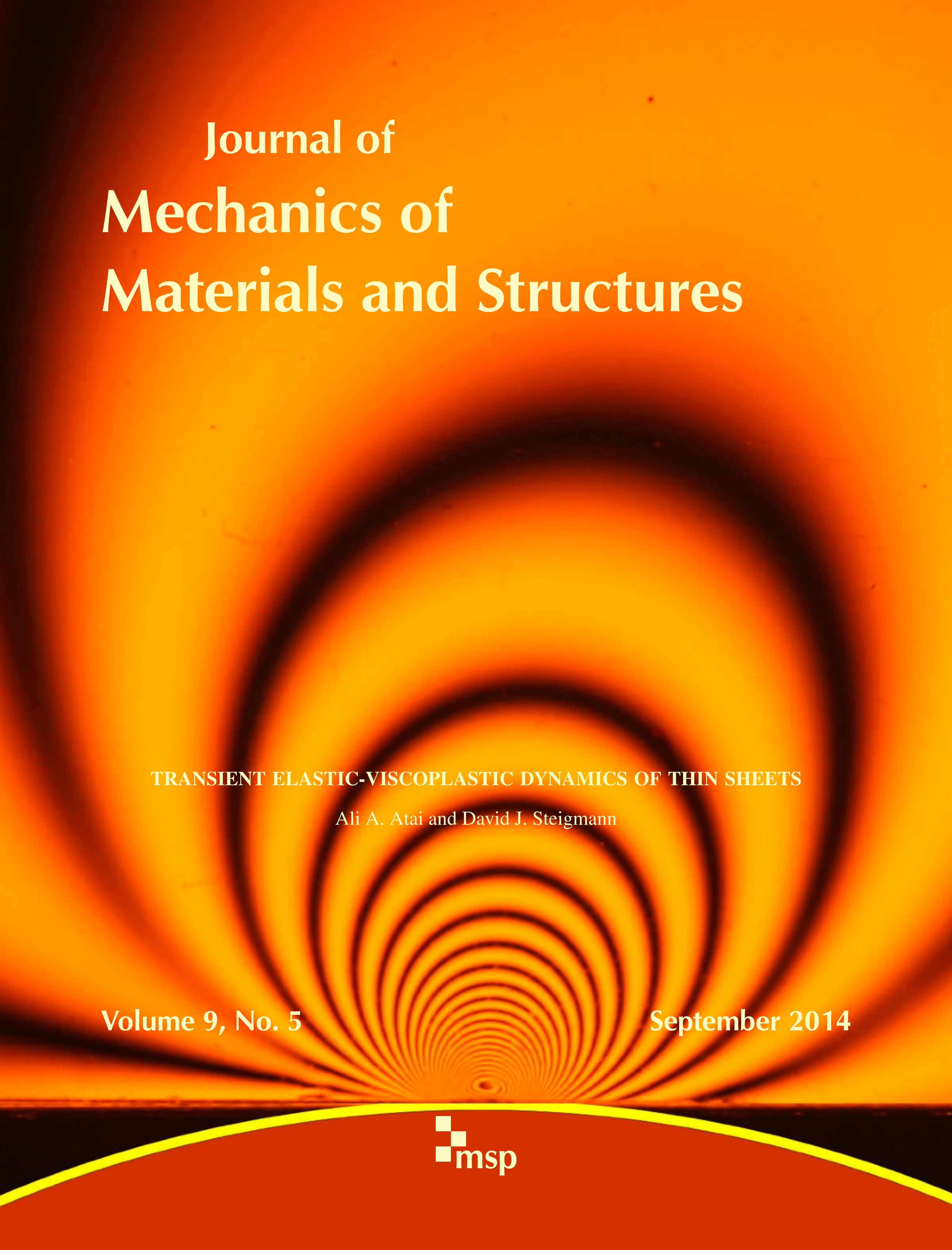

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TRANSIENT ELASTIC-VISCOPLASTIC DYNAMICS OF THIN SHEETS

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Dedicated to José J. A. Rodal

A model for the finite transient viscoplastic response of thin membranes is derived from three-dimensional viscoplasticity theory for isotropic materials. This accommodates large elastic strains and is not limited to conventional Kirchhoff–Love kinematics. In particular, we show that the Kirchhoff–Love hypothesis need not obtain in the presence of plastic deformation. Numerical experiments exhibit large transient elastic strains accompanied by small deviations from Kirchhoff–Love kinematics.

1. Introduction

The problem of the dynamic viscoplastic response of thin metallic sheets to impact and blast loads is thoroughly reviewed in the classic monograph [Cristescu 1967]. In addition to a detailed overview of experimental methods, a description of theoretical and numerical analyses of transient finite axisymmetric motions is given in the setting of rate-independent and rate-dependent response. The present work may be viewed as a further development of this subject, cast in the setting of the modern theory of finite elastic-viscoplastic deformations.

In rate-independent plasticity theory the elastic strain is bounded by the diameter of an *elastic range* associated with a yield criterion. Typically, this implies that elastic strains are small enough to justify the use of classical linear relations between elastic strain and an appropriate measure of stress. In contrast, formulations of viscoplasticity theory to model rate-dependent behavior are characterized by significant excursions of the elastic strain (and associated stress) from the elastic range [Malvern 1951; Prager 1961; Perzyna 1962/1963; 1966]. In this case the elastic strain cannot be regarded as small *a priori* and so there is a need for an extended framework that accommodates finite elastic strain. To this end we adopt the finite-elastic-strain model for isotropic materials proposed in [Krishnan and Steigmann 2014].

The basic framework of finite elastoplasticity theory is recalled in Section 2, both for the sake of completeness and to set the stage for its subsequent application, in Section 3, to the dynamics of thin sheets. We show that the kinematics of the sheet do not conform to classical Kirchhoff–Love kinematics, even in the case of isotropy. In Section 4 we describe numerical experiments conducted using a two-dimensional spatial finite difference scheme based on Green's theorem in conjunction with explicit timewise integration of the equations of motion. This is applied to the simulation of the response of a plane sheet to transverse blast pressure. The predictions exhibit small deviations from Kirchhoff–Love kinematics and substantial transient elastic strains.

From the theoretical point of view it is advantageous to base the theory on the elastic stretch tensor rather than conventional measures of elastic strain; this is explained in Section 2. Typically the use of

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stretch tensors is avoided in numerical work as they are not rational functions of the deformation gradient. However, as shown in Section 4, this issue is easily addressed by using an appropriate version of the Cayley–Hamilton formula.

We use standard notation such as $\boldsymbol{A}^t$, $\boldsymbol{A}^{-1}$, $\boldsymbol{A}^*$, Dev $\boldsymbol{A}$, tr $\boldsymbol{A}$ and J_A. These are, respectively, the transpose, the inverse, the cofactor, the deviatoric part, the trace and the determinant of a tensor $\boldsymbol{A}$, regarded as a linear transformation from a three-dimensional vector space to itself, the latter being identified with the translation space of the usual three-dimensional Euclidean point space. We also use Sym to denote the linear space of symmetric tensors, Sym^+ the positive-definite symmetric tensors and Orth^+ the group of rotation tensors. If $J_A > 0$ then we have the unique polar decompositions $\boldsymbol{A} = \boldsymbol{R}_A \boldsymbol{U}_A = \boldsymbol{V}_A \boldsymbol{R}_A$, with $\boldsymbol{R}_A \in \text{Orth}^+$ and $\boldsymbol{U}_A, \boldsymbol{V}_A \in \text{Sym}^+$. The tensor product of 3-vectors is indicated by interposing the symbol $\otimes$, and the Euclidean inner product of tensors $\boldsymbol{A}, \boldsymbol{B}$ is denoted and defined by $\boldsymbol{A} \cdot \boldsymbol{B} = \text{tr}(\boldsymbol{A}\boldsymbol{B}^t)$; the associated norm is $|\boldsymbol{A}| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$. Finally, the notation $F_{\boldsymbol{A}}$, with a bold subscript, stands for the tensor-valued derivative of a scalar-valued function $F(\boldsymbol{A})$.

2. Elastoviscoplasticity theory for isotropic materials

In this section we recount the basic elements of the three-dimensional theory underpinning this work. Further background may be found in [Cleja-Ţigoiu and Soós 1990; Epstein and Elżanowski 2007; Rajagopal and Srinivasa 1998; Gurtin et al. 2010; Gupta et al. 2007; Bigoni 2012].

2.1. ***Basic theory.*** The fields to be determined are the motion $\boldsymbol{\chi}(\boldsymbol{x}, t)$ and the *plastic* deformation tensor $\boldsymbol{K}(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ is the position of a material point in a fixed reference placement κ_r of the body. Commonly, the plastic deformation is identified with $\boldsymbol{G} = \boldsymbol{K}^{-1}$. The values $\boldsymbol{y} = \boldsymbol{\chi}(\boldsymbol{x}, t)$ are the positions of material points at time t and generate the current placement κ_t of the body as $\boldsymbol{x}$ ranges over κ_r. The deformation gradient, denoted by $\boldsymbol{F}$, is the gradient of $\boldsymbol{\chi}$ with respect to $\boldsymbol{x}$; this is assumed to be invertible with $J_F > 0$. These variables combine to yield the *elastic* deformation

$$\boldsymbol{H} = \boldsymbol{F}\boldsymbol{K}. \tag{1}$$

Here we impose $J_H > 0$ and thus require that $J_K > 0$.

The basic elements of this framework were laid down by Bilby [1960], Kröner [1960], Noll [1967/1968] and Lee [1969]. Reference may be made to [Gurtin et al. 2010] for a thorough and modern account of the subject.

The strain energy of the body is

$$E = \int_{\kappa_t} \psi(\boldsymbol{H})\, dv, \tag{2}$$

where ψ is the spatial strain-energy density. We consider materials that are uniform in the sense that this function, and further constitutive functions to be discussed, do not involve $\boldsymbol{x}$ explicitly.

The local equations of motion, in the absence of body forces, are

$$\text{Div}\, \boldsymbol{P} = \rho_r \ddot{\boldsymbol{y}}, \quad \boldsymbol{P}\boldsymbol{F}^t \in \text{Sym} \quad \text{in } \kappa_r, \tag{3}$$

where $\boldsymbol{P}$ is the Piola stress, ρ_r is the fixed referential mass density, Div is the referential divergence (i.e., the divergence with respect to $\boldsymbol{x}$), and superposed dots are used to denote material derivatives ($\partial/\partial t$ at fixed $\boldsymbol{x}$). The Piola stress is defined in terms of the Cauchy stress $\boldsymbol{T}$ by

$$\boldsymbol{P} = \boldsymbol{T}\boldsymbol{F}^*. \tag{4}$$

The strain-energy function referred to the local *intermediate* configuration κ_i is

$$W(\boldsymbol{H}) = J_H \psi(\boldsymbol{H}), \tag{5}$$

and it generates the Cauchy stress via the formula [Gupta et al. 2007]

$$\boldsymbol{T}\boldsymbol{H}^* = W_{\boldsymbol{H}}. \tag{6}$$

Accordingly,

$$W_{\boldsymbol{H}} = \boldsymbol{P}\boldsymbol{K}^* \tag{7}$$

Necessary and sufficient for the symmetry condition $(3)_2$ is that W depend on $\boldsymbol{H}$ through the *elastic* Cauchy–Green deformation tensor

$$\boldsymbol{C}_H = \boldsymbol{H}^t\boldsymbol{H}. \tag{8}$$

Thus,

$$W(\boldsymbol{H}) = \widehat{W}(\boldsymbol{C}_H). \tag{9}$$

The associated second Piola–Kirchhoff stress, referred to κ_i, is $\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{C}_H)$, where

$$\hat{\boldsymbol{S}}(\boldsymbol{C}_H) = 2\widehat{W}_{\boldsymbol{C}_H}. \tag{10}$$

We assume the local configuration κ_i to be *natural* in the sense that $\hat{\boldsymbol{S}}(\boldsymbol{I}) = \boldsymbol{0}$; realistic constitutive hypotheses [Gupta et al. 2007] then associate it with an *undistorted* state, in the sense of an undistorted crystal lattice. The relevant symmetry group is thus a subgroup of the proper orthogonal group; i.e.,

$$W(\boldsymbol{H}) = W(\boldsymbol{H}\boldsymbol{R}), \tag{11}$$

where $\boldsymbol{R} \in \mathrm{Orth}^+$. For isotropic materials the symmetry group coincides with the proper orthogonal group, implying that this is satisfied for *all* $\boldsymbol{R} \in \mathrm{Orth}^+$.

The sum of the kinetic and strain energies of an arbitrary part $p \subset \kappa_t$ of the body is

$$\int_\pi \Phi\, dv; \qquad \Phi = \Psi + \tfrac{1}{2}\rho_r |\dot{\boldsymbol{y}}|^2, \tag{12}$$

where π is the region occupied by p in κ_r, and

$$\Psi(\boldsymbol{F}, \boldsymbol{K}) = J_K^{-1} W(\boldsymbol{F}\boldsymbol{K}) \tag{13}$$

is the referential strain-energy density. In terms of this the Piola stress is given simply by [Epstein and Elżanowski 2007; Gupta et al. 2007]

$$\boldsymbol{P} = \Psi_{\boldsymbol{F}}. \tag{14}$$

The dissipation $\mathfrak{D}$ is the difference between the mechanical power P supplied to p and the rate of change of the total energy in p. Thus,

$$\mathfrak{D} = P - \frac{d}{dt}\int_\pi \Phi\, dv, \tag{15}$$

which may be reduced to [Gupta et al. 2011]

$$\mathcal{D} = \int_\pi D\,dv, \tag{16}$$

where

$$D = \mathscr{E} \cdot \dot{\boldsymbol{K}}\boldsymbol{K}^{-1}, \tag{17}$$

in which

$$\mathscr{E} = \Psi\boldsymbol{I} - \boldsymbol{F}^t\boldsymbol{P} \tag{18}$$

is Eshelby's energy-momentum tensor.

The dissipation is thus nonnegative for every sub-body if and only if $D \geq 0$. It is convenient to use (17) in the form

$$J_K D = \mathscr{E}' \cdot \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}, \tag{19}$$

where

$$\mathscr{E}' = J_K \boldsymbol{K}^t \mathscr{E} \boldsymbol{K}^{-t} \tag{20}$$

is the pushforward of the Eshelby tensor to κ_i. This satisfies [Gupta et al. 2007]

$$\mathscr{E}' = \widehat{W}(\boldsymbol{C}_H)\boldsymbol{I} - \boldsymbol{C}_H\hat{\boldsymbol{S}}(\boldsymbol{C}_H), \tag{21}$$

and is therefore purely elastic in origin.

In [Gupta et al. 2007] it is further demonstrated that if the strain energy and the dissipation are invariant under superposed rigid-body motions with uniform rotation $\boldsymbol{Q}(t)$, and if plastic evolution is *strictly dissipative* in the sense that $D > 0$ if and only if $\dot{\boldsymbol{K}} \neq \boldsymbol{0}$, then the transformation rules for the elastic and plastic deformations under superposed rigid-body motions are

$$\boldsymbol{H} \to \boldsymbol{Q}(t)\boldsymbol{H} \quad \text{and} \quad \boldsymbol{K} \to \boldsymbol{K}. \tag{22}$$

These imply that $\boldsymbol{C}_H$, $\boldsymbol{S}$ and $\mathscr{E}'$ are invariant.

In this work we consider strain-energy functions Ψ that satisfy the strong ellipticity condition

$$\boldsymbol{a}\otimes\boldsymbol{b} \cdot \Psi_{\boldsymbol{FF}}(\boldsymbol{F}, \boldsymbol{K})[\boldsymbol{a}\otimes\boldsymbol{b}] > 0 \quad \text{for all} \quad \boldsymbol{a}\otimes\boldsymbol{b} \neq 0, \tag{23}$$

and for all deformations. This is equivalent to [Sfyris 2011; Steigmann 2014]

$$\boldsymbol{a}\otimes\boldsymbol{m} \cdot W_{\boldsymbol{HH}}(\boldsymbol{H})[\boldsymbol{a}\otimes\boldsymbol{m}] > 0 \quad \text{for all} \quad \boldsymbol{a}\otimes\boldsymbol{m} \neq 0, \quad \text{with } \boldsymbol{m} = \boldsymbol{K}^t\boldsymbol{b}. \tag{24}$$

It follows that Ψ is strongly elliptic at $\boldsymbol{F}$ if and only if W is strongly elliptic at $\boldsymbol{H} = \boldsymbol{F}\boldsymbol{K}$.

2.2. ***Isotropy.*** Isotropy of the constitutive response implies that the strain-energy function satisfies (11) for all rotations $\boldsymbol{R}$. The choice $\boldsymbol{R} = \boldsymbol{R}_H^t$ yields the necessary condition $W(\boldsymbol{H}) = W(\boldsymbol{V}_H)$. Invariance of W under superposed rigid-body motions imposes the further restriction (see $(22)_1$) $W(\boldsymbol{V}_H) = W(\boldsymbol{Q}\boldsymbol{V}_H\boldsymbol{Q}^t)$ for all rotations $\boldsymbol{Q}$, which is satisfied if and only if W is expressible in the form

$$W(\boldsymbol{H}) = w(h_1, h_2, h_3), \tag{25}$$

where h_i are the principal invariants of the elastic left-stretch $\boldsymbol{V}_H$. Because these are isotropic functions, they are also the invariants of the elastic right-stretch $\boldsymbol{U}_H$; i.e.,

$$h_1 = \operatorname{tr} \boldsymbol{U}_H, \quad h_2 = \operatorname{tr} \boldsymbol{U}_H^*, \quad h_3 = \det \boldsymbol{U}_H = J_H, \tag{26}$$

and this of course is a special case of (9) because $\boldsymbol{U}_H$ is determined by $\boldsymbol{C}_H$. Equation (25) thus yields the canonical representation of an isotropic, frame-invariant strain-energy function. A representation formula derived in [Steigmann 2002] may then be used to arrive at

$$W_{\boldsymbol{H}} = \boldsymbol{R}_H \boldsymbol{\sigma}, \tag{27}$$

where

$$\boldsymbol{\sigma} = (w_1 + h_1 w_2)\boldsymbol{I} - w_2 \boldsymbol{U}_H + w_3 \boldsymbol{U}_H^*, \tag{28}$$

in which

$$w_i = \frac{\partial w}{\partial h_i} \tag{29}$$

is the Biot stress based on the use of κ_i as reference [Ogden 1997]. Using (7), (10) and (27) we have $\boldsymbol{C}_H \boldsymbol{S} = \boldsymbol{U}_H \boldsymbol{\sigma}$, yielding (21) in the form

$$\mathscr{E}' = W\boldsymbol{I} - \boldsymbol{U}_H \boldsymbol{\sigma} = (w - h_3 w_3)\boldsymbol{I} - (w_1 + h_1 w_2)\boldsymbol{U}_H + w_2 \boldsymbol{U}_H^2. \tag{30}$$

Our preference for a framework based on the stretch tensor and associated invariants is due to the availability of simple sufficient conditions, expressed in terms of these variables, for the condition of polyconvexity. This in turn guarantees strong ellipticity, which plays an important role in the considerations of Section 3.

To elaborate, we write the strain energy of the body in the form

$$E = \int_{\kappa_r} \Psi(\nabla \boldsymbol{\chi}, \boldsymbol{K})\, dv, \tag{31}$$

where (cf. (13))

$$\Psi(\nabla \boldsymbol{\chi}, \boldsymbol{K}) = J_K^{-1} W((\nabla \boldsymbol{\chi})\boldsymbol{K}). \tag{32}$$

The function $\Psi(\,\cdot\,, \boldsymbol{K})$, with $\boldsymbol{K}(\boldsymbol{x})$ fixed, delivers the strain energy of an inhomogeneous elastic body.

Polyconvexity is the condition that there exists a (possibly nonunique) function $\Phi_{(\boldsymbol{x})}(\boldsymbol{F}, \boldsymbol{F}^*, J_F)$, jointly convex in each argument, such that $\Phi_{(\boldsymbol{x})}(\boldsymbol{F}, \boldsymbol{F}^*, J_F) = \Psi(\boldsymbol{F}, \boldsymbol{K}(\boldsymbol{x}))$. This in turn is equivalent to the polyconvexity of the function $\Phi'_{(\boldsymbol{x})} = W$, where $\Phi'_{(\boldsymbol{x})}(\boldsymbol{H}, \boldsymbol{H}^*, J_H) = J_K \Phi_{(\boldsymbol{x})}(\boldsymbol{H}\boldsymbol{K}^{-1}, (\boldsymbol{H}\boldsymbol{K}^{-1})^*, J_H J_K^{-1})$ with $\boldsymbol{K}$ fixed [Neff 2003]. In the case of isotropy, sufficient conditions are [Steigmann 2003]

(i) $w(h_1, h_2, h_3)$ is a convex function of all three arguments jointly,

(ii) w is an increasing function of h_1 and h_2; (33)

that is,

$$w(\bar{h}_1, \bar{h}_2, \bar{h}_3) - w(h_1, h_2, h_3) > (\bar{h}_1 - h_1)\frac{\partial w}{\partial h_1} + (\bar{h}_2 - h_2)\frac{\partial w}{\partial h_2} + (\bar{h}_3 - h_3)\frac{\partial w}{\partial h_3}, \tag{34}$$

together with

$$\frac{\partial w}{\partial h_1} > 0 \quad \text{and} \quad \frac{\partial w}{\partial h_2} > 0, \tag{35}$$

in which h_k and $\bar{h}_k$, respectively, are the invariants associated with $\boldsymbol{H}$ and the comparison deformation $\bar{\boldsymbol{H}}$, and the derivatives are evaluated at h_k.

For example, strain energies of the form

$$w = c_1(h_1 - 3) + c_2(h_2 - 3) + g(h_3), \tag{36}$$

with c_1 and c_2 constant, have yielded explicit solutions to some boundary-value problems [Carroll 1988] in the setting of finite elasticity theory. These satisfy (33) — and thus yield polyconvexity — if and only if $c_1 > 0$, $c_2 > 0$ and $g''(h_3) > 0$. An example of such a function $g(\,\cdot\,)$ is

$$g(h_3) = -c_3 \log h_3, \tag{37}$$

with c_3 a positive constant, and the associated Biot stress is (cf. (28))

$$\boldsymbol{\sigma} = (c_1 + h_1 c_2)\boldsymbol{I} - c_2 \boldsymbol{U}_H - c_3 \boldsymbol{U}_H^{-1}. \tag{38}$$

To ensure that the intermediate configuration κ_i is stress-free, as required, and that the asymptotic formula

$$\boldsymbol{\sigma} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon} + o(|\boldsymbol{\epsilon}|) \tag{39}$$

is satisfied for small $|\boldsymbol{\epsilon}|$, where

$$\boldsymbol{\epsilon} = \boldsymbol{U}_H - \boldsymbol{I} \tag{40}$$

is the elastic Biot strain and where λ and μ are the usual Lamé moduli, we impose

$$c_3 = c_1 + 2c_2, \quad c_2 = \lambda \quad \text{and} \quad \tfrac{1}{2}(c_1 + c_2) = \mu. \tag{41}$$

The polyconvexity criteria (33) then imply the classical inequalities $\mu > 0$ and $\kappa > 0$, where μ and $\kappa\big(= \lambda + \frac{2}{3}\mu\big)$, respectively, are the conventional shear and bulk moduli. Accordingly, for materials possessing a positive modulus λ, equations (36), (37) and (41) provide a simple nonlinear, polyconvex extension of the strain-energy function of linear elasticity theory to finite elastic strains.

Finally, following [Krishnan and Steigmann 2014], we adopt the isotropic viscoplastic flow rule

$$\nu\dot{\boldsymbol{G}}\boldsymbol{G}^{-1} = F \operatorname{Dev}(\boldsymbol{\sigma}\boldsymbol{U}_H); \quad F > 0, \quad \text{where } F = 1 - \frac{\sqrt{2}K}{|\operatorname{Dev}(\boldsymbol{\sigma}\boldsymbol{U}_H)|}, \tag{42}$$

$\nu(> 0)$ is a material viscosity and K is the static yield stress in shear. This is an extension of the classical over-stress models pioneered by Malvern [1951], Prager [1961] and Perzyna [1962/1963; 1966]. We impose $\dot{\boldsymbol{G}} = 0$ if $\boldsymbol{U}_H$ belongs to the elastic range, defined by $F \le 0$. In the case of nontrivial plastic evolution the elastic stretch is not bounded by the elastic range, in contrast to the situation in rate-independent plasticity. For this reason it is generally necessary to account for finite elastic strain in the presence of viscoplasticity.

We note that the skew part of $\dot{\boldsymbol{G}}\boldsymbol{G}^{-1}$, the so-called *plastic spin*, is suppressed in (42). That this entails no loss of generality in the case of isotropic response has been established in [Epstein and Elżanowski 2007] and [Gurtin et al. 2010]. A general framework for addressing issues of this kind is discussed in [Steigmann and Gupta 2011].

Using $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = -\dot{\boldsymbol{G}}\boldsymbol{G}^{-1}$, the dissipation D, defined by (19), is easily shown to vanish if $F \leq 0$, and to satisfy

$$D > 2FK^2/\nu \quad \text{if } F > 0, \tag{43}$$

ensuring that plastic evolution is indeed strictly dissipative.

3. Membrane dynamics

We identify the reference configuration κ_r with the prismatic region $\Omega \times \left[-\frac{1}{2}h, \frac{1}{2}h\right]$, where Ω is the midplane of the sheet and h is its thickness. To obtain the relevant model we restrict the exact equations of motion (3) to Ω and invoke the exact traction conditions at the lateral surfaces $\varsigma = \pm\frac{1}{2}h$. The resulting system delivers equations of motion for the sheet valid to leading order in h [Steigmann 2009].

3.1. ***Equations of motion.*** Let $\boldsymbol{k}$ be a unit vector that orients the plane Ω. The associated projection operator is

$$\mathbf{1} = \boldsymbol{I} - \boldsymbol{k} \otimes \boldsymbol{k}, \tag{44}$$

where $\boldsymbol{I}$ is the identity for 3-space. This generates the orthogonal decomposition

$$\boldsymbol{P} = \boldsymbol{P}\mathbf{1} + \boldsymbol{P}\boldsymbol{k} \otimes \boldsymbol{k} \tag{45}$$

of the Piola stress, which may be used to cast the equation of motion (3) in the form

$$\operatorname{div}(\boldsymbol{P}\mathbf{1}) + \boldsymbol{P}'\boldsymbol{k} = \rho_r \ddot{\boldsymbol{y}}, \tag{46}$$

where $\operatorname{div}(\,\cdot\,)$ is the two-dimensional divergence with respect to position $\boldsymbol{u} \in \Omega$, and $(\,\cdot\,)' = \partial(\,\cdot\,)/\partial\varsigma$ is the derivative with respect to the linear coordinate ς orthogonal to Ω in the representation $\boldsymbol{x} = \boldsymbol{u} + \varsigma\boldsymbol{k}$.

This holds at all points in the interior of the body and hence at the midplane Ω defined by $\varsigma = 0$; thus,

$$\operatorname{div}(\boldsymbol{P}_0\mathbf{1}) + \boldsymbol{P}_0'\boldsymbol{k} = \rho_r \ddot{\boldsymbol{r}}, \tag{47}$$

where $\boldsymbol{r} = \boldsymbol{\chi}_0$ and $(\,\cdot\,)_0$ is used to denote the restriction of the enclosed variable to Ω. Here,

$$\boldsymbol{P}_0 = \Psi_{\boldsymbol{F}}(\boldsymbol{F}_0, \boldsymbol{K}_0), \tag{48}$$

where

$$\boldsymbol{F}_0 = \nabla\boldsymbol{r} + \boldsymbol{d} \otimes \boldsymbol{k}, \tag{49}$$

in which $\boldsymbol{d} = \boldsymbol{\chi}_0'$ is the membrane director field and $\nabla(\,\cdot\,)$ is the two-dimensional gradient with respect to $\boldsymbol{u}$. This follows easily from the decomposition $\boldsymbol{F} = \boldsymbol{F}\mathbf{1} + \boldsymbol{F}\boldsymbol{k} \otimes \boldsymbol{k}$.

Equation (47) is the exact equation of motion for the midplane Ω. Approximations arise when using it to represent material response in $\Omega \times \left[-\frac{1}{2}h, \frac{1}{2}h\right]$. Here we seek the leading-order model for small thickness h, which is assumed to be much smaller than any other length scale in the considered problem. The smallest of these is used as the unit of length, so that $h \ll 1$ when nondimensionalized. To this end we impose lateral traction data at $\varsigma = \pm\frac{1}{2}h$, obtaining [Steigmann 2009]

$$\boldsymbol{p}^+ + \boldsymbol{p}^- = h\boldsymbol{P}_0'\boldsymbol{k} + o(h) \quad \text{and} \quad \boldsymbol{p}^+ - \boldsymbol{p}^- = 2\boldsymbol{P}_0\boldsymbol{k} + o(h), \tag{50}$$

where $\boldsymbol{p}^\pm = \pm\boldsymbol{P}^\pm\boldsymbol{k}$ are the tractions at the major surfaces.

In this work we prescribe

$$\boldsymbol{p}^{+}=\mathbf{0} \quad \text{and} \quad \boldsymbol{p}^{-}=p(\boldsymbol{F}^{*})^{-}\boldsymbol{k}, \quad \text{with } p=hP+o(h) \quad \text{and} \quad P=O(1), \tag{51}$$

corresponding to a pressure p of order $O(h)$ acting at $\varsigma=-\frac{1}{2}h$. This is reconciled with (50) if and only if

$$\boldsymbol{P}_0\boldsymbol{k}=\mathbf{0} \quad \text{and} \quad \boldsymbol{P}_0'\boldsymbol{k}=P\alpha\boldsymbol{n}+o(h)/h, \tag{52}$$

where $\boldsymbol{n}$ is the unit normal to the image $\omega=\boldsymbol{\chi}(\Omega,t)$ of the midplane after deformation and α is the areal stretch, defined by Nanson's formula

$$\alpha\boldsymbol{n}=\boldsymbol{F}_0^{*}\boldsymbol{k}, \quad \text{with} \quad \alpha=|\boldsymbol{F}_0^{*}\boldsymbol{k}|. \tag{53}$$

Here we have used the fact — valid for smooth deformations — that $(\boldsymbol{F}^{*})^{-}$ is approximated by $\boldsymbol{F}_0^{*}$ with an error of order $O(h)$.

Substitution of (52) into (47) delivers

$$\operatorname{div}(\boldsymbol{P}_0\mathbf{1})+P\alpha\boldsymbol{n}+o(h)/h=\rho_r\ddot{\boldsymbol{r}}, \tag{54}$$

and passage to the limit yields the leading-order differential-algebraic system

$$\operatorname{div}(\boldsymbol{P}_0\mathbf{1})+P\alpha\boldsymbol{n}=\rho_r\ddot{\boldsymbol{r}} \quad \text{and} \quad \boldsymbol{P}_0\boldsymbol{k}=\mathbf{0}. \tag{55}$$

Combining this with (14) and the restriction to Ω of the flow rule (42), we arrive at a system for the determination of the plastic deformation $\boldsymbol{K}_0$, the midplane motion $\boldsymbol{r}$ and the director field $\boldsymbol{d}$.

To address the requirement $J_{F_0}>0$, we observe that

$$J_{F_0}=\boldsymbol{F}_0^{*}\boldsymbol{k}\cdot\boldsymbol{F}_0\boldsymbol{k}. \tag{56}$$

Accordingly, $J_{F_0}>0$ if and only if

$$\boldsymbol{d}\cdot\boldsymbol{n}>0. \tag{57}$$

Henceforth we work exclusively with functions defined on Ω and thus drop the subscript $(\cdot)_0$ for the sake of convenience.

3.2. ***Elimination of the director field.*** The second equation in (55) requires that Ω be in a state of plane stress. Using (7), this is seen to be equivalent to

$$\{W_{\boldsymbol{H}}(\boldsymbol{H})\}\boldsymbol{l}=0, \quad \text{where } \boldsymbol{l}=\boldsymbol{K}^{t}\boldsymbol{k}. \tag{58}$$

To prove that this system yields a unique $\boldsymbol{d}$, we first show that any solution, $\bar{\boldsymbol{d}}$ say, minimizes W. To this end we fix $\nabla\boldsymbol{r}$ and $\boldsymbol{K}$ and define $R(\boldsymbol{d})=W(\boldsymbol{H})$ with $\boldsymbol{H}=(\nabla\boldsymbol{r}+\boldsymbol{d}\otimes\boldsymbol{k})\boldsymbol{K}$. Let $\boldsymbol{d}(t)$ be a one-parameter family belonging to the half-space $S_{+}(\boldsymbol{d})$ defined by $\boldsymbol{d}\cdot\boldsymbol{n}>0$. This is the admissible set associated with the restriction $J_F>0$. The derivatives of $\sigma(t)=R(\boldsymbol{d}(t))$ are

$$\dot{\sigma}=\dot{\boldsymbol{d}}\cdot\{W_{\boldsymbol{H}}(\boldsymbol{H})\}\boldsymbol{l}=\dot{\boldsymbol{d}}\cdot R_{\boldsymbol{d}} \tag{59}$$

and

$$\ddot{\sigma}=\ddot{\boldsymbol{d}}\cdot\{W_{\boldsymbol{H}}(\boldsymbol{H})\}\boldsymbol{l}+\dot{\boldsymbol{d}}\otimes\boldsymbol{l}\cdot W_{\boldsymbol{H}\boldsymbol{H}}(\boldsymbol{H})[\dot{\boldsymbol{d}}\otimes\boldsymbol{l}]=\ddot{\boldsymbol{d}}\cdot R_{\boldsymbol{d}}+\dot{\boldsymbol{d}}\cdot(R_{\boldsymbol{dd}})\dot{\boldsymbol{d}}. \tag{60}$$

It follows that

$$R_{\boldsymbol{d}}(\bar{\boldsymbol{d}}) = \{W_{\boldsymbol{H}}(\bar{\boldsymbol{H}})\}\boldsymbol{l}, \tag{61}$$

with $\bar{\boldsymbol{H}} = (\nabla \boldsymbol{r} + \bar{\boldsymbol{d}} \otimes \boldsymbol{k})\boldsymbol{K}$, vanishes, and that $R_{\boldsymbol{dd}}(\boldsymbol{d})$ satisfies

$$\{R_{\boldsymbol{dd}}(\boldsymbol{d})\}\boldsymbol{v} = \{W_{\boldsymbol{HH}}(\boldsymbol{H})[\boldsymbol{v} \otimes \boldsymbol{l}]\}\boldsymbol{l}, \tag{62}$$

implying, by the strong ellipticity (cf. (24)) of the polyconvex energy (36), that it is positive definite.

Because $S_+(\boldsymbol{d})$ is a convex set, it contains the straight line $\boldsymbol{d}(t) = t\boldsymbol{d}_2 + (1-t)\boldsymbol{d}_1$ with $\boldsymbol{d}_{1,2} \in S_+(\boldsymbol{d})$ and $t \in [0, 1]$. We have $\ddot{\sigma} > 0$ on this line and hence $\dot{\sigma}(t) > \dot{\sigma}(0)$ for $t \in (0, 1]$. Then $\sigma(1) - \sigma(0) > \dot{\sigma}(0)$, implying that the function $R(\boldsymbol{d})$ is convex on S_+; that is,

$$R(\boldsymbol{d}_2) - R(\boldsymbol{d}_1) > R_{\boldsymbol{d}}(\boldsymbol{d}_1) \cdot (\boldsymbol{d}_2 - \boldsymbol{d}_1). \tag{63}$$

Because such functions have unique stationary points, it follows that there exists a unique solution $\boldsymbol{d} = \bar{\boldsymbol{d}}(\nabla \boldsymbol{r}, \boldsymbol{K})$ to (58), corresponding to the global minimizer of $R(\boldsymbol{d})$.

Plastic deformation generally prevents the solution from conforming to classical Kirchhoff–Love kinematics with $\boldsymbol{d} = \mu \boldsymbol{n}$, where μ is the thickness distension. This stands in contrast to the ubiquitous imposition of the constraint of Kirchhoff–Love kinematics throughout the literature on theories for the plastic deformation of membranes and shells derived from three-dimensional considerations [Cristescu 1967; Lubliner 2008].

To prove the claim we invoke (58) in the form (cf. (27))

$$\boldsymbol{\sigma} \boldsymbol{l} = \boldsymbol{0}, \tag{64}$$

where $\boldsymbol{\sigma}$ is the Biot stress. Thus, $\boldsymbol{l}$ is an eigenvector of $\boldsymbol{\sigma}$ with vanishing eigenvalue. Henceforth we normalize $\boldsymbol{l}$ to be a unit vector without loss of generality. It follows from (38) that $\boldsymbol{\sigma}$ and $\boldsymbol{U}_H$ are coaxial in the case of isotropy, and hence that $\boldsymbol{l}$ is also an eigenvector of $\boldsymbol{U}_H$. The standard representation $\boldsymbol{H} = \sum \lambda_i \boldsymbol{m}_i \otimes \boldsymbol{l}_i$, where $\lambda_i (> 0)$ are the eigenvalues of $\boldsymbol{U}_H$, $\boldsymbol{l}_i$ are the associated (orthonormal) eigenvectors and $\boldsymbol{m}_i = \boldsymbol{R}_H \boldsymbol{l}_i$, follows from the polar decomposition theorem and yields $\boldsymbol{H}^* = \sum \lambda_i^* \boldsymbol{m}_i \otimes \boldsymbol{l}_i$, where $\lambda_i^* = \lambda_i / J_H$.

Because $\boldsymbol{l} \in \{\boldsymbol{l}_i\}$ we have $\boldsymbol{H}^* \boldsymbol{l} = \lambda^* \boldsymbol{m}$, with $\lambda^* \in \{\lambda_i^*\}$ and $\boldsymbol{m} \in \{\boldsymbol{m}_i\}$. Using $\boldsymbol{H}^* = \boldsymbol{F}^* \boldsymbol{K}^*$ we obtain $\lambda^* \boldsymbol{m} = (J_K / |\boldsymbol{K}^t \boldsymbol{k}|) \boldsymbol{F}^* \boldsymbol{k}$, which furnishes $\lambda^* = \alpha J_K / |\boldsymbol{K}^t \boldsymbol{k}|$ and $\boldsymbol{m} = \boldsymbol{n}$. Accordingly, it follows from $\lambda \boldsymbol{m} = \boldsymbol{H} \boldsymbol{l}$ and $\boldsymbol{d} = \boldsymbol{F} \boldsymbol{k}$ (cf. (49)) that

$$(\lambda / |\boldsymbol{K}^t \boldsymbol{k}|) \boldsymbol{n} = \boldsymbol{F} \boldsymbol{K} \boldsymbol{K}^t \boldsymbol{F}^{-1} \boldsymbol{d}, \tag{65}$$

implying that $\boldsymbol{d}$ is not generally aligned with $\boldsymbol{n}$. Exceptionally, such alignment occurs — and the director then conforms to Kirchhoff–Love kinematics — if $\boldsymbol{K}$ is a rotation composed with a dilation, including the case of pure elasticity; i.e., $\boldsymbol{K} = \boldsymbol{I}$.

Equation (65) does not account fully for the restrictions embodied in (64). In Section 4, (64) is solved directly for $\boldsymbol{d}$ by using an iterative method.

The present formulation does not yield energetically optimal solutions in the specialization to equilibrium problems. This is due to the potential of the constitutive relations to supply a compressive state of (plane) stress, in violation of a necessary condition for the existence of an energy-minimizing membrane deformation [Pipkin 1986]. In such circumstances the model may be replaced by its quasiconvexification

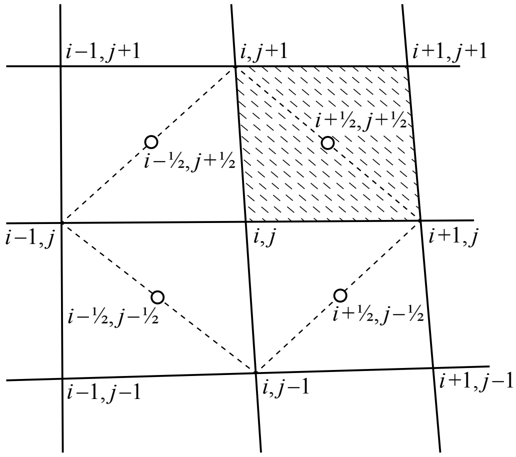

Figure 1. A unit cell of the finite-difference mesh.

[Dacorogna 1989], yielding states that automatically satisfy relevant necessary conditions for energy minimizers. This is the conceptual basis of *tension-field theory* [Steigmann 1990], a model that accommodates finely wrinkled equilibrium states in the context of membrane theory. The same model follows from the method of gamma convergence [Le Dret and Raoult 1996] in the zero-thickness limit. We forego such modifications here, however, as we are concerned exclusively with dynamical states, which of course are not energy minimizers. In particular, the method of gamma convergence is not applicable in this setting. A similar point of view was adopted in [Yokota et al. 2001] in connection with a model for the dynamics of nonlinearly elastic strings. There, dynamics characterized by transient compressive states of stress were obtained via a direct numerical simulation, despite the intermittently nonhyperbolic character of the equations. The present work proceeds in the same spirit.

4. Numerical experiments

4.1. ***Finite differences.*** We discretize $(55)_1$ using a finite-difference scheme derived from Green's theorem. Applications to plane-strain problems in nonlinear elasticity theory are described in [Silling 1988]. The method was first applied by Wilkins [1963] to the simulation of plane plastic flow. Its application to static problems in membrane theory is developed in [Haseganu and Steigmann 1994; Atai and Steigmann 1998]. Here, we present a brief outline of the method and its adaptation to the description of three-dimensional viscoplastic membrane deformations.

The reference plane Ω is covered by a grid consisting of cells of the kind depicted in Figure 1. Nodes are labeled using integer superscripts (i, j). Thus, $u_\alpha^{i,j}$ are the referential Cartesian coordinates of node (i, j), where $u_\alpha = \boldsymbol{u} \cdot \boldsymbol{i}_\alpha$; $\alpha = 1, 2$, and $\{\boldsymbol{i}_\alpha\}$ is an orthonormal basis in Ω. The four regions adjoining a node and its nearest neighbors are called *zones*. Zone-centered points, identified by open circles in the figure, are labeled using half-integer superscripts.

Green's theorem may be stated in the form

$$\int_D \sigma_{\alpha,\alpha}\, da = e_{\alpha\beta} \int_{\partial D} \sigma_\alpha\, du_\beta, \tag{66}$$

where $\sigma_\alpha(u_1, u_2)$ is a smooth two-dimensional vector field, $e_{\alpha\beta}$ is the two-dimensional permutation symbol ($e_{12} = +1$, etc.), D is an arbitrary simply connected subregion of Ω and commas followed by subscripts are partial derivatives with respect to the coordinates. To approximate the divergence $\sigma_{\alpha,\alpha}$ at node (i, j) we identify D with the quadrilateral bounded by the dashed contour of Figure 1. The left-hand side of (66) is estimated as the nodal value of the integrand multiplied by the area of D; the right-hand side as the zone-centered values of the integrand on each of the four edges of ∂D multiplied by the appropriate length. Thus [Silling 1988],

$$2A^{i,j}(\sigma_{\alpha,\alpha})^{i,j} = e_{\alpha\beta}\big[\sigma_\alpha^{i+1/2,j+1/2}(u_\beta^{i,j+1} - u_\beta^{i+1,j}) + \sigma_\alpha^{i-1/2,j+1/2}(u_\beta^{i-1,j} - u_\beta^{i,j+1}) + \sigma_\alpha^{i-1/2,j-1/2}(u_\beta^{i,j-1} - u_\beta^{i-1,j}) + \sigma_\alpha^{i+1/2,j-1/2}(u_\beta^{i+1,j} - u_\beta^{i,j-1})\big], \quad (67)$$

where

$$A^{i,j} = \tfrac{1}{4}\big[(u_2^{i-1,j} - u_2^{i+1,j})(u_1^{i,j+1} - u_1^{i,j-1}) - (u_1^{i-1,j} - u_1^{i+1,j})(u_2^{i,j+1} - u_2^{i,j-1})\big] \quad (68)$$

is one half the area of the quadrilateral.

We also require gradients of functions at zone-centered points. These are derived from the integral formula

$$\int_D \sigma_{,\alpha}\, da = e_{\alpha\beta}\int_{\partial D} \sigma\, du_\beta. \quad (69)$$

We now identify D with the shaded region in the figure. The left-hand side is estimated as the product of the shaded area with the integrand, evaluated at the zone-centered point, and the four edge contributions to the right-hand side are approximated by replacing the integrand in each with the average of the nodal values at the endpoints. This furnishes [Silling 1988]

$$2A^{i+1/2,j+1/2}(\sigma_{,\alpha}^{i+1/2,j+1/2}) = e_{\alpha\beta}\big[(\sigma^{i+1,j+1} - \sigma^{i,j})(u_\beta^{i,j+1} - u_\beta^{i+1,j}) - (\sigma^{i,j+1} - \sigma^{i+1,j})(u_\beta^{i+1,j+1} - u_\beta^{i,j})\big], \quad (70)$$

where

$$A^{i+1/2,j+1/2} = \tfrac{1}{2}\big[(u_2^{i,j+1} - u_2^{i+1,j})(u_1^{i+1,j+1} - u_1^{i,j}) - (u_1^{i,j+1} - u_1^{i+1,j})(u_2^{i+1,j+1} - u_2^{i,j})\big]. \quad (71)$$

The term $\alpha\boldsymbol{n}$ in (55) associated with the applied pressure may be expressed as a divergence on Ω [Taylor and Steigmann 2009]. Thus, $\boldsymbol{n} = n_k\boldsymbol{i}_k$, with $\boldsymbol{i}_3 = \boldsymbol{k}$, where

$$\alpha n_k = \tfrac{1}{2}e_{ijk}e_{\alpha\beta}r_{i,\alpha}r_{j,\beta} = G_{k\beta,\beta} \quad (72)$$

and

$$G_{k\beta} = \tfrac{1}{2}e_{ijk}e_{\alpha\beta}r_{i,\alpha}r_j, \quad (73)$$

in which e_{ijk} is the three-dimensional permutation symbol ($e_{123} = +1$). For uniformly distributed pressures (55) is thus equivalent to the system

$$T_{k\alpha,\alpha} = \rho_r\ddot{r}_k, \quad \text{where } T_{k\alpha} = P_{k\alpha} + P(t)G_{k\alpha}, \quad (74)$$

where $P_{k\alpha} = \boldsymbol{P}\cdot\boldsymbol{i}_k\otimes\boldsymbol{i}_\alpha$ are the components of $\boldsymbol{P1}$ and $r_k = \boldsymbol{i}_k\cdot\boldsymbol{r}$ are the Cartesian coordinates of a material point after deformation.

Each of the equations (74) is of the form

$$\sigma_{\alpha,\alpha} = \rho_r \ddot{\sigma}, \tag{75}$$

where $\sigma_\alpha = T_{k\alpha}$ and $\sigma = r_k$; $k = 1, 2, 3$. This is integrated over the region containing the node, enclosed by the quadrilateral of Figure 1, yielding

$$\Sigma^{i,j,n} = m^{i,j} \ddot{\sigma}^{i,j,n}, \tag{76}$$

where n is the time step,

$$\Sigma^{i,j,n} = 2A^{i,j} (\sigma_{\alpha,\alpha})^{i,j,n}, \tag{77}$$

and

$$m^{i,j} = 2A^{i,j} \rho_r \tag{78}$$

is the nodal mass.

The right-hand side of (76) is evaluated in terms of the zone-centered values of σ_α via (67). The latter depend constitutively on corresponding zone-centered values of the gradients $\sigma_{,\alpha}$ which, in turn, are determined via (70) by the values of σ at the nodes located at the vertices of the shaded region of Figure 1. The scheme requires one degree of differentiability less than that required by the local differential equations. Discussions of the associated truncation errors are given in [Silling 1988; Herrmann and Bertholf 1983].

We observe that the matrix $G_{k\beta}$ associated with lateral pressure (not to be confused with the plastic deformation) is evaluated at zone-centered points (cf. (72)). However, this involves the deformation r_k (cf. (73)), a nodal variable; in place of this we substitute the average of the deformations at the four adjacent nodes.

The time derivatives in (76) are approximated by the central differences

$$\dot{\sigma}^n = \tfrac{1}{2}(\dot{\sigma}^{n+1/2} + \dot{\sigma}^{n-1/2}), \quad \ddot{\sigma}^n = \tfrac{1}{\epsilon}(\dot{\sigma}^{n+1/2} - \dot{\sigma}^{n-1/2}), \quad \dot{\sigma}^{n-1/2} = \tfrac{1}{\epsilon}(\sigma^n - \sigma^{n-1}), \tag{79}$$

where ϵ is the time increment and the node label (i, j) has been suppressed. Substitution into (76) furnishes the explicit, decoupled system

$$\begin{aligned} m^{i,j} \dot{\sigma}^{i,j,n+1/2} &= m^{i,j} \dot{\sigma}^{i,j,n-1/2} + \epsilon \Sigma^{i,j,n}, \\ \sigma^{i,j,n+1} &= \sigma^{i,j,n} + \epsilon \dot{\sigma}^{i,j,n+1/2}, \end{aligned} \tag{80}$$

which is used to advance the solution in time node-by-node.

The starting procedure is derived from the quiescent initial conditions $\boldsymbol{r} = \boldsymbol{u}$ and $\dot{\boldsymbol{r}} = \boldsymbol{0}$ for $t \le 0$, and the initial values of the director and plastic deformation fields are $\boldsymbol{d} = \boldsymbol{k}$ and $\boldsymbol{K} = \boldsymbol{I}$, corresponding to $\boldsymbol{\chi} = \boldsymbol{x}$ and $\boldsymbol{H} = \boldsymbol{F} = \boldsymbol{I}$; the constitutive equations then require that the initial value of the Biot stress vanish. The boundary condition is $\boldsymbol{r}(\boldsymbol{u}, t) = \boldsymbol{u}$ on $\partial\Omega$, for all t.

Stability of the scheme is ensured by using sufficiently small time steps selected on the basis of successive trials based on a sequence of values of ϵ.

Our procedure presumes a degree of regularity for the solution that is not consistent with the existence of shocks. Accordingly, we do not append associated discontinuity relations. The inclusion of such conditions would be appropriate in a numerical scheme based on the method of characteristics, such as described in [Cristescu 1967] in the setting of axisymmetry involving a single spatial dimension (the radius). In contrast, the present procedure is a direct numerical simulation in the spirit of conventional structural dynamics. A similar approach was used to describe the potentially nonhyperbolic dynamics of elastic strings in [Yokota et al. 2001], where, with sufficient mesh refinement, it was shown to furnish close approximations to solutions containing genuine shocks obtained by characteristic-based methods [Beatty and Haddow 1985]. Nevertheless this issue furnishes a logical point of departure for further study, and it is in this sense that our simulations may be regarded as preliminary numerical experiments.

4.2. ***Updating the director, the stress and the plastic deformation.*** Given the nodal deformation $\boldsymbol{r}(\boldsymbol{u}, t_n)$, we use (70) to compute $\nabla\boldsymbol{r}(\boldsymbol{u}, t_n)$ at zone-centered points. The zone-centered values of $\boldsymbol{K}(\boldsymbol{u}, t_n)$ are combined with (1) to express the elastic deformation $\boldsymbol{H}(\boldsymbol{u}, t_n)$ in terms of $\boldsymbol{d}(\boldsymbol{u}, t_n)$, which remains to be determined. To this end we form the elastic Cauchy–Green deformation $\boldsymbol{C}_H$ and compute the associated invariants $H_1 = \operatorname{tr}\boldsymbol{C}_H$, $H_2 = \frac{1}{2}[I_1^2 - \operatorname{tr}(\boldsymbol{C}_H)^2]$ and $H_3 = \det\boldsymbol{C}_H$; these are used to obtain the *elastic* principal stretches [Rivlin 2004]

$$\lambda_k = \frac{1}{\sqrt{3}}\bigl\{H_1 + 2A\cos\bigl[\tfrac{1}{3}(\phi - 2\pi k)\bigr]\bigr\}^{1/2}; \quad k = 1, 2, 3, \tag{81}$$

where

$$\begin{aligned} A &= (H_1^2 - 3H_2)^{1/2} \quad \text{and} \\ \phi &= \cos^{-1}\Bigl[\frac{1}{2A^3}(2H_1^3 - 9H_1H_2 + 27H_3)\Bigr], \end{aligned} \tag{82}$$

and then the invariants

$$\begin{aligned} h_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ h_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ h_3 &= \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{83}$$

of the *elastic* right stretch tensor $\boldsymbol{U}_H$.

With these in hand we form the strain-energy function $w(h_1, h_2, h_3)$ from (36) and (37). The resulting expression is identified with the function $R(\boldsymbol{d})$ of Section 3. We have shown there that a solution to the plane-stress condition $(55)_2$ (or (64)) furnishes the unique minimizer of this function. Being strictly convex, the latter meets the hypotheses of convergence theorems for iterative gradient minimization algorithms such as the Cauchy–Goldstein method of steepest descents [Saaty and Bram 1964; Goldstein 1962].

With $\boldsymbol{d}(\boldsymbol{u}, t_n)$ thus determined, we use the Cayley–Hamilton formula [Steigmann 2002]

$$h\boldsymbol{U}_H = h_1h_3\boldsymbol{I} + (h_1^2 - h_2)\boldsymbol{C}_H - \boldsymbol{C}_H^2 \tag{84}$$

to compute the elastic stretch directly, where

$$h = h_1h_2 - h_3 = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3), \tag{85}$$

which is strictly positive. This is used to compute the zone-centered value of $\boldsymbol{\sigma}^n\boldsymbol{U}_H^n$ via (38) and (41). We then check the sign of the function $F^n = F(t_n)$ in (42) and decide accordingly whether or not to update the plastic deformation. If $F^n \le 0$ we set $\boldsymbol{K}^{n+1} = \boldsymbol{K}^n$ at zone-centered points; otherwise we evaluate $\boldsymbol{K}^{n+1} = \boldsymbol{K}^n + \epsilon \dot{\boldsymbol{K}}^n$, where (cf. (42) with $\boldsymbol{G}\boldsymbol{K} = \boldsymbol{I}$)

$$\dot{\boldsymbol{K}}^n = -(F^n/\nu)\boldsymbol{K}^n \operatorname{Dev}(\boldsymbol{\sigma}^n \boldsymbol{U}_H^n). \tag{86}$$

4.3. ***Examples.*** We conclude by presenting the results of some simulations. Our purpose is to demonstrate that the present model is amenable to computation, and that it furnishes realistic predictions. We make no effort to simulate actual experimental data or to benchmark our predictions against alternative simulations. Indeed, we have not found any alternative simulations in the literature.

In the examples considered the membrane is subjected to a suddenly applied spatially uniform pressure that decays exponentially in time. A spanwise dimension, L, of the reference plane Ω is used to define dimensionless initial and final position vectors $\boldsymbol{u}$ and $\boldsymbol{r}$. Nondimensional time is defined by $\bar{t} = \sqrt{\lambda/\rho_r}(t/L)$, where t is physical time, λ is a Lamé modulus (cf. (41)) and ρ_r is the referential mass density occurring in $(55)_1$. All stress measures are nondimensionalized by λ, and the dimensionless shear modulus is $\bar{\mu} = \mu/\lambda$. We also use the dimensionless yield stress $\bar{K} = K/\lambda$ and viscosity $\bar{\nu} = \sqrt{\rho_r/\lambda}(\nu L/\lambda)$ in the flow rule. The term P in $(55)_1$, representing the actual pressure divided by initial membrane thickness h, is given by $P(t) = (\lambda/L)\bar{P}(\bar{t})$, where $\bar{P}$ is a dimensionless function; the actual pressure is $p(t) = \lambda(h/L)\bar{P}(\bar{t})$, to leading order in h/L. Here, for illustrative purposes, we impose $\bar{P}(\bar{t}) = \bar{P}_0 \exp(-\bar{t})$, where $\bar{P}_0$ is a constant.

Figures 2 and 3 depict the response of an initially square membrane. Here L is taken to be the length of a side, and $L/h = 100$. The selected parameter values are $\bar{K} = 4.93 \times 10^{-4}$, $\bar{\mu} = 0.470$ and $\bar{\nu} = 5.00 \times 10^5$, and the pressure intensity is $\bar{P}_0 = 7.04$. This is sufficient to induce substantial deformation, shown in Figure 2 for an interval spanning peak positive and negative vertical displacements, corresponding to roughly one half of the initial period of oscillation. The pressure induces a wave emanating from the boundaries of the domain and converging toward the center, followed by an interaction phase and a subsequent reversal of the direction of motion over most of the domain. The transient elastic strain is seen to be quite substantial (left image in Figure 3) and well beyond the range of validity of the classical linear relations typically assumed between stress and elastic strain. Also shown (right image in Figure 3) is the history of the norm of the cross product $\boldsymbol{d} \times \boldsymbol{n}$; this is nonzero whenever the Kirchhoff–Love hypothesis fails. The substantial plastic distortion generated in this example is such as to lead to a slight deviation from Kirchhoff–Love kinematics.

We emphasize the fact that the parameters of the model may require adjustment to enhance the simulations from the quantitative standpoint. Here we have simply chosen the parameter values for the purpose of illustrating the general nature of the transient response predicted by the model.

In the second example a circular disc of radius L is subjected to the same pressure distribution, but of a smaller intensity $\bar{P}_0 = 2.82$. All other parameters are as in the first example, and again $L/h = 100$. Snapshots of the motion and the histories of the norms of the elastic strain and $\boldsymbol{d} \times \boldsymbol{n}$ are displayed in Figures 4 and 5. The elastic strain is again seen to be substantial, but the deviation from Kirchhoff–Love kinematics is reduced, due to the diminished plastic distortion attending the smaller pressure pulse.

Acknowledgments

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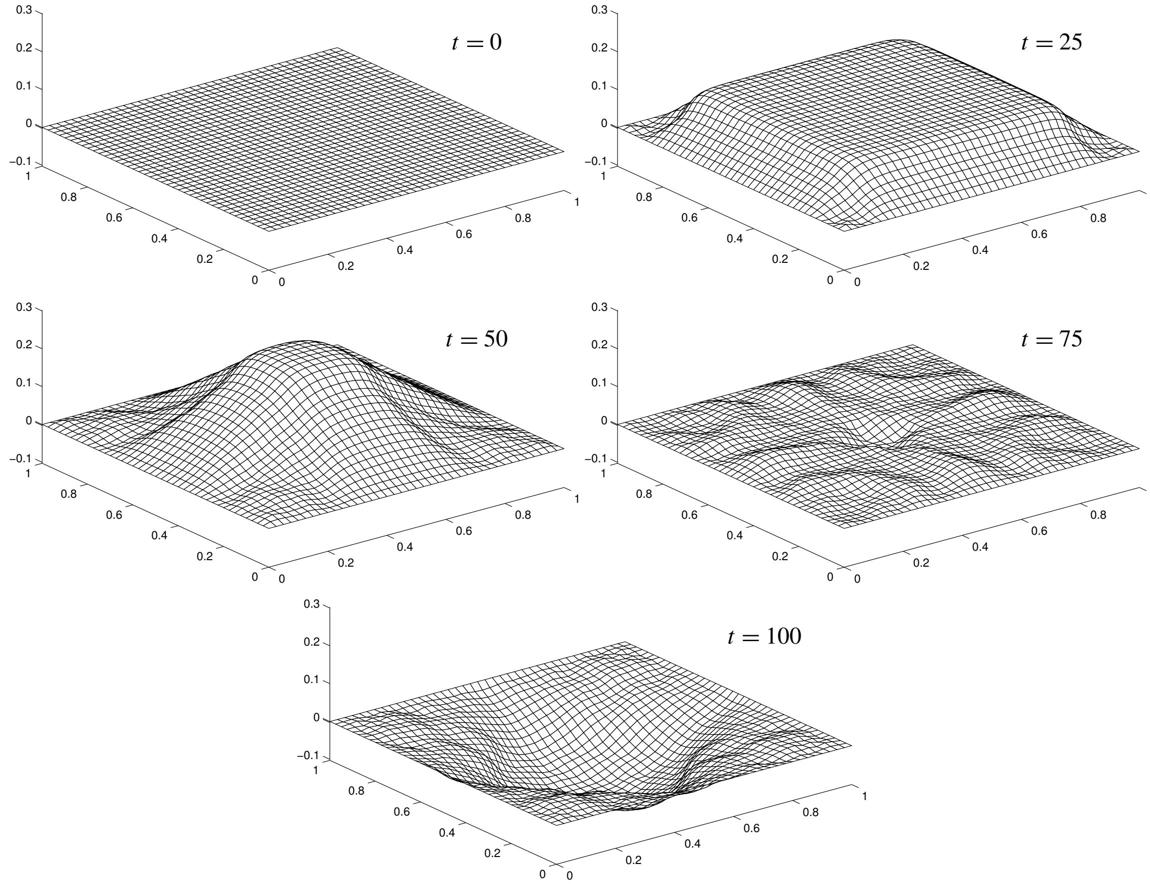

Figure 2. Configurations of a square membrane subjected to blast pressure at a sequence of times.

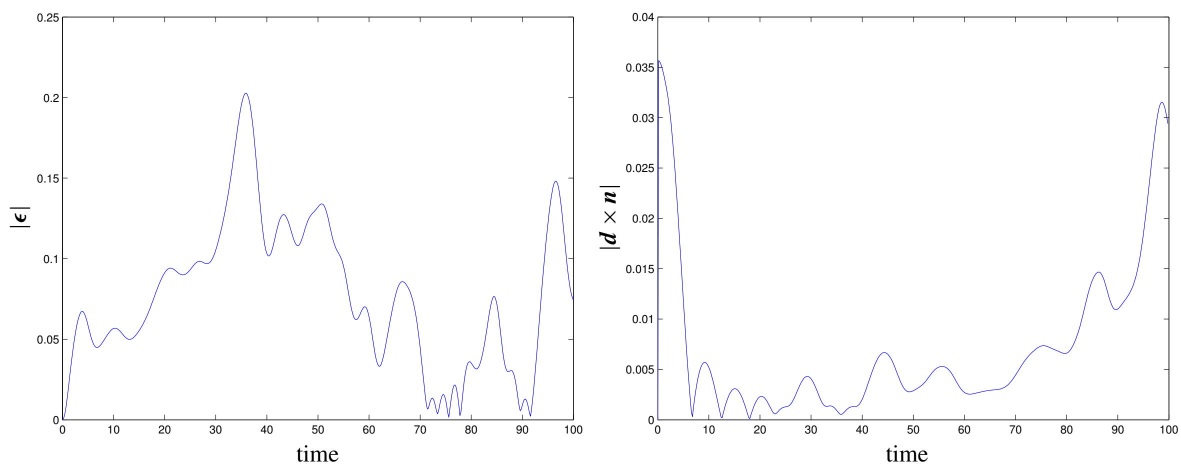

Figure 3. Left: norm of transient Biot strain midway along an edge of the square. Right: norm of $\boldsymbol{d} \times \boldsymbol{n}$ midway along an edge of the square.

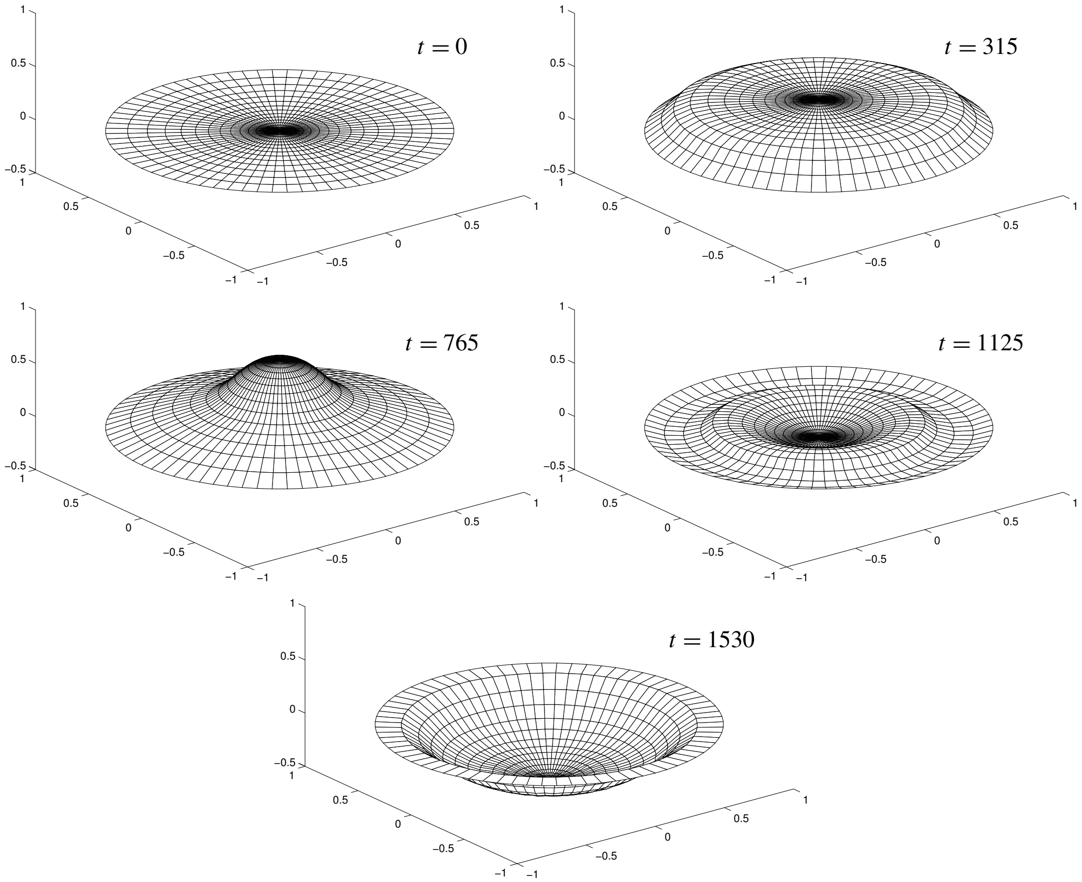

Figure 4. Configurations of a circular membrane subjected to blast pressure at a sequence of times.

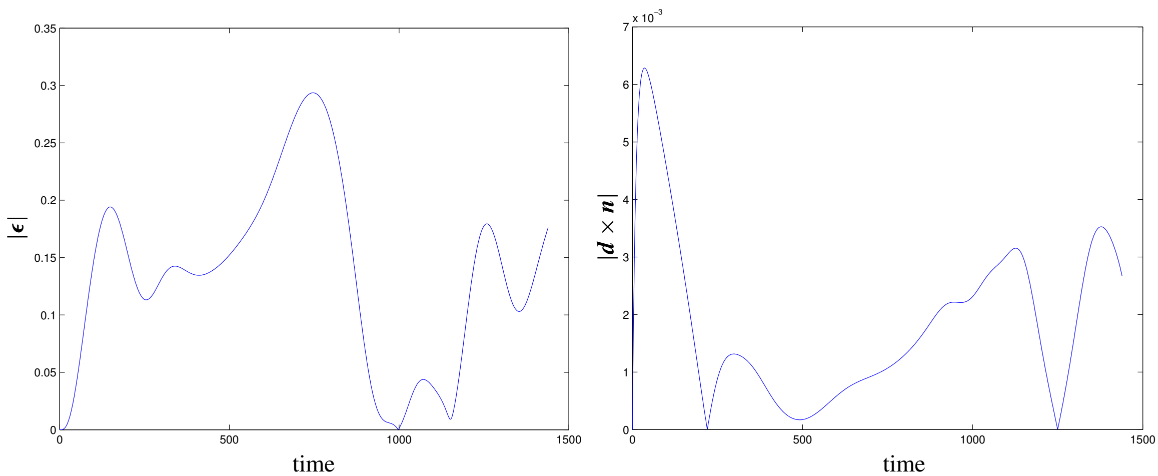

Figure 5. Left: norm of transient Biot strain at the edge of the circle. Right: norm of $\boldsymbol{d} \times \boldsymbol{n}$ at the edge of the circle.

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