathcal{P}^k[(\mathrm{S}^{i^T}\mathcal{P}^j\dot{f}^{j,i}+\mathrm{T}^i\mathcal{P}^j\dot{m}^{j,i})-\mathcal{P}^{j\prime}\omega^{j,i}-(\tilde{\mathrm{k}}^i+\widetilde{\mathrm{S}^{i^T}\mathcal{P}^l f^{l,i}}+\widetilde{\mathrm{T}^i\mathcal{P}^l m^{l,i}})\mathcal{P}^j\omega^{j,i}]dx^i \\ +\mathcal{P}^k(0)[\mathcal{P}^j(1)\omega^{j,i-1}-\mathcal{P}^j(0)\omega^{j,i}]=0. \quad (17)$$

In these equations R^i, S^i, and T^i are the cross-sectional flexibility coefficients for the i-th element, G^i, K^i, I^i are the cross-sectional inertia coefficients for the i-th element, k^i is the initial curvature for the i-th element, and f^i and m^i define the loading for the i-th element. In the equations, summation is assumed over indices j and l. Thus we have a set of equation for each i (element) and k (order), giving us a total of $12(m+1)n$ equations for as many unknowns.

We need to calculate the above integrals so as to obtain the equations in a form suitable for solution. For demonstration, we assume that the cross-sectional properties, the initial twist and curvature, and the distributed loading are all constant within each element. With the above assumptions, the FEM equations for the i-th element can be derived and we obtain the discretized equations of motion as

$$\mathcal{A}^{kj}\mathrm{L}^i(\mathrm{G}^i\dot{v}^{j,i}+\mathrm{K}^i\dot{\omega}^{j,i})+\mathcal{C}^{kjl}\mathrm{L}^i\tilde{\omega}^{l,i}(\mathrm{G}^i v^{j,i}+\mathrm{K}^i\omega^{j,i})-\mathcal{B}^{kj}f^{j,i} \\ -\mathcal{A}^{kj}\mathrm{L}^i\tilde{\mathrm{k}}^i f^{j,i}-\mathcal{C}^{kjl}\mathrm{L}^i(\widetilde{\mathrm{S}^{i^T}f^{l,i}}+\widetilde{\mathrm{T}^i m^{l,i}})f^{j,i}-\mathcal{D}^k\mathrm{L}^i\mathrm{f}^i+\mathcal{P}_1^k\mathcal{P}_1^j f^{j,i}-\mathcal{P}_1^k\mathcal{P}_0^j f^{j,i+1}=0, \quad (18)$$

$$\mathcal{A}^{kj}\mathrm{L}^i(\mathrm{K}^{i^T}\dot{v}^{j,i}+I^i\dot{\omega}^{j,i})+\mathcal{C}^{kjl}\mathrm{L}^i\tilde{\omega}^{l,i}(\mathrm{K}^{i^T}v^{j,i}+I^i\omega^{j,i})+\mathcal{C}^{kjl}\mathrm{L}^i\tilde{v}^{l,i}(\mathrm{G}^i v^{j,i}+\mathrm{K}^i\omega^{j,i})-\mathcal{B}^{kj}m^{j,i} \\ -\mathcal{A}^{kj}\mathrm{L}^i\tilde{\mathrm{k}}^i m^{j,i}-\mathcal{C}^{kjl}\mathrm{L}^i(\widetilde{\mathrm{S}^{i^T}f^{l,i}}+\widetilde{\mathrm{T}^i m^{l,i}})m^{j,i}-\mathcal{A}^{kj}\mathrm{L}^i\tilde{\mathrm{e}}_1 f^{j,i}-\mathcal{C}^{kjl}\mathrm{L}^i(\widetilde{\mathrm{R}^i f^{l,i}}+\widetilde{\mathrm{S}^i m^{l,i}})f^{j,i} \\ -\mathcal{D}^k\mathrm{L}^i\mathrm{m}^i+\mathcal{P}_1^k\mathcal{P}_1^j m^{j,i}-\mathcal{P}_1^k\mathcal{P}_0^j m^{j,i+1}=0, \quad (19)$$

$$\mathcal{A}^{kj}\mathrm{L}^i(\mathrm{R}^i\dot{f}^{j,i}+\mathrm{S}^i\dot{m}^{j,i})-\mathcal{B}^{kj}v^{j,i}-\mathcal{A}^{kj}\mathrm{L}^i\tilde{\mathrm{k}}^i v^{j,i}-\mathcal{C}^{kjl}\mathrm{L}^i(\widetilde{\mathrm{S}^{i^T}f^{l,i}}+\widetilde{\mathrm{T}^i m^{l,i}})v^{j,i} \\ -\mathcal{A}^{kj}\mathrm{L}^i\tilde{\mathrm{e}}_1\omega^{j,i}-\mathcal{C}^{kjl}\mathrm{L}^i(\widetilde{\mathrm{R}^i f^{l,i}}+\widetilde{\mathrm{S}^i m^{l,i}})\omega^{j,i}-\mathcal{P}_0^k\mathcal{P}_0^j v^{j,i}+\mathcal{P}_0^k\mathcal{P}_1^j v^{j,i-1}=0, \quad (20)$$

$$\mathcal{A}^{kj}\mathrm{L}^i(\mathrm{S}^{i^T}\dot{f}^{j,i}+\mathrm{T}^i\dot{m}^{j,i})-\mathcal{B}^{kj}\omega^{j,i}-\mathcal{A}^{kj}\mathrm{L}^i\tilde{\mathrm{k}}^i\omega^{j,i}-\mathcal{C}^{kjl}\mathrm{L}^i(\widetilde{\mathrm{S}^{i^T}f^{l,i}}+\widetilde{\mathrm{T}^i m^{l,i}})\omega^{j,i} \\ -\mathcal{P}_0^k\mathcal{P}_0^j\omega^{j,i}+\mathcal{P}_0^k\mathcal{P}_1^j\omega^{j,i-1}=0. \quad (21)$$

In the above equations summation is implied over indices j and l, and $\mathcal{A}^{kj}$, $\mathcal{B}^{kj}$, $\mathcal{C}^{kjl}$, and $\mathcal{D}^k$ are dimensionless integrals, given by

$$\begin{aligned} \mathcal{A}^{kj}&=\int_0^1\mathcal{P}^k(\bar{x})\mathcal{P}^j(\bar{x})\,d\bar{x}, & \mathcal{B}^{kj}&=\int_0^1\mathcal{P}^k(\bar{x})\big(\mathcal{P}^j(\bar{x})\big)'d\bar{x}, \\ \mathcal{C}^{kjl}&=\int_0^1\mathcal{P}^k(\bar{x})\mathcal{P}^j(\bar{x})\mathcal{P}^l(\bar{x})\,d\bar{x}, & \mathcal{D}^k&=\int_0^1\mathcal{P}^k(\bar{x})\,d\bar{x}. \end{aligned} \quad (22)$$

Now, representing all the system unknowns of the i-th element as

$$q^i(t) = \begin{Bmatrix} v^{0,i}(t) \\ \omega^{0,i}(t) \\ f^{0,i}(t) \\ m^{0,i}(t) \\ \vdots \\ v^{m,i}(t) \\ \omega^{m,i}(t) \\ f^{m,i}(t) \\ m^{m,i}(t) \end{Bmatrix} \quad \text{and} \quad \boldsymbol{q}(t) = \begin{Bmatrix} q^1(t) \\ q^2(t) \\ \vdots \\ q^n(t) \end{Bmatrix}, \tag{23}$$

the complete system consists of $N = 12n(m+1)$ equations and unknowns. The equations can then be written in the form

$$\boldsymbol{A}_{ji}\dot{\boldsymbol{q}}_i + \boldsymbol{B}_{ji}\boldsymbol{q}_i + \boldsymbol{C}_{jik}\boldsymbol{q}_i\boldsymbol{q}_k + \boldsymbol{D}_j = 0, \tag{24}$$

where summation is assumed over the indices i and k.

5. Results

The equations were solved using the variable-order FEM for a simple prismatic beam case with these data:

Span	16 m
Chord	1 m
Mass per unit length	0.75 kg/m
Mom. inertia (50% chord)	0.1 kg m
Spanwise elastic axis	50% chord
Center of gravity	50% chord
Bending rigidity	2×10^4 N m^2
Torsional rigidity	1×10^4 N m^2
Bending rigidity (chordwise)	4×10^6 N m^2
Shear/extensional rigidity	∞

Three examples are presented: (1) a beam undergoing large deformation due to a tip follower force, (2) the natural frequencies and modeshapes of a beam, and (3) the natural frequencies and modeshapes of a rotating beam (this involves calculation of the nonlinear steady state and linearizing the system about this nonlinear steady state).

Let us first consider a beam loaded at the tip with a transverse follower force. As the force increases, the deformation increases; and due to the large deformation, the force direction as well as its position relative to the beam root changes. The correct nonlinear solution thus has a lower root bending moment as compared to the linear solution. The exact solution to the problem satisfies the transcendental equation

$$\frac{1}{6}\bar{m}\sqrt{4-\bar{m}^4\bar{p}^2} + \frac{2\sqrt{2}}{3\sqrt{\bar{p}}}F\left[\sin^{-1}\left(\frac{\bar{m}\sqrt{\bar{p}}}{\sqrt{2}}\right)\middle| -1\right] = 1-\bar{x}, \tag{25}$$

where $\bar{m}$ is the bending moment made dimensionless by PL, $\bar{p}$ is the tip force made dimensionless by EI/L^2, $\bar{x}$ is the axial coordinate made dimensionless by L, and $F(\phi|k)$ is the elliptic integral of the first kind with $k = -1$ and $\sin\phi = \bar{m}\sqrt{\bar{p}}/\sqrt{2}$.

Figure 1 shows the root bending moment calculated using variable-order finite elements compared to exact results for $\bar{p} = 3$. Figure 1a shows the convergence of the root bending moment as the order of the system increases. The red line corresponds to the *h*-version, and the blue line corresponds to the *p*-method. Finally, the green dots correspond to the *hp*-method. As expected, for this simple case, the Galerkin approximation (the *p*-version) is the best of the three. It should be noted that when the Galerkin approximation is applied to beams with properties varying along the span, it may not be possible to calculate the integrals exactly. Thus, the Galerkin approach may become computationally intensive for a general configuration. Figure 1b shows the convergence of the various methods. The Galerkin approximation is seen to reach the exact result with error of the order of machine precision using twelfth-order polynomials. The linear, first-order finite elements have about third-order convergence, that is, for every doubling of the number of finite elements, the error decreases by a factor of eight (that is, 2^3). Figure 1c shows the convergence of the error for finite elements of various orders. The quadratic finite elements exhibit fifth-order convergence, and the cubic finite elements have seventh-order convergence. Finally, the quartic finite elements show a whopping ninth-order convergence, that is, for every doubling of the

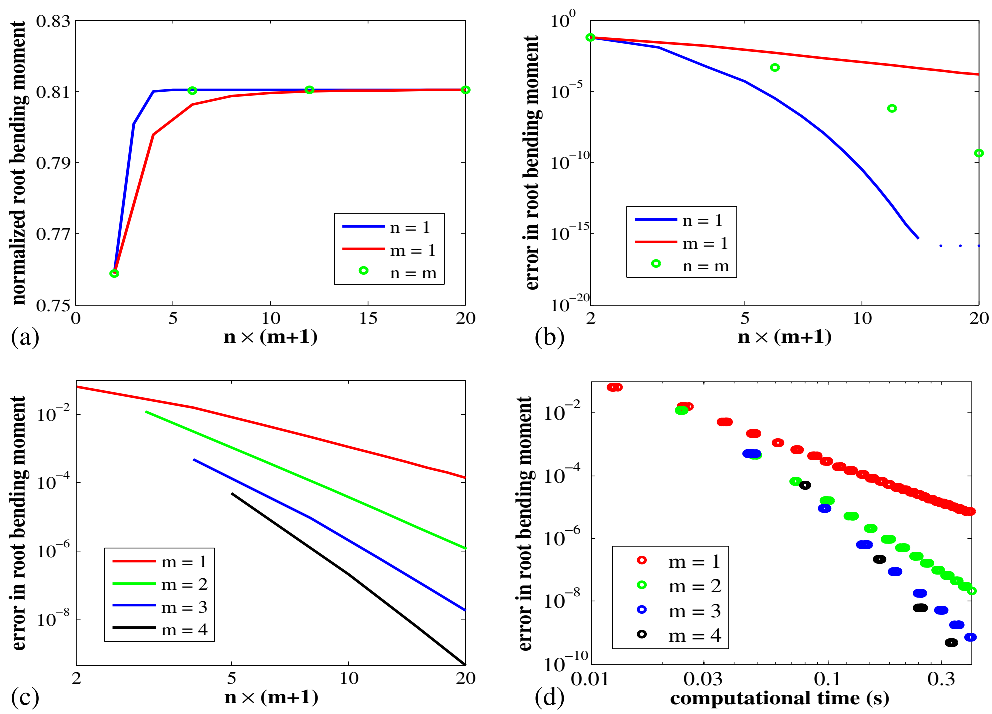


Figure 1. Results for the root bending moment due to follower force at the tip: (a) convergence of root bending moment; (b) error in root bending moment; (c) error in root bending moment for various model orders; (d) error in root bending moment relative to computational time.

number of finite elements there is a reduction in the error by a factor of 512. In other words, the element of n-th order has a convergence of order $2n+1$, so that doubling of the number of elements decreases the error by a factor of 2^{2n+1}.

To further understand the effect of order on the accuracy and computational time, we have plotted the error in results relative to computational time for various order finite elements. It should be noted that the computational time used in this plot does not include time for calculation of the analytical integrals, which can be calculated once for constant properties and stored. For variable properties one will have to recalculate the integrals which can take considerable time. In the present case for a constant property beam, for the same amount of time (0.4 s) we can use 54×12 variables (27 elements) for $m=1$, 45×12 variables (15 elements) for $m=2$, 32×12 variables (8 elements) for $m=3$, and between 20×12 and 25×12 variables (4–5 elements) for $m=4$. One can see that in terms of computational time (even without recalculation of integrals), the various order finite elements are much closer together. The linear finite elements still have relatively slow convergence, but the quadratic finite elements have behavior more like cubic ones, and the cubic finite elements are still more like the quartic ones. The calculations were done using the Matlab linear equation solver for sparse matrices. So, we can see that sparsity matters and we can analyze more variables with lower order finite elements, but the order of accuracy of higher-order finite elements increases at a faster rate than the computational time and thus they are in general preferable. For beams with varying properties one may increase efficiency by using more elements so that the preference will be for lower-order finite elements. One can recalculate the integrals and matrices for higher-order elements analytically or by Gauss integrations by taking into account the variations, but it may increase substantially the computational effort. Thus, one should use very high-order finite elements or the Galerkin form only if model order reduction is important (for example, in control design) or if we are reusing the calculated integrals multiple times (for example, for time marching). Finally, for problems with multiple (stiffness or load) discontinuities, we would have no choice but to place nodes at the discontinuities, as the solution using higher-order elements (without nodes at discontinuities) will not give good results near the discontinuities. Again, *hp* finite elements are good in that one can pick and choose an appropriate order depending on the problem at hand.

Now consider the frequencies of nonrotating as well as rotating beams. Table 1 lists the calculated frequencies and compares them with exact results from [Wright et al. 1982]. The frequency predictions

	Cantilevered beam $\omega=0,\ v=0$				Rotating cantil. beam $\omega=3.189$ rad/s, $v=0$				Rot. cantil. beam with offset ω=3.189 rad/s, v=51.03 m/s			
Mode	Exact	$n=9$ $m=1$	$n=3$ $m=3$	$n=1$ $m=9$	Exact	$n=9$ $m=1$	$n=3$ $m=3$	$n=1$ $m=9$	Exact	$n=9$ $m=1$	$n=3$ $m=3$	$n=1$ $m=9$
1st bending	2.243	2.243	2.243	2.243	4.114	4.114	4.114	4.114	5.703	5.703	5.703	5.703
2nd bending	14.06	14.03	14.06	14.06	16.23	16.21	16.23	16.23	18.72	18.69	18.72	18.72
3rd bending	39.36	39.22	39.38	39.36	41.59	41.44	41.62	41.59	44.50	44.33	44.53	44.50
1st torsion	31.05	31.05	31.05	31.05								
2nd torsion	93.14	93.17	93.14	93.14								

Table 1. Beam structural frequencies (rad/s).

for a nonrotating as well as a rotating beam using the present approach with $mn = 9$ are shown. For $m = 9$ and $n = 1$, that is, a single, high-order element, the frequencies are obtained to three significant digits for both the bending and torsion modes. This approach is equivalent to the Galerkin approach discussed in [Patil and Althoff 2006] and to formulations commonly referred to as p-version [Babuška et al. 1981]. On the other hand, for $m = 1$ and $n = 9$, the maximum number of the crudest possible elements for $mn = 9$, the solution is not as accurate, leading to errors greater than 0.3% for the third bending mode. This approach corresponds to the lowest-order FEM method, commonly referred to as h-version, consisting of linear shape functions. Finally, the case of $m = 3$ and $n = 3$ is a good balance between these two extreme approaches. Here cubic polynomials are used to represent the variables in each of three elements. The results are quite good with negligible errors.

The left column of Figure 2 presents the convergence of results for the first bending mode of a nonrotating beam. Figure 2a shows the convergence of frequencies with increase in the order of the system. As expected, again the Galerkin approximation is the best of the three. Figure 2b shows the convergence of the various methods. The Galerkin approximation is seen to reach the exact result with error of the order of machine precision using eighth-order polynomials. Figure 2c shows the convergence of the error in the first frequency for finite elements of various orders. Similar to the nonlinear solution results, for the linearized perturbation results the linear finite element shows third-order convergence, quadratic finite elements show a fifth-order convergence, cubic finite elements show a seventh-order convergence, and quartic finite elements show a ninth-order convergence. Finally, Figure 2d shows the convergence of mode shape for three implementations. The mode shape obtained by the Galerkin approach is the closest to the exact mode shape. The mode shape predicted by the lowest-order, linear finite elements shows deviation from the exact mode shape and is discontinuous as expected, because the continuity conditions are weakly satisfied. Similarly, the mode shape does not satisfy the boundary conditions exactly because of weak satisfaction of the boundary conditions.

The right column of Figure 2 shows the corresponding plots for the first torsion mode. For the torsional frequencies the Galerkin approximation again leads to the most accurate results while the accuracy increases with the increase in the order of the finite element. The order of the relative error is third-order, fifth-order, seventh-order and, ninth-order for the linear, quadratic, cubic, and quartic elements respectively.

Figure 3 shows the results for the first bending mode of a rotating beam with root velocity. For the rotating beam, the static steady-state solution is nontrivial. Thus, the accuracy of the frequencies obtained from linearizing about the nonlinear steady state is dependent on the accuracy of the steady-state solution and the accuracy of the linearized perturbation. We obtain the exact steady-state solution for finite elements of second and higher order. The errors are in general higher for the rotating beam as compared to the nonrotating beam. Furthermore, the rate of convergence for the rotating beam is slightly slower than that of the nonrotating beam.

We have addressed the impact of bandedness or sparsity of the matrices in the variable-order finite element formulation. For low-order elements, the coefficient matrices are very sparse and one can take advantage of this. In addition, the coefficient matrices for low-order elements can be calculated using lower-order Gauss integration thus further reducing the computational time. The higher-order element, though more accurate, may not be computationally as efficient as the low-order element. This aspect has not been addressed in the present paper. For example, a global Galerkin approximation requires

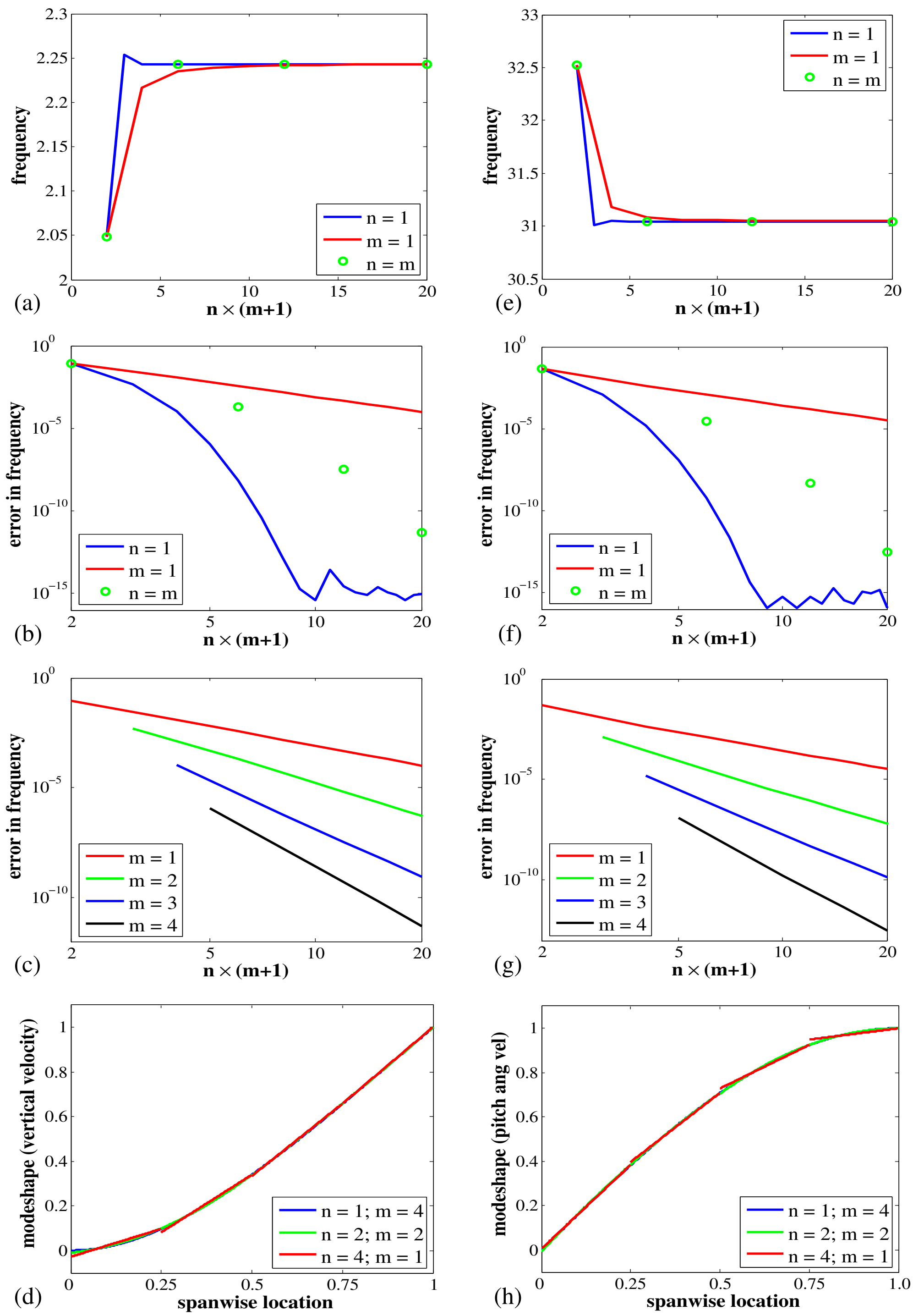


Figure 2. Results for the first bending mode (left column) and the first torsion mode (right column) of a nonrotating cantilevered beam: (a,e) convergence of frequency; (b,f) error in frequency; (c,g) error in frequency; (d,h) convergence of mode shape.

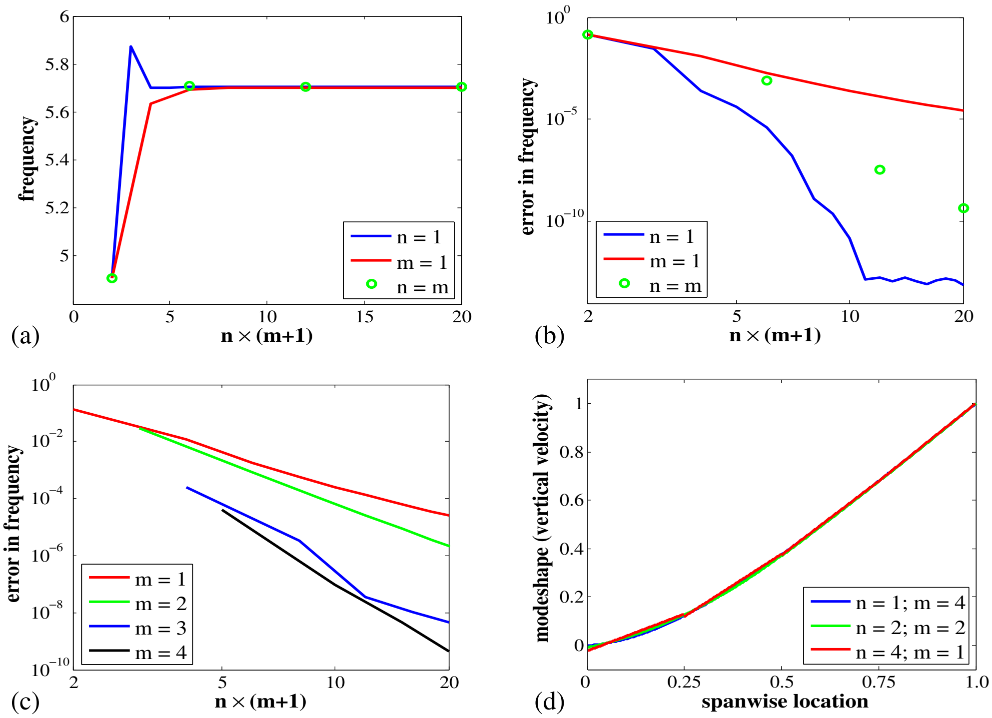


Figure 3. Results for the first bending mode of a rotating cantilevered beam with offset: (a) convergence of first bending frequency; (b) error in first bending frequency; (c) error in first bending frequency; (d) convergence of first bending mode shape.

calculation of complex integrals and will lead to fully populated matrices. But, if one is interested in using nonlinear beam analysis as a part of a multidisciplinary analysis (aeroelasticity), preliminary design (HALE aircraft, helicopter flight mechanics), and control synthesis, the reduced number of degrees of freedom is essential. We have presented a way to calculate the nonlinear dynamics of the beam accurately using a low number of degrees of freedom that will lead to significant advancement in nonlinear aeroelastic calculations (which lead to loss of sparsity anyway), flight dynamics simulations (with complete geometrically nonlinear aeroelasticity), and design optimization. A detailed assessment of the computational cost of the method vis-à-vis the accuracy for various types of beams and various applications will be addressed in a later paper.

6. Conclusions

A variable-order finite element technique is presented and applied to beams with uniform properties along the span. This technique is based on a geometrically exact, fully intrinsic formulation. The results presented show that one can obtain approximately third-order, fifth-order, seventh-order, and ninth-order convergence for the linear, quadratic, cubic, and quartic finite elements. It is recommended that one use quadratic- or higher-order finite elements for a better approximation of mode shapes. The cubic finite elements provide an especially good balance of accuracy, computational efficiency, and applicability to general configurations. Additional work needs to be done to assess the method's accuracy when applied to beams with properties varying along the span.

References

[Abramowitz and Stegun 1964] M. Abramowitz and I. A. Stegun (editors), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series **55**, U.S. Government Printing Office, Washington, DC, 1964.

[Babuška et al. 1981] I. Babuška, B. A. Szabo, and I. N. Katz, "The p-version of the finite element method", *SIAM J. Numer. Anal.* **18**:3 (1981), 515–545.

[Bauchau and Kang 1993] O. A. Bauchau and N. K. Kang, "A multibody formulation for helicopter structural dynamic analysis", *J. Am. Helicopter Soc.* **38**:2 (1993), 3–14.

[Borri and Mantegazza 1985] M. Borri and P. Mantegazza, "Some contributions on structural and dynamic modeling of helicopter rotor blades", *Aerotec. Missili Spaz.* **64**:9 (1985), 143–154.

[Cesnik and Hodges 1997] C. E. S. Cesnik and D. H. Hodges, "VABS: a new concept for composite rotor blade cross-sectional modeling", *J. Am. Helicopter Soc.* **42**:1 (1997), 27–38.

[Danielson 1970] D. A. Danielson, "Simplified intrinsic equations for arbitrary elastic shells", *Int. J. Eng. Sci.* **8**:3 (1970), 251–259.

[Giavotto et al. 1983] V. Giavotto, M. Borri, P. Mantegazza, G. Ghiringhelli, V. Carmaschi, G. C. Maffioli, and F. Mussi, "Anisotropic beam theory and applications", *Comput. Struct.* **16**:1-4 (1983), 403–413.

[Green and Laws 1966] A. E. Green and N. Laws, "A general theory of rods", *Proc. R. Soc. Lond. A* **293**:1433 (1966), 145–155.

[Gui and Babuška 1986] W. Gui and I. Babuška, "The *h, p* and *h-p* versions of the finite element methods in 1 dimension, III: The adaptive *h-p* version", *Numer. Math.* **49**:6 (1986), 659–683.

[Hegemier and Nair 1977] G. A. Hegemier and S. Nair, "A nonlinear dynamical theory for heterogeneous, anisotropic, elastic rods", *AIAA J.* **15**:1 (1977), 8–15.

[Hodges 1990] D. H. Hodges, "A mixed variational formulation based on exact intrinsic equations for dynamics of moving beams", *Int. J. Solids Struct.* **26**:11 (1990), 1253–1273.

[Hodges 2003] D. H. Hodges, "Geometrically exact, intrinsic theory for dynamics of curved and twisted anisotropic beams", *AIAA J.* **41**:6 (2003), 1131–1137.

[Hodges 2006] D. H. Hodges, *Nonlinear composite beam theory*, Progress in Astronautics and Aeronautics **213**, AIAA, Reston, VA, 2006.

[Hodges 2009] D. H. Hodges, "Erratum: geometrically exact, intrinsic theory for dynamics of curved and twisted anisotropic beams", *AIAA J.* **47**:5 (2009), 1308–1309.

[Hodges and Patil 2004] D. H. Hodges and M. J. Patil, "Technical note: correlation of geometrically-exact beam theory with the Princeton data", *J. Am. Helicopter Soc.* **49**:3 (2004), 357–360.

[Patil and Althoff 2006] M. J. Patil and M. Althoff, "Energy-consistent, Galerkin approach for the nonlinear dynamics of beams using mixed, intrinsic equations", in *47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference* (Newport, RI, 2006), AIAA, Reston, VA, 2006.

[Patil and Hodges 2006] M. J. Patil and D. H. Hodges, "Flight dynamics of highly flexible flying wings", *J. Aircraft* **43**:6 (2006), 1790–1799.

[Patil and Johnson 2005] M. J. Patil and E. R. Johnson, "Cross-sectional analysis of anisotropic, thin-walled, closed-section beams with embedded strain actuation", in *Proceedings of the 46th AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics and materials conference* (Austin, TX, 2005), AIAA, Reston, VA, 2005.

[Reissner 1973] E. Reissner, "On one-dimensional large-displacement finite-strain beam theory", *Stud. Appl. Math.* **52**:2 (1973), 87–95.

[Simo and Vu-Quoc 1988] J. C. Simo and L. Vu-Quoc, "On the dynamics in space of rods undergoing large motions: a geometrically exact approach", *Comput. Methods Appl. Mech. Eng.* **66**:2 (1988), 125–161.

[Simo and Vu-Quoc 1991] J. C. Simo and L. Vu-Quoc, "A geometrically-exact rod model incorporating shear and torsion-warping deformation", *Int. J. Solids Struct.* **27**:3 (1991), 371–393.

[Sotoudeh and Hodges 2011] Z. Sotoudeh and D. H. Hodges, "Modeling beams with various boundary conditions using fully intrinsic equations", *J. Appl. Mech.* **78**:1 (2011), article 031010.

[Traugott et al. 2006] J. P. Traugott, M. J. Patil, and F. Holzapfel, "Nonlinear modeling of integrally actuated beams", *Aerosp. Sci. Technol.* **10**:6 (2006), 509–518.

[Volovoi and Hodges 2000] V. V. Volovoi and D. H. Hodges, "Theory of anisotropic thin-walled beams", *J. Appl. Mech.* (*ASME*) **67**:3 (2000), 453–459.

[Wright et al. 1982] A. D. Wright, C. E. Smith, R. W. Thresher, and J. L. C. Wang, "Vibration modes of centrifugally stiffened beams", *J. Appl. Mech.* (*ASME*) **49**:1 (1982), 197–202.

[Yu et al. 2002] W. Yu, D. H. Hodges, V. V. Volovoi, and C. E. S. Cesnik, "On Timoshenko-like modeling of initially curved and twisted composite beams", *Int. J. Solids Struct.* **39**:19 (2002), 5101–5121.

[Yu et al. 2005] W. Yu, D. H. Hodges, V. V. Volovoi, and E. D. Fuchs, "A generalized Vlasov theory for composite beams", *Thin-Walled Struct.* **43**:9 (2005), 1493–1511.

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