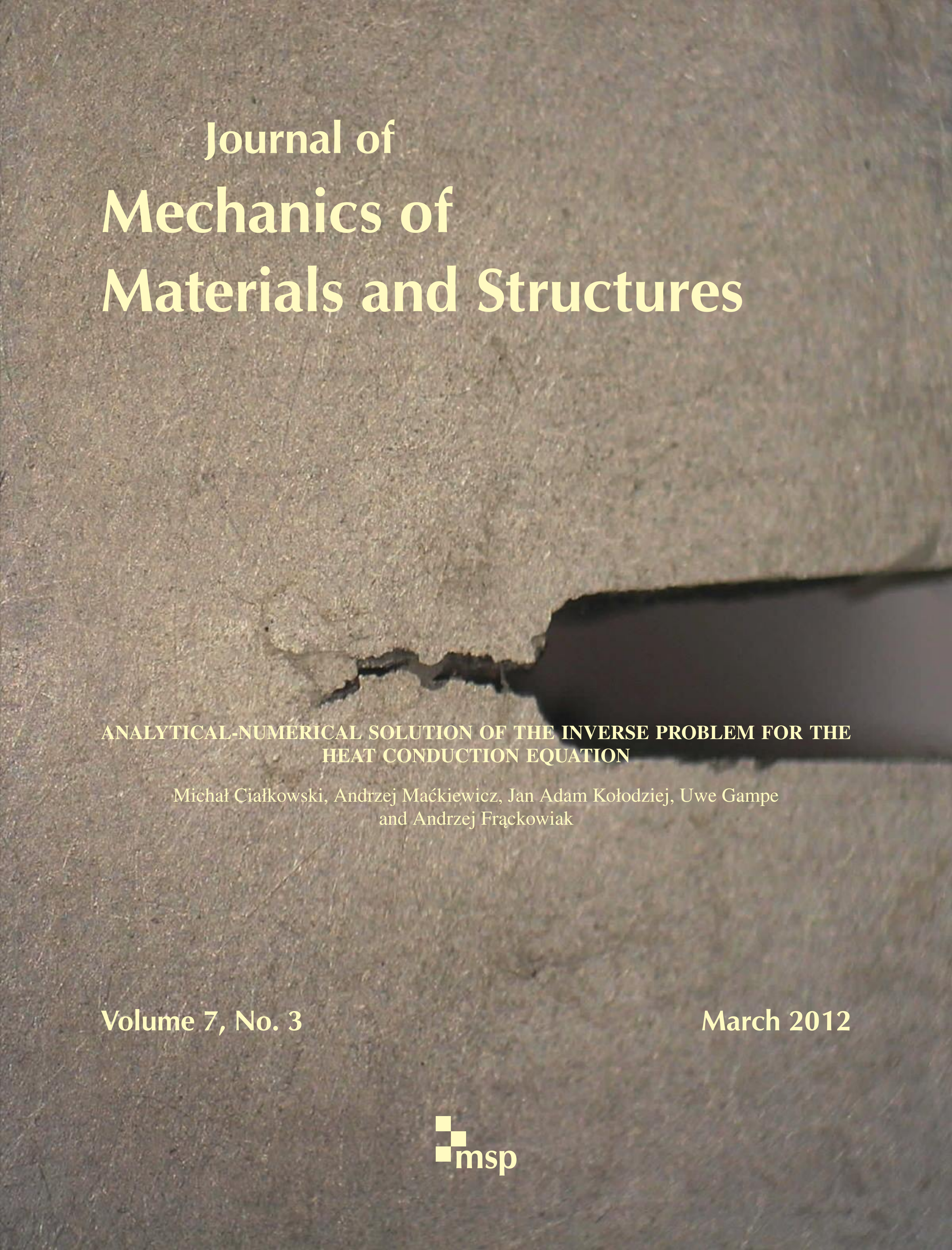

Journal of
Mechanics of
Materials and Structures

ANALYTICAL-NUMERICAL SOLUTION OF THE INVERSE PROBLEM FOR THE HEAT CONDUCTION EQUATION

Michał Ciałkowski, Andrzej Maćkiewicz, Jan Adam Kołodziej, Uwe Gampe
and Andrzej Frąckowiak

Volume 7, No. 3

March 2012

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 7, No. 3, 2012
dx.doi.org/10.2140/jomms.2012.7.239

ANALYTICAL-NUMERICAL SOLUTION OF THE INVERSE PROBLEM FOR THE HEAT CONDUCTION EQUATION

MICHAŁ CIAŁKOWSKI, ANDRZEJ MAĆKIEWICZ,
JAN ADAM KOŁODZIEJ, UWE GAMPE AND ANDRZEJ FRĄCKOWIAK

The solution of the inverse problem for the transient heat transfer equation is considered. The partial differential equation was discretized with respect to the space variable, and a system of ordinary differential equations of first order was obtained. The solution of the system of equations has a wave form. The numerical results obtained from the solution of the inverse problem confirm the effectiveness of the proposed method.

1. Introduction

Numerical solutions to linear partial differential equations can be obtained by different methods depending on the number of variables and the shape of the domain. When the unknown variable is a function of time, one can use the finite difference method, finite element method, boundary element method or other methods in order to conduct discretization with respect to the space variables. Consider a differential equation of heat transfer with initial and boundary conditions as follows:

$$\rho c \frac{\partial T}{\partial t} = \lambda \nabla T \quad x, y, z \in \Omega \subset R^3 \quad t \in (0, \infty), \tag{1-1}$$

with an initial and boundary condition

$$T(x, y, z, 0) = g(x, y, z) \quad T(x, y, z, t)\,|_{x,y,z\in\Gamma} = T_\Gamma(t), \quad t > 0, \tag{1-2}$$

The solution of the matrix equation using the initial condition has the form

$$\{\frac{dT(t)}{dt}\} = [A]\{T(t)\} + [B]\{T_\Gamma(t)\} \tag{1-3}$$

(see [Athans and Falb 1969]), which integrates to

$$\{T(t)\} = e^{[A]t}\{g\} + \int_0^t e^{[A](t-p)}[B]\{T_\Gamma(p)\}dp, \quad t \geq 0. \tag{1-4}$$

The purpose of this paper is to investigate solutions to the inverse problem in the form proposed above by using the different approximation of space-dependent temperature.

The inverse problem considered in this paper is a boundary type problem, which means seeking an unknown boundary condition based on temperature measured in some chosen points inside the domain. In engineering practice, the number of measured points is limited, and in the case of turbines it is typically

The work was partially supported by Grant 3134/B/T02/2007/33 and 4917/B/T02/2010/39.
Keywords: inverse problems, stability, matrix exponential, sensitivity analysis, transient heat transfer equation.

only one or two points, as a result of the occurrence of stress concentration surrounding the thermoelement. That is why we decided to investigate the one-dimensional problem.

During the last decades, some numerical techniques have been proposed to solve a 1-D IHCP. In [Al-Khalidy 1998] the control volume algorithm has been combined with a digital filter method to estimate temperature and heat flux values on the surface of a body based on the temperature measurement inside the body. The accuracy of the method was verified by comparison with a direct analytical solution of the problem. Lesnic and Elliott [1999] have used the Adomian's decomposition approach for solving the inverse heat conduction problem in which temperature and heat flux histories on the left boundary were estimated based on temperature and heat flux on the right boundary. The mollification method was applied to deal with noisy input data and to obtain a stable approximate solution. In [Shen 1999] two kinds of boundary element method were employed to solve IHCP, namely a collocation method and a weighted method. The author has used the Tikhonov's regularization method and the truncated singular value decomposition method for stabilization results. The conjugate gradient method supported by Fourier analysis has been applied in [Prud'homme and Hguyen 1999] to solve the IHCP among others for the 1-D case. It was found that an unknown time-dependent heat flux may be recovered satisfactorily using a single sensor inside the region. A Kalman filter combined with a variable forgetting factor as a weighting function in a recursive least-squares algorithm was applied in [Lee et al. 2000] to estimate impulsive heat flux. In this method, a spatial derivative in the 1-D heat equation was approximated by finite differences. In [Jonas and Louis 2000] some versions of mollification method have been used to solve the 1-D IHCP. Usually the mollification is done in the data space, but in this paper the mollification is performed in the solution space. In [Taler and Duba 2001] the 1-D non-linear IHCP was solved by means of the method of lines. This method is based on replacing the partial differential equation of heat conduction by the system of ordinary differential equations through discretization of the space derivative or the time derivative. Instead of other optimization techniques, the maximum entropy method was used in [Kim and Lee 2002] in the solution of the IHCP. The presented results showed considerable enhancement in the resolution of the inverse problem and bias reduction in comparison with the conventional methods. One of the most popular methods for solving IHCP is a sequential function specification method proposed at first by Beck [1970]. In [Lin et al. 2004] a modification of Beck's method was presented to estimate the heat source in the 1-D case. One base of this proposition is an application of a finite difference method for approximation the spatial and time derivatives. Almost all of the methods mentioned above are mesh methods which need some kind of mesh. In [Hon and Wei 2004] the meshless method, namely the method of fundamental solution, has been used for solution the 1-D IHCP. For regularization of the results, the authors used Tikhonov's regularization technique equipped with the L-curve method. The IHCP in which surface heat flux is estimated based on moving measurements inside the body was presented in [Shidfar and Karamali 2005]. The authors use an integral equation method and a linear least-squares method. The sequential function specification method supported by singular value decomposition was discussed in [Cabeza et al. 2005]. In [Shidfar and Pourgholi 2006; Pourgholi et al. 2009], the ill-posed IHCP is transformed to Cauchy's problem by means of a linear transformation. Next, Cauchy's problem is solved successfully by applying Legendre polynomials. The numerical approach combining the use of the finite difference method with the solution of ordinary differential equations has been proposed in [Ebrahimian et al. 2007] for solving the 1-D IHCP. The least-squares method has been used to determine the unknown boundary condition [Cabeza et al. 2005]. Four different versions of the variable metric method for solving the 1-D IHCP [Luksan and Spedicato 2000] with a symmetric rank-one update are compared in [Pourshaghaghy et al. 2007]. The results indicate that the accuracy of these versions do not differ significantly from each other. In [Deng and Hwang 2007] the 1-D IHCP is solved by means of a Kalman filter-enhanced Bayesian back propagation neutral network. The results show that the proposed method can predict the unknown parameters in inverse problems with acceptable error.

In [Grysa 2010] an application of the Trefftz function for solving the inverse heat conduction problem is considered for 1-D problems, among others.

The use of hyperbolic spline functions to approximate the solution with respect to the space variable is novel in our paper. Thanks to it, the second derivative of the approximate solution is continuous.

2. Formulation of the problem

The linear equation of heat conduction has the form

$$\frac{\rho c}{\lambda}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1-2\gamma}{r}\frac{\partial T}{\partial r}, \qquad R_i < r < R_a, \tag{2-1}$$

where

$$\gamma = \begin{cases} -\frac{1}{2} & \text{for a spherical layer,} \\ 0 & \text{for a cylindrical layer,} \\ \frac{1}{2} & \text{for a plane layer,} \end{cases}$$

is a parameter describing the shape of the domain (see Figure 1). The following conditions are imposed:

– The initial condition,

$$T(r, 0) = T_0(r), \qquad R_i \le r \le R_a. \tag{2-2}$$

– The boundary condition at the surface $r = R_a$,

$$-\lambda \frac{\partial T}{\partial r}\bigg|_{r=R_a} = \alpha(T - T_{\text{fluid}}). \tag{2-3}$$

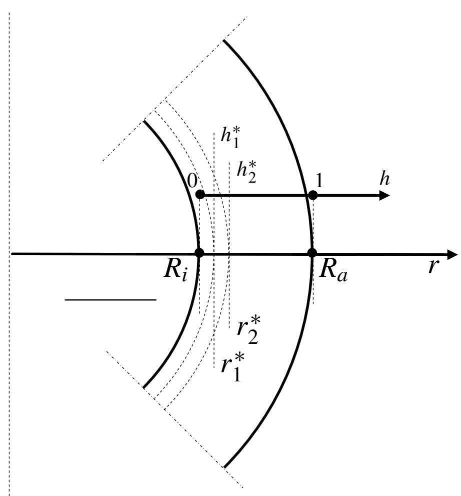

Figure 1. Notation.

– Additional conditions resulting from the measurement of temperature at the inner points of the region r_k^*, $k = 1, 2, \ldots, M$:

$$T(r_k^*, t) = f_k(t). \tag{2-4}$$

We will apply the substitutions

$$\begin{gathered} r = R_i + \eta(R_a - R_i), \quad 0 \le \eta \le 1, \\ \vartheta = \frac{T}{T_m}, \quad T_m = \max(T_0(r), T_{\text{fluid}}), \quad \tau = \frac{\lambda}{\rho c} \frac{t}{(R_a - R_i)^2}, \quad a = \frac{R_i}{R_a - R_i} \end{gathered} \tag{2-5}$$

to obtain the nondimensional counterparts of (2-1)–(2-4):

– The equation of heat transfer,

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \eta^2} + \frac{1 - 2\gamma}{\eta + a} \frac{\partial \vartheta}{\partial \eta}, \quad \eta \in (0, 1), \quad \tau > 0. \tag{2-6}$$

– The initial condition,

$$\vartheta(\eta, 0) = \frac{T_0(r, \eta)}{T_m} = g(\eta), \quad \eta \in [0, 1]. \tag{2-7}$$

– The boundary condition at surface $r = R_a$,

$$\frac{\partial \vartheta}{\partial \eta}\Big|_{\eta=1} = \frac{\alpha(R_a - R_i)}{\lambda}(\vartheta - \vartheta_{\text{fluid}}) = Bi_a(\vartheta - \vartheta_{\text{fluid}}), \quad \tau > 0. \tag{2-8}$$

– The additional conditions in inner points η_k^*, $k = 1, \ldots, M$:

$$\vartheta(\eta_k^*, \tau) = f_k(\tau), \quad \tau > 0. \tag{2-9}$$

The solution of (2-6) can be expressed as

$$\vartheta(\eta, \tau) = \sum_{i=0}^{N} \vartheta_i(\tau) \varphi_i(\eta) \tag{2-10}$$

where the interpolation basis functions $\varphi_i(\eta)$ are specified on a grid of points (see Figure 2).

Taking into consideration the character of the temperature function in r direction (the character of a hyperbolic function), the spline hyperbolic function is chosen as a base function. The unknown functions $\vartheta_i(\tau)$, $i = 0, \ldots, N$ are sought by solving (2-6) in the interior marked points of the segment [0, 1]; that

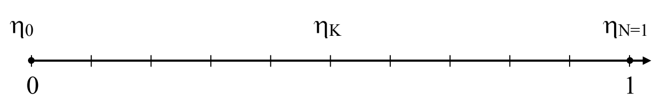

Figure 2. Grid of points for interpolation of spline function.

is, we substitute (2-10) into (2-6) to obtain $N-1$ equations:

$$\sum_{i=0}^{N} \frac{\vartheta_i(\tau)}{d\tau}\varphi_i(\eta_k) = \sum_{i=0}^{N} \vartheta_i(\tau)\left(\frac{d^2\varphi_i(\eta)}{d\eta^2} + \frac{1-2\gamma}{\eta_k+a}\frac{d\varphi_i(\eta)}{d\eta}\right) = \sum_{i=0}^{N} \vartheta_i(\tau)\psi_i(\eta_k), \quad k=1,\ldots,N-1. \tag{2-11}$$

The successive equations result from the conditions (2-8) and (2-9); that is, by substituting the solution (2-10) to the condition (2-8) we obtain

$$-\sum_{i=0}^{N} \vartheta_i(\tau)\varphi_i'(1) = Bi_a\left(\sum_{i=0}^{N} \vartheta_i(\tau)\varphi_i(1) - \vartheta_{\text{fluid}}\right) = Bi_a(\vartheta_N(\tau) - \vartheta_{\text{fluid}}),$$

which we rewrite in the form

$$-\sum_{i=0}^{N-1} \vartheta_i(\tau)\varphi_i'(1) - \vartheta_N(\tau)(\varphi_N'(1) + Bi_a) = -Bi_a\vartheta_{\text{fluid}}(\tau). \tag{2-12}$$

Introducing

$$b_i = \varphi_i'(1), \quad i = 0, 1, \ldots, N-1,$$

and

$$b_N = \varphi_N'(N) + Bi_a,$$

Equation (2-12) becomes

$$\sum_{i=0}^{N} \vartheta_i(\tau)b_i = Bi_a\vartheta_{\text{fluid}}(\tau). \tag{2-13}$$

The additional conditions for the temperature function result from condition (2-9) and have the form

$$\sum_{i=0}^{N} \vartheta_i(\tau)\varphi_i(\eta_k^*) = f_k(\tau), \quad i = 1, 2, \ldots, M. \tag{2-14}$$

3. The solution of the direct problem

The idea for solving the inverse problem is based on the ability to express the solution of the direct problem in parametric form with the boundary conditions attached. Equation (2-11) leads to the following system of equations:

$$\begin{bmatrix} \varphi_0(\eta_1) & \cdots & \varphi_N(\eta_1) \\ \vdots & & \vdots \\ \varphi_0(\eta_k) & \cdots & \varphi_N(\eta_k) \\ \vdots & & \vdots \\ \varphi_0(\eta_{N-1}) & \cdots & \varphi_N(\eta_{N-1}) \end{bmatrix} \begin{bmatrix} \frac{d\vartheta_0}{d\tau} \\ \vdots \\ \frac{d\vartheta_{N-1}}{d\tau} \\ \frac{d\vartheta_N}{d\tau} \end{bmatrix} = \begin{bmatrix} \psi_0(\eta_1) & \cdots & \psi_N(\eta_1) \\ \vdots & & \vdots \\ \psi_0(\eta_k) & \cdots & \psi_N(\eta_k) \\ \vdots & & \vdots \\ \psi_0(\eta_{N-1}) & \cdots & \psi_N(\eta_{N-1}) \end{bmatrix} \begin{bmatrix} \vartheta_0 \\ \vdots \\ \vartheta_{N-1} \\ \vartheta_N \end{bmatrix} \tag{3-1}$$

The system (3-1) has dimension $(N-1)*(N+1)$. In the direct problem, a boundary condition in point $\eta=1$ is attached, which leads to an equation involving an $(N-1)\times(N-1)$ matrix:

$$\begin{bmatrix} \varphi_1(\eta_1) & \cdots & \varphi_{N-1}(\eta_1) \\ \vdots & & \vdots \\ \varphi_1(\eta_k) & \cdots & \varphi_{N-1}(\eta_k) \\ \vdots & & \vdots \\ \varphi_1(\eta_{N-1}) & \cdots & \varphi_{N-1}(\eta_{N-1}) \end{bmatrix} \begin{bmatrix} \dfrac{d\vartheta_1}{d\tau} \\ \vdots \\ \dfrac{d\vartheta_{N-1}}{d\tau} \end{bmatrix} = \begin{bmatrix} \psi_1(\eta_1) & \cdots & \psi_{N-1}(\eta_1) \\ \vdots & & \vdots \\ \psi_1(\eta_k) & \cdots & \psi_{N-1}(\eta_k) \\ \vdots & & \vdots \\ \psi_1(\eta_{N-1}) & \cdots & \psi_{N-1}(\eta_{N-1}) \end{bmatrix} \begin{bmatrix} \vartheta_0 \\ \vdots \\ \vartheta_{N-1} \end{bmatrix}$$
$$- \begin{bmatrix} \varphi_0(\eta_1) \\ \vdots \\ \varphi(\eta_{N-1}) \end{bmatrix} \cdot \frac{d\vartheta_0^{(\tau)}}{d\tau} + \begin{bmatrix} \psi_0(\eta_1) \\ \vdots \\ \psi(\eta_{N-1}) \end{bmatrix} \cdot \psi_0(\tau) - \begin{bmatrix} \varphi_N(\eta_1) \\ \vdots \\ \varphi(\eta_{N-1}) \end{bmatrix} \cdot \frac{d\vartheta_N^{(\tau)}}{d\tau} + \begin{bmatrix} \psi_N(\eta_1) \\ \vdots \\ \psi(\eta_{N-1}) \end{bmatrix} \cdot \psi_N(\tau). \quad (3\text{-}2)$$

The base functions φ_i, $i=0,1,\ldots,N$, are interpolation functions with the property

$$\varphi_i(\eta_k) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k, \end{cases}$$

so the vectors $d\vartheta_0/d\tau$ and $d\vartheta_N/d\tau$ on the right-hand side of the (3-2) disappear, and the matrix of coefficients matching vector $\{d\vartheta/d\tau\}$ is diagonal. Equation (3-2) can be then written as

$$\left\{\frac{d\vartheta(\tau)}{d\tau}\right\} = [\psi]\{\vartheta(\tau)\} + \{\psi_0\}\cdot\vartheta_0(\tau) + \{\psi_N\}\cdot\vartheta_N(\tau). \quad (3\text{-}3)$$

The solution of (3-3) is equal to (see for example [Athans and Falb 1969])

$$\{\vartheta(\tau)\} = e^{[\psi]\tau}\cdot\{g\} + \int_0^\tau e^{[\psi](\tau-p)}\big[\{\psi_0\}\vartheta_0(p) + \{\psi_N\}\vartheta_N(p)\big]\,dp. \quad (3\text{-}4)$$

The diagonalizable matrix $e^{[\psi]}$ takes the form

$$e^{[\psi]} = [Z]\cdot[\mathrm{diag}(e^{\lambda})][Z]^{-1} \quad (3\text{-}5)$$

where $[\mathrm{diag}(e^{\lambda})]$ is the diagonal matrix with elements e^{λ_i}, $i=1,\ldots,N-1$ on the main diagonal and the numbers λ_i are eigenvalues of the matrix [Shen 1999]. Determining the integral in expression (3-4) for any moment of time τ requires each time the integration over the whole interval $[0,\tau]$. Determining the temperature $\vartheta(\tau)$ in subsequent moments of time τ and $\tau+\Delta\tau$ based on the dependence (3-4) we get

$$
\begin{aligned}
&\{\vartheta(\tau+\Delta\tau)\}\\
&\quad = e^{[\psi](\tau+\Delta\tau)}\cdot\{g\}+\int_0^{\tau+\Delta\tau} e^{[\psi](\tau+\Delta\tau-p)}\big[\{\psi_0\}\vartheta_0(p)+\{\psi_N\}\vartheta_N(p)\big]\,dp\\
&\quad = e^{[\psi](\Delta\tau)}\cdot\left(e^{[\psi](\tau+\Delta\tau)}\cdot\{g\}+\int_0^{\tau} e^{[\psi](\tau-p)}\big[\{\psi_0\}\vartheta_0(p)+\{\psi_N\}\vartheta_N(p)\big]\,dp\right)\\
&\qquad\qquad +\int_\tau^{\tau+\Delta\tau} e^{[\psi](\tau+\Delta\tau-p)}\big[\{\psi_0\}\vartheta_0(p)+\{\psi_N\}\vartheta_N(p)\big]\,dp\\
&\quad = e^{[\psi](\Delta\tau)}\cdot\{\vartheta(\tau)\}+\int_\tau^{\tau+\Delta\tau} e^{[\psi](\tau+\Delta\tau-p)}[\{\psi_0\}\vartheta_0(p)+\{\psi_N\}\vartheta_N(p)]dp\\
&\quad = e^{[\psi](\Delta\tau)}\cdot\{\vartheta(\tau)\}+\Delta t\int_0^1 e^{[\psi]\Delta(1-t)}\{\psi_0\}\vartheta_0(\tau+\Delta\tau t)dt\cdot\int_0^1 e^{[\psi]\Delta(1-t)}\{\psi_N\}\vartheta_N(\tau+\Delta\tau t)\,dt. \qquad (3\text{-}6)
\end{aligned}
$$

In engineering practice the measurement of temperature is usually done with constant time steps $\Delta\tau$. Taking $\tau=\tau_m=n\cdot\Delta\tau$ and $\tau+\Delta\tau=\tau_{n+1}$, then ϑ_0 and ϑ_N are known at the measured points $\vartheta_0(\tau+\Delta\tau)=\vartheta_0(\tau_{n+1})=\vartheta_0^{n+1}$, $\vartheta_0(\tau)=\vartheta_0^n$ and similarly for $\vartheta_N(\tau)$. How to calculate the integral in (3-6) is an important problem. We propose three ways to do this:

(a) Approximate the function ϑ by its average value (depending on the parameter ξ):

$$
\begin{aligned}
\Delta\tau\cdot\int_0^1 e^{[\psi]\Delta\tau(1-t)}\cdot w(\tau_n+\Delta\tau\cdot t) &= \Delta\tau\cdot\int_0^1 e^{[\psi]\Delta\tau(1-t)}\cdot dt[w^n\xi+w^{n+1}(1-\xi)]\\
&= w^n[S(\Theta)]+w^{n+1}[T(\Theta)]. \qquad (3\text{-}7)
\end{aligned}
$$

(b) Use the mean value theorem for definite integrals:

$$
\begin{aligned}
\Delta\tau\cdot\int_0^1 e^{[\psi]\Delta\tau(1-t)}\cdot w(\tau_n+\Delta\tau\cdot t) &= \Delta\tau\cdot\int_0^\xi e^{[\psi]\Delta\tau(1-t)}\cdot dt\cdot w^n+\Delta\tau\cdot\int_\xi^1 e^{[\psi]\Delta\tau(1-t)}\cdot dt\cdot w^{n\pm1}\\
&= w^n[S(\xi)]+w^{n+1}[T(\xi)],\quad 0<\xi<1, \qquad (3\text{-}8)
\end{aligned}
$$

(c) Approximate the function ϑ by its asymptotic expansion [Ciałkowski 2008]:

$$
\begin{aligned}
\Delta\tau\cdot\int_0^1 e^{[\psi]\Delta\tau(1-t)}\cdot w(\tau_n+\Delta\tau\cdot t) &= \Delta\tau\cdot\int_0^1 e^{[\psi]\Delta\tau(1-t)}\cdot[w^n(1-t^\xi)+w^{n+1}t^\xi]dt\\
&= w^n[S]+w^{n+1}[T],\quad 0<\xi<1. \qquad (3\text{-}9)
\end{aligned}
$$

The function in these equations is replaced with ϑ_0 or ϑ_N. In the integration process the matrix $\exp([\psi]\cdot\Delta\tau)$ expressed in the form (3-5) is used. The choices (3-7) and (3-9) guarantee that the solution of the inverse problem is stable. After this operation the expressions (3-7) and (3-9) take on the form

$$
\begin{aligned}
\{\vartheta^{n+1}\} &= e^{[\psi]\Delta\tau}\cdot\{\vartheta^n\}+(\vartheta_0^n[S]+\vartheta_0^{n+1}[T])\{\psi_0\}+(\vartheta_N^n[S]+\vartheta_N^{n+1}[T])\{\psi_N\}\\
&= e^{[\psi]\Delta\tau}\cdot\{\vartheta^n\}+\{S0\}\cdot\vartheta_0^n+\{T0\}\cdot\vartheta_0^{n+1}+\{SN\}\cdot\vartheta_N^n+\{TN\}\cdot\vartheta_N^{n+1}. \qquad (3\text{-}10)
\end{aligned}
$$

The matrix $\exp([\psi]\cdot\Delta\tau)$ is a matrix of stability for the direct problem with a boundary condition of the first type at the point $\eta=0$ and $\eta=1$. The dependence (3-10) can be written more compactly as

$$\begin{aligned}\{\vartheta^{n+1}\} &= [\{S\}, e^{[\psi]\Delta\tau}, \{SN\}]\{\bar{\vartheta}_n\} + \{T0\}\vartheta_0^{n+1} + \{TN\}\vartheta_N^{n+1}\\ &= [S\psi]\{\bar{\vartheta}_n\} + \{T0\}\vartheta_0^{n+1} + \{TN\}\vartheta_N^{n+1}, \qquad \{\bar{\vartheta}\} = \{\vartheta_0, \vartheta_1, \dots \vartheta_N\},\end{aligned} \tag{3-11}$$

where $\dim[S\psi] = (N-1)*(N+1)$. Supplementing the vector (ϑ^{n+1}) with the elements ϑ_0^{n+1} and ϑ_N^{n+1} we obtain

$$\begin{aligned}\{\bar{\vartheta}^{n+1}\} &= \begin{bmatrix}\{0\}^T\\ [S\psi]\\ \{0\}^T\end{bmatrix}\{\bar{\vartheta}^n\} + \begin{bmatrix}1\\ \{T0\}\\ 0\end{bmatrix}\{\vartheta_0^{n+1}\} + \begin{bmatrix}0\\ \{TN\}\\ 1\end{bmatrix}\{\vartheta_N^{n+1}\}\\ &= [\bar{S\psi}]\{\bar{\vartheta}^n\} + + \{P0\}\vartheta_0^{n+1} + \{PN\}\vartheta_N^{n+1} = [\bar{S\psi}]\{\bar{\vartheta}^n\} + [TT]\cdot\{\vartheta_0^{n+1}, \vartheta_N^{n+1}\}^T,\end{aligned} \tag{3-12}$$

for $n = 0, 1, \dots$, or, in a form which takes into account the values of temperature ϑ_0^{n+1}, ϑ_N^{n+1} at the boundary in the subsequent moments of time:

$$\{\bar{\vartheta}^{n+1}\} = [\psi]^{n+1}\cdot\sum_{k=0}^{n-1}[\psi]^{n-1+k}\cdot\{P0\}\vartheta_0^{n+k} + \sum_{k=0}^{n-1}[\psi]^{n-1+k}\cdot\{PN\}\vartheta_N^{k+1} \tag{3-13}$$

This gives the solution of the direct problem with a boundary condition of the first type. This series is convergent, if the spectral radius of matrix ψ, fulfills condition $\rho_s[\psi] < 1$. The dependence (3-13) gives a convenient method for solving the global inverse problem [Ciałkowski and Grysa 2010].

4. The solution of the inverse problem

In many practical situations it is impossible to determine temperature ϑ at the surface $\eta = 0$. However, it is possible to measure the temperature $\vartheta(\eta,\tau)$ at points $\eta = \eta_k^*$, $0 < \eta_k^* < 1$, $k = 1, \dots, M_{\text{stern}}$, $\tau = \tau_{n+1}$, $n = 0, 1, 2, \dots$. The dependence (2-9) takes the form

$$\vartheta(\eta_k^*, \tau_{n+1}) = \vartheta^{n+1}(\eta_k^*) = \sum_{i=0}^{N}\varphi_i(\eta_k^*)\vartheta_i^{n+1} = f_k^{n+1}, \quad k = 1, 2, \dots, M_{\text{stern}} \tag{4-1}$$

or separate in (4-1) temperatures ϑ_0^{n+1} and ϑ_N^{n+1} at the ends of a segment [0, 1] we have

$$\varphi_0(\eta_k^*)\vartheta_0^{n+1} + \sum_{i=1}^{N-1}\varphi_i(\eta_k^*)\vartheta_i^{n+1} + \varphi_N(\eta_k^*)\vartheta_N^{n+1} = f_k^{n+1}$$

or

$$\varphi_0(\eta_k^*)\vartheta_0^{n+1} + \{\varphi(\eta_k^*)\}^T\{\vartheta^{n+1}\} + \varphi_N(\eta_k^*)\vartheta_N^{n+1} = f_k^{n+1}. \tag{4-2}$$

Substituting relationship (3-11) into (4-2) we obtain

$$\varphi_0(\eta_k^*)\vartheta_0^{n+1} + \{\varphi(\eta_k^*)\}^T\cdot([S\psi]\{\bar{\vartheta}_n\} + \{T0\}\vartheta_0^{n+1} + \{TN\}\vartheta_N^{n+1}) + \varphi_N(\eta_k^*))\vartheta_N^{n+1} = f_k^{n+1}$$

or

$$(\varphi_0(\eta_k^*)+\{\varphi(\eta_k^*)\}^T\{T0\})\cdot\vartheta_0^{n+1}+(\varphi_n(\eta_k^*)+\{\varphi(\eta_k^*)\}^T\{TN\})\cdot\vartheta_N^{n+1}=f_k^{n+1}-\{\varphi(\eta_k^*)\}^T[S\psi]\cdot\{\bar{\vartheta}^n\}, \quad (4\text{-}3)$$

for $k = 1, 2, \dots, M_{\text{stern}}$. The boundary condition (2-13) is simplified to a similar form:

$$b_0\cdot\vartheta_0^{n+1}+\{b\}^T\{\vartheta^{n+1}\}+b_N\cdot\vartheta_N^{n+1}=Bi_a\cdot\vartheta_{f_a}^{n+1} \quad (4\text{-}4)$$

and after introducing the dependence (3-11) we have:

$$b_0\cdot\vartheta_0^{n+1}+\{b\}^T([S\psi]\{\bar{\vartheta}_n\}+\{T0\}\vartheta_0^{n+1}+\{TN\}\vartheta_N^{n+1})+b_N\cdot\vartheta_N^{n+1}=Bi_a\cdot\vartheta_{f_a}^{n+1},$$

or

$$(b_0+\{b\}^T\{T0\})\cdot\vartheta_0^{n+1}+(b_N+\{b\}^T\{TN\})\cdot\vartheta_N^{n+1}=Bi_a\cdot\vartheta_{f_a}^{n+1}-\{b\}^T[S\psi]\cdot\{\bar{\vartheta}^n\}. \quad (4\text{-}5)$$

Equations (4-3) and (4-4) create a system of equations with the unknowns ϑ_0^{n+1} and ϑ_N^{n+1} which can be written as follows:

$$\begin{bmatrix} \varphi_0(\eta_1^*)+\{\varphi(\eta_1^*)\}^T\{T0\} & \cdots & \varphi_N(\eta_1^*)+\{\varphi(\eta_1^*)\}^T\{TN\} \\ \vdots & & \vdots \\ \varphi_0(\eta_M^*)+\{\varphi(\eta_M^*)\}^T\{T0\} & \cdots & \varphi_N(\eta_M^*)+\{\varphi(\eta_M^*)\}^T\{TN\} \\ b_0+\{b\}^T\{T0\} & \cdots & b_N+\{b\}^T\{TN\} \end{bmatrix}\cdot\begin{bmatrix}\vartheta_0^{n+1}\\ \vdots \\ \vartheta_N^{n+1}\end{bmatrix}$$
$$=\begin{bmatrix} f_1^{n+1} \\ \vdots \\ f_M^{n+1} \\ Bi_a\cdot\vartheta_{f_n}^{n+1}\end{bmatrix}-\begin{bmatrix}\{\varphi(\eta_1^*)\}^T[S\psi] \\ \vdots \\ \{\varphi(\eta_M^*)\}^T[S\psi] \\ \{b\}^T[S\psi]\end{bmatrix}\cdot\{\bar{\vartheta}_n\}, \quad M=M_{\text{stern}},$$

or

$$[FIB]\begin{bmatrix}\vartheta_0^{n+1}\\ \vartheta_N^{n+1}\end{bmatrix}=\{FBiot^{n+1}\}-[SB]\{\bar{\vartheta}^n\}, \quad \dim[FIB]=\dim[SB]=2(M_{\text{stern}}+1), \quad (4\text{-}6)$$

and finally

$$\begin{bmatrix}\vartheta_0^{n+1}\\ \vartheta_N^{n+1}\end{bmatrix}=[FIB]^+\{FBiot^{n+1}\}-[FIB]^+[SB]\{\bar{\vartheta}^n\}. \quad (4\text{-}7)$$

The solution (3-13) takes the form:

$$\begin{aligned}\{\bar{\vartheta}^{n+1}\}&=[\bar{S\psi}]\{\bar{\vartheta}^n\}+[TT]([FIB]^+\{FBiot^{n+1}\}-[FIB]^+[SB]\{\bar{\vartheta}^n\})\\ &=([\bar{S\psi}]-[TT][FIB]^+[SB])\{\bar{\vartheta}^n\}+[TT][FIB]^+\{FBiot^{n+1}\}\\ &=[\psi_{int}]\{\bar{\vartheta}^n\}+[TF]\{FBiot^{n+1}\}, \quad n=1,2,\dots, \end{aligned} \quad (4\text{-}8)$$

where

$$\{FBiot^{n+1}\}=\{f^{n+1}(\eta_1^*),\dots,f^{n+1}(\eta_M^*),Bi_a\cdot\vartheta_{f_n}^{n+1}\}^T, \quad f^{n+1}=f(\tau_{n+1}).$$

5. The stability of the inverse problem

For the subsequent time steps $\tau = \tau_n$, $n = 0, 1, 2, \dots$ we write (4-8) as

$$\begin{aligned}
\{\bar\vartheta^1\} &= [\psi_{\text{inv}}]\{\bar\vartheta^0\} + [FIB]^+\{FBiot^1\},\\
\{\bar\vartheta^2\} &= [\psi_{\text{inv}}]\{\bar\vartheta^1\} + [FIB]^+\{FBiot^2\},\\
\{\bar\vartheta^3\} &= [\psi_{\text{inv}}]\{\bar\vartheta^2\} + [FIB]^+\{FBiot^3\},\\
\{\bar\vartheta^n\} &= [\psi_{\text{inv}}]^n\{\bar\vartheta^0\} + \sum_{k=0}^{n-1}[\psi_{\text{inv}}]^k\cdot[FIB]^+\{FBiot^{n-k}\}.
\end{aligned} \tag{5-1}$$

If the spectral radius ρ_s of the matrix $[\psi_{\text{inv}}]$ satisfies $\rho_s([\psi_{\text{inv}}]) < 1$, the Neumann series is convergent and the solution of the inverse Equation (4-8) is stable. For the initial temperature vector $\{\bar\vartheta_0\}$ disturbed by $\{\delta\bar\vartheta_0\}$ and the data vector $\{FBiot^n\}$ disturbed by value $\{\delta FBiot^n\}$, the value of disturbance $\{\delta\bar\vartheta\}$ of temperature by $\{\bar\vartheta^n\}$ is determined from the dependence (5-1), namely

$$\{\bar\vartheta^n + \delta\bar\vartheta^n\} = [\psi_{\text{inv}}]^n\{\bar\vartheta^0 + \delta\bar\vartheta^0\} + \sum_{k=0}^{n-1}[\psi_{\text{inv}}]^k\cdot[TF]\{FBiot^{n-k} + \delta FBiot^{n-k}\}. \tag{5-2}$$

Subtracting (5-2) from the dependence (5-1), we have:

$$\{\delta\bar\vartheta^n\} = [\psi_{\text{inv}}]\{\delta\bar\vartheta^0\} + \sum_{k=0}^{n-1}[\psi_{\text{inv}}]^k\cdot[TF]\{\delta FBiot^{n-k}\}, \quad n = 1, 2, \dots \tag{5-3}$$

The dependence (5-3) determines the propagation of the measurement errors of the initial temperature and of the vector of measured temperatures in points $\eta_1^*, \dots, \eta_M^*$, $M = M_{\text{stern}}$. The distance between the first thermoelement and the boundary $\eta = 0$ has an essential influence on the value of the spectral radius ρ_s, and so does the method of determining the integral — (3-7), (3-8), or (3-9).

6. Numerical calculation

To investigate the numerical properties of the proposed method, the calculation was carried out for a circular ring with inner radius $R_i = 0.1$ m and outer radius $R_a = 0.209$ m. We further took $r_1^* = 0.108$ m and $r_2^* = 0.157$ m. The thermophysical properties were chosen as $c = 500$ J/(kg K), $\rho = 7800$ kg/m^3, and $\lambda = 47.76$ W/mK. For test values, we took temperatures in cylindrical ring with different heat transfer coefficient at inner surface of ring $\alpha_i = \{5000, 2500, 1000, 500, 100\}$ W/m^{2}K, outer surface isolation $q = 0$ and the temperature $T_{f_i} = 535$°C, initial temperature $T_0 = 0$°C. Figure 3 presents the distribution of heat transfer coefficient α_I obtained from the solution of the inverse problem for the data mentioned above.

The distribution of heat transfer coefficient is obtained from the solution of the inverse problem by means of a different way of calculating the integral in (3-6). Integration according to the method expressed by the formula (3-8) or (3-9) leads to a nonoscillating solution of the inverse problem.

For $\zeta = 0.75$ and method of integration (3-7), the spectral radius of the matrix is given in Figure 4 for positions of thermoelements at $s_1 = 8$ mm and $s_2 = 57$ mm from the inner surface. The influence the number of spline functions on the heat transfer coefficient is given in Figure 5. Figure 6 presents inverse

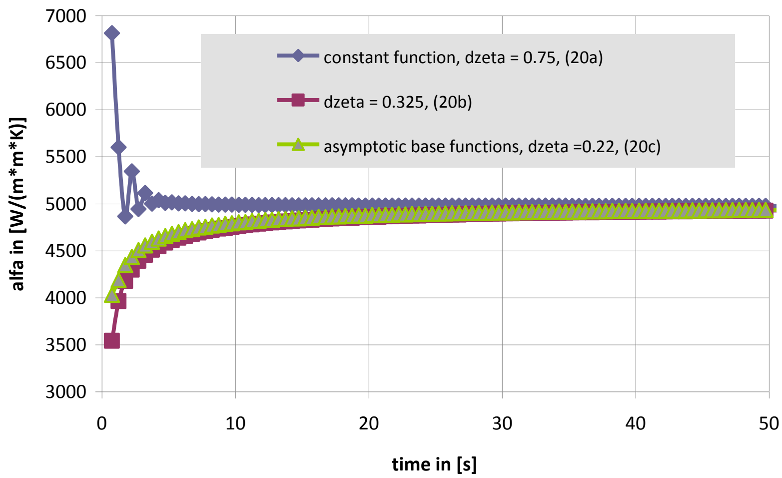

Figure 3. Comparison of methods of calculating the integral in formula (3-6).

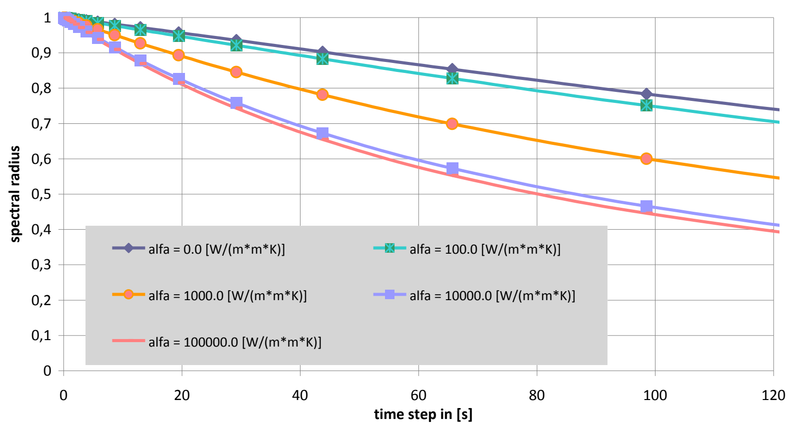

Figure 4. Influence of time step on the spectral radius of the stability matrix of the inverse problem ($\zeta = 0.75$) for given values of heat transfer coefficient at the outer surface.

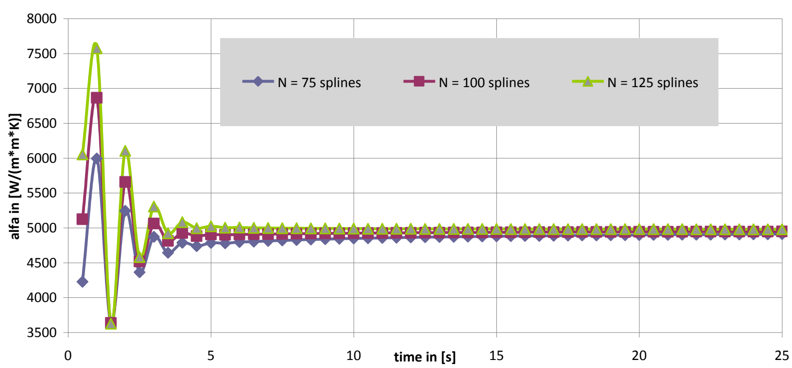

Figure 5. Influence of spline number on the calculation of heat transfer coefficient. All curves tend asymptotically to the exact value. ($\zeta = 0.75$.)

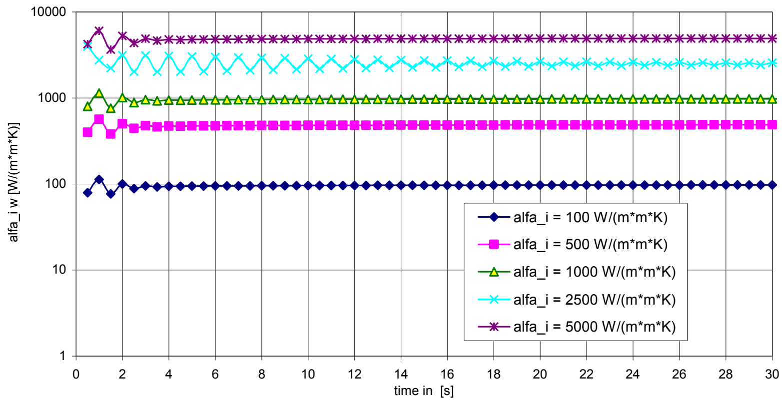

Figure 6. Solution of the inverse problem for different measurements (analytical values). (ζ = 0.75.)

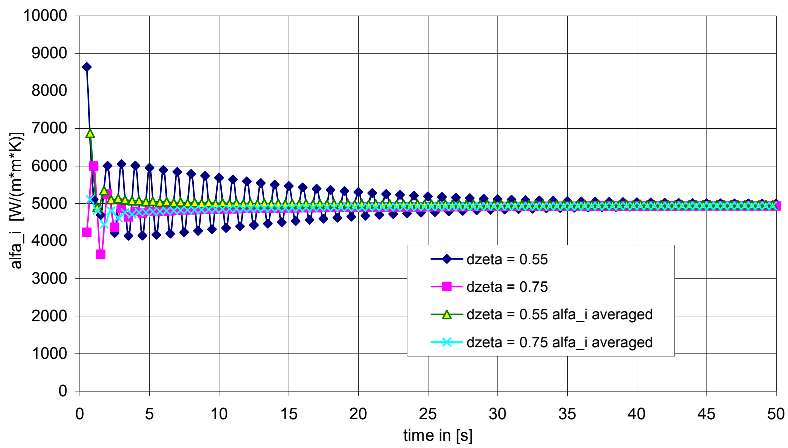

Figure 7. Solution of the inverse problem for different values of parameter ζ in (3-7).

problem solution for the data taken from the exact solution. Stability of the inverse problem depends on a length of time interval, a way of integration and the value of heat transfer coefficient on the outer side of cylinder. The time step was equal to 0.5 s. The use of (3-7) with parameter $\zeta = 0.75$ in our calculations guarantees the stability of the inverse problem while at the same time identifying the value of the heat transfer coefficient with high precision in a large time interval. Decreasing the value of parameter ζ leads to oscillation of the heat transfer coefficient obtained from the solution of the inverse problem, Figure 7. The arithmetic average of the subsequent values for $\zeta = 0.55$ brings the results very near to the values for parameter $\zeta = 0.75$. It can then be concluded that the interval of appropriate values for the parameter ζ is relatively broad and allows one to obtain stable results for the solution of the inverse problem using the arithmetic average. Figure 8 demonstrates the distribution of the measured temperature in a fluid at two inner points of a ring with radius $R_i = 0.1$m placed $s_1 = 8$ mm and $s_2 = 57$ mm from the inner surface.

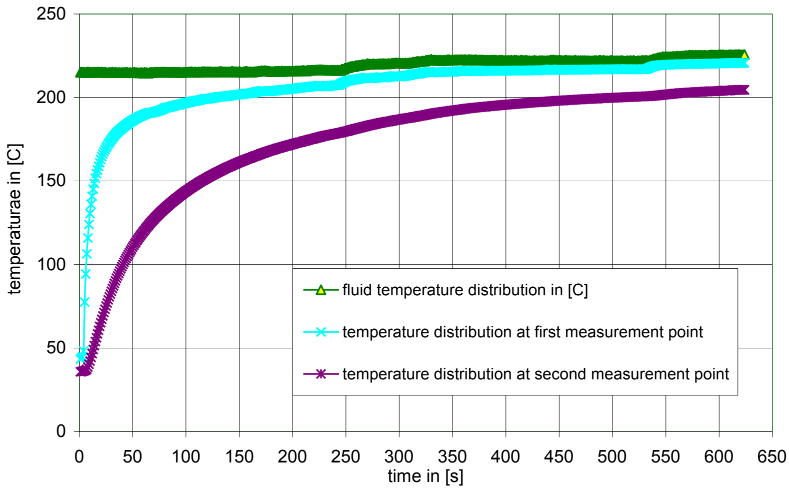

Figure 8. Distribution of measured temperature.

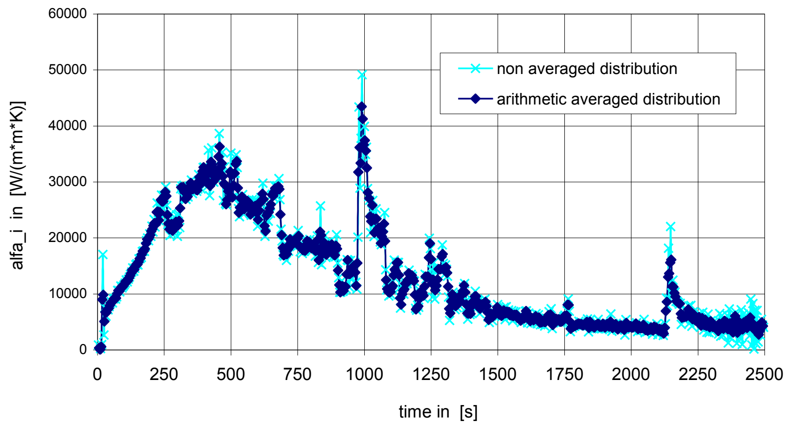

Figure 9. Solution to the inverse problem for real measurement. The thermocouples were placed at $s_1 = 8$ mm and $s_2 = 57$ mm from the inner surface.

These distributions are used to solve the inverse problem and to determine the heat transfer coefficient at the surface $r = R_i$. The values of the heat transfer coefficient are shown on Figure 9, where the time step is equal to $\Delta t = 4$ s.

7. Conclusions

We have presented a method for solving the inverse problem for the transient linear heat transfer equation by approximating the solution with respect to the space variable with hyperbolic spline functions. The unknown coefficients of the linear combination of spline functions were determined by fulfilling the differential equation in the points of the net (Figure 2).

The essence of the proposed method is based on the approximation of the solution of differential equations with respect to the space variable by a twice-differentiable function. Hyperbolic spline functions were chosen because using a backward quotient difference for the time variable transforms the heat transfer to Helmholtz's equation with source function, the solution of which contains a hyperbolic function. The smoothness of spline functions guarantees continuity of heat flux in each point of region which do not have place in the finite element method (there are jumps of flux between elements). The proposed method is appropriate for continuous temperature monitoring of casings with cylindrical or spherical geometries. This assumption usually follows from the lack of possibility of placing a large number of thermoelements because each hole leads to greater thermal stress.

References

[Al-Khalidy 1998] N. Al-Khalidy, "On the solution of parabolic and hyperbolic inverse heat conduction problems", *Int. J. Heat Mass Transf.* **41** (1998), 3731–3740.

[Athans and Falb 1969] M. Athans and P. L. Falb, *Sterownie optymalne*, Wydawnictwa Naukowo-Techniczne, Warsaw, 1969.

[Beck 1970] J. V. Beck, "Nonlinear estimation applied to the nonlinear inverse heat conduction problem", *Int. J. Heat Mass Transf.* **13** (1970), 703–716.

[Cabeza et al. 2005] J. M. G. Cabeza, J. A. M. Garcia, and A. C. Rodriguez, "A sequential algorithm of inverse heat conduction problems using singular value decomposition", *Int. J. Thermal Sci.* **44** (2005), 235–244.

[Ciałkowski 2008] M. Ciałkowski, "Zastosowanie asymptotycznych funkcji bazowych do aproksymacji niestacjonarmych przebiegów temperatur", *Zeszyty Nauk. Politech. Poznań. Maszyny Robocze i Transport* **63** (2008), 27–43.

[Ciałkowski and Grysa 2010] M. Ciałkowski and K. Grysa, "A sequential and global method of solving an inverse problem of heat conduction equation", *J. Theor. Appl. Mech.* (*Warsaw*) **48**:1 (2010), 111–134.

[Deng and Hwang 2007] S. Deng and Y. Hwang, "Solution of inverse heat conduction problems using Kalman filter-enhanced Bayesian back propagation neural network data fusion", *Int. J. Heat Mass Transf.* **50** (2007), 2089–2100.

[Ebrahimian et al. 2007] M. Ebrahimian, R. Pourgholi, M. Emamjome, and P. Reihani, "A numerical solution of an inverse parabolic problem with unknown boundary conditions", *Appl. Math. Comput.* **189** (2007), 228–234.

[Grysa 2010] K. Grysa, *Funkcje Trefftza i ich zastosowania w rozwiązywaniu zagadnień odwrotnych*, Monografie, Studia, Rozprawy **M13**, Politechnika Świętokrzyska, Kielce, 2010.

[Hon and Wei 2004] Y. C. Hon and T. Wei, "A fundamental solution method for inverse heat conduction problem", *Eng. Anal. Bound. Elem.* **28** (2004), 489–495.

[Jonas and Louis 2000] P. Jonas and A. K. Louis, "Approximate inverse for a one-dimensional inverse heat conduction problem", *Inverse Problems* **16** (2000), 175–185.

[Kim and Lee 2002] S. K. Kim and W. I. Lee, "Solution of inverse heat conduction problems using maximum entropy method", *Int. J. Heat Mass Transf.* **45** (2002), 381–391.

[Lee et al. 2000] W.-S. Lee, Y.-H. Ko, and C.-C. Ji, "A study of an inverse method for the estimation of impulsive heat flux", *J. Franklin Inst.* **337** (2000), 661–671.

[Lesnic and Elliot 1999] D. Lesnic and L. Elliot, "The decomposition approach to inverse heat conduction", *J. Math. Anal. Appl.* **232** (1999), 82–98.

[Lin et al. 2004] S.-M. Lin, C.-K. Chen, and Y.-T. Yang, "A modified sequential approach for solving inverse heat conduction problems", *Int. J. Heat Mass Transf.* **47** (2004), 2669–2680.

[Luksan and Spedicato 2000] L. Luksan and E. Spedicato, "Variable metric methods for unconstrained optimisation and nonlinear least squares", *J. Comput. Appl. Math.* **124** (2000), 61–95.

[Pourgholi et al. 2009] R. Pourgholi, N. Azizi, Y. S. Gasimov, F. Aliev, and H. K. Khalafi, "Removal of numerical instability in the solution of an inverse heat conduction problem", *Comm. Nonlin. Sci. Numer. Simul.* **14** (2009), 2664–2669.

[Pourshaghaghy et al. 2007] A. Pourshaghaghy, F. Kowsary, and A. Behbahaninia, "Comparison of four different versions of the variable metric method for solving inverse heat conduction problems", *Heat Mass Transf.* **43** (2007), 285–294.

[Prud'homme and Hguyen 1999] M. Prud'homme and T. H. Hguyen, "Fourier analysis of conjugate gradient method applied to inverse heat conduction problems", *Int. J. Heat Mass Transf.* **42** (1999), 4447–4460.

[Shen 1999] S.-Y. Shen, "A numerical study of inverse heat conduction problems", *Comput. Math. Appl.* **38** (1999), 173–188.

[Shidfar and Karamali 2005] A. Shidfar and G. R. Karamali, "Numerical solution of inverse heat conduction problem with nonstationary measurements", *Appl. Math. Comput.* **168** (2005), 540–548.

[Shidfar and Pourgholi 2006] A. Shidfar and R. Pourgholi, "Numerical approximation of solution of an inverse heat conduction problem based on Legendre polynomials", *Appl. Math. Comput.* **175** (2006), 1366–1374.

[Taler and Duba 2001] J. Taler and P. Duba, "Solution of non-linear inverse heat conduction problems using the method of lines", *Heat Mass Transf.* **37** (2001), 147–155.

Received 14 Oct 2010. Revised 16 Jan 2012. Accepted 3 Apr 2012.

MICHAŁ CIAŁKOWSKI: michal.cialkowski@put.poznan.pl
Chair of Thermal Engineering, Poznan University of Technology, ul. Jana Pawla II 24, 60-965 Poznan, Poland

ANDRZEJ MAĆKIEWICZ: andrzej.mackiewicz@put.poznan.pl
Institute of Mathematics, Poznan University of Technology, ul. Jana Pawla II 24, 60-965 Poznan, Poland

JAN ADAM KOŁODZIEJ: jan.kolodziej@put.poznan.pl
Institute of Applied Mechanics, Poznan University of Technology, ul. Jana Pawla II 24, 60-965 Poznan, Poland

UWE GAMPE: uwe.gampe@tu-dresden.de
Institut für Energietechnik, Technische Universität Dresden, Helmholtzstraße 14, 01069 Dresden, Germany

ANDRZEJ FRĄCKOWIAK: andrzej.frackowiak@put.poznan.pl
Chair of Thermal Engineering, Poznan University of Technology, 60-965 Poznan, Poland

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

jomms.net

Founded by Charles R. Steele and Marie-Louise Steele

See http://jomms.net for submission guidelines.

JoMMS (ISSN 1559-3959) is published in 10 issues a year. The subscription price for 2012 is US $555/year for the electronic version, and $735/year (+ $60 shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720–3840.

JoMMS peer-review and production is managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

Journal of Mechanics of Materials and Structures

Volume 7, No. 3 March 2012

Special issue
Trends in Continuum Physics (TRECOP 2010)

Preface BOGDAN T. MARUSZEWSKI, WOLFGANG MUSCHIK, JOSEPH N. GRIMA and KRZYSZTOF W. WOJCIECHOWSKI 225

The inverse determination of the volume fraction of fibers in a unidirectionally reinforced composite for a given effective thermal conductivity MAGDALENA MIERZWICZAK and JAN ADAM KOŁODZIEJ 229

Analytical-numerical solution of the inverse problem for the heat conduction equation MICHAŁ CIAŁKOWSKI, ANDRZEJ MAĆKIEWICZ, JAN ADAM KOŁODZIEJ, UWE GAMPE and ANDRZEJ FRĄCKOWIAK 239

Analysis of stress-strain distribution within a spinal segment ZDENKA SANT, MARIJA CAUCHI and MICHELLE SPITERI 255

A mesh-free numerical method for the estimation of the torsional stiffness of long bones ANITA USCILOWSKA and AGNIESZKA FRASKA 265

Rayleigh-type wave propagation in an auxetic dielectric ANDRZEJ DRZEWIECKI 277

Local gradient theory of dielectrics with polarization inertia and irreversibility of local mass displacement VASYL KONDRAT and OLHA HRYTSYNA 285

Electromagnetoelastic waves in a vortex layer of a superconductor BOGDAN T. MARUSZEWSKI, ANDRZEJ DRZEWIECKI AND ROMAN STAROSTA 297

1559-3959(2012)7:3;1-B