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THE RECIPROCITY LIKELIHOOD MAXIMIZATION: A VARIATIONAL APPROACH OF THE RECIPROCITY GAP METHOD

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We introduce a new concept allowing the recasting of the reciprocity gap method into a variational method. The reciprocity likelihood functional maximization gives rise to nested approximation properties when performed on minimization spaces with increasing dimensions and leads to direct identification methods grounded on the reciprocity property. Application to the identification of point sources is given for illustration of the solution procedure of identification, and an analysis of the effect of noisy data shows that the proposed methods exhibit very good robustness.

1. Identification problems and results with the reciprocity gap method

The kind of identification or inverse problems addressed here is the following:

given: a solid Ω and a physical phenomenon described by a linear "equilibrium" operator A acting on vector fields u, defined in the solid, and provided a pair of data $u = U^m$ and $Bu \cdot \boldsymbol{n} = F^m$ is known on the whole external boundary of the solid $\partial\Omega$, with external outside unit normal $\boldsymbol{n}$, for a field satisfying $Au = 0$,

determine: the eventual sources or objects buried inside the solids (cracks, flaws, holes, inclusions, etc.).

Examples of such equilibrium operators are the Laplace operator, describing isotropic conduction of heat or electricity, the Lamé operator in the elastostatic framework, the Helmholtz operator in acoustics, the Darcy operator for saturated porous media, etc. Although the reciprocity gap method can be applied to nonsymmetric or non-self-adjoint operators, where the reciprocity property does not directly hold, the reciprocity likelihood concept is developed for symmetric or self-adjoint ones.

B is the natural (or Neumann) boundary condition operator associated with A. The data usually comes from an experiment performed on the solid, for example, by prescribing a flux or an external force on the solid and measuring the response on the boundary. Unlike other kinds of inverse problems, the whole Neumann-to-Dirichlet map or Poincaré–Steklov operator is not supposed to be known, even if one can have access to more than one single experiment (or data pair). On the other hand, data on the whole boundary is supposed to be available. These data form a redundant data pair with respect to the operator A (namely Dirichlet and Neumann boundary data available simultaneously on the boundary).

Practical identification is achieved through identification procedures. Various propositions appeared in the literature; they can be separated into two broad classes that we shall quickly review in the particular case of crack-identification problems with the Laplace equation although similar methods and results have been obtained for other operators, namely acting on vector fields, as in elasticity.

Keywords: inverse problems, reciprocity gap, sources identification.

First are direct iterative methods grounded on a variational property of the solution: after designing a functional of the geometrical parameters describing the cracked domain, iterative resolutions of Laplace problems are performed in order to minimize (or get a stationary point of) the functional. To this class belong classical least-squares methods or more sophisticated functional-minimization methods as adaptations of the functional of [Kohn and Vogelius 1984]. This class relies on a lot of resolutions of elliptic problems, which are generally costly and always need the introduction of the point of departure of the algorithm (the initial guess), the choice of which is difficult and can significantly alter the speed of convergence or even the converged solution itself. Moreover, for the identification of geometries, a remeshing of the solid is necessary for each iteration.

The second class of methods uses families of particular fields and avoids any resolution of the PDEs underlying the physical phenomenon used in the identification or uses only a few: auxiliary field methods [Bui 2011]. The reciprocity gap method belongs to this last class of methods sometimes called sampling or probe methods in the inverse scattering community because no resolution of any PDE is needed.

To obtain or to approximate the solution of this kind of identification problem, the reciprocity gap concept has been introduced first for the Laplace operator [Andrieux and Ben Abda 1992; 1993], the concept being suited to symmetric operators as it relies on the reciprocity property. More precisely, the reciprocity gap RG is a linear form acting on auxiliary fields defined on the safe domain (without flaws or source distribution) and satisfying the operator equation. For each auxiliary field picked out of the auxiliary field subspace, the computation of the action of the form RG supplies scalar information on the “difference” between the actual solid and the safe one. The concept has been extended to general operators by using auxiliary fields, which are solutions of the adjoint (or conjugate) operator equation [Andrieux 1995]. A comprehensive introduction to the reciprocity gap concept can be found in [Andrieux and Bui 2011].

Numerous identification results, both theoretical and constructive (identification formulas), have been obtained by using appropriate auxiliary field families. For planar crack identification, results were obtained for the Laplace and Lamé operators [Andrieux and Ben Abda 1992; 1996; Andrieux et al. 1999], for the heat equation [Ben Abda and Bui 2001], for viscoelastic media, in inverse scattering [Bui et al. 1999; Ben Abda et al. 2005; Colton and Haddar 2005], and in elastodynamics with the concept of the instantaneous reciprocity gap [Bui et al. 2004; 2005]. Mention must be made of [Ikehata 1999] using similar concepts but with a different use. For source distribution identification, [El Badia and Ha Duong 1998; El Badia et al. 2000] gave identification algorithms for point sources with the Laplace equation and [Alves and Silvestre 2004] for the Stokes equation.

Recently, Shifrin and Shushpannikov [2010; 2011; 2013b; 2013a] derived results in elasticity for the identification of inclusions in 3D for a single traction experiment, based on approximation of the solution for an infinite medium. This use of the reciprocity gap with an approximate solution is close to the assumption of infinite medium taking advantage of the Eshelby results [1957] proposed for inclusion identification by [Andrieux et al. 2006].

2. Recollection of the reciprocity gap method

Let us recall briefly the reciprocity gap definition and the associated identification method for the simplest case of a scalar isotropic conduction equation. A is the Laplace operator, u is a scalar field, and $\boldsymbol{B}$ is the gradient operator so that on the boundary $\boldsymbol{B}u \cdot \boldsymbol{n} = \nabla u \cdot \boldsymbol{n}$. The bilinear form a associated to the operator A is

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, d\Omega. \tag{1}$$

A collection of cracks $\{\Gamma_i : i = 1, N\}$ can be buried inside the solid, and some source distribution $s(x)$ can also appear so that we have, inside the domain occupied by the solid, the equations

$$\begin{cases} -\Delta u = s & \text{in } \Omega \setminus \Gamma_i,\ i = 1, \dots, n, \\ \nabla u \cdot \boldsymbol{n}_i = 0 & \text{on } \Gamma_i,\ i = 1, \dots, n. \end{cases} \tag{2}$$

The problem of identification of the crack geometry and the source distribution is set provided a pair of boundary data is available on the whole external boundary of the domain Ω:

$$\nabla u \cdot n = F^m, \quad u = U^m \text{ on } \partial\Omega. \tag{3}$$

Definition. For any field v belonging to the Sobolev space $H^1(\Omega)$, the reciprocity gap RG is the linear form defined by

$$\mathrm{RG}(v) = \int_{\partial\Omega} (F^m v - \nabla v \cdot \boldsymbol{n} U^m) \, dS. \tag{4}$$

The RG definition uses only the known quantities of the identification problem. The term "reciprocity gap" has been coined after the Betti–Maxwell reciprocity property: for two harmonic fields u and v in the domain Ω, one has

$$\int_{\partial\Omega} (\nabla u \cdot \boldsymbol{n} v - \nabla v \cdot \boldsymbol{n} u) \, dS = 0. \tag{5}$$

The idea here is that the form RG is vanishing on every harmonic field v in Ω if there are neither cracks nor source distributions in the actual solid; it is on the contrary nonzero if some cracks or source distributions exist, measuring then in some sense the "difference" between the safe and the actual solids. More precisely, the following fundamental property of the reciprocity gap holds:

Fundamental property. *For every harmonic field v in Ω, the reciprocity gap has the interpretation*

$$\mathrm{RG}(v) = \int_\Omega s v \, d\Omega + \sum_{i=1}^{N} \int_{\Gamma_i} \nabla v \cdot \boldsymbol{n}_i [u] \, d\Gamma_i. \tag{6}$$

In (5), $[u]$ stands for the jump of the field u across the cracks.

The reciprocity gap method consists of selecting appropriate auxiliary harmonic fields v, each of them giving one piece of scalar information about the sources and the cracks or more precisely on the actual field u on the support of the cracks or of the source distribution. For planar cracks lying in a single plane Π with normal $\boldsymbol{N}_\Pi$, and with $s = 0$, it is easy to see that, using linear fields x^i as auxiliary fields, the normal $\boldsymbol{N}_\Pi$ can be recovered with the explicit formula [Andrieux and Ben Abda 1992]

$$\boldsymbol{N}_\Pi = \frac{\mathrm{RG}(x^i)\boldsymbol{e}_i}{\sqrt{\mathrm{RG}(x^1) + \mathrm{RG}(x^2) + \mathrm{RG}(x^3)}}. \tag{7}$$

The formula involves only a straightforward computation of the linear form RG on three fields and does not require any resolution of a partial differential equation. The identification results mentioned in the introduction have been obtained by such ad hoc choices for the auxiliary fields. For example, the complete identification results for infinite-parameter identification such as geometry of the cracks relies on the choice of a Hilbertian basis of the space of the auxiliary fields fulfilling the operator equation defined by A (here harmonic fields) in the sound domain. The reciprocity gap method is then a constructive method of identification, but a general method for choosing the relevant auxiliary fields is still lacking. That is why we turn to an alternative way of using the reciprocity gap linear form. Before that, a general abstract setting is introduced in the next section.

3. Abstract general form of the reciprocity gap

The abstract form of the reciprocity gap is built from the weak formulation of the physical problem at hand. We suppose thus that we have

- a symmetric bilinear form $a(u, v)$, continuous on a functional (Hilbert) space $W(\Omega \setminus \Gamma)$, where Γ is a collection of (unknown) cracks, and
- two linear and continuous forms, corresponding respectively to natural boundary conditions on the external boundary of Ω and to the interior source distribution,

$$l_f(v) = \int_{\partial\Omega} f v \, dS, \qquad l_s(v) = \int_{\Omega} s v \, d\Omega \tag{8}$$

so that the weak form of the operator equation is

$$u \in W(\Omega \setminus \Gamma), \qquad a(u, v) = l_f(v) + l_s(v) \quad \text{for all } v \in W(\Omega \setminus \Gamma). \tag{9}$$

It is assumed here that external solicitation is exerted on neither the crack lips nor the special constitutive equation linking the jumps of the field u to the dual quantity $\boldsymbol{B}u \cdot \boldsymbol{n}$ so that $\boldsymbol{B}u \cdot \boldsymbol{n} = 0$ on the crack lips. The bilinear form a can vanish on a finite-dimensional subspace R of W (rigid body motions in mechanics, constant fields for stationary heat conduction, etc.):

$$r \in R \implies a(r, v) = 0 \quad \text{for all } v \in W(\Omega \setminus \Gamma). \tag{10}$$

With the bilinear form a coercive on the quotient space of W by R and the operator B on the boundary defined via a Green formula, the solution of (8) is determined up to a field belonging to R and uniqueness is obtained by adding a finite set of linear conditions on the solution. Existence of solutions is ensured provided a compatibility condition on the solicitations f and s is fulfilled:

$$l_f(r) + l_s(r) = 0 \quad \text{for all } r \in R. \tag{11}$$

For the identification problem, let us assume that the domain $\Omega \setminus \Gamma$, the bilinear form a, and the source distribution s are parametrized by a (possibly infinite) set of parameters p belonging to a parameter space P (we have to assume that when the parameters p are in the space P the coerciveness of a is preserved). The identification problem addressed here is then to determine a set (or several sets) of parameters, provided the value of the field u and its dual counterpart $\boldsymbol{B}u \cdot \boldsymbol{n}$, (U^m, F^m), are given on the whole external boundary $\partial\Omega$ of the solid:

$$\text{determine } p \in P \quad \text{such that} \quad \begin{cases} a(u, v; p) = l_{F^m}(v) + l_{s(p)}(v) & \text{for all } v \in W(p), \\ l_{F^m}(r) + l_{s(p)}(r) = 0 & \text{for all } r \in R, \\ u|_{\partial\Omega} = U^m. \end{cases} \tag{12}$$

This formulation encompasses the cases of crack identification (the parameters p are describing the geometry of the cracks), source distribution identification (parameters p describe both geometry of the support and intensity), and inclusion identification (the parameters p describe the geometry of the support and value of the contrast of material properties in the bilinear form a).

In this context, a enjoys a reciprocity property

$$\left.\begin{array}{ll} a(u_1, v) = l_1(v) & \text{for all } v, \\ a(u_2, v) = l_2(v) & \text{for all } v \end{array}\right\} \implies l_1(u_2) = l_2(u_1)$$

and the reciprocity gap is defined by the residual of the reciprocity property:

$$\mathrm{RG}(v) = l_{F^m}(v) - \int_{\partial\Omega} \boldsymbol{B}v \cdot \boldsymbol{n} U^m \, dS. \tag{13}$$

The RG form does not depend on the parameters p as it is assumed that all the heterogeneities or sources buried inside the solid are strictly in the interior of the domain (the case of an emerging crack has been addressed in [Andrieux et al. 1998]). The space of auxiliary fields to be used in the reciprocity gap method is simply $W(\Omega)$, and the fundamental property reads:

Fundamental property. *For every v in $W(\Omega)$ fulfilling the equation*

$$a(v, w) = l_{\boldsymbol{B}v\cdot\boldsymbol{n}}(w) \quad \textit{for all } w \in W(\Omega), \tag{14}$$

the reciprocity gap has the interpretation

$$\mathrm{RG}(v) = \sum_{i=1}^{N} \int_{\Gamma_i} \boldsymbol{B}v \cdot \boldsymbol{n}_i [u] \, d\Gamma_i - l_{s(p)}(v) - \Delta a(u, v; p), \tag{15}$$

where $\Delta a(u, v; p) = a(u, v; p) - a(u, v; 0)$ with the convention that $p = 0$ corresponds to the sound solid.

This property is grounded on the symmetry of the bilinear form a. The interpretation of the RG linear form makes more precise the nature of the information brought by its calculation for every auxiliary field. The identification procedure is the same as in the preceding section: select an appropriate family of auxiliary fields and exploit the set of scalar equations obtained by computing the RG form for each field of the family in order to gain information on the parameters p. As already said, there is not yet a systematic way to choose the family and to exploit the information gained on p.

4. General definition and properties of the reciprocity likelihood

The general idea is to search for a variational usage of the reciprocity gap linear form in order to derive the minimization process as the "systematic" way for the identification procedure. For that purpose, we revisit the exploitation of the reciprocity property by using the reciprocity gap not between the actual and the safe solids but between the actual solid and a solid where some flaws exist, corresponding here to a given set of parameters p, say q. It is clear that, if the q parameters coincide with the "true" values p_0, then the reciprocity property is recovered between boundary data and the values on boundary of the auxiliary fields because they are acting on the same solid. Then the reciprocity gap is the null linear form on the subspace of auxiliary fields satisfying the operator equation in the solid Ω with parameters q. Conversely, if the reciprocity gap is not the null form, then a parameter set leading to a lower value of the norm of RG must be preferred to a parameter set leading to higher values of the norm. The opposite of the square of the norm of RG will be called the reciprocity likelihood, and the proposed identification procedure will consist of maximizing the reciprocity likelihood over the parameter space P (that is, minimizing the norm of RG).

Let us define the vector space V_q of auxiliary fields fulfilling the equilibrium equation in the solid $\Omega(q)$ parametrized by the set of parameters q and furthermore canceling the linear form related to the source distribution $s(q)$, W_0, the space of fields with a null trace on the external boundary:

$$V_q = \{v \in W[\Omega(q)] : a(v, w; q) = 0 \text{ for all } w \in W_0[\Omega(q)],\ l_{s(q)}(v) = 0\}, \tag{16a}$$

$$W_0(q) = \{h \in W[\Omega(q)] : h|_{\partial\Omega} = 0\}. \tag{16b}$$

It is now clear that, for any given parameters q, the reciprocity gap does not vanish on V_q, but the following property establishes that the set of parameters q causing the RG form to vanish on the V_q space is exactly the set of parameters that are likely with respect to the data or measurements at hand:

Optimality of the reciprocity gap. *Let p_0 be a set of parameters such that the reciprocity gap vanishes on V_{p_0}.*

If the source distribution is zero or if the source distribution is nonzero but there exists no field of R except zero that vanishes on the external boundary $\partial\Omega$ of the solid,

$$(r \in R \text{ and } r|_{\partial\Omega} = 0) \implies r = 0, \tag{17}$$

then there exists a field v_0 in $W(p_0)$ satisfying

$$\begin{cases} a(v_0, w; p_0) = l_{s(p_0)}(w) & \text{for all } w \in W_0(p), \\ v_0|_{\partial\Omega} = U^m, \ \boldsymbol{B}v_0 \cdot \boldsymbol{n}|_{\partial\Omega} = F^m. \end{cases} \tag{18}$$

This property means that, as soon as the reciprocity gap vanishes on the space V_{p_0}, there exist a collection of cracks, a source distribution, a set of inclusions (described by p_0), and a field v_0 that fulfills the equilibrium equation and meets exactly the values of the given boundary condition pair (U^m, F^m). The proof of this property is given in Appendix A.

The optimality property of RG allows us to propose a new formulation for the identification problem addressed here: search for the space V_{p_0} where the reciprocity gap linear form is zero. To obtain a variational formulation of this problem, i.e., to define a functional or a function on the space P whose minimization or maximization will furnish candidates for the identified parameters, a natural way is to use the norm of the linear form RG on the linear space V_p. The lower the value of this norm, the greater the likelihood of the parameters p with respect to the data at hand so that we shall define the opposite of the square of the RG norm as the *reciprocity likelihood functional* (as a function of p). The solution procedure will be to maximize it over P. In order to give meaning to the norm of RG on the space V_p, it is necessary to establish the following topological properties:

Topological properties of RG and V_p. *Assume the following for every p in P:*

(i) *$W(p)$ is a Hilbert space with scalar product and norm denoted by*

$$\langle u, v\rangle_{W(p)}, \qquad \|u\|_{W(p)} = \sqrt{\langle u, u\rangle_{W(p)}}.$$

(ii) *The trace operator $\gamma : W(p) \to T_{W(p)}$ with $v \mapsto \gamma v = v|_{\partial\Omega}$ is continuous*: $\|\gamma v\|_{T_{W(p)}} \le c\|v\|_{W(p)}$.

(iii) *The bilinear form a is coercive and continuous on V_p:*

there exist $\alpha(p)$ and $\beta(p)$, $\quad 0 < \alpha < \beta < +\infty,$

$$\textit{such that,}\quad \textit{for all } (u, v) \in V_p^2, \quad \begin{cases} \alpha\|v\|^2_{W(p)} \le a(v, v;\, p), \\ a(v, u;\, p) \le \beta\|v\|_{W(p)}\|u\|_{W(p)}. \end{cases}$$

(iv) *The linear form $l_{s(p)}$ is continuous on V_p:*

$$\textit{there exists } \eta(p) > 0 \quad \textit{such that,} \quad \textit{for all } v \in V_p, \quad |l_{s(p)}(v)| \le \eta\|v\|_{W(p)}.$$

Then the linear subspace V_p of $W(p)$ is closed, and the linear form RG *is continuous on V_p.*

The proof of this property is given in Appendix B. We can now define the *reciprocity likelihood* using the classical definition of the norm of a linear form.

Definition. For p belonging to P, the reciprocity likelihood $\mathrm{RL}(p)$ is the opposite of the square of the norm of the linear form RG on the space V_p:

$$\mathrm{RL}(p) = -\|\mathrm{RG}\|^2_{V_p} \tag{19}$$

with

$$\begin{aligned} V_p &= \{v \in W(p) : a(v, w;\, p) = 0 \text{ for all } w \in W_0(p),\ l_{s(p)}(v) = 0\}, \\ W_0(p) &= \{w \in W(p) : \gamma(p) = 0\}, \\ \mathrm{RG}(v) &= \int_{\partial\Omega} (F^m v - \boldsymbol{B}v \cdot \boldsymbol{n} U^m)\, dS, \\ \|\mathrm{RG}\|_{V_p} &= \sup_{v \in V_p - \{0\}} \frac{\mathrm{RG}(v)}{\|v\|_{W(p)}}. \end{aligned}$$

Thanks to the optimality property of RG, the identification problem (12) is equivalent to the maximization of the reciprocity likelihood, and the optimal value of RL is zero:

$$p = \operatorname*{ArgMax}_{q \in P} \mathrm{RL}(q) \iff p \in P,\ \begin{cases} a(u, v;\, p) = l_{F^m}(v) + l_{s(p)}(v) & \text{for all } v \in W(p), \\ l_{F^m}(r) + l_{s(p)}(r) = 0 & \text{for all } r \in R, \\ u|_{\partial\Omega} = U^m. \end{cases} \tag{20}$$

As it can be seen, the variational formulation of the identification problem is quite general. Some illustrations on the specific identification problems for point source distributions will be given in the sequel. Let us have two preliminary remarks.

Remark 1. The reciprocity likelihood maximization method enables one to determine a solution to the identification problem even if the model used (bilinear form a and linear form $l_{s(p)}$) or the parameter space chosen (geometry of the cracks or inclusions, etc.) are only approximations that are not totally compatible with the experimental data (U^m, F^m). Indeed, the maximization will produce a set of parameters that are the most likely in the actual context even if the maximum will not be zero. Moreover, one can determine nested approximations for the parameters p (with growing reciprocity likelihood) by using nested subsets of the global parameter space:

$$\left.\begin{aligned} &P_1^h \subset P_2^h \subset P, \\ &p_i^h = \mathrm{ArgMax}_{q \in P_i^h} \mathrm{RL}(q) \end{aligned}\right\} \implies \mathrm{RL}(p_1^h) \le \mathrm{RL}(p_2^h) \implies p_2^h \text{ is a better identification than } p_1^h.$$

Remark 2. The practical calculation of the RL functional is generally not achievable because the space V_p is generally non-finite-dimensional so that the norm of the reciprocity gap cannot be computed exactly in most cases. Nevertheless, it is possible to compute an approximation of RL by using a finite-dimensional space with dimension $n_d V_p^{n_d}$ approximating V_p:

$$\mathrm{RL}_{n_d}(p) = -\left[\sup_{v \in V_p^{n_d} - \{0\}} \frac{\mathrm{RG}(v)}{\|v\|_{W(p)}}\right]^2. \tag{21}$$

When the space V_p is approximated by the finite-dimensional space $V_p^{n_d}$, the reciprocity property will not be fulfilled (for the optimal parameter set) for all the auxiliary fields but for only on a subspace of the auxiliary fields space. But again the nested approximation spaces for V_p (with increasing dimension) lead to increasing quality of the identification results.

In current applications, two approximations are then made: one on the parameters space and the other on the linear space of the auxiliary fields, approximated by a finite-dimensional space, so that the RL maximization would generally read

$$p_{n_d}^h = \operatorname*{ArgMax}_{q \in P^h} \mathrm{RL}_{n_d}(q). \tag{22}$$

The following result can prove useful in the applications when computing the norm of RL_{n_d}:

Norm of a linear form on a finite-dimensional vector space. *Let V^n be a vector space of dimension n with scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$, $(\phi^\alpha)_{\alpha=1,\dots,n}$ a linearly independent family of vectors of V^n, and l a linear form on V^n. The norm of l is given by*

$$\|l\|_{V^n} = \sqrt{{}^t l M^{-1} l}, \qquad M^{\alpha\beta} = \langle \phi^\alpha, \phi^\beta \rangle, \qquad l^\alpha = l(\phi^\alpha). \tag{23}$$

The reciprocity likelihood maximization method can then be summarized by:

RLM method. (1) Choose a finite set of independent auxiliary fields $(\phi^\alpha)_{\alpha=1,\dots,n_d}$ satisfying

$$a(\phi^\alpha, w; p) = 0 \quad \text{for all } w \in W_0(p), \qquad l_{s(p)}(\phi^\alpha) = 0,$$

providing a vector basis for the approximated space of auxiliary fields $V_p^{n_d}$.

(2) Maximize the approximated reciprocity likelihood function defined by

$$p^{\mathrm{opt}} = \operatorname*{ArgMin}_{q \in P} M_{\alpha\beta}^{-1}(q)\, \mathrm{RG}(\phi^\alpha(q))\, \mathrm{RG}(\phi^\beta(q)), \qquad M^{\alpha\beta}(q) = \langle \phi^\alpha(q), \phi^\beta(q) \rangle.$$

5. Application to the identification of point source distributions for a conduction equation

This 2D problem has been addressed by [El Badia and Ha Duong 1998; El Badia et al. 2000] with the reciprocity gap functional, by a direct approach, similar to the one described in Section 2. Consider the problem

$$\begin{cases} -\Delta u = \lambda \sum_{i=1}^{i=S} \delta_{X_i} & \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} = F^m & \text{on } \partial\Omega, \\ u = U^m & \text{on } \partial\Omega. \end{cases} \tag{24}$$

The inverse problem consists in determining the location X_i of the sources whose intensity λ is known from the redundant boundary data (U^m, F^m). El Badia and Ha-Duong proved that, if an upper bound of the number of sources S is known, then the redundant boundary data pair determines exactly the source locations. They also give an identification algorithm. The reciprocity gap is given here by (4) as in Section 2.

An alternative algorithm based on the reciprocity likelihood method is the following. First, the intensity λ of the point sources being known, the number of sources is determined by computing the reciprocity gap on the constant (harmonic) field $v = 1$:

$$S = -\frac{1}{\lambda}\,\mathrm{RG}(1). \tag{25}$$

From now on, λ is set at the value $\lambda = 1$. The space V_p is parametrized by the positions of the sources $p = (x_i, y_i)_{i=1,\ldots,S}$

$$\begin{aligned} V_p &= \{v \in W[\Omega(p)] : a(v, w; p) = 0 \text{ for all } w \in W_0[\Omega(p)],\ l_{s(p)}(v) = 0\},\\ W_0(p) &= \{h \in W[\Omega(p)] : h|_{\partial\Omega} = 0\}, \end{aligned}$$

namely here

$$V_q = \left\{ v \in H^1(\Omega) : \int_\Omega \nabla v \nabla w \, d\Omega = 0 \text{ for all } w \in H_0^1(\Omega),\ \sum_{i=1}^{S} v(x_i, y_i) = 0 \right\}. \tag{26}$$

To build a finite-dimensional approximation $V_p^{n_d}$ of this space, we have to choose a linearly independent family of n_d functions $(\phi^\alpha)_{\alpha=1,\ldots,n_d}$ in V_p. For that, it suffices to select a family of n_d independent harmonic fields $(\psi^\alpha)_{\alpha=1,\ldots,n_d}$ and to take as the $(\phi^\alpha)_{\alpha=1,\ldots,n_d}$ family

$$\begin{cases} \phi^\alpha(x, y) = \psi^\alpha(x, y) - \sigma^\alpha(q),\\ \sigma^\alpha(q) = \dfrac{1}{S}\displaystyle\sum_{i=1}^{S} \psi^\alpha(x_i, y_i). \end{cases} \tag{27}$$

Remark that the family $(\psi^\alpha)_{\alpha=1,\ldots,n}$ is independent of the parameters q. To derive the matrix M appearing in the RLM for finite-dimensional V_q spaces (see the RLM method), we just calculate

$$\begin{aligned} M^{\alpha\beta} &= \langle \phi^\alpha, \phi^\beta \rangle\\ &= \int_\Omega \nabla\phi^\alpha \cdot \Delta\phi^\beta + \phi^\alpha\phi^\beta\\ &= \int_\Omega \psi^\alpha\psi^\beta + \frac{1}{2}\int_{\partial\Omega} (\nabla\psi^\alpha \cdot \boldsymbol{n}\psi^\beta + \nabla\psi^\beta \cdot \boldsymbol{n}\psi^\alpha) + \sigma^\alpha\sigma^\beta - \int_\Omega \sigma^\alpha\psi^\beta + \sigma^\beta\psi^\alpha. \end{aligned} \tag{28}$$

Only the last two terms (involving the σ quantities) in the last expression are functions of the parameters p. Once the matrix M is built, the RL function

$$\mathrm{RL}(p) = -M_{\alpha\beta}^{-1}(p)\,\mathrm{RG}(\phi^\alpha(p))\,\mathrm{RG}(\phi^\beta(p)) \tag{29}$$

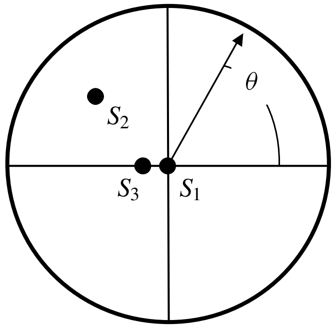

Figure 1. Unit disc with three point sources.

is computed by the following scheme, which requires solving the same linear system (p-dependent) n_d times:

$$X^\alpha(p) = \mathrm{RG}(\phi^\alpha(p)), \qquad M(p)Y(p) = X(p), \qquad \mathrm{RL}(p) = -X(p) \cdot Y(p). \tag{30}$$

The computation of the RL function is then very cost-effective because, as for the reciprocity gap methods, no resolution of any PDE is needed.

As an example, consider the identification of a source distribution in the unit disc as depicted in Figure 1.

Choose the $(\psi^\alpha)_{\alpha=1,\dots,2n_d}$ family in the polar coordinate system centered at the disc center:

$$\begin{aligned}\psi^\alpha(r,\theta) &= \sqrt{\frac{2\pi(\alpha+1)}{1+2\alpha(\alpha+1)}}\, r^\alpha \cos(\alpha\theta), \quad \alpha = 1,\dots,n_d,\\ \psi^\alpha(r,\theta) &= \sqrt{\frac{2\pi(\beta+1)}{1+2\beta(\beta+1)}}\, r^\beta \sin(\beta\theta), \quad \alpha = n_d+1,\dots,2n_d \quad \text{with } \beta = \alpha - n_d.\end{aligned} \tag{31}$$

These functions are obviously harmonic, and because they form a orthonormal family for the $H^1(\Omega)$ scalar product, the M matrix and its inverse turn out to have very simple expressions:

$$M = I_{n_d} + \sigma \otimes \sigma, \qquad M^{-1} = I_{n_d} - \frac{1}{1+\|\sigma\|^2}\sigma \otimes \sigma. \tag{32}$$

It is thus possible to have a closed-form expression for the RL functional

$$-\mathrm{RL}(p) = \sum_{\alpha=1}^{2n} (\mathrm{RG}(\psi^\alpha) + S\sigma^\alpha)^2 - \sum_{\alpha,\beta} \frac{\sigma^\alpha\sigma^\beta}{1+\|\sigma\|^2}(\mathrm{RG}(\psi^\alpha) + S\sigma^\alpha)(\mathrm{ER}(\psi^\beta) + S\sigma^\beta), \tag{33}$$

where only the σ terms depend on the parameters p.

To study the overall performance of the reciprocity likelihood maximization method, three cases are examined with one, two, and six sources. The data (U^m, F^m) on the boundary of the unit disc are provided by the closed-form solution to the Laplace equation with point source distribution

$$u(X) = \frac{1}{2\pi}\sum_{i=1}^{S} \log\|X - X_i\|. \tag{34}$$

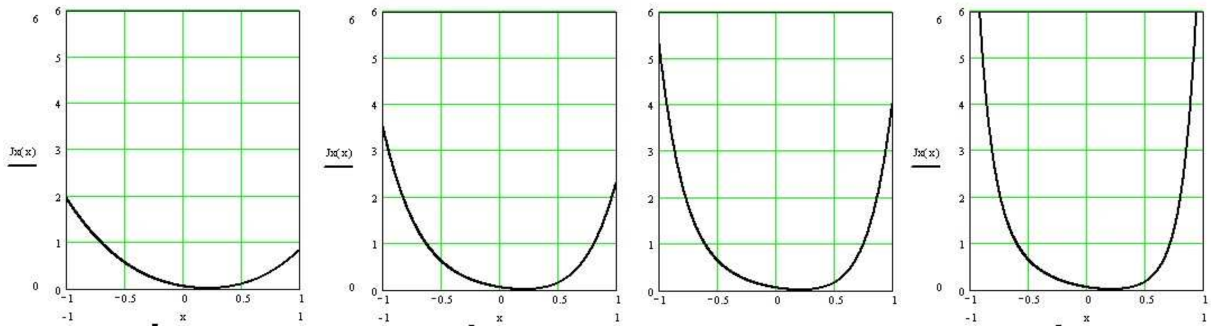

Figure 2. The function $J(x) = -\mathrm{RL}(x)$ for dimensions $n_d = 1, 2, 3, 6$, respectively, of the space $V_p^{n_d}$.

Identification of a single point source. The source is located on the Ox axis with $x = 0.21$. The parameters p reduce to the abscissa x of the source. Figure 2 displays the function $J(x) = -\mathrm{RL}(x)$ when the dimension $2n_d$ of the approximation space of V_p is varied from 2 to 12, that is, when one to six auxiliary functions are used or when the dimension of the matrix M follows the same way (2×2 to 12×12). The symmetry of the solution leads to a symmetry that can be observed on the data on the boundary so that the reciprocity gap vanishes on the half-part of the function set. The effective dimension used for the approximation space for V_p is only n_d and can then take odd values.

The behavior of the $J = -\mathrm{RL}$ function is very good even for a single auxiliary function ($n_d = 1$) where the computation time is negligible, the matrix M being reduced to a single scalar. When the number of functions increases, the computation of the RL function, which remains very quick, leads to sharper minima and a stabilization of the local form of the function around its minimum (maximum of the RL function).

Identification of two point sources. Let us now consider the case of two point sources in the unit disc: $S = 2$, $(x_1, y_1) = (0.21, 0.0)$, and $(x_2, y_2) = (-0.70, 0.0)$, which corresponds to the temperature field plotted in Figure 3. If the ordinates of the sources are known or more simply by exploiting the symmetry of the measurements with respect to the Oy axis, the RL function becomes a function of the abscissae of the sources $\mathrm{RL} = \mathrm{RL}(x_1, x_2)$. Because the identification problems have families of solutions that

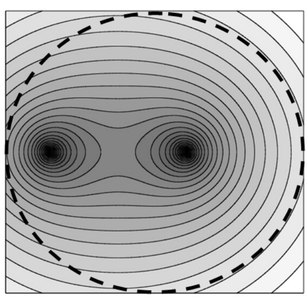

Figure 3. Level lines of the temperature within the unit disc with the two point sources to be identified.

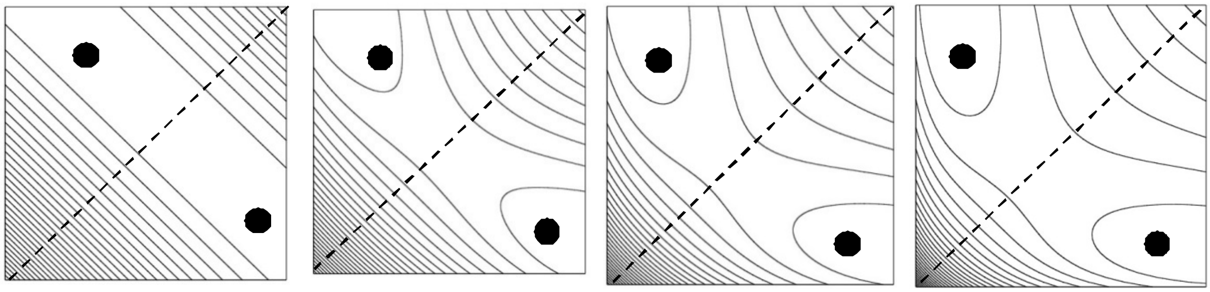

Figure 4. Level lines of the reciprocity likelihood function on the square $(x_1, x_2) = [-0.8, 0.3] \times [-0.8, 0.3]$ for dimensions $n = 2, 4, 8, 12$, respectively, of the space V_p^n (number of ψ functions). The dots correspond to the exact locations of the sources.

are symmetric with respect to the line $x_1 = x_2$ in the parameter space (x_1, x_2), the maximization of the reciprocity likelihood function (or the minimization of its opposite J) has to be conducted in only one half of the square, say $x_1 < x_2$.

One can notice here again an almost convex behavior of the RL function on each triangle except for the first case ($n = 1$), where the use of only one auxiliary field ($\dim V_p^n = 1$) is obviously insufficient to capture the details of the temperature field. Here the information given by the boundary data is evidently underexploited.

In the other cases, the identification via an optimization procedure is quite easy and the addition of new auxiliary fields seems to have no significant effect for $n > 4$, showing then the sharpness of the reciprocity likelihood concept.

***Identification of point sources by direct maximization of* RL.** The general problem of identification of S sources by direct maximization of the reciprocity likelihood is now addressed, that is, the determination of the locations of the sources $q = [X_k]_{k=1,\dots,S} = [(x_k, y_k)]_{k=1,\dots,S}$. As seen in the preceding case, the order of the sources is irrelevant so that there exist several maxima in $[-1, 1]^{2S}$ corresponding to the same family of sources with the same value of the maximum $\mathrm{RL}(p_{\max})$. The selected (converged) family depends on the starting point of the descent algorithm (initial guess). Here the departure locations of the six sources are disposed on a spiral curve in order to have a good initial distribution (Figure 5).

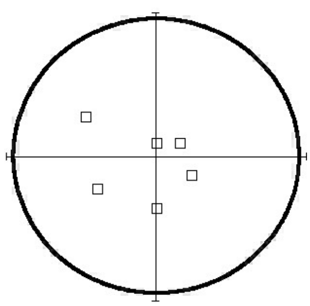

Figure 5. Initial distribution of the six sources to be identified by the RLM.

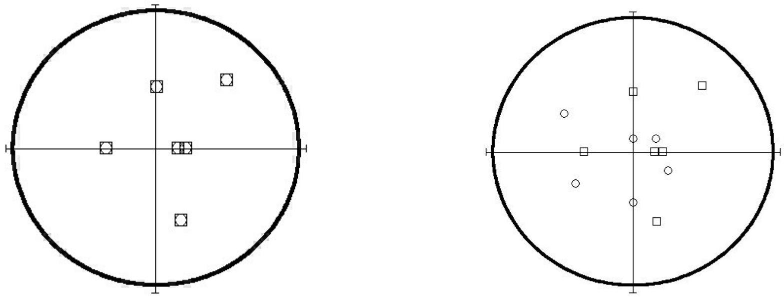

Figure 6. Identified (circles) and real (squares) locations of six sources with $2n_d = 12$, 8 auxiliary functions, respectively.

The gradient of the RL functional with respect to the source locations is straightforward to calculate and is given by

$$\begin{aligned}\nabla\,\mathrm{RL}(q) = &-2\sum_{\alpha=1}^{n}(\mathrm{RL}(\psi^\alpha)+S\sigma^\alpha)\nabla\sigma^\alpha\\ &-2\sum_{\alpha,\beta}\frac{\sigma^\alpha}{1+\|\sigma\|^2}\big[(\mathrm{RL}(\psi^\alpha)+S\sigma^\alpha)(\mathrm{RL}(\psi^\beta)+S\sigma^\beta)+S\sigma^\beta(\mathrm{RL}(\psi^\alpha)+S\sigma^\alpha)\big]\nabla\sigma^\beta\\ &+2\Big[\sum_{\alpha,\beta}\sigma^\alpha\sigma^\beta(\mathrm{RL}(\psi^\alpha)+S\sigma^\alpha)(\mathrm{RL}(\psi^\beta)+S\sigma^\beta)\Big]\frac{\sum_{\alpha=1}^{n}\sigma^\alpha\nabla\sigma^\alpha}{[1+\|\sigma\|^2]^2}\end{aligned}\tag{35}$$

with $\nabla\sigma^\beta = \sum_{i=1}^{m}\nabla\psi^\beta(S_i)$.

The computation of the function RL and its gradients is very cost-effective as it requires only integrals over the boundary of the unit disc of products of the data and the trace of the auxiliary fields. For six sources, the number of parameters to be identified is twelve. Figure 6 (left) displays the converged locations of the sources in the unit disc when the initial guess is the spiral distribution of Figure 5. The dimension of the space $V_p^{n_d}$ is also 12, that is, $n_d = 6$ (results are identical with $n_d = 7$). The locations are perfectly recovered with only one data pair (U^m, F^m). The number of iterations with the line search option of MATLAB [2000] is 123 (194 for $n = 7$) with a tolerance on the gradient of 10^{-6}.

We shall note in particular that the two very close sources situated on the horizontal axis are very well separated. On the other hand, with a lower number of auxiliary functions, eight for example ($n_d = 4$), the maximization stops after 36 iterations because of a too low value of the gradient: the source locations are very badly estimated as seen in Figure 6 (right).

In this case, the RL functional does not contain enough information (or enough probing of the reciprocity property) so that it is too "flat" in a large vicinity of the eventual maxima.

6. Analysis of the effect of noisy data

In this part, the effect of noise on the Cauchy data (u and $\partial_r u$) on the boundary is analyzed for the last example (identification of a six-source distribution). The noise is added to the data with the form

$$f^{\mathrm{noisy}}(\theta) = f(\theta)(1+\epsilon^{\mathrm{noise}}\,\mathrm{rand}\,\theta),\tag{36}$$

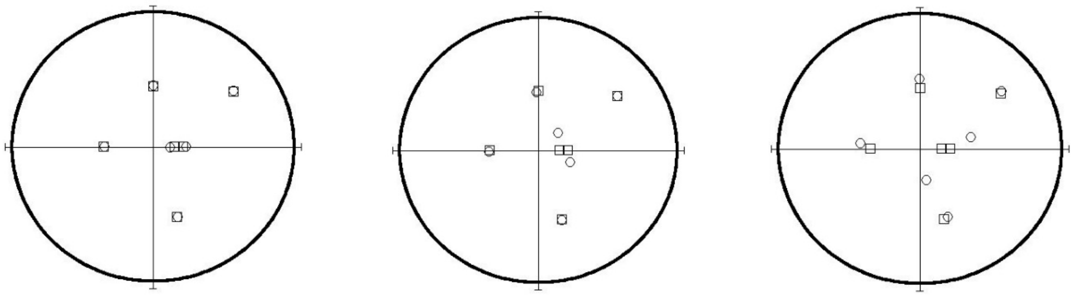

Figure 7. Identified (circles) and real (squares) location of six sources with noisy data (1%, 10%, and 25% noise, respectively).

where rand is the uniform distribution on the segment $[-0.5, 0.5]$ and ϵ^{noise} is the noise level.

One can observe that up to 25% noise the sources are very well recovered except for the two close sources situated on the Ox axis, where the estimation of the location of these sources degrades with the increase of the noise in a symmetric way with regard to the axis Ox. This robustness can be explained by Figure 8. Indeed, the method uses computations of the RG form on auxiliary fields that are very regular and with increasing "space frequency" (with α). It is then clear that the product of the noisy data fields (u and $\partial_r u$) with these auxiliary fields is insensitive to the high-frequency noise for the lowest value of α in (30) and more and more sensitive with growing values of α.

Then the low value of the number of auxiliary fields needed for the identification (as mentioned in Section 5) ensure the good robustness of the identification procedure. For this reason, significantly better

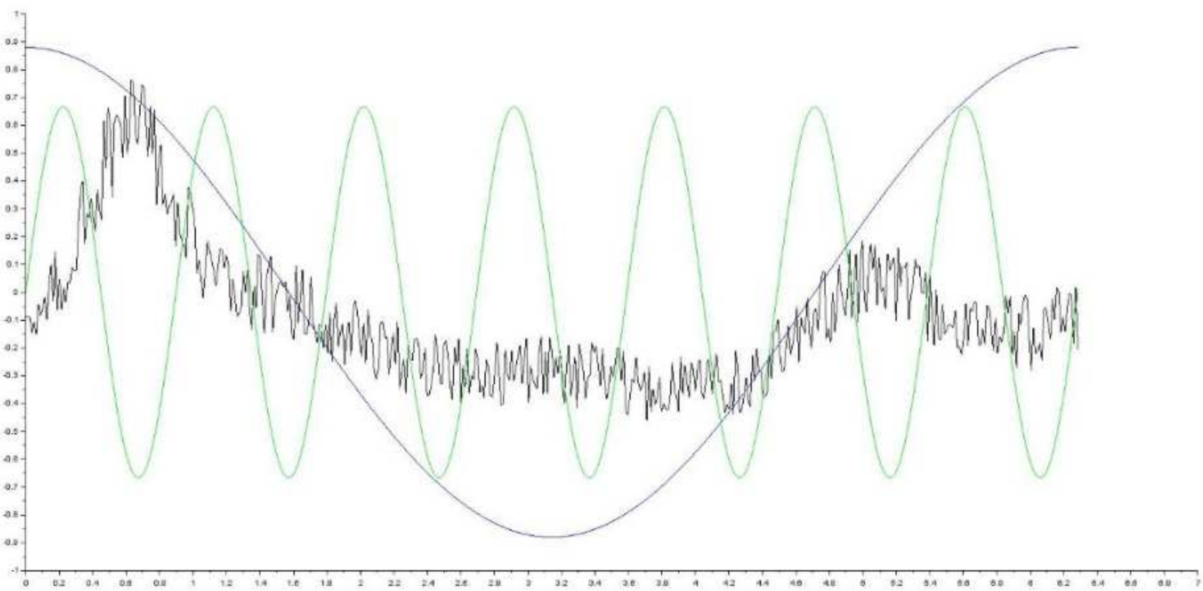

Figure 8. Plot of the noisy data (15% noise) and of two auxiliary fields ($\alpha = 1$ and $\alpha = 5$) on the external boundary.

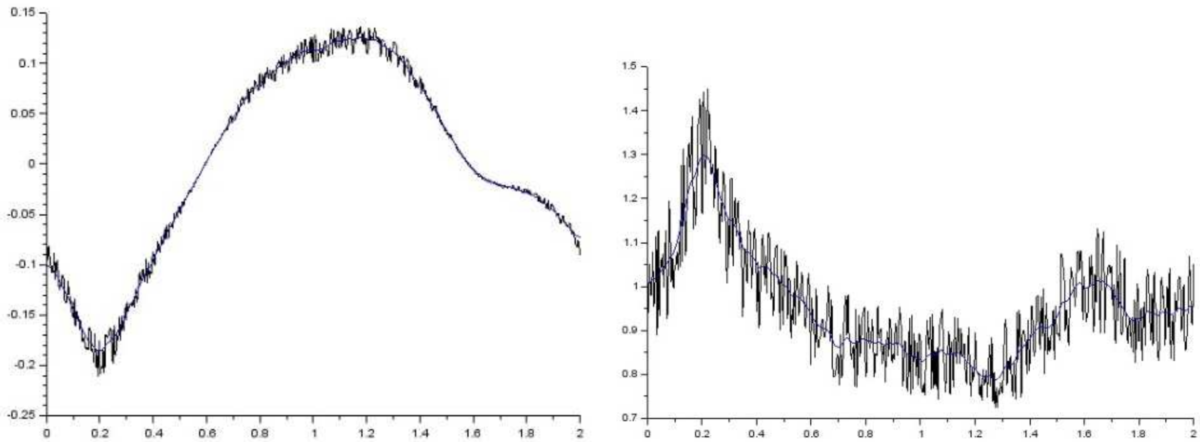

Figure 9. Plots of the noisy data (25% noise) and the regularized data obtained by a Tikhonov regularization. Left: U^m. Right: F^m.

results cannot be expected when using a smoothing of the noisy data with a regularization before processing it with the reciprocity likelihood method. This regularization enables one to compute a smoothed version f^r of a function f over an interval I via the regularized projection

$$f^r = \operatorname*{ArgMin}_g \int_I [f(x) - g(x)]^2 \, dx + \eta S(g), \tag{37}$$

where $S(g)$ is a stabilizing functional (Tikhonov [Tikhonov and Arsenin 1977] or total variation [Rudin et al. 1992]) and η the regularization parameter. Here a Tikhonov regularization must be preferred to the total variation regularization because the real data exhibits no discontinuities or strong gradient zones, and only oscillations due to the noise on each measurement points have to be damped out. An example of Tikhonov regularization is displayed in Figure 9 for a noise level of 25%.

The comparison of the results with and without prior regularization shows that they are indeed quite similar.

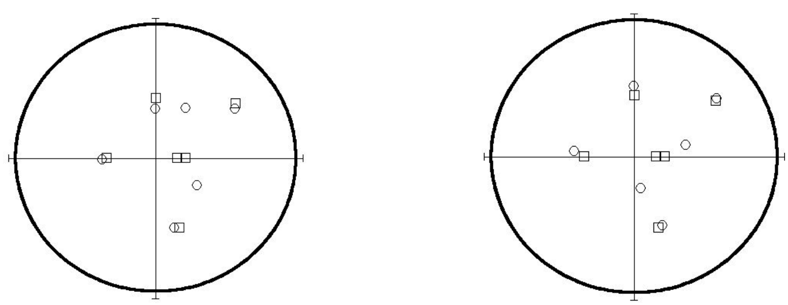

Figure 10. Identified (circles) and real (squares) location of six sources with noisy data. Left: 25% noise with prior Tikhonov regularization. Right: 25% noise without prior regularization.

Nevertheless, the prior regularization would prove useful for the case of more numerous sources because the identification procedure would have to involve more auxiliary fields and then more computations of the RG linear form on fields with higher spatial wave number.

7. Conclusion

We derived a new identification method with the reciprocity gap that produces a systematic way for addressing the identification problem of sources and cracks of inclusion in a solid provided redundant data are available on the whole external boundary. The concept of reciprocity likelihood has been designed and is applicable to any symmetric elliptic operator. The minimization of the reciprocity likelihood function provides a systematic way for using the reciprocity gap concept by avoiding the choice of auxiliary fields and the design of a method for exploiting the information given by the value of the reciprocity gap on it. On the other hand, it is necessary to carefully define the parametrization of objects to be identified. Very good performance and robustness with respect to the noise in the data have been observed for the identification of point source distributions. Applications to crack detection and inclusion identification are in progress.

Appendix A: Proof of the optimality property of the reciprocity gap

Let v_0 be the field in $W(p_0)$ meeting the Dirichlet boundary condition on $\partial\Omega$ and satisfying the equilibrium condition with the source distribution $s(p_0)$. The existence of v_0 is ensured by the Lax–Milgram theorem [Brezis 2011] applied to the linear problem

$$\begin{cases} a(v_0, w; p_0) = l_{s(p_0)}(w) & \text{for all } w \in W_0(p_0), \\ v_0|_{\partial\Omega} = U^m. \end{cases} \tag{A-1}$$

To prove the optimality property of RG, it remains to show that v_0 also meets the Neumann boundary condition: $\boldsymbol{B}v_0 \cdot \boldsymbol{n} = F^m$ on $\partial\Omega$. For that purpose, let us remark that the following reciprocity property holds, thanks to the symmetry of the bilinear form $a(\,\cdot\,,\cdot\,;p_0)$ and to the fact that the fields in the space V_{p_0} are canceling the linear form $l_{s(p_0)}$:

$$\int_{\partial\Omega} \boldsymbol{B}v_0 \cdot \boldsymbol{n} v \, dS = \int_{\partial\Omega} \boldsymbol{B}v \cdot \boldsymbol{n} v_0 \, dS \quad \text{for all } v \in V_{p_0}. \tag{A-2}$$

Indeed,

$$\left.\begin{array}{ll} a(v_0, w; p) = l_{s(p_0)}(w) + l_{\boldsymbol{B}v_0\cdot\boldsymbol{n}}(w) & \text{for all } w \in W(p_0), \\ a(v, w; p) = l_{\boldsymbol{B}v_0\cdot\boldsymbol{n}}(w) & \text{for all } w \in W(p_0) \text{ and } v \in V_{p_0}, \\ l_{s(p_0)}(v) = 0 & \text{for all } v \in V_{p_0} \end{array}\right\} \\ \implies l_{\boldsymbol{B}v_0\cdot\boldsymbol{n}}(v) = l_{\boldsymbol{B}v\cdot\boldsymbol{n}}(v_0) \quad \text{for all } v \in V_{p_0}. \tag{A-3}$$

Because of the condition (A-1), the reciprocity property involves the data U^m:

$$\int_{\partial\Omega} \boldsymbol{B}v_0 \cdot \boldsymbol{n} v \, dS = \int_{\partial\Omega} \boldsymbol{B}v \cdot \boldsymbol{n} U^m \, dS \quad \text{for all } v \in V_{p_0}. \tag{A-4}$$

Using now the assumption that the reciprocity gap vanishes on the space V_{p_0},

$$\int_{\partial\Omega} F^m v\, dS = \int_{\partial\Omega} \boldsymbol{B} v \cdot \boldsymbol{n} U^m\, dS \quad \text{for all } v \in V_{p_0}, \tag{A-5}$$

one obtains by subtracting the two last equalities

$$\int_{\partial\Omega} (F^m - \boldsymbol{B} v_0 \cdot \boldsymbol{n}) v\, dS = 0 \quad \text{for all } v \in V_{p_0}. \tag{A-6}$$

The result $\boldsymbol{B} v_0 \cdot \boldsymbol{n} = F^m$ would follow, by duality arguments, if equality (A-6) can be extended to all the external traces g of fields in $W(p_0)$, namely if

$$\text{for all } g \in T_{W(p_0)} \text{ there exists } v_g \in W(p_0) \quad \text{such that} \quad \begin{cases} a(v_g, w; p) = 0 & \text{for all } w \in W_0(p_0), \\ l_{s(p_0)}(v_g) = 0, \\ v_g|_{\partial\Omega_{\text{ext}}} = g. \end{cases} \tag{A-7}$$

To prove this last property, and for a given g in $T_{W(p_0)}$, let us define the unique field v^g in $W(p_0)$ to be the solution of the problem

$$\begin{cases} a(v^g, w; p_0) = l_{s(p_0)}(w) & \text{for all } w \in W_0(p_0), \\ v^g|_{\partial\Omega} = g. \end{cases} \tag{A-8}$$

Existence and uniqueness of v^g follow again by the Lax–Milgram theorem. Two cases are now possible:

(i) $l_{s(p_0)}(v^g) = 0$; then v^g is in V_{p_0} and we can take $v_g = v^g$.

(ii) $l_{s(p_0)}(v^g) \neq 0$; then define v_g by $v_g = v^g - (l_{s(p_0)}(v^g)/l_{s(p_0)}(v^0))v^0$.

For this last case, it is straightforward to verify that v_g belongs to in V_{p_0} and that its trace is g:

$$l_{s(p_0)}(v_g) = l_{s(p_0)}(v^g) - l_{s(p_0)}(v^g)\frac{l_{s(p_0)}(v^0)}{l_{s(p_0)}(v^0)} = 0, \qquad v_g|_{\partial\Omega} = v^g|_{\partial\Omega} - \frac{l_{s(p_0)}(v^g)}{l_{s(p_0)}(v^0)} v^0|_{\partial\Omega} = g - 0 = g. \tag{A-9}$$

Furthermore, $l_{s(p_0)}(v^0)$ cannot vanish because $l_{s(p_0)}(v^0) = a(v^0, v^0; p_0)$ so that the only possibility would be that v^0 belongs to R. With the assumption that only the null field in R can vanish on the boundary, this leads to a contradiction because the source distribution is not zero.

If there is no source, only case one is relevant.

Appendix B: Proof of the topological properties of V_p and RG

The proof of the closure of V_p contains two steps. First, we build a characterization of this space. Let $T^{*e(p)}_{W(p)}$ be the closed subspace of the dual of $T_{W(p)}$, where the compatibility conditions (10) for the equilibrium are met:

$$T^{*e(p)}_{W(p)} = \{g \in T^*_{W(p)} : l_g(r) + l_{s(p)}(r) = 0 \text{ for all } r \in R\}. \tag{B-1}$$

Define the function L as

$$L : T^{*e(p)}_{W(p)} \to V_p \text{ with } g \mapsto v(g) \quad \text{such that} \quad a(v, w; p) = l_g(w) + l_{s(p)}(w) \quad \text{for all } w \in W(p). \tag{B-2}$$

This function associates with any loading g on $\partial\Omega$ the equilibrium solution in the solid $\Omega(p)$ submitted to the source $s(p)$. L is surjective (just take $g = \boldsymbol{B}v \cdot n$ for v in V_p; g fulfills the compatibility condition because $R \subset W$) and injective by the Lax–Milgram theorem.

The second step is to show that L is continuous, leading then to the conclusion. Indeed, for every convergent sequence v_n in $W(p)$ of vectors of V_p, one can associate to L a convergent sequence g_n in $T^{*e(p)}_{W(p)}$ because strong convergence in $W(p)$ implies the weak convergence and then the convergence of $a(v_n, w)$ for every w. The limit of g_n is necessarily in $T^{*e(p)}_{W(p)}$ as it is a closed space. Finally, the image of this limit of L is in V_p and corresponds to the limit of v_n. So the limits of convergent sequences of V_p are in V_p, which shows that it is closed.

Let us now show that L is continuous. For any g_1 and g_2 in $T^{*e(p)}_{W(p)}$, we have with $v_i = L(g_i)$

$$a(v_1 - v_2, w; p) = l_{g_1 - g_2}(w) \quad \text{for all } w \in W(p). \tag{B-3}$$

Thanks to the coerciveness of a and the continuity of the trace operator γ, one can obtain the inequality

$$\begin{aligned} \alpha \|v_1 - v_2\|^2_{W(p)} \le a(v_1 - v_2, v_1 - v_2; p) &\le \beta \|g_1 - g_2\|_{T*_{W(p)}} \|\gamma v_1 - \gamma v_2\|_{T_{W(p)}} \\ &\le \beta c \|g_1 - g_2\|_{W(p)} \|v_1 - v_2\|_{T_{W(p)}}, \end{aligned} \tag{B-4}$$

showing that L is continuous. The continuity of RG results from the Schwartz inequality

$$|\mathrm{RG}(v)| \le \|F^m\|_{T*_{W(p)}} \|v\|_{T_{W(p)}} + \|\boldsymbol{B}v \cdot \boldsymbol{n}\|_{T*_{W(p)}} \|U^m\|_{T_{W(p)}} \tag{B-5}$$

as well as the continuity of g, a and $l_{s(p)}$ because we have

$$l_{\boldsymbol{B}v\cdot\boldsymbol{n}}(w) = a(v, w; p) - l_{s(p)}(w) \quad \text{for all } w \in W(p). \tag{B-6}$$

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