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ON THE OCCURRENCE OF LUMPED FORCES AT CORNERS IN CLASSICAL PLATE THEORIES: A PHYSICALLY BASED INTERPRETATION

LAURA GALUPPI AND GIANNI ROYER-CARFAGNI

The paradigmatic example of a twisted square plate is here considered. An *equivalent* partition of the plate in a grid of beams *à la* Grashof is found such that, as the number of beams tends to infinity, the grid exhibits the same deflection of the plate. This scheme is used to interpret, through the distinction between Euler–Bernoulli and Timoshenko beam theories, the different types of natural boundary conditions that can arise in the Kirchhoff–Love and Mindlin–Reissner theories of plates. A physically based interpretation for the occurrence of lumped forces at the plate corners through the formation of a boundary layer is provided.

1. Introduction

It is well known that the solution of the biharmonic equation governing the bending of plates in Kirchhoff–Love theory is compatible with only two distinct conditions at each boundary point, whereas in general three boundary data can be independently assigned on an unconstrained border. This contradiction for the order of the equation is a two-hundred-year-old problem. The paradox arose when the three-boundary-data statement by Poisson [1829] was criticized by Kirchhoff [1850], who obtained only two natural conditions at the border within a variational framework, using a static equivalence sometimes referred to as the "Kirchhoff transformation" [Vasil'ev 2012]. This result arose from the first variation of the energy functional, but it was not corroborated by any physically based interpretation. A long discussion ensued among the most eminent scientists of the period with the purpose of reconciling the Poisson and Kirchhoff theories. The dispute culminated with the elementary interpretation by Thomson and Tait [1883], who showed how to reduce the torque per unit length on the contour to a shear transverse force. Friedrichs and Dressler [1961] and Gol'Denveiser and Kolos [1965] have independently shown that the plate theory is the leading term of the expansion solution (in a small thickness parameter) for the linear elastostatics of thin, flat, isotropic bodies. As expected, this leading term alone is unable to satisfy arbitrarily prescribed edge conditions.

There has been a renewed interest during the last years in the fundamental problem of understanding the relationship between the three-dimensional elasticity theory and theories for lower-dimensional objects (plates, shells, rods). Due to the availability of sophisticated methods of variational convergence [Ciarlet 1997], important achievements have been obtained by showing that various theories of plates arise as a rigorous variational limit (or Γ-limit) of the equations of three-dimensional elasticity as the

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thickness tends to zero, both in the linear and in the nonlinear case [Friesecke et al. 2006]. An approach of this kind allows one to rigorously recover the Kirchhoff–Thomson–Tait boundary conditions.

In particular, the Kirchhoff transformation results in the appearance of lumped forces at corners of rectangular plates, which are sometimes suspiciously treated as a drawback of the theory [Alfutov 1992]. Mutually exclusive interpretations either admit the existence of actual supporting reactions associated with the Kirchhoff transformation, possibly due to internal constraints [Podio-Guidugli 1989], or completely deny the physical meaning of this approach [Zhilin 1995]. Vasil'ev [2012] has discussed the applicability of the Kirchhoff transformation, concluding that, for plates with fixed contour, the reduction of twisting moments to shear forces can be performed only approximately: in general, it is not applicable for static boundary conditions where the torque is prescribed on the plate contour. In such cases, one has to consider higher-order theories like Mindlin–Reissner theory [Mindlin 1951; Reissner 1945], accounting for the boundary effect due to shear strains.

The aim of this note is to give an elementary, physically based, interpretation for the occurrence of lumped forces at plate corners, predicted by the Kirchhoff transformation, through the paradigmatic example of a twisted square plate. The approach is somehow dual to the customary derivation of plate theory as a downgrade limit of the equations of three-dimensional elasticity: here, plate theory is considered as a proper *upgrade* of lower-order beam theory. A partition *à la* Grashof [1878] of the plate as a grid of beams will provide an immediate interpretation of the diffusion of stress from the corners, where the forces are applied, to the interior of the body. Such diffusion strongly depends upon the shear stiffness of the constituting beams. Different types of responses can be obtained if one assumes for the beams either the Euler–Bernoulli or the Timoshenko [1940] models. This distinction is at the base of the different types of boundary conditions arising in the Kirchhoff–Love or Mindlin–Reissner theories, which somehow represent the counterparts, for plates, of the Euler–Bernoulli and the Timoshenko one-dimensional approaches, respectively.

2. Practice

Let $\mathcal{B} \equiv \Omega \times [-h, h] \subset \mathbb{R}^3$, $\Omega \subset \mathbb{R}^2$, denote the undistorted reference configuration of a flat plate, with boundary $\partial\Omega \times [-h, h]$ supposed piecewise regular. Introduce a right-handed orthogonal reference system $\boldsymbol{x} = (x_1, x_2, x_3)$, with $x_1, x_2 \in \Omega$ and x_3 at a right angle to them, and let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ denote the associated unit vectors. It is customary [Timoshenko and Woinowsky-Krieger 1959] to define the stress state inside the plate through thickness-averaged descriptors of the Cauchy stress field $\tau_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, $i, j = 1, 2, 3$. In particular, the shear forces per unit length Q_α and the moments per unit length $M_{\alpha\beta}$, $\alpha, \beta = 1, 2$, are defined as

$$Q_\alpha = \int_{-h}^{h} \tau_{\alpha 3}\, dx_3, \qquad M_{\alpha\beta} = \int_{-h}^{h} x_3 \tau_{\alpha\beta}\, dx_3, \qquad \alpha, \beta = 1, 2. \tag{2-1}$$

In this way the problem becomes two-dimensional, and definable in the domain Ω and its boundary $\partial\Omega$.

Let $p\boldsymbol{e}_3$, with $p = \hat{p}(x_1, x_2)$, represent the force per unit area acting orthogonally to Ω. With a little abuse of notation, define

$$\boldsymbol{Q} = \sum_{\alpha=1}^{2} Q_\alpha \boldsymbol{e}_\alpha, \qquad \boldsymbol{M} = \sum_{\alpha,\beta=1}^{2} M_{\alpha\beta}\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta, \qquad \alpha, \beta = 1, 2, \tag{2-2}$$

where clearly $\boldsymbol{M} = \boldsymbol{M}^T$. The equilibrium in the x_3 direction and the equilibrium of moments around any axis parallel to Ω read, respectively,

$$\operatorname{div} \boldsymbol{Q} = -p, \quad \operatorname{div} \boldsymbol{M} = \boldsymbol{Q}, \tag{2-3}$$

where "div" denotes the divergence operator in $\mathbb{R}^2$. By combining the aforementioned relationships one readily obtains the equilibrium equation $\operatorname{div}(\operatorname{div} \boldsymbol{M}) = -p$.

Denoting with a comma partial differentiation with respect to the indicated variable, the Kirchhoff–Love kinematical hypothesis [Timoshenko and Woinowsky-Krieger 1959] consists in assuming that the displacement field $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$ has the form

$$u_3(\boldsymbol{x}) = w(x_1, x_2), \quad u_\alpha(\boldsymbol{x}) = -x_3 w_{,\alpha}(x_1, x_2), \quad \alpha = 1, 2, \tag{2-4}$$

where we have not considered (for simplicity, and because it is here irrelevant) the membrane strain due to forces in the plate middle-plane. Consequently, the strain components ε_{ij}, $i, j = 1, 2, 3$, read

$$\varepsilon_{\alpha\beta} = -x_3\, w_{,\alpha\beta}(x_1, x_2), \quad \varepsilon_{\alpha 3} = 0, \quad \varepsilon_{33} = 0, \quad \alpha, \beta = 1, 2. \tag{2-5}$$

If the material is homogeneous and isotropic, denoting by E the Young's modulus and by ν the Poisson's ratio, one finds

$$\begin{cases} M_{11} = -\dfrac{2h^3 E}{3(1-\nu^2)}[w_{,11} + \nu\, w_{,22}], \\ M_{22} = -\dfrac{2h^3 E}{3(1-\nu^2)}[w_{,22} + \nu\, w_{,11}], \\ M_{12} = -\dfrac{2h^3 E}{3(1+\nu)}\, w_{,12}. \end{cases} \tag{2-6}$$

This theory, as is clear from (2-5), neglects shear deformations, but the shear strains Q_α of (2-2) can be recovered from just the equilibrium considerations from (2-3).

The strains due to shear are accounted for in the Mindlin–Reissner theory of moderately thick plates [Reissner 1945; Mindlin 1951], where the displacement field is assumed of the form

$$u_3(\boldsymbol{x}) = w(x_1, x_2), \quad u_\alpha(\boldsymbol{x}) = -x_3 \varphi_\alpha(x_1, x_2), \quad \alpha = 1, 2, \tag{2-7}$$

where φ_α is the rotation of fibers parallel to $\boldsymbol{e}_\alpha$ with, in general, $\varphi_\alpha \neq w_{,\alpha}$. The strain components thus become

$$\varepsilon_{\alpha\beta} = -\tfrac{1}{2}x_3(\varphi_{\alpha,\beta} + \varphi_{\beta,\alpha}), \quad \varepsilon_{\alpha 3} = \tfrac{1}{2}(w_{,\alpha} - \varphi_\alpha), \quad \varepsilon_{33} = 0; \quad \alpha, \beta = 1, 2. \tag{2-8}$$

From these, the constitutive equations read

$$\begin{cases} M_{11} = -\dfrac{2h^3 E}{3(1-\nu^2)}\, [\varphi_{1,1} + \nu\varphi_{2,2}], \\ M_{22} = -\dfrac{2h^3 E}{3(1-\nu^2)}\, [\varphi_{2,2} + \nu\varphi_{1,1}], \\ M_{12} = -\dfrac{h^3 E}{3(1+\nu)}\, [\varphi_{1,2} + \varphi_{2,1}], \end{cases} \tag{2-9}$$

$$\begin{cases} Q_1 = -\kappa G h\, [w_{,1} - \varphi_1], \\ Q_2 = -\kappa G h\, [w_{,2} - \varphi_2], \end{cases} \tag{2-10}$$

where κ is the shear correction factor, usually assumed equal to $\frac{5}{6}$.

Both in the Kirchhoff–Love and the Mindlin–Reissner theories, the governing field equations in terms of displacements are obtained by inserting the constitutive equations (2-6), or (2-9)–(2-10), into the equilibrium equations (2-3). The first theory gives rise to the well-known biharmonic equation in w, whereas the second theory produces two differential equations in w and φ_α.

At the boundary $\partial\Omega$ define the orthogonal right-handed triad of unit vectors $(\boldsymbol{m}, \boldsymbol{t}, \boldsymbol{n})$, with $\boldsymbol{n}$ parallel and in the same direction as $\boldsymbol{e}_3$, while $\boldsymbol{m} = m_1\boldsymbol{e}_1 + m_2\boldsymbol{e}_2$ is the outward unit normal to $\partial\Omega$ and, consequently, $\boldsymbol{t} = t_1\boldsymbol{e}_1 + t_2\boldsymbol{e}_2 = -m_2\boldsymbol{e}_1 + m_1\boldsymbol{e}_2$ is tangent to $\partial\Omega$. Introduce then a curvilinear abscissa s, parametrized by arc length and oriented as $\boldsymbol{t}$. The static state at $\partial\Omega$ is defined by the bending moment $M_m\boldsymbol{t}$, by the torque $-M_{mt}\boldsymbol{m}$ and by the shear force $Q_m\boldsymbol{n} = Q_m\boldsymbol{e}_3$, all of them per unit length of the border. One has

$$M_m = \boldsymbol{M}\boldsymbol{m}\cdot\boldsymbol{m}, \qquad M_{mt} = \boldsymbol{M}\boldsymbol{m}\cdot\boldsymbol{t}, \qquad Q_m = \boldsymbol{Q}\cdot\boldsymbol{m}. \tag{2-11}$$

The boundary conditions are substantially different in the two aforementioned theories of plates, in agreement with the order of the governing differential equations.

In Mindlin–Reissner theory, the geometric boundary conditions may involve three quantities: the displacement u_3 and the two rotation components in both the normal direction $\boldsymbol{m}$ (i.e., $\varphi_m = \varphi_1 m_1 + \varphi_2 m_2$) and in the tangential direction $\boldsymbol{t}$ (i.e., $\varphi_t = \varphi_1 t_1 + \varphi_2 t_2$). The corresponding natural boundary conditions involve, respectively, Q_m, M_m and M_{mt}, that is, the shear force, the bending moment and the torque (per unit length), defined in (2-11).

On the other hand, it is well-known that in Kirchhoff–Love theory the three quantities Q_m, M_m and M_{mt} cannot be prescribed independently. In fact, the Kirchhoff transformation defines the *effective* shear force per unit length

$$V_m\boldsymbol{n} = V_m\boldsymbol{e}_3 = [Q_m + (M_{mt})_{,s}]\boldsymbol{n}, \tag{2-12}$$

which is dual in energy with the vertical displacement at the boundary. Therefore, on $\partial\Omega$ the geometric boundary conditions prescribe either the displacement w in the direction $\boldsymbol{e}_3$ or its derivative $w_{,m}$ with respect to the outward unit normal $\boldsymbol{m}$, to which correspond the natural boundary conditions on the effective shear force V_m, defined as per (2-12), and on the bending moment M_m, given by $(2\text{-}11)_1$.

If the boundary presents a corner at $s = s_0$, denote by $\boldsymbol{m}(s_0^+)$, $\boldsymbol{t}(s_0^+)$ and $\boldsymbol{m}(s_0^-)$, $\boldsymbol{t}(s_0^-)$ the normal and tangential unit vectors at $s = s_0^+$ and $s = s_0^-$, respectively. Then, the Kirchhoff transformation implies the occurrence of lumped forces $F(s_0)\boldsymbol{e}_3$ at the corner given by

$$F(s_0) = \boldsymbol{M}(s_0)\boldsymbol{m}(s_0^+)\cdot\boldsymbol{t}(s_0^+) - \boldsymbol{M}(s_0)\boldsymbol{m}(s_0^-)\cdot\boldsymbol{t}(s_0^-) = M_{mt}(s_0^+) - M_{mt}(s_0^-), \tag{2-13}$$

which are usually considered to be physically justified by the presence of unbalanced torques.

3. A paradigmatic example

The following example can be found in most textbooks (see, e.g., [Timoshenko and Woinowsky-Krieger 1959, Section 11] or [Belluzzi 1986, exercise 1187]). With reference to Figure 1, let Ω be the square

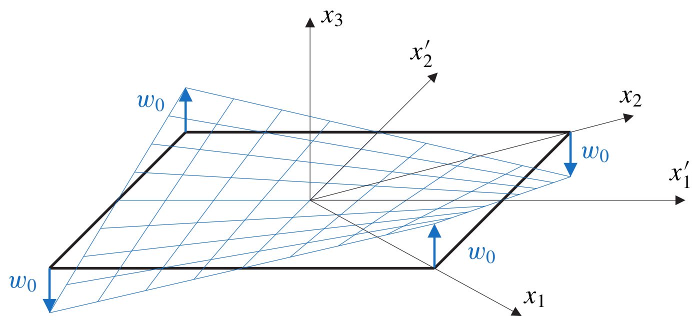

Figure 1. Reference and deformed configurations of the square plate.

defined by the vertices $(x_1, x_2) = (d/2, 0)$, $(0, d/2)$, $(-d/2, 0)$, $(0, -d/2)$, where d is the length of the diagonal. Our aim here is to determine states of stress that are compatible with a vertical displacement of the type

$$w(x_1, x_2) = 4w_0 \frac{x_1^2 - x_2^2}{d^2}, \tag{3-1}$$

which represents a hyperbolic paraboloid.

3.1. ***State of stress and boundary conditions.*** It is easy to verify that in a Kirhhoff–Love plate one has from (2-6), $(2\text{-}3)_2$ and $(2\text{-}11)_2$ that

$$M_{11} = -M_{22} = -\frac{16h^3 E}{3(1+\nu)d^2} w_0, \qquad M_{12} = 0, \qquad Q_1 = Q_2 = 0. \tag{3-2}$$

Then, clearly, from (2-12), $V_m = 0$. But, from (2-13), four concentrated forces are acting at the four corners of the plate and, more precisely,

$$F_0 \boldsymbol{e}_3 \quad \text{at } (d/2, 0),\ (-d/2, 0); \qquad -F_0 \boldsymbol{e}_3 \quad \text{at } (0, d/2),\ (0, -d/2); \qquad F_0 = \frac{32h^3 E}{3(1+\nu)d^2} w_0. \tag{3-3}$$

Denoting by (x_1', x_2') an auxiliary reference system rotated by $\pi/4$ with respect to (x_1, x_2), as represented in Figure 1, it is possible to verify that

$$M_{1'1'} = M_{2'2'} = 0, \qquad M_{1'2'} = \frac{\boldsymbol{M}}{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2) \cdot (-\boldsymbol{e}_1 + \boldsymbol{e}_2) = -\frac{16h^3 E}{3(1+\nu)d^2} w_0. \tag{3-4}$$

Therefore, whereas the four borders of the plate are stress free, in the immediate neighborhood, on lines parallel to each border, the torque per unit length is not zero. This turns out to be an apparent contradiction of the theory [Alfutov 1992].

Consider, on the other hand, a Mindlin–Reissner plate. We look for a solution which is associated with a null shear deformation: the purpose of this choice is to find, within the framework of this higher-order theory, the same state of stress predicted by Kirchhoff–Love theory. In fact, if in (2-8) $\varepsilon_{\alpha 3} = 0$, $\alpha = 1, 2$, then $\varphi_\alpha = w_{,\alpha}$ and, consequently, one finds from (2-9) the field (3-2). At the boundary the only nonzero component of stress is the torque per unit length, which is equal to $M_{1'2'}$ of (3-4).

In conclusion, in a twisted plate deformed according to (3-1) with no shear, both Kirchhoff–Love and Mindlin–Reissner theories prescribe the same state of stress *inside* the plate, but the associated boundary

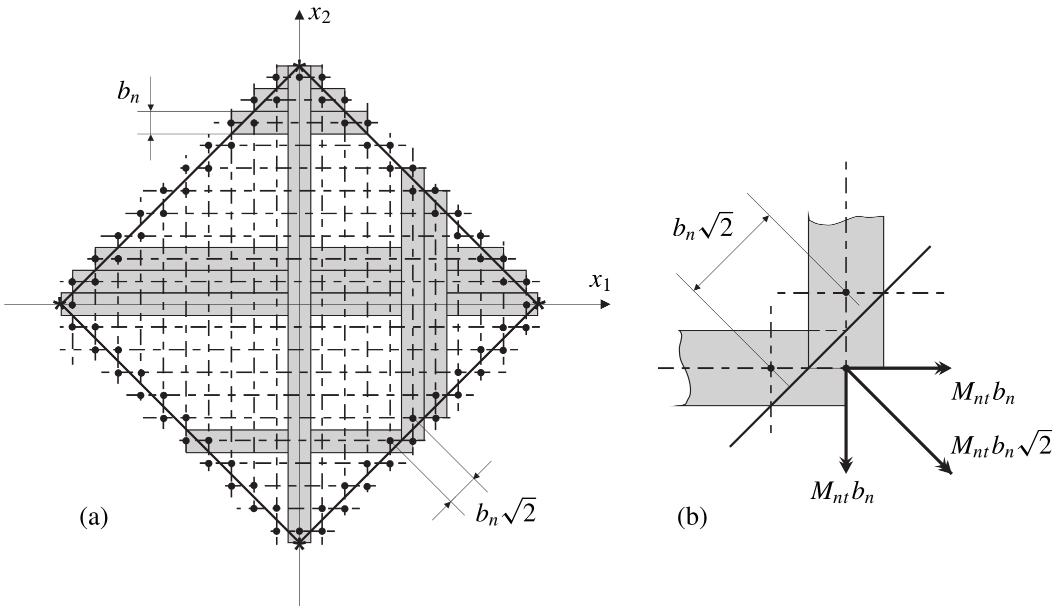

Figure 2. Partition *à la* Grashof of the square plate. (a) Ideal division into beams; (b) decomposition of the torque at the border into two bending moments for the beams.

conditions are completely different since they predict lumped forces at the corners in the first case and uniformly distributed torque per unit length in the second case. The aim of the next section is to give an elementary physically based explanation of this finding.

3.2. ***Partition* à la *Grashof.*** The Grashof approximation [1878], according to which plates are considered as grids of beams, is commonly used in practical applications. Here, we will discuss a partition which is *exact*, in the sense that, when the number of beams tends to infinity, one recovers the same deflection of the plate. To do so, the body of Figure 1 is ideally divided by imaginary cuts into $(2n+1)$ beams of the same width in the x_1 direction, and by an equal partition in the x_2 direction. Consequently, as represented in Figure 2(a), the width b_n of each beam and the corresponding cross-sectional moment of inertia I_n read, respectively,

$$b_n = \frac{d}{2n+1}, \qquad I_n = \frac{2b_n h^3}{3} = \frac{2d\,h^3}{3(2n+1)}. \tag{3-5}$$

Consider first the Mindlin–Reissner solution described in the previous section. The border of the plate is loaded by a uniformly distributed torque per unit length, which can be distributed to each beam according to the corresponding partition of influence, of length $b_n\sqrt{2}$ as represented in Figure 2(b). Clearly, x_1 and x_2 are axes of geometric and loading symmetry for the structure. The moment resultant can then be decomposed in the two components in the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions, which represent two bending moments for the beams.

Observe that the partition of the twisting moment into two bending moments only, with no torsion moments for the beams, is not arbitrary. In fact, each beam should carry torsion moments of opposite signs at the ends in order to satisfy equilibrium, but such a distribution violates the symmetry of the problem.[1]

In conclusion, each beam in the x_1 (resp. x_2) direction is uniformly bent by the moment $M_1^{(n)}$ (resp. $M_2^{(n)}$) given by

$$M_1^{(n)} = -M_2^{(n)} = -\frac{d}{2n+1}\,\frac{16h^3E}{3(1+\nu)d^2}\,w_0. \tag{3-6}$$

The corresponding curvatures χ_1 and χ_2, taking into account that the transversal strain of each composing beam is restrained by the flexure of the orthogonal sets of beams, read

$$\chi_1 \simeq w_{,11} = -\frac{M_1^{(n)}}{E\,I_n/(1+\nu)} = 8\frac{w_0}{d^2}, \quad \chi_2 = -\frac{M_2^{(n)}}{E\,I_n/(1+\nu)} = -\chi_1 \simeq w_{,22}, \tag{3-7}$$

which clearly coincides with the expectation from the assumed deformation (3-1).

It should be noticed that in this case each beam is subject to pure bending. Therefore, the flexure of (3-7) remains the same whether one assumes the Euler–Bernoulli model or the Timoshenko model for the beam, i.e., whether one does or does not neglect shear deformations.

Consider now the case in which the boundary is stress-free apart from the four lumped forces F_0 of (3-3), acting at the points marked with stars in Figure 2(a). Let us suppose that the beams are connected by spherical hinges only at those points marked with dots in Figure 2(a), i.e., at those points which are closer to the border of the reference domain Ω. It will be verified, *a posteriori*, that in the limit $n \to \infty$ the deformation of the beam lattice associated with the aforementioned static state is compatible, in the sense that the deflection of each nodal point is the same if it belongs to either one of the two orthogonal beams passing through it.

Then, with symmetry considerations, the grid is statically determined and the forces acting in each beam can be directly calculated. Three possible conditions, as represented in Figure 3, need to be distinguished:

- Each one of the two longest beams of length $(2n+1)b_n = d$ in Figure 3, whose axes coincide with one of the diagonals, is bent by the applied loads F_0 (at the plate corners), and by the reaction forces of the two orthogonal short beams hinged to them, which by statics are also equal to F_0. Such forces form pairs with lever arm $b_n/2$.
- Consequently, each one of the four shortest beams in proximity to the corners, denoted by $i = 1$ in Figure 3, of length $2b_n$, is loaded by a concentrated force F_0 in the middle and transfers two concentrated forces $F_0/2$ to the orthogonal beams of length $2nb_n$.
- One can repeat the same construction and derive that the other beams, of length $4b_n \le l \le 2nb_n$, are bent by pairs of forces at the extremities, each one formed by two forces $F_0/2$ with arm b_n.

The maximal bending moment acting in each beam is, in absolute value, equal to

$$M_F = \tfrac{1}{2}b_nF_0 = \frac{16h^3E}{3(2n+1)(1+\nu)d}\,w_0, \tag{3-8}$$

[1]If the plate deformation was represented by a beam lattice with the beams parallel to the edges, the only distribution of loads that could respect the symmetry of this new partition would be a state of zero bending (all beams remain straight) and pure torsion.

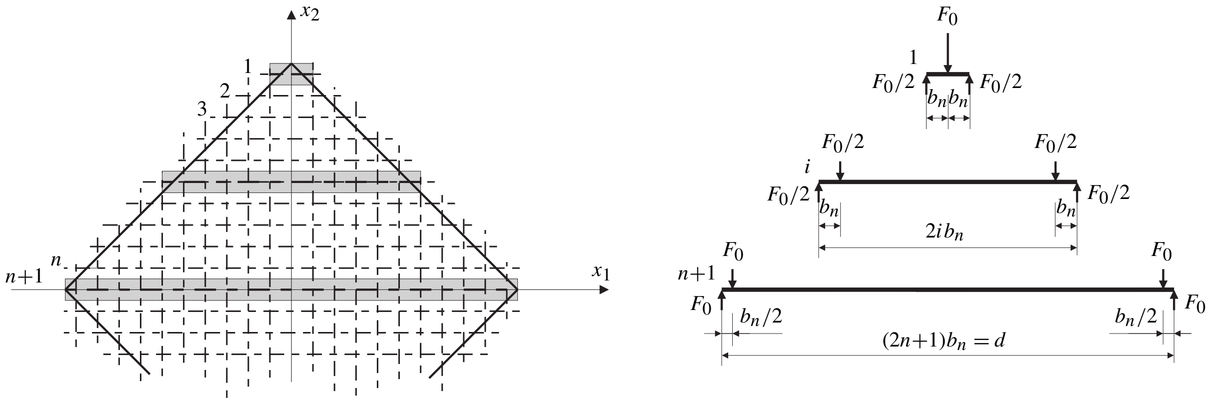

Figure 3. Bending of beams in the Grashof partition: shortest beams of length $2b_n$, intermediate beams, beams along the diagonals.

which coincides with the value prescribed by (3-6). It is then clear that in the limit of an infinite partition ($n \to \infty$), one recovers the same curvature as in (3-7). In fact, since $b_n \to 0$, the arm of the pair of forces tends to zero, but the reduction of the corresponding bending moment is exactly compensated by the reduction of the cross-sectional inertia, according to (3-5)$_2$.

It is then a simple exercise to show that if the curvature of each beam is of the form (3-7) then the deflections of the nodal points of the beam lattice accommodate one another and the resulting deformed shape is given by (3-1). Therefore, the static state that has been derived from the assumed distribution of internal constraints for the constituting elementary beams of Figure 2 is balanced and compatible.

However, one should notice that the two longest diagonal beams present a shear equal to F_0 in portions in proximity to each end of length $b_n/2$, whereas all the other beams undergo a shear equal to $F_0/2$ at the extremal portions of length b_n. As $b_n \to 0$, the corresponding shear stress tends to infinity because the width of each beam tends to zero. Denoting by γ_F and $\gamma_{F/2}$ the shear strain in the diagonal beams and in the other beams, respectively, one finds that there are relative displacements δ_F and $\delta_{F/2}$ associated with such a shear strain. These read

$$\delta_F = \gamma_F \cdot \tfrac{1}{2} b_n = \kappa \frac{F_0}{G\,2hb_n} \cdot \tfrac{1}{2} b_n = \kappa \frac{F_0}{G\,4h}, \qquad \delta_{F/2} = \gamma_{F/2} b_n = \kappa \frac{F_0/2}{G\,2hb_n} b_n = \kappa \frac{F_0}{G\,4h}, \tag{3-9}$$

and are independent of n. Thus, as $n \to \infty$, a shear dislocation remains at the beam extremities.

Consequently, if the constituting elements are beams *à la* Timoshenko, one can no longer recover, with the aforementioned partition *à la* Grashof, the deformation of the plate prescribed by (3-1). The counterpart of Timoshenko beam theory for plates is Mindlin–Reissner theory. The proposed elementary example thus illustrates why Mindlin–Reissner theory cannot account for the possibility of concentrated forces at the plate corners compatibly with the assumed displacement (3-1).

On the other hand, Euler–Bernoulli beam theory cannot account for shear strain. Consequently, if one assumes this model for the Grashof partition, the concentrated displacement due to slip is null and one recovers the same curvature prescribed by (3-7). The resulting deformation is again compatible with an expression of the form (3-1). Remarkably, there is a “transformation” of the *bending with shear* produced by the lumped forces into *pure bending*. Such a transformation, made possible by the shear-insensitivity of the constituting beams, occurs in a boundary layer whose thickness is of the order of b_n, which tends to zero as $n \to \infty$.

From this example it is clear what role is played by the Kirchhoff transformation, which regulates the substitution of the torque per unit length with lumped forces at the corners due to the assumed shear-stiffness of the constitutive model. Such a substitution takes place in a boundary layer of evanescent thickness. On the other hand, in Mindlin–Reissner theory this transformation is not allowed because of the different types of deformation that are associated with the two systems of forces due to the shear deformability of the plate.

It should also be remarked that the Kirchhoff transformation is not required, but it simply states the static and kinematic equivalence of diverse equipollent system of actions as boundary conditions. In fact, the deformation indicated by (3-1) is perfectly compatible with a Kirchhoff–Love plate twisted by lumped forces, but it is also compatible with other boundary data, e.g., uniformly distributed torque per unit length applied at the border. This model cannot distinguish between the two static distributions because their difference produces shear stress only, which are associated with a null deformation. Indeed, infinite boundary data that are statically and kinematically equivalent can be found. For example, as shown in [Fosdick and Royer-Carfagni 2015], it is sufficient to take just a part of the applied forces and, for that, use the Kirchhoff transformation, while maintaining the remaining part unaltered.

4. Discussion and conclusions

Despite its simplicity, the elementary example just discussed gives an immediate, physically based, interpretation of the Kirchhoff transformation. The static and kinematic equivalence of various systems of forces and torques at the border, obtained through the notion of effective shear, is a straightforward consequence of the basic assumptions that shear deformations in Kirchhoff–Love plates are null. Such an equivalence cannot be established in Mindlin–Reissner plates, because although the aforementioned equivalent systems have the same resultant and the same moment-resultant, they are associated with different types of shear deformations that this model can detect.

Indeed, there are infinite boundary data that are compatible with the same deformation of Kirchhoff–Love plates, i.e., those which give the same result when the Kirchhoff transformation is applied. This is somehow a limit of the theory, but it would be erroneous to conclude, as is sometimes done in the technical literature [Vasil'ev 2012], that this theory is compatible with the only boundary datum that results from Kirchhoff transformation. The elementary example just illustrated shows that Kirchhoff transformation simply establishes an equivalence of various systems of forces, but does not select among these a privileged one.

In particular, the shear-stiffness assumption of Kirchhoff–Love plates allows for the possibility of lumped forces at the corners, but this is not a paradox of the theory [Alfutov 1992]. The partition *à la* Grashof allows one to recognize that there is a thin layer in proximity to the boundary where, due to the aforementioned shear stiffness, there is a transformation of the bending with shear (produced by the concentrated forces) into pure bending in the neighboring internal portions. Therefore, there is no paradox in the classical solution of a Kirchhoff–Love plate twisted by concentrated forces at the corners, where the borders are stress-free, but the torque per unit length is nonzero on fibers parallel to the borders, at an arbitrarily small distance.

However, the example has shown that this transformation is possible only at the price of infinite shear stresses occurring in a boundary layer of evanescent thickness in proximity to the borders. Indeed, the assumption of Kirchhoff–Love theory is that (transverse) shear deformations are negligibly small and, accordingly, they are assumed to be null: this implies the plate to be shear-rigid. The latter hypothesis is certainly correct in most cases of the practice, where shear stresses remain finite, but in the case of concentrated forces the shear stress becomes infinite.

Therefore, Kirchhoff–Love theory cannot consistently be applied when the border presents sharp corners with concentrated forces, because these would generate infinite shear stress, regardless of the thickness of the plate. In fact, the formation of a boundary layer [Lobkovsky 1996] cannot be neglected. For such cases a more refined theory, possibly accounting for shear deformations like Mindlin and Reissner's, appears to be necessary to reproduce the actual "diffusion" of such forces inside the body.

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