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CONTINUOUS CONTACT PROBLEM FOR TWO ELASTIC LAYERS RESTING ON AN ELASTIC HALF-INFINITE PLANE

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The continuous contact problem for two elastic layers resting on an elastic half-infinite plane and loaded by means of a rigid stamp is presented. The elastic layers have different heights and elastic constants. An external load is applied to the upper elastic layer by means of a rigid stamp. The problem is solved under the assumptions that all surfaces are frictionless, body forces of elastic layers are taken into account, and only compressive normal tractions can be transmitted through the interfaces. General expressions of stresses and displacements are obtained by using the fundamental equations of the theory of elasticity and the integral transform technique. Substituting the stress and the displacement expressions into the boundary conditions, the problem is reduced to a singular integral equation, in which the function of contact stresses under the rigid stamp is unknown. The integral equation is solved numerically by making use of the appropriate Gauss–Chebyshev integration formula for circular and rectangular stamp profiles. The contact stresses under the rigid stamp, contact areas, initial separation loads, and initial separation distances between the two elastic layers and the lower-layer elastic half-infinite plane are obtained numerically for various dimensionless quantities and shown in graphics and tables.

1. Introduction

Contact problems have been widely carried out in the literature. Their areas of application include pavements of highways and airfields, foundations, railway ballasts, foundation grillages, roller bearings, joints, and support elements (see, for example, [Garrido and Lorenzana 1998; Birinci and Erdöl 2001; Ozsahin 2007]). General methods for contact problems may be seen in [Hertz 1895; Galin 1961; Uffliand 1965]. Keer et al. [1972] analyzed the smooth receding contact problem between an elastic layer and a half-space formulated under the assumptions of plane stress, plane strain, and axisymmetric conditions. The plane smooth contact problem for an elastic layer lying on an elastic half-space with a compressive load applied to the layer through a frictionless rigid stamp was considered in [Ratwani and Erdogan 1973]. Civelek and Erdogan [1975] investigated the continuous and discontinuous contact problems between an elastic layer and a rigid half-plane for the case of a single load in tension. Geçit [1980] analyzed a tensionless contact without friction between an elastic layer and an elastic foundation. Geçit [1981] also studied the axisymmetric contact problem for an elastic layer and an elastic foundation. Çakıroğlu et al. [2001] analyzed the continuous and discontinuous contact problems of two elastic layers resting on an elastic semiinfinite plane. Dini and Nowell [2004] considered the problem of plane elastic contact between a thin strip and symmetric flat and rounded punches. El-Borgi et al. [2006] analyzed a receding contact plane problem between a functionally graded layer and a homogeneous substrate. The contact problem for multilayered composite structures was studied in [Ke and Wang 2006; 2007]. Kahya et al.

Keywords: continuous contact, elastic layer, integral equation, rigid stamp, theory of elasticity.

[2007] investigated a frictionless receding contact problem between an anisotropic elastic layer and an anisotropic elastic half-plane, when the two bodies were pressed together by means of a rigid circular stamp. A receding contact axisymmetric problem between a functionally graded layer and a homogeneous substrate was examined in [Rhimi et al. 2009]. Rhimi et al. [2011] studied a double receding contact axisymmetric problem between a functionally graded layer and a homogeneous substrate. A frictional contact problem for a rigid cylindrical stamp and an elastic layer resting on a half-plane was solved in [Çömez 2010]. Argatov and Mishuris [2010] examined an axisymmetric contact problem for a biphasic cartilage layer with allowance for tangential displacements on the contact surface. Chen et al. [2011] investigated the singular integral equation method for a contact problem of rigidly connected punches on an elastic half-plane. The contact problem for a layer was studied for the case when the elastic properties of the medium are arbitrary continuously differentiable functions of its thickness in Trubchik et al. [2011]. Adibnazari et al. [2012] investigated the contact of an asymmetrical rounded apex wedge with a half-plane. The two-dimensional contact problem of a rigid cylinder indenting an elastic half-space with surface tension was examined in [Long et al. 2012]. Aleksandrov [2012] solved the axisymmetric contact problem for a prestressed incompressible elastic layer. Chidlow et al. [2013] analyzed the two-dimensional solutions of both adhesive and nonadhesive contact problems involving functionally graded materials. Kumar and DasGupta [2012] studied the mechanics of contact of an inflated spherical nonlinear hyperelastic membrane pressed between two rigid plates. A quadratic boundary element formulation for continuously nonhomogeneous, isotropic, and linear elastic functionally graded material contact problems was carried out in Gun and Gao [2014]. Vollebregt [2014] presented a new solver, called BCCG+FAI, for solving elastic normal contact problems.

Although there is much research available in the literature related to contact problems of multiple layers and half-planes, there are not enough studies about initial separation loads and distances in contact mechanics. This paper aims to obtain initial separation loads and initial separation distances between two layers and a lower-layer elastic half-plane for a continuous contact problem. This paper also presents the contact stresses under the rigid stamp, and the contact areas.

2. General expressions for stresses and displacements

Figure 1 shows two elastic, homogeneous, isotropic layers with different elastic constants and heights, resting on an elastic half-infinite plane and subjected to a concentrated load with magnitude P by means of a rigid stamp. The thickness in the z direction is taken to be unit. Since $x = 0$ is the symmetry plane, it is sufficient to consider the problem in the region $0 \leq x < \infty$ only. For numerical calculations, two types of stamp profiles are used, circular and rectangular.

Consider a plane strain problem and let $\rho_1 g$ and $\rho_2 g$ be body forces acting vertically in the layers. The body force of the elastic half-infinite plane is neglected. The stress and the displacement components may be obtained as

$$u_i(x, y) = u_{ip}(x) + u_{ih}(x, y), \tag{1a}$$

$$v_i(x, y) = v_{ip}(y) + v_{ih}(x, y), \tag{1b}$$

$$\sigma_{ix}(x, y) = \sigma_{ixp}(y) + \sigma_{ixh}(x, y), \tag{2a}$$

$$\sigma_{iy}(x, y) = \sigma_{iyp}(y) + \sigma_{iyh}(x, y), \tag{2b}$$

$$\tau_{ixy}(x, y) = \tau_{ixyh}(x, y), \tag{2c}$$

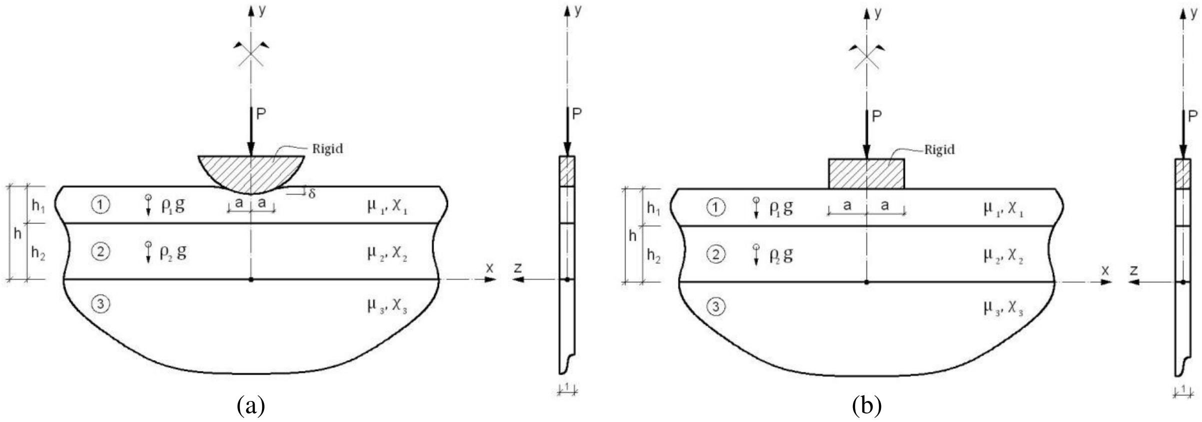

Figure 1. Geometry of the problem and loading condition. (a) Circular and (b) rectangular stamp cases.

where $i = 1, 2, 3$, the subscripts p and h refer to the particular part of the stress and the displacement components corresponding only to existing body forces, and the components of displacements and stresses for the layers and half-infinite plane without body forces, respectively. The particular part of the stress and the displacement components corresponding to $\rho_1 g$ and $\rho_2 g$ for the layers and the elastic half-infinite plane may be obtained as [Çakıroğlu 1990]

$$u_{1p}(x) = \left(\frac{3-\chi_1}{8\mu_1}\right)\left(\frac{\rho_1 g h_1}{2}\right)x, \tag{3a}$$

$$v_{1p}(y) = -\frac{\rho_1 g y}{2\mu_1}\left[\frac{1+\chi_1}{8}h_1 + \frac{\chi_1 - 1}{\chi_1 + 1}(h_2 + h - y)\right], \tag{3b}$$

$$u_{2p}(x) = \left(\frac{3-\chi_2}{8\mu_2}\right)\left(\frac{\rho_2 g h_2}{2} + \rho_1 g h_1\right)x, \tag{3c}$$

$$v_{2p}(y) = \frac{\chi_2 - 1}{\chi_2 + 1}\,\frac{\rho_2 g y}{2\mu_2}(y - h_2) - \frac{1+\chi_2}{8\mu_2}y\left(\rho_1 g h_1 + \rho_2 g\frac{h_2}{2}\right), \tag{3d}$$

$$u_{3p}(x) = \left(\frac{3-\chi_3}{8\mu_3}\right)(\rho_2 g h_2 + \rho_1 g h_1)x, \tag{3e}$$

$$v_{3p}(y) = -\frac{1+\chi_3}{8\mu_3}(\rho_1 g h_1 + \rho_2 g h_2)y, \tag{3f}$$

$$\sigma_{1xp}(y) = \frac{3-\chi_1}{1+\chi_1}\,\frac{\rho_1 g}{2}(2y - h - h_2), \tag{3g}$$

$$\sigma_{1yp}(y) = \rho_1 g(y - h), \tag{3h}$$

$$\sigma_{2xp}(y) = \frac{3-\chi_2}{1+\chi_2}\,\frac{\rho_2 g}{2}(2y - h_2), \tag{3i}$$

$$\sigma_{2yp}(y) = -\rho_1 g h_1 + \rho_2 g(y - h_2), \tag{3j}$$

$$\sigma_{3yp}(y) = -(\rho_1 g h_1 + \rho_2 g h_2), \tag{3k}$$

$$\sigma_{3xp} = \tau_{1xyp} = \tau_{2xyp} = \tau_{3xyp} = 0, \tag{3l}$$

where $u = u(x, y)$ and $v = v(x, y)$ represent displacement components in the x and y directions, respectively; μ_i is the shear modulus, χ_i is an elastic constant, with $\chi_i = (3-4\nu_i)$ for plane strain, and ν_i is the Poisson's ratio ($i = 1, \dots, 3$). The subscripts 1, 2, and 3 refer to the upper layer, the lower layer, and the elastic half-infinite plane, respectively.

The components of the displacements and stresses for the layers and the half-infinite plane without body forces may be expressed as follows [Çakıroğlu 1990]:

$$u_{ih}(x, y) = \frac{2}{\pi}\int_0^\infty \{[A_i + B_i y]e^{-\alpha y} + [C_i + D_i y]e^{\alpha y}\}\sin(\alpha x)\,d\alpha, \tag{4a}$$

$$v_{ih}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[A_i + B_i\left(\frac{\chi_i}{\alpha} + y\right)\right]e^{-\alpha y} + \left[-C_i + D_i\left(\frac{\chi_i}{\alpha} - y\right)\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4b}$$

$$\frac{1}{2\mu_i}\sigma_{ixh}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[\alpha(A_i + B_i y) - \left(\frac{3-\chi_i}{2}\right)B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) + \left(\frac{3-\chi_i}{2}\right)D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4c}$$

$$\frac{1}{2\mu_i}\sigma_{iyh}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_i + B_i y) + \left(\frac{1+\chi_i}{2}\right)B_i\right]e^{-\alpha y} + \left[-\alpha(C_i + D_i y) + \left(\frac{1+\chi_i}{2}\right)D_i\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4d}$$

$$\frac{1}{2\mu_i}\tau_{ixyh}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha(A_i + B_i y) + \left(\frac{\chi_i-1}{2}\right)B_i\right]e^{-\alpha y} + \left[\alpha(C_i + D_i y) - \left(\frac{\chi_i-1}{2}\right)D_i\right]e^{\alpha y}\right\}\sin(\alpha x)\,d\alpha, \tag{4e}$$

$$u_{3h}(x, y) = \frac{2}{\pi}\int_0^\infty \{[C_3 + D_3 y]e^{\alpha y}\}\sin(\alpha x)\,d\alpha, \tag{4f}$$

$$v_{3h}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[-C_3 + D_3\left(\frac{\chi_3}{\alpha} - y\right)\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4g}$$

$$\frac{1}{2\mu_3}\sigma_{3xh}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[\alpha(C_3 + D_3 y) + \left(\frac{3-\chi_3}{2}\right)D_3\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4h}$$

$$\frac{1}{2\mu_3}\sigma_{3yh}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[-\alpha(C_3 + D_3 y) + \left(\frac{1+\chi_3}{2}\right)D_3\right]e^{\alpha y}\right\}\cos(\alpha x)\,d\alpha, \tag{4i}$$

$$\frac{1}{2\mu_3}\tau_{3xyh}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[\alpha(C_3 + D_3 y) - \left(\frac{\chi_3-1}{2}\right)D_3\right]e^{\alpha y}\right\}\sin(\alpha x)\,d\alpha, \tag{4j}$$

where A_i, B_i, C_i, D_i ($i = 1, 2$) and C_3, D_3 are unknown coefficients which will be determined from the boundary conditions prescribed for $y = 0$, $y = h_2$, and $y = h$.

3. Boundary conditions and solution of the singular integral equation

The continuous contact problem for two elastic layers resting on an elastic half-infinite plane and subjected to a concentrated load with magnitude P by means of a rigid stamp will be investigated. The contact stresses under the rigid stamp, the contact areas, the distribution of the contact stresses between the layers and the lower-layer half-infinite plane until the occurrence of the initial separation, the initial separation loads, and the initial separation distances will be examined.

The boundary conditions for the frictionless contact problem outlined above can be defined as follows:

$$\tau_{1xy}(x, h) = 0, \qquad 0 \le x < \infty, \tag{5a}$$

$$\sigma_{1y}(x, h) = \begin{cases} -p(x), & 0 \le x < a, \\ 0, & a \le x < \infty, \end{cases} \tag{5b}$$

$$\tau_{1xy}(x, h_2) = 0, \qquad 0 \le x < \infty, \tag{5c}$$

$$\tau_{2xy}(x, h_2) = 0, \qquad 0 \le x < \infty, \tag{5d}$$

$$\sigma_{1y}(x, h_2) = \sigma_{2y}(x, h_2), \qquad 0 \le x < \infty, \tag{5e}$$

$$\frac{\partial}{\partial x}[v_1(x, h_2) - v_2(x, h_2)] = 0, \qquad 0 \le x < \infty, \tag{5f}$$

$$\tau_{2xy}(x, 0) = 0, \qquad 0 \le x < \infty, \tag{5g}$$

$$\tau_{3xy}(x, 0) = 0, \qquad 0 \le x < \infty, \tag{5h}$$

$$\sigma_{2y}(x, 0) = \sigma_{3y}(x, 0), \qquad 0 \le x < \infty, \tag{5i}$$

$$\frac{\partial}{\partial x}[v_2(x, 0) - v_3(x, 0)] = 0, \qquad 0 \le x < \infty, \tag{5j}$$

$$\frac{\partial}{\partial x}[v_1(x, h)] = f(x), \qquad 0 \le x < a, \tag{5k}$$

where a is the half-width of the contact area between the rigid stamp and the upper layer, $p(x)$ is the unknown contact stress under the rigid stamp, and $f(x)$ is the derivative of the function $F(x)$ which characterizes profile of the rigid stamp. In the case of a circular stamp, $f(x)$ can be obtained as follows:

$$F(x) = h - \delta - [(R^2 - x^2)^{1/2} - R], \tag{6a}$$

$$f(x) = \frac{d}{dx}[F(x)] = -\frac{x}{(R^2 - x^2)^{1/2}}, \tag{6b}$$

where δ is the maximum displacement, which occurs on the layer under the stamp on the axis of symmetry ($x = 0$), and R is the radius of the rigid circular stamp.

In the case of rectangular stamp, because $F(x)$ is equal to a constant, $f(x)$ can be obtained as follows:

$$f(x) = \frac{d}{dx}[F(x)] = 0. \tag{6c}$$

Applying the boundary conditions (5a)–(5j) to the stress and displacement expressions, the coefficients A_i, B_i, C_i, D_i ($i = 1, 2$), C_3, and D_3 can be determined in terms of the unknown contact stress $p(x)$; by substituting these coefficients into (5k), after some routine manipulations and using the symmetry condition $p(x) = p(-x)$, one may obtain the following singular integral equation for $p(x)$:

$$\int_{-a}^{a} \Big[\frac{1}{t-x} + k(x, t)\Big] p(t)\, dt = -\frac{4\pi\mu_1}{1+\chi_1} f(x), \qquad -a < x < a, \tag{7}$$

where the kernel $k(x, t)$ is given by (A.1). The equilibrium condition of the problem may be expressed as

$$\int_{-a}^{a} p(t)\, dt = P. \tag{8}$$

In order to obtain the initial separation load and initial separation distance between the two elastic layers and the lower-layer half-infinite plane, the contact stresses $\sigma_{1y}(x, h_2)$ and $\sigma_{2y}(x, 0)$ need to be determined. Substituting the values of A_i, B_i, C_i, and D_i $(i = 1, 2)$ as evaluated in terms of $p(x)$ into (2b) and after some algebraic manipulation, the contact stresses are obtained as follows:

$$\sigma_{1y}(x,h_2) = -\rho_1 g h_1 - \frac{1}{\pi h}\int_{-a}^{a} k_2^*(x,t)p(t)\,dt, \qquad 0 \le x < \infty, \tag{9a}$$

$$\sigma_{2y}(x,0) = -\rho_1 g h_1\left[1+\frac{\rho_2 h_2}{\rho_1 h_1}\right] - \frac{\mu_2}{\mu_1}\frac{1}{\pi h}\int_{-a}^{a} k_3^*(x,t)p(t)\,dt, \quad 0 \le x < \infty, \tag{9b}$$

where the kernels $k_2^*(x, t)$ and $k_3^*(x, t)$ are given by (A.2) and (A.3). In order to simplify the solution of the singular integral equation, the following dimensionless quantities are introduced:

$$x = as, \quad t = ar, \quad \phi(r) = \frac{p(ar)}{P/h}, \quad M(s) = \frac{m(as)}{P/h}, \quad m(as) = -\frac{4\pi\mu_1}{1+\chi_1} f(as). \tag{10}$$

Substituting from (10), then (7), (8), (9a), and (9b) may be obtained as follows:

$$\int_{-1}^{1}\left[\frac{1}{r-s} + N(s,r)\right]\phi(r)\,dr = M(s), \quad -1 < s < 1, \qquad N(s,r) = ak(as,ar), \tag{11a}$$

$$\frac{a}{h}\int_{-1}^{1}\phi(r)\,dr = 1, \tag{11b}$$

$$\frac{\sigma_{1y}(x,h_2)}{P/h} = -\frac{1}{\lambda} - \frac{1}{\pi}\frac{a}{h}\int_{-1}^{1} k_2(x,ar)\Phi(r)\,dr, \quad 0 \le x < \infty, \tag{11c}$$

$$\frac{\sigma_{2y}(x,0)}{P/h} = -\frac{1}{\lambda}\left[1+\frac{\rho_2 h_2}{\rho_1 h_1}\right] - \frac{\mu_2}{\mu_1}\frac{1}{\pi}\frac{a}{h}\int_{-1}^{1} k_3(x,ar)\Phi(r)\,dr, \quad 0 \le x < \infty, \tag{11d}$$

where λ is called the load factor, and defined as

$$\lambda = \frac{P}{\rho_1 g h h_1}. \tag{12}$$

3.1. ***Circular stamp case.*** The contact stress $p(x)$ vanishes at the ends because of the smooth contact at the end points, and therefore the index of the integral equation (11a) is -1. Noting this, the solution of integral equation can be found as follows [Erdogan and Gupta 1972]:

$$\phi(r) = g(r)(1-r^2)^{1/2}, \quad -1 < r < 1. \tag{13}$$

Using the appropriate Gauss–Chebyshev integration formula, (11a) and (11b) may be reduced to the following forms:

$$\begin{gathered}\sum_{i=1}^{n}(1-r_i^2)\left[\frac{1}{r_i - s_j} + N(s_j, r_i)\right]g(r_i) = \frac{n+1}{\pi}M(s_j), \quad j = 1,\dots,n+1,\\ \frac{a}{h}\sum_{i=1}^{n}(1-r_i^2)g(r_i) = \frac{n+1}{\pi},\end{gathered} \tag{14}$$

where

$$r_i = \cos\Big(\frac{i\pi}{n+1}\Big), \qquad i = 1, \dots, n, \tag{15a}$$

$$s_j = \cos\Big(\frac{2j-1}{n+1}\frac{\pi}{2}\Big), \quad j = 1, \dots, n+1. \tag{15b}$$

The extra equation in (14) corresponds to the consistency condition of the original integral equation in (11a). In this case, the $(n+1/2)$-th equation in (14) is satisfied automatically. Hence, the equations in (14) constitute a system of $n+1$ equations for $n+1$ unknowns. Note that the system is highly nonlinear in a and an interpolation scheme is required to determine this unknown. Solving this system of equations and using (13), $\phi(r)$, the normalized contact stress distribution, and a half-width of the contact area are obtained. By using (13), substituting the results into (11c) and (11d) and using the Gauss integration formula, the contact stresses $\sigma_{1y}(x, h_2)$ and $\sigma_{2y}(x, 0)$ are determined. In order to be valid for the singular integral equation given in (11a), the contact stresses $\sigma_{1y}(x, h_2)$ and $\sigma_{2y}(x, 0)$ must be compressive everywhere and no sign changing is allowed. So, the critical load value can be calculated numerically by equating (11c) and (11d) to zero. Then λ_{cr} (the initial separation load) and x_{cr} (the initial separation distance) can be obtained.

3.2. *Rectangular stamp case.* Since the contact stress under the rigid stamp goes to infinity at the corners, that is, $g(\pm 1) \to \infty$, the index of the singular integral equation is $+1$. Assuming the solution of integral equation as [Erdogan and Gupta 1972]

$$\phi(r) = g(r)(1-r^2)^{-1/2}, \quad -1 < r < 1, \tag{16a}$$

and using the appropriate Gauss–Chebyshev integration formula, (11a) and (11b) may then be replaced by

$$\sum_{i=1}^{n} W_i\Big(\frac{1}{r_i - s_j} + N(s_j, r_i)\Big)g(r_i) = 0, \quad j = 1, \dots, n-1, \tag{16b}$$

$$\frac{a}{h}\sum_{i=1}^{n} W_i g(r_i) = 1, \tag{16c}$$

where

$$W_1 = W_n = \frac{\pi}{2n-2}, \quad W_i = \frac{\pi}{n-1}, \quad i = 2, \dots, n-1, \tag{16d}$$

$$r_i = \cos\Big(\frac{i-1}{n-1}\pi\Big), \qquad i = 1, \dots, n, \tag{16e}$$

$$s_j = \cos\Big(\frac{2j-1}{n-1}\frac{\pi}{2}\Big), \qquad j = 1, \dots, n-1. \tag{16f}$$

Equations (16b) and (16c) constitute n linear algebraic equations for n unknowns, $g(r_i)$, $i = 1, \dots, n$. Solution of these algebraic equations and use of (16a) gives the unknown contact stress function under the rigid stamp, $p(x)$. In order to obtain the initial separation load and distance in the case of a rectangular stamp, the same method as in the case of the circular stamp is followed.

4. Results and discussion

Some of the calculated results obtained from the solution of the continuous contact problem described in the previous sections for various dimensionless quantities such as R/h, $\mu_1/(P/h)$, μ_2/μ_1, and μ_3/μ_2 are shown in Tables 1–5 and Figures 2–6. Table 1 and Figure 2 show the variation of the half-width of the contact area for a circular stamp with $\mu_1/(P/h)$ and R/h. As it can be seen in Table 1 and Figure 2, the half-width of the contact area a/h increases with increasing R/h, but decreases with increasing load ratio $\mu_1/(P/h)$. The variation of the contact stress under the rigid stamp for a rectangular stamp with μ_2/μ_1 is given in Figure 3. It may be observed in this figure that as μ_2/μ_1 increases, the normalized contact stress increases in the interior region of the rigid stamp and decreases in the region close to the corners. Figure 4 shows the variation of the contact stress distribution under a rigid circular stamp with μ_2/μ_1. It appears that the maximum value of the contact stress is always at $x = 0$, and it increases with increasing μ_2/μ_1.

In Figure 5, the contact stress distribution under the rectangular stamp is given. As expected, the contact stresses become infinite at the corners of the stamp. The normalized contact stress increases with decreasing (a/h). Variation of the initial separation load λ_{cr} and the initial separation distance x_{cr} between the layers and the lower-layer half-infinite plane with μ_2/μ_1 for various values of μ_3/μ_2 in case

	a/h						
$\mu_1/(P/h)$	$R/h=10$	$R/h=50$	$R/h=100$	$R/h=250$	$R/h=500$	$R/h=750$	$R/h=1000$
10	0.6541	1.3043	1.7382	2.5392	3.3912	4.0241	4.5486
50	0.3069	0.6541	0.8861	1.3043	1.7382	2.0552	2.3149
100	0.21806	0.4762	0.6541	0.9747	1.3043	1.5433	1.7382
250	0.13815	0.3069	0.4287	0.6541	0.8861	1.0529	1.1883
500	0.0977	0.21806	0.3069	0.4762	0.6541	0.7822	0.8861
750	0.07978	0.17824	0.25142	0.39304	0.5442	0.6541	0.7429
1000	0.0691	0.15443	0.21806	0.3421	0.4762	0.5743	0.6541

Table 1. Variation of the half-width of the contact area for a circular stamp with $\mu_1/(P/h)$ and R/h ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $\mu_2/\mu_1 = 2$, and $\mu_3/\mu_2 = 2$).

	$\mu_3/\mu_2=0.5$		$\mu_3/\mu_2=1$		$\mu_3/\mu_2=2$	
μ_2/μ_1	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}
0.1	4.10	166.95	3.31	118.343	2.80	83.5115
0.25	3.10	139.777	2.53	99.2449	2.17	69.8606
0.5	2.59	139.172	2.11	94.9923	1.82	64.8572
4	1.21	96.3381	1.16	55.4769	1.13	42.5852
10	1.04	43.3536	1.03	37.0934	1.02	33.6179
50	0.97	29.9315	0.97	29.1694	0.97	28.6559

Table 2. Variation of the initial separation load λ_{cr} and the initial separation distance x_{cr} between layers for a circular stamp with μ_2/μ_1 and μ_3/μ_2 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $R/h = 100$, $\mu_1/(P/h) = 500$, and $\rho_2/\rho_1 = 1$).

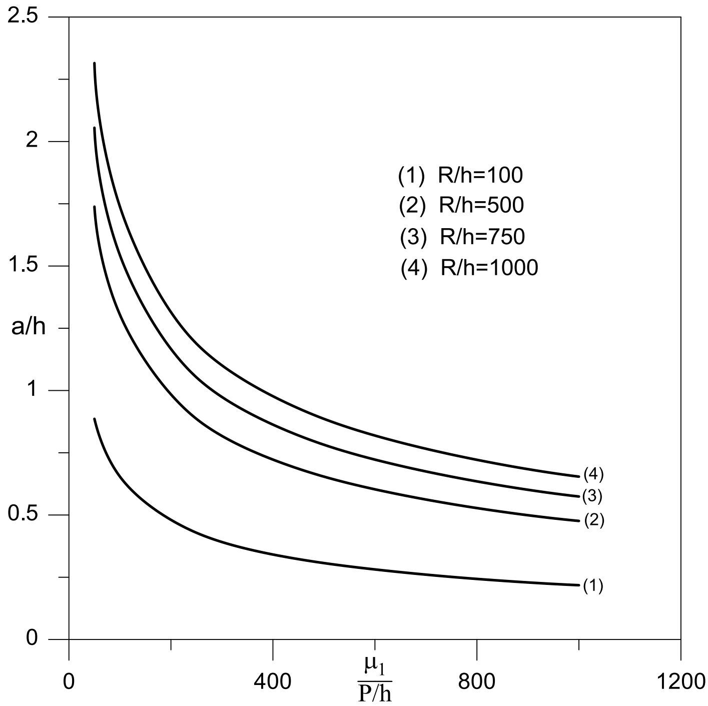

Figure 2. Variation of the half-width of the contact area for a circular stamp with $\mu_1/(P/h)$ ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $\mu_2/\mu_1 = 2$, and $\mu_3/\mu_2 = 2$).

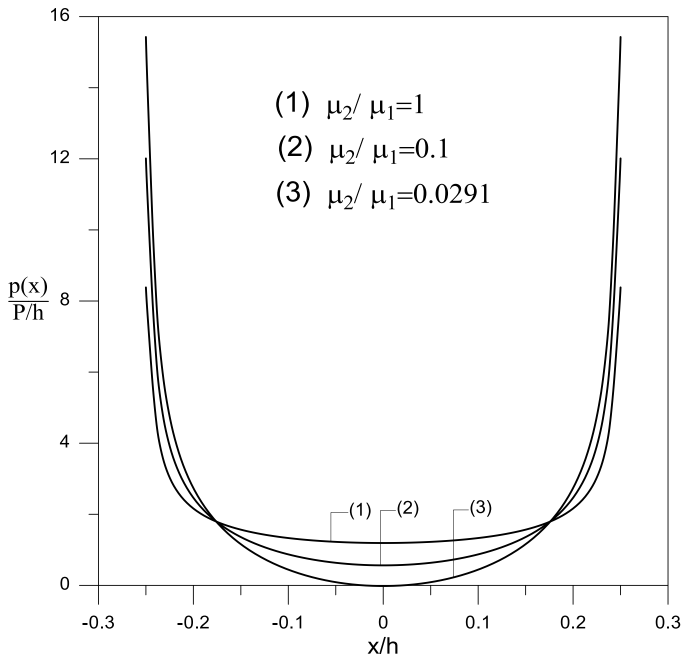

Figure 3. Variation of the contact stress distribution under the rigid stamp for a rectangular stamp with μ_2/μ_1 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $a/h = 0.7$, and $\mu_3/\mu_2 = 1$).

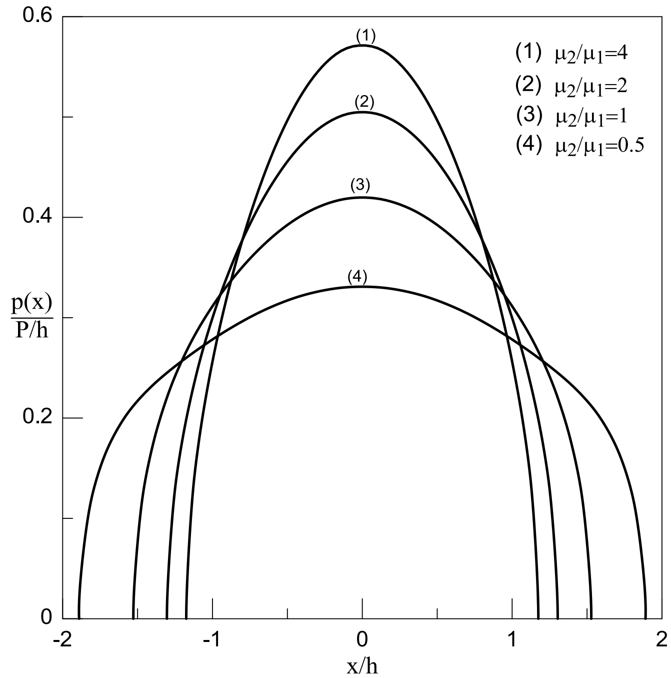

Figure 4. Variation of the contact stress distribution under the rigid stamp for a circular stamp with μ_2/μ_1 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $\mu_3/\mu_2 = 2$, $R/h = 500$, and $\mu_1(P/h) = 100$).

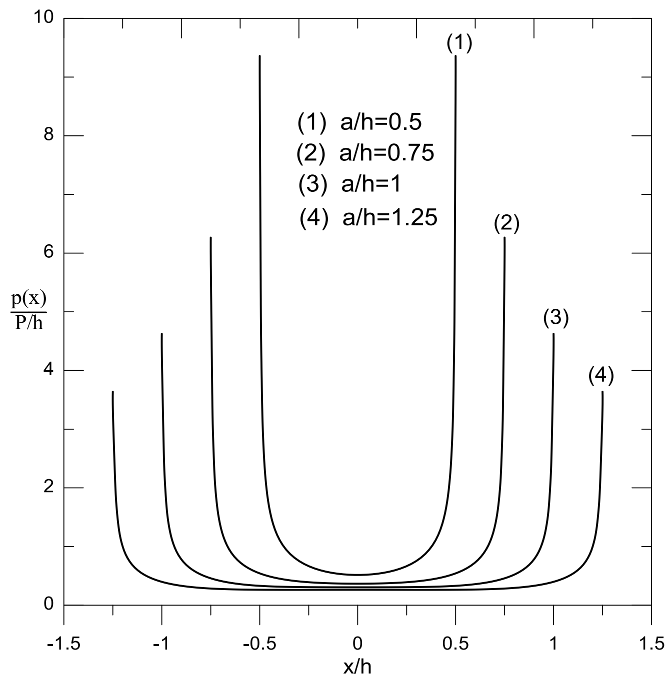

Figure 5. Variation of the contact stress distribution under the rigid stamp for a rectangular stamp with a/h ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $\mu_3/\mu_2 = 1$, and $\mu_2/\mu_1 = 1$).

	$\mu_3/\mu_2 = 0.5$		$\mu_3/\mu_2 = 1$		$\mu_3/\mu_2 = 2$	
μ_2/μ_1	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}
0.1	4.11	306.572	3.32	219.487	2.81	156.831
0.25	3.11	229.880	2.55	167.793	2.19	121.864
0.5	2.61	195.927	2.15	142.534	1.86	103.65
4	1.92	150.064	1.58	102.83	1.40	73.6904
10	1.82	143.238	1.51	96.6480	1.34	69.1730
50	1.77	139.115	1.47	92.8876	1.30	67.53

Table 3. Variation of the initial separation load λ_{cr} and the initial separation distance x_{cr} between the lower layer and the elastic half-infinite plane for a circular stamp with μ_2/μ_1 and μ_3/μ_2 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $R/h = 100$, $\mu_1/(P/h) = 500$, and $\rho_2/\rho_1 = 1$).

	$\mu_3/\mu_2 = 0.5$		$\mu_3/\mu_2 = 1$		$\mu_3/\mu_2 = 2$	
μ_2/μ_1	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}
0.1	3.76	91.2134	3.64	76.8386	3.16	35.5079
0.25	3.44	87.0030	2.91	63.6723	2.56	32.8831
0.5	2.98	80.4193	2.50	54.4106	2.22	32.6244
1	2.57	70.5130	2.15	46.2382	1.94	32.5297
4	1.67	49.8831	1.61	32.118	1.58	26.0751
10	1.51	27.9614	1.50	24.2309	1.49	22.3332

Table 4. Variation of the initial separation load λ_{cr} and the initial separation distance x_{cr} between layers for a rectangular stamp with μ_2/μ_1 and μ_3/μ_2 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $a/h = 0.7$, and $\rho_2/\rho_1 = 1$).

	$\mu_3/\mu_2 = 0.5$		$\mu_3/\mu_2 = 1$		$\mu_3/\mu_2 = 2$	
μ_2/μ_1	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}	x_{cr}	λ_{cr}
0.1	3.86	96.1742	3.65	86.8571	3.17	66.6502
0.25	3.47	93.7733	2.93	73.7035	2.59	57.2691
0.5	2.99	84.9483	2.55	66.6905	2.27	52.1141
1	2.65	78.6212	2.28	61.3007	2.05	48.06
4	2.29	70.7674	1.99	54.4808	1.82	43.0774
10	2.19	68.2357	1.91	52.4519	1.76	41.6836

Table 5. Variation of the initial separation load λ_{cr} and the initial separation distance x_{cr} between the lower layer and the elastic half-infinite plane for a rectangular stamp with μ_2/μ_1 and μ_3/μ_2 ($\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $a/h = 0.7$, and $\rho_2/\rho_1 = 1$).

of a circular stamp is given in Tables 2 and 3. As it can be seen in these tables, the initial separation load and distance between the layers and the lower-layer half-infinite plane decrease with increasing μ_2/μ_1 and μ_3/μ_2.

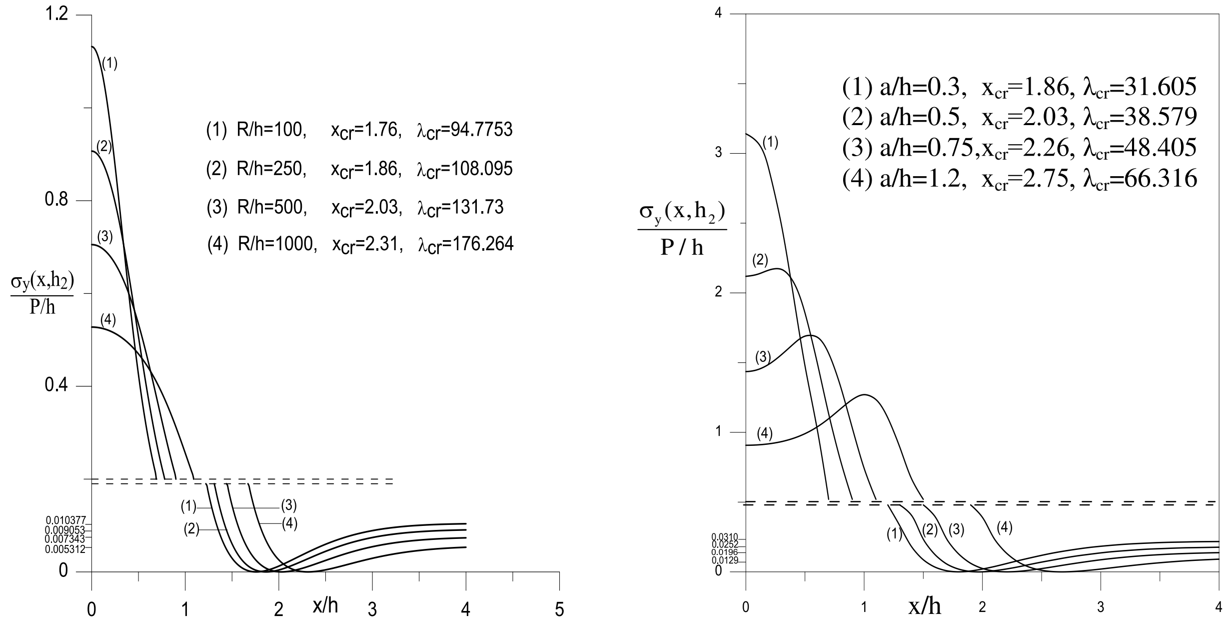

Figure 6. Variation of the contact stress distribution between two elastic layers. Left: variation with R/h for a circular stamp with $\chi_1 = \chi_2 = \chi_3 = 2$, $\mu_1/(P/h) = 500$, $h_2/h_1 = 1$, $\mu_2/\mu_1 = 2$, $\mu_3/\mu_2 = 2$, and $\rho_2/\rho_1 = 1$. Right: Variation with a/h for a rectangular stamp with $\chi_1 = \chi_2 = \chi_3 = 2$, $h_2/h_1 = 1$, $\mu_2/\mu_1 = 1$, $\mu_3/\mu_2 = 1$, and $\rho_2/\rho_1 = 1$.

Figure 6, left, shows the variation of the initial separation load and the initial separation distance between layers for a circular stamp with R/h. It appears that the initial separation load and the initial separation distance increase with increasing R/h. The variation of the initial separation load and the initial separation distance between layers for a rectangular stamp with a/h is shown in Figure 6, right. It appears that the initial separation load and the initial separation distance increase with increasing a/h. In Tables 4 and 5, variations of the initial separation load and distance between the two elastic layers and the lower-layer elastic half-infinite plane for a rectangular stamp with μ_3/μ_2 and μ_2/μ_1 are given. As it can be seen in Tables 4 and 5, the initial separation load and distance decrease with increasing μ_3/μ_2 and μ_2/μ_1.

5. Conclusions

This paper considers the continuous contact problem for two elastic layers resting on an elastic half-infinite plane. The results presented in this paper show that the elastic properties of the layers and intensity of the applied load have considerable effect on the contact stress distribution, the contact areas, the initial separation load, the and the initial separation distance. Additionally, the rigid stamp width (in the rectangular stamp case) and the radius of the rigid stamp (in the circular stamp case) play a very important role in the contact stress distribution, contact areas, initial separation load, and initial separation distance.

Appendix

Expressions of the kernels $k(x,t)$, $k_2^*(x,t)$, and $k_3^*(x,t)$ appearing in (7), (9a), and (9b) are given as follows:

$$\begin{aligned}
k(x,t) = \frac{1}{h}\int_0^\infty \Bigg\{ & \frac{1}{\Delta^{**}}\big[[e^{-5z-4z\xi}(-4z^2\xi^2 e^{z+2z\xi}(m_1(1+\chi_1)(e^{4z}-2e^{2z+2z\xi}+e^{4z\xi}) \\
& +4e^{2z+2z\xi}(1+\chi_2))(1+\chi_3)-4z\xi(1+\chi_2)e^{z+2z\xi}(m_2(-m_1(1+\chi_1)(e^{4z}-2e^{2z+2z\xi}+e^{4z\xi}) \\
& +(1+\chi_2)e^{2z}(1-2e^{2z\xi}+e^{4z\xi}))+(-e^{2z}+e^{4z\xi}-e^{4z}+e^{2z+4z\xi}-4ze^{2z+2z\xi})(1+\chi_3) \\
& +e^{z}\big(m_1(1+\chi_1)(e^{4z}-2e^{2z+2z\xi}+e^{4z\xi})(m_2(1+\chi_2)(-1+e^{4z\xi}) \\
& +(1+\chi_3)(1-2e^{2z\xi}+e^{4z\xi}))+(1+\chi_2)(e^{4z}-e^{4z\xi}+4ze^{2z+2z\xi})(m_2(1+\chi_2)(1-2e^{2z\xi}+e^{4z\xi}) \\
& +(1+\chi_3)(-1+e^{4z\xi})))\big]-1\Bigg\}\sin\Big[(t-x)\frac{z}{h}\Big]\,dz,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
k_2^*(x,t) = \int_0^\infty & \frac{2}{\Delta^{**}}\big[-m_1(1+\chi_1)e^{-4z-5z\xi}(-4(e^{z+6z\xi}+e^{3z+4z\xi})z^3\xi^3(1+\chi_3) \\
& +e^{z+2z\xi}(e^{2z\xi}(-1+z)+e^{2z}(1+z))\big(-m_2(1+\chi_2)(-1+e^{4z\xi}-(1+\chi_3)(1-2e^{2z\xi}+e^{4z\xi})) \\
& +e^{z+2z\xi}z\xi\big(-(e^{2z}+e^{2z\xi}-e^{6z\xi}-e^{2z+4z\xi}+4(z-1)e^{4z\xi}+4(z+1)e^{2z}e^{2z\xi})m_2(1+\chi_2) \\
& +(1-2e^{2z\xi}+e^{4z\xi})(e^{2z}+e^{2z\xi})(1+\chi_3)\big)+4z^2\xi^2e^{z+4z\xi}(m_2(1+\chi_2)(e^{2z}+e^{2z\xi}) \\
& +(e^{2z\xi}(z-1)+e^{2z}(z+1))(1+\chi_3)))\big]\cos\Big[\frac{z}{h}(t-x)\Big]\,dz,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
k_3^*(x,t) = \int_0^\infty & \frac{4}{\Delta^{**}}\big[-e^{-3z-2z\xi}(-1+e^{2z\xi}+z\xi(1+e^{2z\xi}))((1-z)e^{2z\xi} \\
& -(1+z)e^{2z}+z\xi(e^{2z+2z\xi}))m_2(1+\chi_1)(1+\chi_2)\big]\cos\Big[\frac{z}{h}(t-x)\Big]\,dz,
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}
\Delta^{**} = e^{-4z-4z\xi}\big[& 16e^{2z+4z\xi}z^3\xi^3(-(1+\chi_2)+m_1(1+\chi_1))(1+\chi_3)-(-e^{4z}+e^{4z\xi}-4ze^{2z+2z\xi}) \\
& \times m_1(1+\chi_1)((-1+e^{4z\xi})m_2(1+\chi_2)+(1-2e^{2z\xi}+e^{4z\xi})(1+\chi_3))+(e^{4z}+e^{4z\xi}-2e^{2z+2z\xi}) \\
& \times(1+2z^2))(1+\chi_2)((1-2e^{2z\xi}+e^{4z\xi})m_2(1+\chi_2)+(-1+e^{4z\xi})(1+\chi_3)) \\
& -4z^2\big(e^{2z+4z\xi}(1+\chi_2)((1-2e^{2z\xi}+e^{4z\xi})m_2(1+\chi_2)+(-1+e^{4z\xi}-8ze^{2z\xi})(1+\chi_3)) \\
& -m_1(1+\chi_1)(-4e^{2z+4z\xi}m_2(1+\chi_2)+(e^{6z\xi}-e^{4z+2z\xi}-4ze^{2z+4z\xi})(1+\chi_3))\big) \\
& -4z\xi e^{2z\xi}(m_1(1+\chi_1)\big((-e^{4z}+e^{4z\xi}-e^{2z}-4ze^{2z+2z\xi}+e^{2z+4z\xi})m_2(1+\chi_2) \\
& +(e^{2z}(1-2e^{2z\xi}+e^{4z\xi})(1+\chi_3))-(1+\chi_2)\big(2e^{2z}(1-2e^{2z\xi}+e^{4z\xi})m_2z(1+\chi_2) \\
& +(e^{4z}+e^{4z\xi}-2ze^{2z}+2ze^{2z+4z}-2e^{2z+2z\xi}(1+2z^2\xi^2))(1+\chi_3)\big)\big)\big].
\end{aligned}$$

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