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ON ACOUSTOELASTICITY AND THE ELASTIC CONSTANTS OF SOFT BIOLOGICAL TISSUES

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We analyze the acoustoelastic study of material moduli that appear in the constitutive relations that characterize the response of anisotropic nonlinearly elastic bodies, in particular, materials reinforced with one set of fibers along one direction. Studies dealing with acoustoelastic coefficients in incompressible solids modeled by means of strain-energy density functions expanded up to different orders in terms of the Green strain tensor can be found in the literature. In this paper, we connect that analysis and the parallel one developed from the general theory of nonlinear elasticity which is based on strain energies that depend on the right Cauchy–Green deformation tensor. Establishing this relation explicitly will improve understanding of the mechanical properties of soft biological tissues among other materials.

1. Introduction

Determination of the acoustoelastic coefficients in incompressible solids has very recently attracted a lot of attention since these analyses give an opportunity to capture the mechanical properties of such materials, among other applications [Bigoni et al. 2007; 2008; Destrade et al. 2010b].

An incompressible transversely isotropic model has recently been analyzed by Destrade et al. [2010a] in which the strain-energy density is given by

$$W = \mu I_2 + \tfrac{1}{3} A I_3 + \alpha_1 I_4^2 + \alpha_2 I_5 + \alpha_3 I_2 I_4 + \alpha_4 I_4^3 + \alpha_5 I_4 I_5, \tag{1}$$

where

$$I_2 = \mathrm{tr}(\boldsymbol{E}^2), \quad I_3 = \mathrm{tr}(\boldsymbol{E}^3), \quad I_4 = \boldsymbol{M}\cdot(\boldsymbol{E}\boldsymbol{M}), \quad I_5 = \boldsymbol{M}\cdot(\boldsymbol{E}^2\boldsymbol{M}), \tag{2}$$

$\boldsymbol{E}$ is the Green strain tensor, $\boldsymbol{M}$ is the unit vector that gives the undeformed fiber direction, and μ, α_1, α_2 and A, α_3, α_4, α_5 are second and third-order elastic constants, respectively (the order is given by the exponent of $\boldsymbol{E}$). To evaluate the elastic constants Destrade et al. established formulas for the velocity waves. The formulas are given as first-order polynomials in terms of the elongation e_1, which is defined by $\lambda = 1 + e_1$, where λ is the principal stretch in the direction that gives both the fiber direction and the direction of uniaxial tension. The speeds of infinitesimal waves do provide a basis for the acoustoelastic evaluation of the material constants [Destrade and Ogden 2010].

Soft biological tissues are anisotropic solids due to the presence of oriented collagen fiber bundles [Holzapfel et al. 2000; Destrade et al. 2010a]. To make the model (1) more general and able to capture

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soft biological tissue mechanical behavior a fourth-order incompressible strain-energy function has been analyzed in [Vinh and Merodio 2013], namely

$$W = \mu I_2 + \tfrac{1}{3} A I_3 + \alpha_1 I_4^2 + \alpha_2 I_5 + \alpha_3 I_2 I_4 + \alpha_4 I_4^3 \\ + \alpha_5 I_4 I_5 + \alpha_6 I_2^2 + \alpha_7 I_2 I_4^2 + \alpha_8 I_2 I_5 + \alpha_9 I_4^4 + \alpha_{10} I_5^2 + \alpha_{11} I_3 I_4, \quad (3)$$

where $\alpha_6, \dots, \alpha_{11}$ are fourth-order elastic constants. The results show that linear corrections to the acoustoelastic wave speed formulas involve second and third-order constants, and that quadratic corrections involve second, third, and fourth-order constants, in agreement with [Hoger 1999]. Indeed, this is precisely the rationale behind the considered expansions (1) and (3) and the reason to develop the acoustoelastic wave speed formulas in terms of constitutive models that depend on the invariants of the Green strain tensor. Nevertheless, in this paper we develop acoustoelastic wave speed formulas in terms of constitutive models that depend on the invariants of the right Cauchy–Green tensor.

The model (3) has 13 elastic constants. It would be perfectly justifiable to question the efficacy of a model that depends on such a number of elastic constants. It is not easy to determine the structure of these material constants by any correlation with experiments. This makes a careful scrutiny of the nature of these constitutive models necessary in order to determine which of these elastic constants must be retained in the development of models. The models have to be well understood, and we can only do this if we analyze their structure. In passing, we mention that there has been lately in the literature some controversy regarding the use of planar tests to characterize anisotropic nonlinearly elastic materials. For a general discussion refer to [Holzapfel and Ogden 2009].

The more available formulations there exist in the literature to characterize elastic materials the more possibilities researchers have to capture the structure of the constitutive models. Furthermore, while physical acousticians are interested in third-order constants for anisotropic solids, workers in nonlinear elasticity, and, in particular, in soft biological tissue, use finite extensions involving fourth-order constants. In addition, soft biological tissue is modeled using general nonlinear elasticity theory expressed in terms of the right Cauchy–Green deformation tensor (see [Holzapfel et al. 2000]), among other formulations. Therefore, to develop the acoustoelastic wave speed formulas in terms of general nonlinear elasticity theory, that is, in terms of the right Cauchy–Green tensor, may improve our understanding of the mechanical response of soft tissue, among other materials. It is to this aspect of the problem that this study is directed.

In this paper we only address the subtle differences between the two approaches. To the best of our knowledge this relation has not been explored in the literature and we believe that the cumbersome technical details of the analysis are worthy of investigation. Our purpose is twofold: on the one hand, we illustrate the analysis for these nonisotropic elastic energy functions; on the other, we connect the acoustoelastic formulations of both material models, the one depending on the Green strain tensor and the one depending on the right Cauchy–Green tensor.

The layout of the paper is as follows. In Section 2, we introduce briefly the main governing equations. Section 3 is devoted to the acoustoelastic analysis of constitutive models that depend on the invariants of the right Cauchy–Green strain tensor. In particular, the equations governing infinitesimal motions superimposed on a finite deformation have been used to establish formulas for the velocity of (plane homogeneous) shear bulk waves propagating in general soft biological tissues subject to uniaxial tension or compression. Furthermore, the analysis connects with the constitutive model (3). In Section 4 we give some conclusions.

2. Overview of the main equations

We consider an elastic body whose initial geometry defines a *reference configuration*, which we denote by $\mathcal{B}_r$, and a finitely deformed equilibrium configuration $\mathcal{B}_0$. The position vectors of representative particles in $\mathcal{B}_r$ and $\mathcal{B}_0$ are denoted by $\boldsymbol{X}$ and $\boldsymbol{x}$, respectively. It is well known that $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$, where t is time. The deformation gradient tensor associated with the deformation $\mathcal{B}_r \to \mathcal{B}_0$ is denoted by $\boldsymbol{F}$.

2.1. ***Material model.*** Soft tissue is modeled as an incompressible transversely isotropic elastic solid. The most general transversely isotropic nonlinear elastic strain-energy function Ω depends on $\boldsymbol{F}$ through the right Cauchy–Green tensor $\boldsymbol{C}$, which is $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$, and we therefore consider Ω to depend on the invariants of the tensor $\boldsymbol{C}$. It is well known that $\boldsymbol{E} = (\boldsymbol{C} - \boldsymbol{I})/2$, where $\boldsymbol{I}$ is the identity tensor. The isotropic invariants of $\boldsymbol{C}$ most commonly used are the *principal* invariants, defined by

$$I_1^* = \operatorname{tr}\boldsymbol{C}, \quad I_2^* = \tfrac{1}{2}[(\operatorname{tr}\boldsymbol{C})^2 - \operatorname{tr}(\boldsymbol{C}^2)], \quad I_3^* = \det\boldsymbol{C}. \tag{4}$$

The (anisotropic) invariants associated with $\boldsymbol{M}$ and $\boldsymbol{C}$ are usually taken as (see, for instance, [Merodio and Saccomandi 2006])

$$I_4^* = \boldsymbol{M}\cdot(\boldsymbol{C}\boldsymbol{M}), \quad I_5^* = \boldsymbol{M}\cdot(\boldsymbol{C}^2\boldsymbol{M}). \tag{5}$$

It follows that for incompressible materials $\Omega = \Omega(I_1^*, I_2^*, I_4^*, I_5^*)$ since $I_3^* = 1$.

In the Appendix several expressions that are needed in this analysis are given. The corresponding Cauchy stress tensor for Ω using the relations (A.4) and (A.6) yields

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\Omega_1\boldsymbol{B} + 2\Omega_2(I_1^*\boldsymbol{B} - \boldsymbol{B}^2) + 2\Omega_4\boldsymbol{m}\otimes\boldsymbol{m} + 2\Omega_5(\boldsymbol{m}\otimes\boldsymbol{B}\boldsymbol{m} + \boldsymbol{B}\boldsymbol{m}\otimes\boldsymbol{m}), \tag{6}$$

where p is the hydrostatic pressure arising from the incompressibility constraint, $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, and $\boldsymbol{m} = \boldsymbol{F}\boldsymbol{M}$ gives the deformed fiber direction. This expression in indicial notation is

$$\sigma_{ij} = -p\delta_{ij} + 2\Omega_1 B_{ij} + 2\Omega_2(I_1^*\delta_{i j} - B_{i\gamma})B_{j\gamma} + 2\Omega_4 m_i m_j + 2\Omega_5(B_{j\gamma}m_\gamma m_i + B_{i\gamma}m_\gamma m_j). \tag{7}$$

It follows using (7) that

$$p = 2\Omega_1, \quad \Omega_4 + 2\Omega_5 = 0, \tag{8}$$

in the reference configuration where $\boldsymbol{F} = \boldsymbol{I}$ and the Cauchy stress components are zero.

2.2. ***Linearized equations of motion.*** The linearized equations of motion for incompressible materials are summarized below. For a complete derivation see [Ogden and Singh 2011]. From $\mathcal{B}_0$, the linearized incremental form of the equations of motion and the incompressibility constraint, in component form, are written as

$$\mathcal{A}_{0piqj}u_{j,pq} - p^*_{,i} = \rho u_{i,tt}, \quad u_{i,i} = 0, \tag{9}$$

respectively, where $u_i(\boldsymbol{x}, t)$ is the small time-dependent displacement increment, a comma indicates differentiation with respect to the spatial coordinate or with respect to t, p^* is the time-dependent pressure increment, and

$$\mathcal{A}_{0piqj} = F_{p\alpha} F_{q\beta} \frac{\partial^2 \Omega}{\partial F_{i\alpha} \partial F_{j\beta}}.$$

The subscript 0 indicates the so-called pushed-forward quantity from the initial reference configuration to the finitely deformed equilibrium configuration. We give its specialization to the situation in which there is no finite deformation and $\mathcal{B}_0$ coincides with $\mathcal{B}_r$. The elasticity tensor $\mathcal{A}_{0piqj}$ is given in (A.7). Under these conditions, customarily, the subscript 0 on $\mathcal{A}_0$ is omitted, and in what follows we do so.

2.3. ***Homogeneous plane waves.*** We apply the equation of motion and the incompressibility condition to the analysis of homogeneous plane waves. In particular, we consider the incremental displacement $\boldsymbol{u}$ and Lagrange multiplier p^* to have the forms

$$\boldsymbol{u} = f(\boldsymbol{n}\cdot\boldsymbol{x} - vt)\boldsymbol{d}, \quad p^* = g(\boldsymbol{n}\cdot\boldsymbol{x} - vt), \tag{10}$$

where $\boldsymbol{d}$ is a constant unit (polarization) vector, the unit vector $\boldsymbol{n}$ is the direction of propagation of the plane wave, v is the wave speed, f is a function that need not be made explicit but is subject to the restriction $f'' \neq 0$, and g is a function related to f. A prime on f or g indicates differentiation with respect to its argument.

Substitution of (10) into (9) then yields

$$[\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{d} - \rho v^2 \boldsymbol{d}] f'' - g'\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{d}\cdot\boldsymbol{n} = 0, \tag{11}$$

where the (symmetric) *acoustic tensor* $\boldsymbol{Q}(\boldsymbol{n})$ is defined by

$$Q_{ij}(\boldsymbol{n}) = \mathcal{A}_{piqj} n_p n_q. \tag{12}$$

The elasticity tensor $\mathcal{A}_{piqj}$ is given in (A.7). Now, we just give the main result. For a complete derivation see [Ogden and Singh 2011]. It follows that for a given $\boldsymbol{n}$ and $\boldsymbol{d}$ the wave speed v is obtained from

$$\rho v^2 = [\boldsymbol{Q}(\boldsymbol{n})\boldsymbol{d}]\cdot\boldsymbol{d}. \tag{13}$$

3. An approach to finding formulas for the speeds of homogeneous plane waves using general nonlinear elasticity theory

We now describe the loading and geometric case that will be used in the analysis that follows. Consider a rectangular block of a soft transversely isotropic incompressible elastic solid whose faces in the unstressed state are parallel to the (X_1, X_2), (X_2, X_3), and (X_3, X_1)-planes and with the fiber direction $\boldsymbol{M}$ parallel to the X_1-direction. Suppose that the sample is under uniaxial tension or compression with the direction of tension parallel to the X_1-axis. It is easy to see that the sample is subject to $x_1 = \lambda_1 X_1$, $x_2 = \lambda_2 X_2$, and $x_3 = \lambda_3 X_3$, and whence

$$\boldsymbol{F} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \tag{14}$$

in which

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = \lambda^{-1/2}, \quad \lambda > 0, \tag{15}$$

where λ_k are the principal stretches of deformation. Note that the faces of the deformed block are parallel to the (x_1, x_2), (x_2, x_3), and (x_3, x_1)-planes.

The analysis can now focus on different cases. We consider motion in the (x_1, x_2)-plane (a plane that contains the deformed fiber direction) with $n_1 = \cos\theta = c$, $n_2 = \sin\theta = s$, $d_1 = -\sin\theta$, and $d_2 = \cos\theta$, where θ is the angle between $\boldsymbol{n}$ and the x_1-direction. The wave speed is, under these conditions, obtained using (13), (12), and (A.7) and can be written as

$$\begin{aligned}\rho v^2 &= \mathcal{A}_{piqj} n_p n_q d_i d_j \\ &= \mathcal{A}_{1212} c^4 + 2(\mathcal{A}_{1222} - \mathcal{A}_{1112}) c^3 s + (\mathcal{A}_{1111} + \mathcal{A}_{2222} - 2\mathcal{A}_{1122} - 2\mathcal{A}_{1221}) c^2 s^2 \\ &\qquad + 2(\mathcal{A}_{1121} - \mathcal{A}_{2221}) c s^3 + \mathcal{A}_{2121} s^4. \end{aligned} \tag{16}$$

We now assume the situation described in (14) and investigate propagation in the fiber direction, that is, $\boldsymbol{n}$ coincides with the deformed fiber direction, which initially is in the X_1-direction. In this case, the relevant term in (16) is

$$\begin{aligned}\mathcal{A}_{1212} = {} & 2\Omega_1\lambda_1^2 + 2\Omega_2(I_1^*\lambda_1^2 - \lambda_1^2\lambda_2^2 - \lambda_1^4) + 2\Omega_4 m_1^2 + 2\Omega_5[2m_1 B_{1\gamma} m_\gamma + \lambda_2^2 m_1^2 + \lambda_1^2 m_2^2] \\ & + 4\Omega_{44} m_1^2 m_2^2 + 2\Omega_{55}(m_1 B_{2\gamma} m_\gamma + m_2 B_{1\gamma} m_\gamma)^2 + 4\Omega_{45} 2 m_1 m_2 (m_1 B_{2\gamma} m_\gamma + m_2 B_{1\gamma} m_\gamma) \\ & + 2\Omega_1\lambda_1^2 + 2\Omega_2\lambda_1^2\lambda_3^2 + 2\Omega_4 m_1^2 + 2\Omega_5[2m_1 B_{1\gamma} m_\gamma + \lambda_1^2 m_2^2 + \lambda_2^2 m_1^2] \\ & + 4\Omega_{44} m_1^2 m_2^2 + 4\Omega_{55}(m_1 B_{2\gamma} m_\gamma + m_2 B_{1\gamma} m_\gamma)^2 + 8\Omega_{45} m_1 m_2 (m_1 B_{2\gamma} m_\gamma + m_2 B_{1\gamma} m_\gamma).\end{aligned}$$

Considering, further, that in this case the components of $\boldsymbol{m}$ are $m_1 = \lambda_1$ and $m_2 = 0$ the wave speed v_{12} is

$$\rho v_{12}^2 = \mathcal{A}_{1212} = 2\Omega_1\lambda_1^2 + 2\Omega_2\lambda_1^2\lambda_3^2 + 2\Omega_4\lambda_1^2 + 2\Omega_5(2\lambda_1^4 + \lambda_1^2\lambda_2^2). \tag{17}$$

On the other hand, and with a parallel argument, the wave speed for the analysis of propagation in the perpendicular-to-the-fiber direction denoted by v_{21}, that is, $\boldsymbol{n}$ is perpendicular to the deformed fiber direction, which initially is in the X_1-direction, yields using (16)

$$\rho v_{21}^2 = \mathcal{A}_{2121} = 2\Omega_1\lambda_2^2 + 2\Omega_2\lambda_2^2\lambda_3^2 + 2\Omega_5\lambda_1^2\lambda_2^2. \tag{18}$$

It follows using (7), particularized for the (uniaxial) conditions at hand, (17), and (18) that

$$\mathcal{A}_{1212} - \mathcal{A}_{2121} = \sigma_1 - \sigma_2. \tag{19}$$

We do not pursue here the study of these relations. For further details, we refer to [Ogden and Singh 2011], which also establishes connections between the identities given with the formulation developed here and the identities developed by Biot [1965] with his formulation.

Now, we consider $\lambda_1 = 1 + e_1$ for e_1 sufficiently small. By incompressibility $\lambda_2 = \lambda_3 = \lambda_1^{-1/2}$, and it follows that $\lambda_1^2\lambda_3^2 = \lambda_1^2\lambda_2^2 = \lambda_1 = 1 + e_1$. Using these expressions, (4), and (5), the invariants I_i^* in terms of e_1 are

$$\begin{aligned} I_1^* &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1^2 + 2\lambda_2^2 = 1 + 2e_1 + e_1^2 + 2(1+e_1)^{-1} = 3 + 3e_1^2, \\ I_2^* &= \lambda_1^{-2} + 2\lambda_2^{-2} = 1 - 2e_1 + 3e_1^2 + 2(1+e_1) = 3 + 3e_1^2, \\ I_4^* &= \lambda_1^2 = 1 + 2e_1 + e_1^2, \\ I_5^* &= \lambda_1^4 = 1 + 4e_1 + 6e_1^2. \end{aligned} \tag{20}$$

Expansion of (17) requires the expansion of the different derivatives Ω_i, $i = 1, 2, 4, 5$, in (17). One finds using the chain rule and (20) that

$$\begin{aligned}\Omega_1 &= \Omega_1^{(o)} + e_1 \frac{d}{de_1}\Omega_1\Big|_{e_1=0} + \frac{1}{2}e_1^2 \frac{d^2}{de_1^2}\Omega_1\Big|_{e_1=0} \\ &= \Omega_1^{(o)} + e_1\big[2\Omega_{14}^{(o)} + 4\Omega_{15}^{(o)}\big] + \frac{1}{2}e_1^2\frac{d}{de_1}[6\Omega_{11}e_1 + 6\Omega_{12}e_1 + \Omega_{14}(2+2e_1) + \Omega_{15}(4+12e_1)]\Big|_{e_1=0} \\ &= \Omega_1^{(o)} + 2e_1\big(\Omega_{14}^{(o)} + 2\Omega_{15}^{(o)}\big) + \frac{1}{2}e_1^2\big(6\Omega_{11}^{(o)} + 6\Omega_{12}^{(o)} + 2\Omega_{14}^{(o)} + a2\Omega_{15}^{(o)} + 4\Omega_{144}^{(o)} + 16\Omega_{145}^{(o)} + 16\Omega_{155}^{(o)}\big),\end{aligned}$$

where $\Omega_1^{(o)}$ is the value of Ω_1 in the reference configuration. Furthermore, it is easy to obtain that

$$\Omega_i = \Omega_i^{(o)} + 2\big(\Omega_{i4}^{(o)} + 2\Omega_{i5}^{(o)}\big)e_1 + \big(3\Omega_{i1}^{(o)} + 3\Omega_{i2}^{(o)} + \Omega_{i4}^{(o)} + 6\Omega_{i5}^{(o)} + 2\Omega_{i44}^{(o)} + \Omega_{i45}^{(o)} + 8\Omega_{i55}^{(o)}\big)e_1^2, \quad (21)$$

where i can take the values 1, 2, 4, and 5.

Use of (21) and (17) yields for the wave speed, disregarding terms of order higher than 2 in e_1,

$$\begin{aligned}\rho v_{12}^2 = {} & 2\Omega_1^{(o)} + 2\Omega_2^{(o)} + 2\Omega_4^{(o)} + 6\Omega_5^{(o)} + e_1\big[4\Omega_1^{(o)} + 2\Omega_{12}^{(o)} + 4\Omega_4^{(o)} + 18\Omega_5^{(o)} + 4\Omega_{14}^{(o)} + 8\Omega_{15}^{(o)} \\ & + 4\Omega_{24}^{(o)} + 8\Omega_{25}^{(o)} + 4\Omega_{44}^{(o)} + 20\Omega_{45}^{(o)} + 24\Omega_{55}^{(o)}\big] + e_1^2\big[2\Omega_1^{(o)} + 2\Omega_4^{(o)} + 24\Omega_5^{(o)} \\ & + 8\big(\Omega_{14}^{(o)} + 2\Omega_{15}^{(o)}\big) + 4\big(\Omega_{24}^{(o)} + 2\Omega_{25}^{(o)}\big) + 8\big(\Omega_{44}^{(o)} + \Omega_{45}^{(o)}\big) + 36\big(\Omega_{45}^{(o)} + 2\Omega_{55}^{(o)}\big) + 6\Omega_{11}^{(o)} + 12\Omega_{12}^{(o)} \\ & + 8\Omega_{14}^{(o)} + 30\Omega_{15}^{(o)} + 6\Omega_{22}^{(o)} + 8\Omega_{24}^{(o)} + 30\Omega_{25}^{(o)} + 2\Omega_{44}^{(o)} + 18\Omega_{45}^{(o)} + 36\Omega_{55}^{(o)} + 4\Omega_{144}^{(o)} + 16\Omega_{145}^{(o)} \\ & + 16\Omega_{155}^{(o)} + 4\Omega_{244}^{(o)} + 16\Omega_{245}^{(o)} + 16\Omega_{255}^{(o)} + 4\Omega_{444}^{(o)} + 30\Omega_{445}^{(o)} + 64\Omega_{455}^{(o)} + 48\Omega_{555}^{(o)}\big]. \qquad (22)\end{aligned}$$

This formula gives the general acoustoelastic wave speed for constitutive models that depend on the invariants of the right Cauchy–Green tensor. This completes the first purpose of our analysis. Now, we focus on the second purpose, which is to connect this formulation and the one developed for constitutive models that depend on invariants of the Green strain tensor.

The relations between both invariant formulations I_i^* and I_i are established though C and the well-known Cayley–Hamilton theorem, that we write as

$$\boldsymbol{C}^3 = I_1^*\boldsymbol{C}^2 - I_2^*\boldsymbol{C} + \boldsymbol{I}, \quad \operatorname{tr}(\boldsymbol{C}^3) = I_1^*(I_1^{*2} - 2I_2^*) - I_1^*I_2^* + 3.$$

Hence, the invariants I_i in terms of I_i^*

are

$$\begin{aligned} I_2 &= \operatorname{tr}(\boldsymbol{E}^2) = \tfrac{1}{4}\operatorname{tr}(\boldsymbol{C}^2 - 2\boldsymbol{C} + \boldsymbol{I}) = \tfrac{1}{4}(I_1^{*2} - 2I_2^* - 2I_1^* + 3), \\ I_3 &= \operatorname{tr}(\boldsymbol{E}^3) = \tfrac{1}{8}\operatorname{tr}(\boldsymbol{C}^3 - 3\boldsymbol{C}^2 + 3\boldsymbol{C} + \boldsymbol{I}) = \tfrac{1}{8}(I_1^{*3} - 3I_1^*I_2^* - 3I_1^{*2} + 6I_2^* + 3I_1^*), \\ I_4 &= \boldsymbol{M}\cdot(\boldsymbol{E}\boldsymbol{M}) = \tfrac{1}{2}(I_4^* - 1), \\ I_5 &= \boldsymbol{M}\cdot(\boldsymbol{E}^2\boldsymbol{M}) = \tfrac{1}{4}(I_5^* - 2I_4^* + 1). \end{aligned} \qquad (23)$$

Using (23) the constitutive model (3) (or any other model written in terms of the invariants I_i) can be written in terms of the invariants I_i^* and the same constants μ, A, and $\alpha_1, \dots, \alpha_{11}$. Then, (22) for that particular model after a lengthy but straightforward calculation yields

$$\rho v_{12}^2 = \mu + \tfrac{1}{2}\alpha_2 + \big(3\mu + \tfrac{1}{4}A + 2\alpha_1 + \tfrac{5}{2}\alpha_2 + \alpha_3 + \tfrac{1}{2}\alpha_5\big)e \\ + \big(5\mu + \tfrac{7}{4}A + 5\alpha_1 + 5\alpha_2 + 5\alpha_3 + 3\alpha_4 + \tfrac{15}{4}\alpha_5 + 3\alpha_6 + \alpha_7 + \tfrac{7}{4}\alpha_8 + \alpha_{10} + \tfrac{3}{4}\alpha_{11}\big)e^2. \quad (24)$$

This formula was obtained in [Vinh and Merodio 2013]. Furthermore, the result given in [Destrade et al. 2010a] is a special case of the approximation (24) when $\alpha_k = 0$ and $k = \overline{6, 11}$.

4. Conclusions

The motivation behind this analysis is the possibility of capturing the mechanical properties of soft transversely isotropic incompressible nonlinear elastic materials, such as certain soft biological tissues, using acoustoelasticity theory. The constitutive model is given as a strain-energy density function that depends on the invariants of the right Cauchy–Green tensor. The equations governing infinitesimal motions superimposed on a finite deformation have been used in conjunction with the constitutive law to examine the propagation of homogeneous plane waves. The speeds of homogeneous plane waves have been derived. Furthermore, the differences between this theoretical framework and the parallel one obtained for constitutive models that depend on the Green strain tensor have been highlighted. The use of both acoustoelastic wave speed framework formulations may help to scrutinize the nature of the elastic constants as well as to decide which elastic constants must be retained in the development of models.

Appendix: Derivatives of the invariants and the elasticity tensor

The expressions for the stress and elasticity tensors require the calculation of

$$\frac{\partial \Omega}{\partial \boldsymbol{F}} = \sum_{i=1}^{N} \Omega_i \frac{\partial I_i^*}{\partial \boldsymbol{F}} \quad (A.1)$$

and

$$\frac{\partial^2 \Omega}{\partial \boldsymbol{F}\,\partial \boldsymbol{F}} = \sum_{i=1}^{N} \Omega_i \frac{\partial^2 I_i^*}{\partial \boldsymbol{F}\,\partial \boldsymbol{F}} + \sum_{i=1}^{N}\sum_{j=1}^{N} \Omega_{ij} \frac{\partial I_i^*}{\partial \boldsymbol{F}} \otimes \frac{\partial I_j^*}{\partial \boldsymbol{F}}, \quad (A.2)$$

where we have used the shorthand notations $\Omega_i = \partial\Omega/\partial I_i^*$, $\Omega_{ij} = \partial^2\Omega/\partial I_i^*\partial I_j^*$, $i, j = 1, 2, \ldots, N$.

For the considered incompressible material the nominal stress is

$$\boldsymbol{S} = \frac{\partial \Omega}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1} = \sum_{i=1, i\neq 3}^{5} \Omega_i \frac{\partial I_i^*}{\partial \boldsymbol{F}} - p\boldsymbol{F}^{-1}, \quad (A.3)$$

and the corresponding Cauchy stress is

$$\boldsymbol{\sigma} = \boldsymbol{F}\frac{\partial \Omega}{\partial \boldsymbol{F}} - p\boldsymbol{I} = \sum_{\substack{i=1\\ i\neq 3}}^{5} \Omega_i \boldsymbol{F}\frac{\partial I_i^*}{\partial \boldsymbol{F}} - p\boldsymbol{I}. \quad (A.4)$$

The elasticity tensor is given by

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 \Omega}{\partial \boldsymbol{F} \partial \boldsymbol{F}} = \sum_{\substack{1\le i\le 5\\ i\ne 3}} \Omega_i \frac{\partial^2 I_i^*}{\partial \boldsymbol{F}\partial \boldsymbol{F}} + \sum_{\substack{1\le i\le 5\\ i\ne 3}} \sum_{\substack{1\le j\le 5\\ j\ne 3}} \Omega_{ij} \frac{\partial I_i^*}{\partial \boldsymbol{F}} \otimes \frac{\partial I_j^*}{\partial \boldsymbol{F}}. \tag{A.5}$$

This requires expressions for the derivatives of the invariants, which are

$$\begin{aligned}
&\frac{\partial I_1^*}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2^*}{\partial F_{i\alpha}} = 2(I_1^* F_{i\alpha} - F_{k\alpha}F_{k\delta}F_{i\delta}), \quad \frac{\partial I_4^*}{\partial F_{i\alpha}} = 2M_\alpha F_{i\delta} M_\delta,\\
&\frac{\partial I_5^*}{\partial F_{i\alpha}} = 2(M_\alpha F_{i\delta}C_{\delta\gamma}M_\gamma + C_{\alpha\gamma}M_\gamma F_{i\delta}M_\delta), \quad \frac{\partial^2 I_1^*}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta},\\
&\frac{\partial^2 I_2^*}{\partial F_{i\alpha}\partial F_{j\beta}} = 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + C_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - B_{ij}\delta_{\alpha\beta} - C_{\alpha\beta}\delta_{ij}), \quad \frac{\partial^2 I_4^*}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}M_\alpha M_\beta,\\
&\frac{\partial^2 I_5^*}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}(M_\alpha C_{\beta\gamma} + M_\beta C_{\alpha\gamma})M_\gamma + 2B_{ij}M_\alpha M_\beta\\
&\qquad\qquad + 2\delta_{\alpha\beta}F_{i\gamma}M_\gamma F_{j\delta}M_\delta + 2(F_{i\beta}F_{j\gamma}M_\alpha + F_{j\alpha}F_{i\gamma}M_\beta)M_\gamma.
\end{aligned} \tag{A.6}$$

The pushed-forward quantity from the initial reference configuration to the finitely deformed equilibrium configuration of $\boldsymbol{\mathcal{A}}$ is denoted $\boldsymbol{\mathcal{A}}_0$. We give its specialization to the situation in which there is no finite deformation and $\mathcal{B}_0$ coincides with $\mathcal{B}_r$. The components of $\boldsymbol{\mathcal{A}}_0$ in the reference configuration for Ω using (A.6), the chain rule, and the conditions (8) can be arranged in the form

$$\begin{aligned}
&\mathcal{A}_{0piqj}\\
&= F_{p\alpha}F_{q\beta}\frac{\partial^2 \Omega}{\partial F_{i\alpha}\partial F_{j\beta}}\\
&= 2\Omega_1\delta_{ij}B_{pq} + 2\Omega_2(2B_{ip}B_{jq} - B_{iq}B_{jp} + I_1^*\delta_{ij}B_{pq} - B_{ij}B_{pq} - \delta_{ij}(B^2)_{pq}) + 2\Omega_4\delta_{ij}m_pm_q\\
&\quad + 2\Omega_5[\delta_{ij}(m_pB_{qr}m_r + m_qB_{pr}m_r) + B_{ij}m_pm_q + m_im_jB_{pq} + B_{iq}m_jm_p + B_{pj}m_im_q]\\
&\quad + 4\Omega_{11}B_{ip}B_{jq} + 4\Omega_{22}(I_1^*B_{ip} - (B^2)_{ip})(I_1^*B_{jq} - (B^2)_{jq}) + 4\Omega_{44}m_im_jm_pm_q\\
&\quad + 4\Omega_{55}[m_pB_{ir}m_r + m_iB_{pr}m_r][m_qB_{jr}m_r + m_jB_{qr}m_r]\\
&\quad + 4\Omega_{12}[B_{ip}(I_1^*B_{jq} - (B^2)_{jq}) + B_{jq}(I_1^*B_{ip} - (B^2)_{ip})] + 4\Omega_{14}[B_{ip}m_jm_q + B_{jq}m_im_p]\\
&\quad + 4\Omega_{15}\big[B_{ip}[m_qB_{jr}m_r + m_jB_{qr}m_r] + B_{jq}[m_iB_{pr}m_r + m_pB_{ir}m_r]\big]\\
&\quad + 4\Omega_{24}[(I_1^*B_{ip} - (B^2)_{ip})m_jm_q + (I_1^*B_{jq} - (B^2)_{jq})m_im_p]\\
&\quad + 4\Omega_{25}\big[(I_1^*B_{ip} - (B^2)_{ip})[m_qB_{jr}m_r + m_jB_{qr}m_r] + (I_1^*B_{jq} - (B^2)_{jq})[m_iB_{pr}m_r + m_pB_{ir}m_r]\big]\\
&\quad + 4\Omega_{45}\big[m_im_p[m_qB_{jr}m_r + m_jB_{qr}m_r] + m_jm_q[m_iB_{pr}m_r + m_pB_{ir}m_r]\big].
\end{aligned} \tag{A.7}$$

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