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DYNAMIC CONSERVATION INTEGRALS AS DISSIPATIVE MECHANISMS IN THE EVOLUTION OF INHOMOGENEITIES

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By the application of Noether's theorem, conservation laws in linear elastodynamics are derived by invariance of the Lagrangian functional under a class of infinitesimal transformations. The recent work of Gupta and Markenscoff (2012) providing a physical meaning to the dynamic J-integral as the variation of the Hamiltonian of the system due to an infinitesimal translation of the inhomogeneity if linear momentum is conserved in the domain, is extended here to the dynamic M- and L-integrals in terms of the "if" conditions. The variation of the Lagrangian is shown to be equal to the negative of the variation of the Hamiltonian under the above transformations for inhomogeneities, which provides a physical meaning to the dynamic J-, L- and M-integrals as dissipative mechanisms in elastodynamics. We prove that if linear momentum is conserved in the domain, then the total energy loss of the system per unit scaling under the infinitesimal scaling transformation of the inhomogeneity is equal to the dynamic M-integral, and if linear and angular momenta are conserved then the total energy loss of the system per unit rotation under the infinitesimal rotational transformation is equal to the dynamic L-integral.

1. Introduction

Conservation laws can be expressed as dissipative mechanisms related to the variation of the energy of the system due to infinitesimal configurational variations in the inhomogeneities. Eshelby [1951] used the energy momentum tensor to define the *force on an elastic singularity* as a variation of the total energy of the body due to the infinitesimal displacement of the defect. Furthermore, he provided additional insights by extending this idea in a series of papers [Eshelby 1956; 1970; 1975] through his ingenious cutting and rewelding thought experiment. Rice [1968] independently discovered the two-dimensional path-independent J-integral for a crack. Günther [1962] and Knowles and Sternberg [1972] derived two additional nontrivial conservation laws (M- and L-integrals) by applying Noether's theorem [Noether 1918] in linear elastostatics. Rice and Drucker [1967] calculated the energy changes during the growth of voids and cracks. Budiansky and Rice [1973] interpreted these new laws as energy release rates associated with the expansion and the rotation rates of a cavity or a crack. Rice [1985] provided further applications of these integrals to the defects.

Fletcher [1976] extended the application of Noether's theorem to derive the conservation laws in linear elastodynamics, and established the completeness of the corresponding conservation laws under a certain group of the infinitesimal transformations. Hermann [1981; 1982] presented a unified formulation to recover the conservations laws by employing different vector calculus operations on the Lagrangian

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density. Eischen and Herrmann [1987] extended this formulation to account for material inhomogeneity temperature gradients, anisotropy, and body forces. Herrmann and Kienzler [1999] represented these balance laws of continuum mechanics by 4×4 tensors.

Markenscoff [2006] expressed the conservation integrals as a variation of the total energy of the system by extending Eshelby's thought experiment to elastodynamics. In elastostatics, Gupta and Markenscoff [2008] showed that the total energy dissipation due to material translation of the inhomogeneity equals the configurational force (J-integral) times the infinitesimal displacement of the inhomogeneity, if and only if equilibrium is preserved in the domain. They extended the proof to elastodynamics [Gupta and Markenscoff 2012], where the variation of the Lagrangian or the Hamiltonian is equal to the dynamic J-integral if and only if the linear momentum is conserved in the domain.

In elastodynamics, Fletcher [1976] proved that the Lagrangian functional was invariant under a certain group of infinitesimal transformations; Kienzler and Herrmann [2000, p. 66] also have a detailed proof for elastostatics, which we extend to elastodynamics. We impose the scaling transformation to derive the M-integral, and for infinitesimal rotational transformation we derive the dynamic L-integral. Furthermore, we also relate the variation of the Lagrangian to the variation of the Hamiltonian for scaling and rotation of the inhomogeneity. This allows us to give an energy dissipative meaning to the above "if" statements and to the dynamic J-, L-, and M-integrals as dissipated energy by mechanisms not considered in elasticity theory [Eshelby 1951, p. 108].

2. Mathematical framework

We briefly present the mathematical framework of the derivation of the conservation integrals from Noether's theorem in linear elastodynamics.

Consider the Lagrangian functional [Gelfand et al. 2000; Fletcher 1976]

$$\Pi^{\mathscr{L}} = \int_R \mathscr{L}(x_\alpha, u_i, u_{i_{x_\alpha}})\, dx_1\, dx_2\, dx_3\, dx_4, \quad i = 1, 2, 3,\ \alpha = 1, 2, 3, 4, \tag{1}$$

where R is the region of integration. In elastodynamics, the independent variables are the material coordinates x_1, x_2, x_3 and x_4 is the time variable, and the dependent variable u_i is the displacement field. For the infinitesimal transformations on the independent and the dependent variables,

$$x^*_\alpha = x_\alpha + \epsilon\phi_\alpha(x_\beta, u_i, u_{i,\beta}) + O(\epsilon^2), \quad i = 1, 2, 3,\ \alpha, \beta = 1, 2, 3, 4, \tag{2a}$$

$$u^*_j = u_j + \epsilon\psi_j(x_\beta, u_i, u_{i,\beta}) + O(\epsilon^2), \quad i, j = 1, 2, 3,\ \beta = 1, 2, 3, 4, \tag{2b}$$

where ϵ is the infinitesimal transformation parameter. The variation of the functional (1) is written as

$$\delta\Pi^{\mathscr{L}} = \int_{R^*} \mathscr{L}(x^*_\alpha, u^*_i, u^*_{i_{x^*_\alpha}})\, dx^*_1\, dx^*_2\, dx^*_3\, dx^*_4 - \int_R \mathscr{L}(x_\alpha, u_i, u_{i_{x_\alpha}})\, dx_1\, dx_2\, dx_3\, dx_4, \tag{3}$$

where R^* is a new region of integration. In view of equations (2a)–(2b), Equation (3) can be further written as [Gelfand et al. 2000, p. 176]

$$\delta\Pi^{\mathscr{L}} = \int_R \left\{ \frac{\partial \mathscr{L}}{\partial u_j} - \frac{\partial}{\partial x_\alpha} \frac{\partial \mathscr{L}}{\partial u_{j,\alpha}} \right\} \overline{\delta u_j}\, dx_1\, dx_2\, dx_3\, dx_4 + \int_R \frac{\partial}{\partial x_\alpha} \left\{ \frac{\partial \mathscr{L}}{\partial u_{j,\alpha}} \overline{\delta u_j} + \mathscr{L}\delta x_\alpha \right\} dx_1\, dx_2\, dx_3\, dx_4, \tag{4}$$

where (see also [Gelfand et al. 2000, Figure 10, p. 171])

$$\delta u_j = u_j^*(x_\alpha^*) - u_j(x_\alpha) = \{u_j^*(x_\alpha^*) - u_j^*(x_\alpha)\} + \{u_j^*(x_\alpha) - u_j(x_\alpha)\} \approx \frac{\partial u_j^*}{\partial x_\alpha}\delta x_\alpha + \overline{\delta u_j} \approx \frac{\partial u_j}{\partial x_\alpha}\delta x_\alpha + \overline{\delta u_j} \tag{5}$$

or

$$\overline{\delta u_j} = \delta u_j - u_{j,\alpha}\delta x_\alpha. \tag{6}$$

Furthermore, in terms of the transformations ϕ_α and ψ_j, (4) becomes

$$\delta \Pi^{\mathscr{L}} = \epsilon \int_R \left\{ \frac{\partial \mathscr{L}}{\partial u_j} - \frac{\partial}{\partial x_\alpha} \frac{\partial \mathscr{L}}{\partial u_{j,\alpha}} \right\} \overline{\psi}_j \, dx_1\, dx_2\, dx_3\, dx_4 + \epsilon \int_R \frac{\partial}{\partial x_\alpha} \left\{ \frac{\partial \mathscr{L}}{\partial u_{j,\alpha}} \overline{\psi}_j + \mathscr{L}\phi_\alpha \right\} dx_1\, dx_2\, dx_3\, dx_4, \tag{7}$$

where, from relation (6),

$$\overline{\psi}_j = \psi_j - u_{j,\alpha}\phi_\alpha. \tag{8}$$

Note that, above and in the sequel, the partial derivatives with respect to x_i and t, for any $A(x_j, u_j, \dot{u}_j, u_{j,k})$, are defined as

$$\frac{\partial(A)}{\partial x_i} = \frac{\partial(A)}{\partial x_i}\bigg|_{\exp} + \frac{\partial(A)}{\partial u_l} u_{l,i} + \frac{\partial(A)}{\partial \dot{u}_l} \dot{u}_{l,i} + \frac{\partial(A)}{\partial u_{l,m}} u_{l,mi} \tag{9a}$$

and

$$\frac{\partial(A)}{\partial t} = \frac{\partial(A)}{\partial t}\bigg|_{\exp} + \frac{\partial(A)}{\partial u_l} \dot{u}_l + \frac{\partial(A)}{\partial \dot{u}_l} \ddot{u}_l + \frac{\partial(A)}{\partial u_{l,m}} \dot{u}_{l,m}. \tag{9b}$$

Under the infinitesimal transformations (2a)–(2b), the functional $\Pi^{\mathscr{L}}$ is said to be invariant at $\boldsymbol{u}$ if

$$\delta \Pi^{\mathscr{L}} = 0. \tag{10}$$

Furthermore, if $\boldsymbol{u}$ satisfies the Euler–Lagrange equations [Gelfand et al. 2000]

$$\frac{\partial \mathscr{L}}{\partial u_j} - \frac{\partial}{\partial x_\alpha} \frac{\partial \mathscr{L}}{\partial u_{j,\alpha}} = 0, \tag{11}$$

then the first term in (7) vanishes, and it yields

$$\int_R \frac{\partial}{\partial x_\alpha} \left\{ \frac{\partial \mathscr{L}}{\partial u_{j,\alpha}} \overline{\psi}_j + \mathscr{L}\phi_\alpha \right\} dx_1\, dx_2\, dx_3\, dx_4 = 0. \tag{12}$$

Let Ω be a region in three-dimensional space occupied by a linearly elastic solid, undergoing small deformations and containing an inhomogeneity which is a surface of discontinuity in the strain and velocity. Let $u_j(x_i, t)$ denote the displacement, ε_{ij} the small strains, C_{ijkl} the components of the elasticity tensor, ρ the density — which in linear elasticity is assumed constant, independent of time — and $(\dot{\ })$ the time derivative, and denote the Cauchy stress by $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$. The Lagrange density is defined as

$$\mathscr{L} = T - W \tag{13}$$

where the strain energy density is

$$W = \tfrac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \tfrac{1}{2} C_{ijkl} u_{i,j} u_{k,l}, \tag{14}$$

and the specific kinetic energy is

$$T = \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i. \tag{15}$$

We write the total Lagrangian functional for $\Omega \subset \mathbb{R}^3$ and $[0, t] \subset \mathbb{R}$, and assume further $\mathscr{L} \in \mathscr{C}^\infty$, so that $\mathscr{L}$ possesses continuous partial derivatives of all orders with respect to the element of its matrix arguments on its domain of definition:

$$\Pi^{\mathscr{L}}(u_{i,j}, \dot{u}_i) = \int_0^t \int_\Omega \mathscr{L}(u_{i,j}, \dot{u}_i)\, dV\, dt = \int_0^t \int_\Omega \{T(\dot{u}_i) - W(u_{i,j})\}\, dV\, dt. \tag{16}$$

For $\mathscr{L} = T - W$, the Euler–Lagrange equations (11) give

$$\frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial(\rho \dot{u}_j)}{\partial t} = 0, \tag{17}$$

which represents the conservation of the linear momentum. If the Euler–Lagrange equations (11) are satisfied, then (12) should be satisfied in order for the Lagrangian functional $\Pi^{\mathscr{L}}$ to be invariant under the transformations (2a)–(2b). This will give the equations to derive the families ϕ_α and ψ_j of infinitesimal transformations.

Equation (12) is expanded in space and time variables as

$$\int_0^t \int_\Omega \left[\frac{\partial}{\partial x_i} \left\{ \frac{\partial \mathscr{L}}{\partial u_{j,i}} \overline{\psi}_j + \mathscr{L}\phi_i \right\} + \frac{\partial}{\partial t} \left\{ \frac{\partial \mathscr{L}}{\partial \dot{u}_j} \overline{\psi}_j + \mathscr{L}\phi_4 \right\} \right] dV\, dt = 0. \tag{18}$$

Using (13), (18) is written

$$\int_0^t \int_\Omega \left[\frac{\partial}{\partial x_i} \{-\sigma_{ij}\overline{\psi}_j + \mathscr{L}\phi_i\} + \frac{\partial}{\partial t} \{\rho \dot{u}_j \overline{\psi}_j + \mathscr{L}\phi_4\} \right] dV\, dt = 0. \tag{19}$$

The above relation applied to infinitesimal transformations given by equations (2a)–(2b) provides the corresponding conservation laws for translation, scaling and rotation of the inhomogeneities (under which the Lagrangian remains invariant), which are additional field equations and are derived in the following section.

3. Family of infinitesimal transformations and dynamic conservation laws

In this section we extend the work of Kienzler and Herrmann [2000, p. 66] to elastodynamics in order to obtain the family of infinite transformations under which the Lagrangian remains invariant. In this section, for the sake of notational simplicity, we define and use

$$\frac{d}{dx_i} \equiv \frac{\partial}{\partial x_i}, \tag{20a}$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t}, \tag{20b}$$

where the partial derivatives with respect to x_i and t are taken as in equations (9a)–(9b), respectively.

Equation (19) is true for any arbitrary volume Ω and any arbitrary time interval, so we can write

$$\frac{d}{dx_i}\{-\sigma_{ij}\overline{\psi}_j + \mathscr{L}\phi_i\} + \frac{d}{dt}\{\rho \dot{u}_j \overline{\psi}_j + \mathscr{L}\phi_4\} = 0. \tag{21}$$

Next, using $\overline{\psi}_j$ from (8), expanded in space and time variables $\overline{\psi}_j = \psi_j - u_{j,l}\phi_l - \dot{u}_j\phi_4$, we rewrite (21) as

$$\frac{d}{dx_i}\{-\sigma_{ij}(\psi_j - u_{j,l}\phi_l - \dot{u}_j\phi_4) + \mathscr{L}\phi_i\} + \frac{d}{dt}\{\rho\dot{u}_j(\psi_j - u_{j,l}\phi_l - \dot{u}_j\phi_4) + \mathscr{L}\phi_4\} = 0, \tag{22}$$

and we employ linear momentum balance to obtain

$$-\sigma_{ij}\frac{d}{dx_i}(\psi_j - u_{j,l}\phi_l - \dot{u}_j\phi_4) + \frac{d}{dx_i}(\mathscr{L}\phi_l)\delta_{il} + \rho\dot{u}_j\frac{d}{dt}(\psi_j - u_{j,l}\phi_l - \dot{u}_j\phi_4) + \frac{d}{dt}(\mathscr{L}\phi_4) = 0. \tag{23}$$

Differentiating explicitly the terms on the left-hand side of (23) with the derivatives

$$\frac{d\mathscr{L}}{dx_i} = \frac{\partial\mathscr{L}}{\partial x_i} + \frac{\partial\mathscr{L}}{\partial u_{k,j}}\frac{\partial u_{k,j}}{\partial x_i} + \frac{\partial\mathscr{L}}{\partial\dot{u}_k}\frac{\partial\dot{u}_k}{\partial x_i} = -\sigma_{jk}u_{k,ji} + \rho\dot{u}_k\dot{u}_{k,i}, \tag{24a}$$

$$\frac{d\mathscr{L}}{dt} = \frac{\partial\mathscr{L}}{\partial t} + \frac{\partial\mathscr{L}}{\partial u_{k,j}}\frac{\partial u_{k,j}}{\partial t} + \frac{\partial\mathscr{L}}{\partial\dot{u}_k}\frac{\partial\dot{u}_k}{\partial t} = -\sigma_{jk}\dot{u}_{k,j} + \rho\dot{u}_k\ddot{u}_k, \tag{24b}$$

$$\frac{d\phi_j}{dx_i} = \frac{\partial\phi_j}{\partial x_i} + \frac{\partial\phi_j}{\partial u_k}u_{k,i}, \qquad \frac{d\phi_4}{dx_i} = \frac{\partial\phi_4}{\partial x_i} + \frac{\partial\phi_4}{\partial u_k}u_{k,i}, \tag{24c}$$

$$\frac{d\phi_j}{dt} = \frac{\partial\phi_j}{\partial t} + \frac{\partial\phi_j}{\partial u_k}\dot{u}_k, \qquad \frac{d\phi_4}{dt} = \frac{\partial\phi_4}{\partial t} + \frac{\partial\phi_4}{\partial u_k}\dot{u}_k, \tag{24d}$$

$$\frac{d\psi_j}{dx_i} = \frac{\partial\psi_j}{\partial x_i} + \frac{\partial\psi_j}{\partial u_k}u_{k,i} \qquad \frac{d\psi_j}{dt} = \frac{\partial\psi_j}{\partial t} + \frac{\partial\psi_j}{\partial u_k}\dot{u}_k. \tag{24e}$$

Therefore, (23) becomes

$$\begin{aligned}
&-\sigma_{ij}\left(\frac{\partial\psi_j}{\partial x_i} + \frac{\partial\psi_j}{\partial u_k}u_{k,i}\right) + \sigma_{ij}u_{j,l}\left(\frac{\partial\phi_l}{\partial x_i} + \frac{\partial\phi_l}{\partial u_k}u_{k,i}\right) + \phi_l\sigma_{ij}u_{j,li} + \sigma_{ij}\dot{u}_j\left(\frac{\partial\phi_4}{\partial x_i} + \frac{\partial\phi_4}{\partial u_k}u_{k,i}\right)\\
&\quad + \phi_4\sigma_{ij}\dot{u}_{j,i} + \mathscr{L}\left(\frac{\partial\phi_l}{\partial x_i} + \frac{\partial\phi_l}{\partial u_k}u_{k,i}\right)\delta_{il} + \phi_l\delta_{il}(-\sigma_{jk}u_{k,ji} + \rho\dot{u}_k\dot{u}_{k,i}) + \rho\dot{u}_j\left(\frac{\partial\psi_j}{\partial t} + \frac{\partial\psi_j}{\partial u_k}\dot{u}_k\right)\\
&\quad - \rho\dot{u}_ju_{j,l}\left(\frac{\partial\phi_l}{\partial t} + \frac{\partial\phi_l}{\partial u_k}\dot{u}_k\right) - \phi_l\rho\dot{u}_j\dot{u}_{j,l} - \rho\dot{u}_j\dot{u}_j\left(\frac{\partial\phi_4}{\partial t} + \frac{\partial\phi_4}{\partial u_k}\dot{u}_k\right) - \phi_4\rho\dot{u}_j\ddot{u}_j\\
&\quad + \mathscr{L}\left(\frac{\partial\phi_4}{\partial t} + \frac{\partial\phi_4}{\partial u_k}\dot{u}_k\right) + \phi_4(-\sigma_{jk}\dot{u}_{k,j} + \rho\dot{u}_k\ddot{u}_k) = 0.
\end{aligned} \tag{25}$$

Rearranging this equation as in [Kienzler and Herrmann 2000, p. 64] leads to

$$0 = \frac{\partial\phi_l}{\partial u_k}[\sigma_{ij}u_{j,l}u_{k,l} - Wu_{k,i}\delta_{il}] \qquad [\sim u_{i,l}^3] \tag{26a}$$

$$+ \frac{\partial\phi_l}{\partial u_k}[Tu_{k,i}\delta_{il} - \rho\dot{u}_j\dot{u}_ku_{j,l}] \qquad [\sim u_{j,l}\dot{u}_k^2] \tag{26b}$$

$$+ \frac{\partial\phi_4}{\partial u_k}[\sigma_{ij}\dot{u}_ju_{k,l} - W\dot{u}_k] \qquad [\sim u_{i,j}^2\dot{u}_k] \tag{26c}$$

$$+ \frac{\partial\phi_4}{\partial u_k}[T\dot{u}_k - \rho\dot{u}_j\dot{u}_j\dot{u}_k] \qquad [\sim \dot{u}_k^3] \tag{26d}$$

$$+ \frac{\partial\psi_j}{\partial u_k}[-u_{k,i}\sigma_{ij}] + \frac{\partial\phi_l}{\partial x_i}[\sigma_{ij}u_{j,l} - W\delta_{il}] + \frac{\partial\phi_4}{\partial t}[-W] \qquad [\sim u_{j,k}^2] \tag{26e}$$

$$+\frac{\partial\psi_j}{\partial u_k}[\rho\dot{u}_j\dot{u}_k]+\frac{\partial\phi_l}{\partial x_i}[T\delta_{il}]+\frac{\partial\phi_4}{\partial t}[-\rho\dot{u}_j\dot{u}_j+T] \qquad [\sim \dot{u}_k^2] \tag{26f}$$

$$+\frac{\partial\phi_l}{\partial t}[-\rho\dot{u}_j u_{j,l}]+\frac{\partial\phi_4}{\partial x_i}[\sigma_{ij}\dot{u}_j] \qquad [\sim u_{j,l}\dot{u}_j] \tag{26g}$$

$$+\frac{\partial\psi_j}{\partial x_i}[\sigma_{ij}] \qquad [\sim u_{i,j}] \tag{26h}$$

$$+\frac{\partial\psi_j}{\partial t}[-\rho\dot{u}_j]. \qquad [\sim \dot{u}_j] \tag{26i}$$

Setting all the coefficients equal to zero leads to the requirement that the functions ϕ_l, ϕ_4 and ψ_j satisfy an overdetermined system of linear differential equations.

From (26a) it follows that ϕ_l must not be a function of u_j. Thus,

$$\phi_l=\phi_l(x_k,t); \tag{27}$$

with this, part (26b) is also satisfied. From (26c) it follows that ϕ_4 must not be a function of u_k. Thus,

$$\phi_4=\phi_4(x_k,t); \tag{28}$$

with this, part (26d) is also satisfied. From (26i) it follows that ψ_j must not be a function of t. Thus,

$$\psi_j=\psi_j(x_k,u_l). \tag{29}$$

Using relations (27)–(29), from (26e) or (26f) it follows that

$$\frac{\partial\psi_j}{\partial u_k}=h_{jk}(x_l), \tag{30}$$

that is,

$$\psi_j=h_{jk}(x_l)u_k+g_j(x_l). \tag{31}$$

From (26h) it follows that the functions $h_{jk}(x_l)$ are actually constants, and, due to the symmetry of the stress tensor, the terms $\partial g_j/\partial x_i$ form a skew-symmetric constant matrix. Thus,

$$\psi_j=\alpha_{jk}u_k+\Omega_k\varepsilon_{kil}x_i+r_j. \tag{32}$$

Because $\partial\psi_j/\partial u_k$ is matrix of constant coefficients, from (26e) or (26f), we further conclude that ϕ_l must not be a function of t as well; thus,

$$\phi_l=\phi_l(x_k); \tag{33}$$

furthermore, ϕ_4 must not be a function of x_i as well; thus,

$$\phi_4=\phi_4(t). \tag{34}$$

With this, (26g) is also satisfied. Therefore, we can write

$$\psi_j=\alpha_{jk}u_k+\Omega_k\varepsilon_{kil}x_i+r_j, \tag{35a}$$

$$\phi_j=\beta_{jk}x_k+a_j, \tag{35b}$$

$$\phi_4=l_0t+t_0. \tag{35c}$$

Now we split the constant matrices α_{ij} and β_{ij} into symmetric and antisymmetric parts and, further, the symmetric parts into spherical and deviatoric parts, as follows:

$$\beta_{ji} = l\delta_{ij} + \beta'_{ji} + m_n\varepsilon_{nij}, \tag{36a}$$

$$\alpha_{jk} = l\gamma\delta_{kj} + \alpha'_{jk} + \omega_n\varepsilon_{nkj}, \tag{36b}$$

with l, γ, m_n, ω_n, β'_{ji}, α'_{jk} being constant parameters or matrices of constant coefficients, satisfying

$$\beta'_{ji} = \beta'_{ij}, \quad \alpha'_{jk} = \alpha'_{kj}, \quad \beta'_{jj} = \alpha'_{jj} = 0. \tag{37}$$

With this, using (26e) and (26f) we obtain

$$(l\gamma\delta_{kj}+\alpha'_{jk}+\omega_n\varepsilon_{nkj})[-u_{k,i}\sigma_{ij}+\rho\dot{u}_j\dot{u}_k]+(l\delta_{il}+\beta'_{li}+m_n\varepsilon_{nil})[\sigma_{ij}u_{j,l}+\mathscr{L}\delta_{il}]+l_0[-\rho\dot{u}_j\dot{u}_j+\mathscr{L}]=0; \tag{38}$$

after rearranging, we can write

$$\begin{aligned} l(-\gamma\delta_{kj}u_{k,i}\sigma_{ij} + \gamma\delta_{kj}\rho\dot{u}_j\dot{u}_k + \delta_{il}\sigma_{ij}u_{j,l} + \delta_{il}\mathscr{L}\delta_{il}) + l_0(-2T + \mathscr{L}) + \omega_n\varepsilon_{nkj}(-u_{k,i}\sigma_{ij} + \rho\dot{u}_j\dot{u}_k) \\ + m_n\varepsilon_{nil}(\sigma_{ij}u_{j,l} + \mathscr{L}\delta_{il}) + \alpha'_{jk}(-u_{k,i}\sigma_{ij} + \rho\dot{u}_j\dot{u}_k) + \beta'_{il}(\sigma_{ij}u_{j,l} + \mathscr{L}\delta_{il}) = 0, \end{aligned} \tag{39}$$

and we further simplify to write

$$l[-\gamma 2W + \gamma 2T + 2W + n(T - W)] + l_0(-2T + \mathscr{L}) \tag{40a}$$

$$+ \varepsilon_{npq}\sigma_{ip}(\omega_n u_{q,i} + m_n u_{i,q}) \tag{40b}$$

$$+ (\beta'_{il}\sigma_{ij}u_{j,l} - \alpha'_{jk}u_{k,i}\sigma_{ij}) + \alpha'_{jk}\rho\dot{u}_j\dot{u}_k = 0, \tag{40c}$$

where $n = \delta_{ii}$ is the number of space dimensions. If $l_0 = l$, then, for the first term (40a) to vanish, we have

$$-2\gamma W + 2\gamma T + 2W + n(T - W) - 2T + T - W = 0 \quad \Longrightarrow \quad \gamma = \tfrac{1}{2}(1 - n), \tag{41}$$

and the second term (40b) vanishes if

$$m_n = \omega_n, \tag{42}$$

provided that the material is isotropic, i.e., $\varepsilon_{npq}\sigma_{ip}[u_{q,i} + u_{i,q}] = 0$ [Eshelby 1975]. The third term (40c) vanishes only if $\alpha'_{jk} = \beta'_{il} = 0$, which means that

$$\beta_{ji} = l\delta_{ij} + \omega_n\varepsilon_{nij}, \quad \alpha_{ji} = \tfrac{1}{2}l(1-n)\delta_{ij} + \omega_n\varepsilon_{nij} \quad \text{and} \quad \phi_4 = lt + t_0. \tag{43}$$

Hence, we state the suitable infinitesimal transformations

$$\phi_j = \omega_n\varepsilon_{nij}x_i + lx_j + a_j, \tag{44a}$$

$$\phi_4 = lt + t_0, \tag{44b}$$

$$\psi_j = \omega_n\varepsilon_{nij}u_i + \tfrac{1}{2}l(1-n)u_j + \Omega_n\varepsilon_{nij}x_i + r_j, \tag{44c}$$

or

$$x^*_j = x_j + \epsilon(\omega_n\varepsilon_{nij}x_i + lx_j + a_j), \tag{45a}$$

$$t^* = t + \epsilon(lt + t_0), \tag{45b}$$

$$u^*_j = u_j + \epsilon\big(\omega_n\varepsilon_{nij}u_i + \tfrac{1}{2}l(1-n)u_j + \Omega_n\varepsilon_{nij}x_i + r_j\big), \tag{45c}$$

where l, t_0 are constant parameters and ω_n, a_j, Ω_n, r_j are vectors with constant components. The vectors r_j and Ω_n describe a rigid-body translation and rotation, respectively, while a_j and ω_n describe material translation (coordinate translation) and material rotation (coordinate rotation), respectively, and the parameter l represents the scaling. The above family of transformations agrees with [Fletcher 1976] in three dimensions ($n = 3$). Applying the transformations indicated by equations (45a)–(45c) for the material translation, scaling and rotation separately to (19), the conservation laws for elastodynamics are derived in the following subsections.

3.1. ***Invariance of the Lagrangian under translation.*** For the infinitesimal translation of the material, we utilize the transformation [Fletcher 1976] such that the new coordinates are $x_i^* = x_i + \epsilon a_i$ and the new time and displacement field remain invariant ($t^* = t$, $u_i^* = u_i$), where ϵa_i is the infinitesimal translation. After comparing the transformation with equations (45) and (44), we have

$$\phi_i = a_i, \quad \phi_4 = 0, \quad \text{and} \quad \psi_j = 0; \tag{46}$$

therefore, from (8),

$$\overline{\psi}_j = -u_{j,k} a_k. \tag{47}$$

Inserting the above transformation in (19) to obtain the conservation law for translation, we obtain

$$\int_0^t \int_\Omega \left[\frac{\partial}{\partial x_i} \{ (\mathcal{L} \delta_{ik} + \sigma_{ij} u_{j,k}) a_k \} - \frac{\partial}{\partial t} \{ \rho \dot{u}_j u_{j,k} a_k \} \right] dV\, dt = 0. \tag{48}$$

The relation is true for any a_k; therefore, we get

$$\int_0^t \int_\Omega \left[\frac{\partial}{\partial x_i} \{ \mathcal{L} \delta_{ik} + \sigma_{ij} u_{j,k} \} - \frac{\partial}{\partial t} \{ \rho \dot{u}_j u_{j,k} \} \right] dV\, dt = 0. \tag{49}$$

Equation (49) holds true for any arbitrary volume Ω and any arbitrary time interval, so we have

$$\frac{\partial}{\partial x_i} \{ \mathcal{L} \delta_{ik} + \sigma_{ij} u_{j,k} \} - \frac{\partial}{\partial t} \{ \rho \dot{u}_j u_{j,k} \} = 0, \tag{50}$$

which is in agreement with [Fletcher 1976, Equation 3.4]. Equation (50) is an additional field equation valid anywhere in the domain of analyticity. Ni and Markenscoff [2009] have used (50) as a field equation to obtain the logarithmic singularity of the near field of an accelerating (generally moving) dislocation rather than by singular asymptotics of the full solution [Callias and Markenscoff 1988].

Analogously to statics, for linear elastodynamics we define the dynamic J-integral as [Bui 1978; Maugin 1993; Markenscoff 2006]

$$J_k^{\text{dyn}} \equiv -\int_\Omega \left[\frac{\partial}{\partial x_i} \{ \mathcal{L} \delta_{ik} + \sigma_{ij} u_{j,k} \} - \frac{\partial}{\partial t} \{ \rho \dot{u}_j u_{j,k} \} \right] dV. \tag{51}$$

The dynamic J-integral would be zero if the region Ω excludes the inhomogeneity, but it would be nonzero if the volume Ω includes it. The above expression for the dynamic J-integral agrees in the static case with [Eshelby 1959; Günther 1962; Rice 1968; Knowles and Sternberg 1972].

3.1.1. *Relation of J_k^{dyn} with the energy release rate.* If Ω is a region of analyticity excluding the inhomogeneity then, from (50), using relation (13), we can write

$$\int_\Omega \Big[\frac{\partial}{\partial x_i}\{(T-W)\delta_{ik}+\sigma_{ij}u_{j,k}\}-\frac{\partial}{\partial t}\{\rho\dot{u}_j u_{j,k}\}\Big]\,dV=0, \tag{52}$$

which, equivalently, is written as

$$\int_\Omega \Big[\frac{\partial}{\partial x_i}\{(W+T)\delta_{ik}-\sigma_{ij}u_{j,k}\}-2\frac{\partial T}{\partial x_k}+\frac{\partial}{\partial t}\{\rho\dot{u}_j u_{j,k}\}\Big]\,dV=0. \tag{53}$$

We may write this in a form similar to [Gupta and Markenscoff 2012, Equation 10], as

$$\int_\Omega \frac{\partial}{\partial x_i}\{(W+T)\delta_{ik}-\sigma_{ij}u_{j,k}\}\,dV+\int_\Omega[\rho\ddot{u}_j u_{j,k}-\rho\dot{u}_i\dot{u}_{i,k}]\,dV=0. \tag{54}$$

By considering the region of analyticity Ω as $\Omega=\Omega_2-\Omega_1$, i.e., as the difference between two regions Ω_2 and Ω_1 (with $\Omega_1\subset\Omega_2$) that include the inhomogeneity, and by using the divergence theorem to convert the first volume integral into a surface integral, we have

$$\int_{S_1+S_2}\{(W+T)n_k-\sigma_{ij}u_{j,k}n_i\}\,dS+\int_{\Omega_2-\Omega_1}[\rho\ddot{u}_j u_{j,k}-\rho\dot{u}_i\dot{u}_{i,k}]\,dV=0, \tag{55}$$

where n_i is the outward unit normal vector to the surface S_1+S_2. It follows that

$$\begin{aligned}&\int_{S_1}\{(W+T)n_k-\sigma_{ij}u_{j,k}n_i\}\,dS+\int_{\Omega_1}[\rho\ddot{u}_j u_{j,k}-\rho\dot{u}_i\dot{u}_{i,k}]\,dV\\ &=\int_{S_2}\{(W+T)n_k-\sigma_{ij}u_{j,k}n_i\}\,dS+\int_{\Omega_2}[\rho\ddot{u}_j u_{j,k}-\rho\dot{u}_i\dot{u}_{i,k}]\,dV=J_k^{\mathrm{dyn}}.\end{aligned} \tag{56}$$

We now consider the volume Ω_1 to shrink to zero as the contour S_1 shrinks onto the moving inhomogeneity and moves with it. As the volume Ω_1 shrinks to zero, in view of the fact that "the elastic field in the immediate vicinity of the moving inhomogeneity at any instant is indistinguishable from the local field of an appropriate steady state moving inhomogeneity, for which $\partial/\partial t=-v\partial/\partial x$" [Freund 1972], the volume integral in the region Ω_1 vanishes, so that (55) yields the expression for J_k^{dyn} as

$$J_k^{\mathrm{dyn}}=\lim_{S_1\to 0}\int_{S_1}\{(W+T)n_k-\sigma_{ij}u_{j,k}n_i\}\,dS, \tag{57}$$

where S_1 is an arbitrary surface surrounding the inhomogeneity, moving with it and shrinking upon it. The above relation of J_k^{dyn} agrees with [Freund 1990, p. 269] and [Markenscoff 2006, Equation 14]. This expression will relate J_k^{dyn} to the energy release rate for the moving inhomogeneity, as treated in Section 5.1 (see (116)).

3.2. ***Invariance of the Lagrangian under scaling.*** For the self-similar expansion of the material, consider the smooth scaling such that the new coordinates and time are $x_i^*=x_i+\epsilon l x_i$ and $t^*=t+\epsilon l t$, respectively, and the new displacement field is $u_i^*=u_i+\frac{1}{2}(1-n)\epsilon l u_i$, where l is the scaling parameter and n is the number of space dimensions. After comparing the transformation with equations (45) and (44), we have

$$\phi_i=lx_i,\quad \phi_4=lt\quad\text{and}\quad \psi_j=\tfrac{1}{2}(1-n)lu_j; \tag{58}$$

therefore, from (8),

$$\overline{\psi}_j = l\big(\tfrac{1}{2}(1-n)u_j - u_{j,k}x_k - t\dot{u}_j\big). \tag{59}$$

Substituting the above transformation in (19) to obtain the conservation law for scaling, we write

$$\int_0^t \int_\Omega \Big[\frac{\partial}{\partial x_i}\{-\sigma_{ij} l\big(\tfrac{1}{2}(1-n)u_j - u_{j,k}x_k - t\dot{u}_j\big) + \mathscr{L} l x_i\} + \frac{\partial}{\partial t}\{\rho \dot{u}_j l\big(\tfrac{1}{2}(1-n)u_j - u_{j,k}x_k - t\dot{u}_j\big) + \mathscr{L} l t\}\Big]\, dV\, dt = 0. \tag{60}$$

The relation is true for any scaling parameter l, therefore we get

$$\int_0^t \int_\Omega \Big[\frac{\partial}{\partial x_i}\{\mathscr{L} x_i + \sigma_{ij}\big(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\big)\} + \frac{\partial}{\partial t}\{t\mathscr{L} - \rho \dot{u}_j\big(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\big)\}\Big]\, dV\, dt = 0. \tag{61}$$

Equation (61) holds true for any arbitrary volume Ω and any arbitrary time interval, so we have

$$\frac{\partial}{\partial x_i}\{\mathscr{L} x_i + \sigma_{ij}\big(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\big)\} + \frac{\partial}{\partial t}\{t\mathscr{L} - \rho \dot{u}_j\big(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\big)\} = 0. \tag{62}$$

Equation (62) is compared to [Fletcher 1976, Equation 3.5] for a three-dimensional case ($n = 3$) and it is an additional field equation valid anywhere in the domain of analyticity.

Analogously to statics, for linear elastodynamics we define the dynamic M-integral as

$$M^{\mathrm{dyn}} \equiv -\int_\Omega \Big[\frac{\partial}{\partial x_i}\{\mathscr{L} x_i + \sigma_{ij}\big(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\big)\} + \frac{\partial}{\partial t}\{t\mathscr{L} - \rho \dot{u}_j\big(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\big)\}\Big]\, dV. \tag{63}$$

The dynamic M-integral would be zero if the region Ω excludes the inhomogeneity, but it would be nonzero if the volume Ω includes the inhomogeneity. The above expression for the dynamic M-integral agrees in the static case with [Günther 1962; Knowles and Sternberg 1972]. After further rearrangements, we may write the M-integral as

$$M^{\mathrm{dyn}} = -\int_\Omega x_\alpha \Big[\frac{\partial}{\partial x_i}\{\mathscr{L}\delta_{i\alpha} + \sigma_{ij} u_{j,\alpha}\} - \frac{\partial}{\partial t}\{\rho \dot{u}_j u_{j,\alpha}\}\Big]\, dV, \tag{64}$$

where the x_i are the material coordinates for $i = 1, 2, 3$, and $x_4 = t$ (time variable).

3.3. *Invariance of the Lagrangian under rotation.* From the family of transformations we have two types of rotation: one is rigid-body rotation (Ω_n) and the other is material rotation (ω_n). By choosing nonzero physical rotation in equations (45) and (44) we obtain the angular momentum balance law, and by choosing nonzero material rotation we obtain the expression for the dynamic L-integral.

3.3.1. *Rigid-body rotation*: $\Omega_n \neq 0$, $\omega_n = 0$. In the case of a rigid-body rotation of the material, consider the smooth transformation in x_i and u_i such that the coordinates and the time variable remain unchanged ($x_i^* = x_i$, $t^* = t$), and the new displacement field is $u_i^* = u_i + \varepsilon_{ilm}\epsilon\Omega_m x_l$, where $\epsilon\Omega_m$ is the infinitesimal physical rotation. After comparing the transformation with equations (45) and (44), we have

$$\phi_i = 0, \quad \phi_4 = 0 \quad \text{and} \quad \psi_j = \varepsilon_{jlm}\Omega_m x_l; \tag{65}$$

therefore, from (8),

$$\overline{\psi}_j = \Omega_m \varepsilon_{jlm} x_l. \tag{66}$$

Inserting the above transformation in (19) to obtain the conservation law for rotation, we obtain

$$\int_0^t \int_\Omega \Big[\frac{\partial}{\partial t}\{\rho \dot{u}_j \Omega_m \varepsilon_{jlm} x_l\} + \frac{\partial}{\partial x_i}\{-\sigma_{ij}\Omega_m \varepsilon_{jlm} x_l\}\Big]\, dV\, dt = 0. \tag{67}$$

The relation is true for any Ω_m; therefore, the expression for the conservation of angular momentum is

$$\varepsilon_{jlm} \int_0^t \int_\Omega \Big[\frac{\partial}{\partial t}(\rho \dot{u}_j x_l) - \frac{\partial}{\partial x_i}(\sigma_{ij} x_l)\Big]\, dV\, dt = 0. \tag{68}$$

The above equation holds true for any arbitrary volume Ω and arbitrary time interval, so we have

$$\frac{\partial}{\partial t}(\varepsilon_{jlm}\rho \dot{u}_j x_l) - \frac{\partial}{\partial x_i}(\varepsilon_{jlm}\sigma_{ij} x_l) = 0, \tag{69}$$

which is the field equation for the angular momentum balance.

3.3.2. *Material or coordinate rotation*: $\Omega_n = 0,\ \omega_n \neq 0$. In case of the material or coordinate rotation of an *isotropic* material, consider the smooth transformation in x_i and u_i such that the new coordinates are $x_i^* = x_i + \varepsilon_{ilm}\epsilon\omega_m x_l$, new time remains unchanged ($t^* = t$), and the new displacement field is $u_i^* = u_i + \varepsilon_{ilm}\epsilon\omega_m u_l$, where $\epsilon\omega_m$ is the infinitesimal material rotation. After comparing the transformation with equations (45) and (44), we have

$$\phi_i = \varepsilon_{ilm}\omega_m x_l, \quad \phi_4 = 0 \quad \text{and} \quad \psi_j = \varepsilon_{jlm}\omega_m u_l; \tag{70}$$

therefore, from (8),

$$\overline{\psi}_j = \omega_m(\varepsilon_{jlm} u_l - \varepsilon_{klm} u_{j,k} x_l). \tag{71}$$

Inserting the above transformation in (19) to obtain the conservation law for rotation, we obtain

$$\int_0^t \int_\Omega \Big[\frac{\partial}{\partial t}\big\{\rho \dot{u}_j \omega_m(\varepsilon_{jlm} u_l - \varepsilon_{klm} u_{j,k} x_l)\big\} + \frac{\partial}{\partial x_i}\big\{-\sigma_{ij}\omega_m(\varepsilon_{jlm} u_l - \varepsilon_{klm} u_{j,k} x_l) + \mathscr{L}\varepsilon_{ilm}\omega_m x_l\big\}\Big]\, dV\, dt = 0. \tag{72}$$

The relation is true for any ω_m; therefore we get

$$\int_0^t \int_\Omega \Big[\frac{\partial}{\partial x_i}\big(\varepsilon_{mlj} u_l \sigma_{ij} + \varepsilon_{mkl} x_l u_{j,k}\sigma_{ij} - \varepsilon_{mli} x_l \mathscr{L}\big) + \frac{\partial}{\partial t}\big(\rho\varepsilon_{mjl} u_l \dot{u}_j + \rho\varepsilon_{mlk} x_l \dot{u}_j u_{j,k}\big)\Big]\, dV\, dt = 0. \tag{73}$$

Equation (73) holds true for any arbitrary volume Ω and any arbitrary time interval, so we have

$$\frac{\partial}{\partial x_i}\big(\varepsilon_{mlj} u_l \sigma_{ij} + \varepsilon_{mkl} x_l u_{j,k}\sigma_{ij} - \varepsilon_{mli} x_l \mathscr{L}\big) + \frac{\partial}{\partial t}\big(\rho\varepsilon_{mjl} u_l \dot{u}_j + \rho\varepsilon_{mlk} x_l \dot{u}_j u_{j,k}\big) = 0. \tag{74}$$

Equation (74) is compared to [Fletcher 1976, Equation 3.6]; however, Fletcher's expression has a negative sign in front of the second term of the first integrand on the left-hand side. In addition to equations (50) and (62), (74) is an additional field equation of elastodynamics valid anywhere in the domain of analyticity.

Analogously to statics, for linear elastodynamics we define the dynamic L-integral as

$$L_m^{\text{dyn}} \equiv -\int_\Omega \Big[\frac{\partial}{\partial x_i}\big(\varepsilon_{mlj}u_l\sigma_{ij} + \varepsilon_{mkl}x_l u_{j,k}\sigma_{ij} - \varepsilon_{mli}x_l\mathscr{L}\big) + \frac{\partial}{\partial t}\big(\rho\varepsilon_{mjl}u_l\dot{u}_j + \rho\varepsilon_{mlk}x_l\dot{u}_j u_{j,k}\big)\Big]\,dV. \tag{75}$$

The dynamic L-integral would be zero if the region Ω excludes the inhomogeneity, but it would be nonzero if the volume Ω includes the inhomogeneity. The above expression for the dynamic L-integral agrees in the static case with [Günther 1962; Knowles and Sternberg 1972]. After further rearrangements, for an isotropic material, we may write the L-integral as

$$L_m^{\text{dyn}} = -\int_\Omega \varepsilon_{mkl}x_l\Big[\frac{\partial}{\partial x_i}\{\mathscr{L}\delta_{ik} + \sigma_{ij}u_{j,k}\} - \frac{\partial}{\partial t}\{\rho\dot{u}_j u_{j,k}\}\Big]\,dV. \tag{76}$$

In the next section, we present these conservation laws as dissipative mechanisms for the corresponding infinitesimal transformations of translation, scaling and rotation of the inhomogeneities.

4. Conservation integrals as dissipative mechanisms

With the objective of relating the conservation integrals J, M and L to the corresponding energy loss of the system, in this section we express the variation of the Lagrangian in terms of balance laws of linear and angular momenta and the "conserved" integrals. Subsequently, the variation of the Lagrangian will be related to the variation of the Hamiltonian, which, in term, will be related to the total energy loss of the system.

Equation (7) is written, after expanding in space and time variables,

$$\begin{aligned}\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega \Big\{\frac{\partial\mathscr{L}}{\partial u_j} - \frac{\partial}{\partial x_i}\frac{\partial\mathscr{L}}{\partial u_{j,i}} - \frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\dot{u}_j}\Big\}\overline{\psi}_j\,dV\,dt \\ + \epsilon\int_0^t\int_\Omega \Big[\frac{\partial}{\partial x_i}\Big\{\frac{\partial\mathscr{L}}{\partial u_{j,i}}\overline{\psi}_j + \mathscr{L}\phi_i\Big\} + \frac{\partial}{\partial t}\Big\{\frac{\partial\mathscr{L}}{\partial\dot{u}_j}\overline{\psi}_j + \mathscr{L}\phi_4\Big\}\Big]\,dV\,dt,\end{aligned} \tag{77}$$

In view of equations (13)–(15), the term $\partial\mathscr{L}/\partial u_j$ vanishes, $\partial\mathscr{L}/\partial u_{j,i} = -\sigma_{ij}$, and $\partial\mathscr{L}/\partial\dot{u}_j = \rho\dot{u}_j$; therefore, (77) can be written as

$$\begin{aligned}\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega \Big\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\Big\}\overline{\psi}_j\,dV\,dt \\ + \epsilon\int_0^t\int_\Omega \Big[\frac{\partial}{\partial x_i}\{-\sigma_{ij}\overline{\psi}_j + \mathscr{L}\phi_i\} + \frac{\partial}{\partial t}\{\rho\dot{u}_j\overline{\psi}_j + \mathscr{L}\phi_4\}\Big]\,dV\,dt.\end{aligned} \tag{78}$$

Next, (78) is applied to the infinitesimal transformations $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ corresponding to translation, scaling and rotation of the inhomogeneities.

4.1. ***Translation of the inhomogeneity.*** For translation of the inhomogeneity, we utilize the transformation [Fletcher 1976] such that the new coordinates are $x_i^* = x_i + \epsilon a_i$ and the new time and displacement field remain invariant ($u_i^* = u_i$), where ϵa_i is the infinitesimal translation of the inhomogeneity. After comparing the transformation with equations (45) and (44), we have

$$\phi_i = a_i, \quad \phi_4 = 0 \quad \text{and} \quad \psi_j = 0; \tag{79}$$

therefore, from (8),

$$\overline{\psi}_j = \psi_j - u_{j,\alpha}\phi_\alpha = -u_{j,k}a_k. \tag{80}$$

Substituting the above transformation in (78) gives [Gupta and Markenscoff 2012]

$$\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}(-u_{j,k}a_k)\,dV\,dt \\ + \epsilon\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\{(\mathscr{L}\delta_{ik}+\sigma_{ij}u_{j,k})a_k\} + \frac{\partial}{\partial t}\{-\rho\dot{u}_j u_{j,k}a_k\}\right]dV\,dt. \tag{81}$$

Taking the translation vector a_k out of the second integral of the right-hand side, we write

$$\delta\Pi^{\mathscr{L}} = -\epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}u_{j,k}a_k\,dV\,dt \\ + \epsilon a_k\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\{\mathscr{L}\delta_{ik}+\sigma_{ij}u_{j,k}\} - \frac{\partial}{\partial t}\{\rho\dot{u}_j u_{j,k}\}\right]dV\,dt. \tag{82}$$

Taking the time derivative of the above equation, we obtain

$$\delta\dot{\Pi}^{\mathscr{L}} = -\epsilon\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}u_{j,k}a_k\,dV + \epsilon a_k\int_\Omega\left[\frac{\partial}{\partial x_i}\{\mathscr{L}\delta_{ik}+\sigma_{ij}u_{j,k}\} - \frac{\partial}{\partial t}\{\rho\dot{u}_j u_{j,k}\}\right]dV. \tag{83}$$

From (51), the integral in the second term of the right-hand side of (83) is $-J_k^{\mathrm{dyn}}$, so we can rewrite (83) as

$$\delta\dot{\Pi}^{\mathscr{L}} = -\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}u_{j,k}a_k\,dV - \epsilon a_k J_k^{\mathrm{dyn}}. \tag{84}$$

In (84), the term in the curly brackets in the integrand is the linear momentum balance expression (Equation (17)), which vanishes by the Euler–Lagrange equations applied to the Lagrangian.

4.2. ***Scaling of the inhomogeneity.*** For the self-similar expansion, consider the smooth scaling such that the new coordinates and time are $x_i^* = x_i + \epsilon l x_i$ and $t^* = t + \epsilon l t$, respectively, and the new displacement field is $u_i^* = u_i + \frac{1}{2}(1-n)\epsilon l u_i$, where l is the scaling parameter and n is the number of space dimensions. After comparing the transformation with equations (45) and (44), we have

$$\phi_i = l x_i, \quad \phi_4 = l t \quad \text{and} \quad \psi_j = \tfrac{1}{2}(1-n) l u_j; \tag{85}$$

therefore, from (8), we have

$$\overline{\psi}_j = \psi_j - u_{j,\alpha}\phi_\alpha = l\left(\tfrac{1}{2}(1-n)u_j - u_{j,k}x_k - t\dot{u}_j\right). \tag{86}$$

Substituting the above transformation in (78) to obtain the variation of the Lagrangian for scaling of the inhomogeneity, we write

$$\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV\,dt + \epsilon\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\left\{-\sigma_{ij}l\left(\tfrac{1}{2}(1-n)u_j - u_{j,k}x_k - t\dot{u}_j\right) + \mathscr{L}l x_i\right\}\right. \\ \left. + \frac{\partial}{\partial t}\left\{\rho\dot{u}_j l\left(\tfrac{1}{2}(1-n)u_j - u_{j,k}x_k - t\dot{u}_j\right) + \mathscr{L}l t\right\}\right]dV\,dt. \tag{87}$$

Taking the scaling parameter l out of the second integral on the right-hand side, we write

$$\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV\,dt + \epsilon l\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}\left\{\mathscr{L}x_i + \sigma_{ij}\left(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\right)\right\}\right.$$
$$\left. + \frac{\partial}{\partial t}\left\{t\mathscr{L} - \rho\dot{u}_j\left(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\right)\right\}\right]dV\,dt. \quad (88)$$

Taking the time derivative of the above equation, we write

$$\delta\dot{\Pi}^{\mathscr{L}} = \epsilon\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV + \epsilon l\int_\Omega\left[\frac{\partial}{\partial x_i}\left\{\mathscr{L}x_i + \sigma_{ij}\left(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\right)\right\}\right.$$
$$\left. + \frac{\partial}{\partial t}\left\{t\mathscr{L} - \rho\dot{u}_j\left(\tfrac{1}{2}(n-1)u_j + u_{j,k}x_k + t\dot{u}_j\right)\right\}\right]dV. \quad (89)$$

From (63), the integral in the second term of the right-hand side of (89) is $-M^{\text{dyn}}$, so we can rewrite (89) as

$$\delta\dot{\Pi}^{\mathscr{L}} = \int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV - \epsilon l M^{\text{dyn}}. \quad (90)$$

In (90), the term in the curly brackets in the integrand is the linear momentum balance expression (Equation (17)), which vanishes by the Euler–Lagrange equations applied to the Lagrangian.

4.3. *Rotation of the inhomogeneity.* Following the *lever arm* $(u_l + x_l)$ described by Eshelby [1956, p. 106], and taking $\Omega_n = \omega_n$ in (45) for the rotation of the inhomogeneity in an *isotropic* material, we consider the smooth transformation in x_i and u_i such that the new coordinates are $x_i^* = x_i + \varepsilon_{ilm}\epsilon\omega_m x_l$, new time remains unchanged ($t^* = t$), and the new displacement field is $u_i^* = u_i + \varepsilon_{ilm}\epsilon\omega_m(u_l + x_l)$, where ω_m is the rotation vector. After comparing the transformation with equations (45) and (44), we have

$$\phi_i = \varepsilon_{ilm}\omega_m x_l, \quad \phi_4 = 0 \quad \text{and} \quad \psi_j = \varepsilon_{jlm}\omega_m(u_l + x_l); \quad (91)$$

therefore, from (8), we have

$$\overline{\psi}_j = \psi_j - u_{j,k}\phi_k = \omega_m\left(\varepsilon_{jlm}(u_l + x_l) - \varepsilon_{klm}u_{j,k}x_l\right). \quad (92)$$

Substituting the above transformation into (78) to obtain the variation of the Lagrangian for rotation of the inhomogeneity, we write

$$\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV\,dt + \epsilon\int_0^t\int_\Omega\left[\frac{\partial}{\partial t}\left\{\rho\dot{u}_j\omega_m(\varepsilon_{jlm}(u_l + x_l) - \varepsilon_{klm}u_{j,k}x_l)\right\}\right.$$
$$\left. + \frac{\partial}{\partial x_i}\left\{-\sigma_{ij}\omega_m(\varepsilon_{jlm}(u_l + x_l) - \varepsilon_{klm}u_{j,k}x_l) + \mathscr{L}\varepsilon_{ilm}\omega_m x_l\right\}\right]dV\,dt, \quad (93)$$

after collecting the angular momentum balance terms, we write

$$\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV\,dt - \epsilon\omega_m\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}(\varepsilon_{jlm}x_l\sigma_{ij}) - \frac{\partial}{\partial t}(\rho\varepsilon_{jlm}x_l\dot{u}_j)\right]dV\,dt$$
$$+\epsilon\omega_m\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}(-\varepsilon_{jlm}u_l\sigma_{ij} + \varepsilon_{klm}x_l u_{j,k}\sigma_{ij} + \varepsilon_{ilm}x_l\mathscr{L}) + \frac{\partial}{\partial t}(\rho\varepsilon_{jlm}u_l\dot{u}_j - \rho\varepsilon_{klm}x_l\dot{u}_j u_{j,k})\right]dV\,dt, \quad (94)$$

after further rearrangements we obtain

$$\delta\Pi^{\mathscr{L}} = \epsilon\int_0^t\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i}-\frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV\,dt-\epsilon\omega_m\int_0^t\int_\Omega\left[\frac{\partial}{\partial t}(\rho\varepsilon_{mlj}x_l\dot{u}_j)-\frac{\partial}{\partial x_i}(\varepsilon_{mlj}x_l\sigma_{ij})\right]dV\,dt$$
$$+\epsilon\omega_m\int_0^t\int_\Omega\left[\frac{\partial}{\partial x_i}(\varepsilon_{mlj}u_l\sigma_{ij}+\varepsilon_{mkl}x_l u_{j,k}\sigma_{ij}-\varepsilon_{mli}x_l\mathscr{L})+\frac{\partial}{\partial t}(\rho\varepsilon_{mjl}u_l\dot{u}_j+\rho\varepsilon_{mlk}x_l\dot{u}_j u_{j,k})\right]dV\,dt. \quad (95)$$

Taking the time derivative of the above equation, we write

$$\delta\dot{\Pi}^{\mathscr{L}} = \epsilon\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i}-\frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV-\epsilon\omega_m\int_\Omega\left[\frac{\partial}{\partial t}(\rho\varepsilon_{mlj}x_l\dot{u}_j)-\frac{\partial}{\partial x_i}(\varepsilon_{mlj}x_l\sigma_{ij})\right]dV$$
$$+\epsilon\omega_m\int_\Omega\left[\frac{\partial}{\partial x_i}(\varepsilon_{mlj}u_l\sigma_{ij}+\varepsilon_{mkl}x_l u_{j,k}\sigma_{ij}-\varepsilon_{mli}x_l\mathscr{L})+\frac{\partial}{\partial t}(\rho\varepsilon_{mjl}u_l\dot{u}_j+\rho\varepsilon_{mlk}x_l\dot{u}_j u_{j,k})\right]dV. \quad (96)$$

From (75), the integral in the third term of the right-hand side of (96) is $-L_m^{\text{dyn}}$, so we can rewrite (96) as

$$\delta\dot{\Pi}^{\mathscr{L}} = \epsilon\int_\Omega\left\{\frac{\partial\sigma_{ij}}{\partial x_i}-\frac{\partial(\rho\dot{u}_j)}{\partial t}\right\}\overline{\psi}_j\,dV-\epsilon\omega_m\int_\Omega\left[\frac{\partial}{\partial t}(\rho\varepsilon_{mlj}x_l\dot{u}_j)-\frac{\partial}{\partial x_i}(\varepsilon_{mlj}x_l\sigma_{ij})\right]dV-\epsilon\omega_m L_m^{\text{dyn}}. \quad (97)$$

In (97), the term in the curly brackets in the first integrand is the linear momentum balance expression (Equation (17)) and the second integrand on the right-hand side is the angular momentum expression (Equation (69)).

It may be noted that we obtain both the expression for the angular moment balance and the dynamic L-integral from the variation of the Lagrangian functional because the rigid-body rotation (Equation (65)) and the material rotation (Equation (70)) are both considered. The transformation of rigid-body rotation (Section 3.3.1) by itself leads to the expression for the angular momentum balance [Fletcher 1976, Equation 3.3], and the transformation of material rotation (Section 3.3.2) leads to the expression for the dynamic L-integral [Fletcher 1976, Equation 3.6]. By using both together we are able to obtain the dissipative statement (97), as further discussed in the following sections.

5. Relation of the variations of the Lagrangian and Hamiltonian under the transformations of translation, scaling and rotation of inhomogeneities

In the previous sections, Noether's theorem was applied to the Lagrangian functional of the system from which the conservation of linear momentum is derived as the Euler–Lagrange equations (11). In this section, we relate the variation of the Lagrangian to the variation of the Hamiltonian under translation, scaling and rotation of the inhomogeneities so that we can explicitly relate the conservation integrals with energy release rates ([Gupta and Markenscoff 2012]; and private communication with Gupta).

The Hamiltonian density is defined as

$$\mathscr{H} = T + W, \quad (98)$$

where the strain energy density is $W=\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}=\frac{1}{2}C_{ijkl}u_{i,j}u_{k,l}$ and the specific kinetic energy is $T=\frac{1}{2}\rho\dot{u}_i\dot{u}_i$. We consider the total Hamiltonian functional for $\Omega\subset\mathbb{R}^3$ and $[0,t]\subset\mathbb{R}$:

$$\Pi^{\mathscr{H}}(u_{i,j},\dot{u}_i)=\int_0^t\int_\Omega\mathscr{H}(u_{i,j},\dot{u}_i)\,dV\,dt=\int_0^t\int_\Omega\{T(\dot{u}_i)+W(u_{i,j})\}\,dV\,dt. \quad (99)$$

The functional (99) represents the total mechanical energy stored in an arbitrary part Ω of the body during the time interval $[0, t]$. Applying (7) to the Hamiltonian (98) and expanding in space and time variables, we write (similarly to (77) for the Lagrangian $\mathscr{L} = T - W$) the variation of the Hamiltonian functional (99) under the infinitesimal transformation (2a)–(2b) as

$$\delta\Pi^{\mathscr{H}} = \epsilon \int_0^t \int_\Omega \left\{ \frac{\partial \mathscr{H}}{\partial u_j} - \frac{\partial}{\partial x_i}\frac{\partial \mathscr{H}}{\partial u_{j,i}} - \frac{\partial}{\partial t}\frac{\partial \mathscr{H}}{\partial \dot{u}_j} \right\} \overline{\psi}_j \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial x_i} \left\{ \frac{\partial \mathscr{H}}{\partial u_{j,i}} \overline{\psi}_j + \mathscr{H}\phi_i \right\} dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial t} \left\{ \frac{\partial \mathscr{H}}{\partial \dot{u}_j} \overline{\psi}_j + \mathscr{H}\phi_4 \right\} dV\, dt. \quad (100)$$

In view of equations (98), (14) and (15), the term $\partial\mathscr{H}/\partial u_j$ vanishes and $\partial\mathscr{H}/\partial u_{j,i} = \sigma_{ij}$, and $\partial\mathscr{H}/\partial \dot{u}_j = \rho \dot{u}_j$; therefore, the above equation can be written as

$$\delta\Pi^{\mathscr{H}} = \epsilon \int_0^t \int_\Omega \left\{ -\frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial(\rho \dot{u}_j)}{\partial t} \right\} \overline{\psi}_j \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial x_i} \{ \sigma_{ij} \overline{\psi}_j + (W+T)\phi_i \} \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial t} \{ \rho \dot{u}_j \overline{\psi}_j + (W+T)\phi_4 \} \, dV\, dt. \quad (101)$$

Note that the first term on the right-hand side of the above equation is not the same as the first term of the variation of the Lagrangian (Equation (78)), which is the linear momentum balance term. Next, we rearrange the terms so as to produce the linear momentum balance expression in the first integrand and make a connection to the variation of the Lagrangian:

$$\delta\Pi^{\mathscr{H}} = \epsilon \int_0^t \int_\Omega \left\{ -\frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial(\rho \dot{u}_j)}{\partial t} \right\} \overline{\psi}_j \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial x_i} \{ \sigma_{ij} \overline{\psi}_j + (W+T)\phi_i \} \, dV\, dt + \epsilon \int_0^t \int_\Omega \left\{ -\overline{\psi}_j \frac{\partial}{\partial t}(\rho \dot{u}_j) + \rho \dot{u}_j \frac{\partial}{\partial t} \overline{\psi}_j + \frac{\partial}{\partial t}[(W+T)\phi_4] \right\} dV\, dt. \quad (102)$$

We further rearrange as to produce terms with $(W - T)$ in the remaining terms on the right-hand side:

$$\delta\Pi^{\mathscr{H}} = \epsilon \int_0^t \int_\Omega \left\{ -\frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial(\rho \dot{u}_j)}{\partial t} \right\} \overline{\psi}_j \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial x_i} \{ \sigma_{ij} \overline{\psi}_j + (W-T)\phi_i \} \, dV\, dt + \epsilon \int_0^t \int_\Omega \left\{ -\overline{\psi}_j \frac{\partial}{\partial t}(\rho \dot{u}_j) - \rho \dot{u}_j \frac{\partial}{\partial t} \overline{\psi}_j + \frac{\partial}{\partial t}[(W-T)\phi_4] \right\} dV\, dt + 2\epsilon \int_0^t \int_\Omega \left\{ \frac{\partial}{\partial x_i}(T\phi_i) + \frac{\partial}{\partial t}(T\phi_4) + \rho \dot{u}_j \frac{\partial}{\partial t} \overline{\psi}_j \right\} dV\, dt. \quad (103)$$

We further rewrite this expression using (13), so that the expression in the variation of the Hamiltonian involves the Lagrangian:

$$\delta\Pi^{\mathscr{H}} = \epsilon \int_0^t \int_\Omega \left\{ -\frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial(\rho \dot{u}_j)}{\partial \partial t} \right\} \overline{\psi}_j \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{\partial}{\partial x_i} \{ \sigma_{ij} \overline{\psi}_j - \mathscr{L}\phi_i \} \, dV\, dt + \epsilon \int_0^t \int_\Omega \frac{d}{dt} \{ -\rho \dot{u}_j \overline{\psi}_j - \mathscr{L}\phi_4 \} \, dV\, dt + 2\epsilon \int_0^t \int_\Omega \left\{ \frac{\partial}{\partial x_i}(T\phi_i) + \frac{\partial}{\partial t}(T\phi_4) + \rho \dot{u}_j \frac{\partial}{\partial t} \overline{\psi}_j \right\} dV\, dt. \quad (104)$$

Using (78), which is the expression for the variation of the Lagrangian, we have

$$\delta\Pi^{\mathscr{H}} = -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \left\{ \frac{\partial}{\partial x_i}(T\phi_i) + \frac{\partial}{\partial t}(T\phi_4) + \rho\dot{u}_j \frac{\partial}{\partial t}\overline{\psi}_j \right\} dV\, dt, \tag{105}$$

which can be written, using (8), as

$$\delta\Pi^{\mathscr{H}} = -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \left\{ \frac{\partial}{\partial x_i}(T\phi_i) + \frac{\partial}{\partial t}(T\phi_4) + \rho\dot{u}_j \frac{\partial}{\partial t}(\psi_j - u_{j,i}\phi_i - \dot{u}_j\phi_4) \right\} dV\, dt, \tag{106}$$

Now we employ the relation (106) of the variations of the Lagrangian and Hamiltonian to the corresponding infinitesimal transformations of translation, rotation, and scaling of the inhomogeneity.

5.1. *Translation of the inhomogeneity.* In this case we use the transformation such that $\phi_i = a_i$, $\phi_4 = 0$ and $\psi_j = 0$, i.e., translation of the inhomogeneity. Inserting it in (106) gives

$$\begin{aligned} \delta\Pi^{\mathscr{H}} &= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \left\{ \frac{\partial}{\partial x_i}(Ta_i) + \rho\dot{u}_j \frac{\partial}{\partial t}(-u_{j,i}a_i) \right\} dV\, dt \\ &= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \{\rho\dot{u}_k\dot{u}_{k,i}a_i - \rho\dot{u}_j\dot{u}_{j,i}a_i\}\, dV\, dt \\ &= -\delta\Pi^{\mathscr{L}}. \end{aligned} \tag{107}$$

Thus, under an infinitesimal translation of the inhomogeneity, the variation of the Lagrangian is equal to the negative variation of the Hamiltonian, which was already shown by Gupta and Markenscoff [2012]. Taking the time derivative of (107), we can write

$$\delta\dot{\Pi}^{\mathscr{H}} = -\delta\dot{\Pi}^{\mathscr{L}}, \tag{108}$$

which, using (84), can be written as

$$\delta\dot{\Pi}^{\mathscr{H}} = -\delta\dot{\Pi}^{\mathscr{L}} = \epsilon \int_\Omega \left\{ \frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t} \right\} u_{j,k}a_k\, dV + \epsilon a_k J_k^{\text{dyn}}, \tag{109}$$

where J_k^{dyn} is defined by (51). Considering the definition of the Hamiltonian $\Pi^{\mathscr{H}}(u_{i,j}, \dot{u}_i)$ according to (99), we define $\delta\mathscr{E}^{\text{tot}}$ as

$$\delta\mathscr{E}^{\text{tot}} \equiv \delta\dot{\Pi}^{\mathscr{H}}, \tag{110}$$

where $\delta\mathscr{E}^{\text{tot}}$ is the change of the total energy in the volume Ω under the infinitesimal transformations of (45), evaluated at time t. The external forces are assumed to be absent. Now, from equations (110) and (109) we can write

$$\delta\mathscr{E}^{\text{tot}} = \epsilon \int_\Omega \left\{ \frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t} \right\} u_{j,k}a_k\, dV + \epsilon a_k J_k^{\text{dyn}}. \tag{111}$$

In (111), the term in the curly brackets in the integrand is the linear momentum balance expression (Equation (17)), which will vanish due to conservation of linear momentum. So, if linear momentum is conserved in the whole domain, then (111) can be written as

$$\delta\mathscr{E}^{\text{tot}} = \epsilon a_k J_k^{\text{dyn}}. \tag{112}$$

Moreover, as shown in [Gupta and Markenscoff 2012], if $\delta\mathscr{E}^{\text{tot}} = \epsilon a_k J_k^{\text{dyn}}$ then the first term on the right-hand side of (111) will vanish, and, if $u_{j,k}$ is invertible, then the term in the curly brackets (linear momentum balance expression) will vanish since the integral is valid for any arbitrary volume Ω. Therefore, (111) can be stated as the proposition that, under an infinitesimal translation of the inhomogeneity (transformation (79)), the change of the total energy of the system per unit infinitesimal translation of the inhomogeneity is equal to the dynamic J-integral *if and only if* linear momentum is conserved in the whole domain [Gupta and Markenscoff 2012], provided that $u_{j,k}$ is invertible.

If the inhomogeneity is moving with the velocity $\epsilon\dot{a}_k \equiv v_k$, then we can write the rate of the total energy change $\delta\dot{\mathscr{E}}^{\text{tot}}$ as

$$\delta\dot{\mathscr{E}}^{\text{tot}} = v_k J_k^{\text{dyn}}. \tag{113}$$

The above equation agrees in the static case with [Budiansky and Rice 1973; Lubarda and Markenscoff 2007]. With the expression for J_k^{dyn} given in (57), Equation (113) yields

$$\delta\dot{\mathscr{E}}^{\text{tot}} = \lim_{S_d\to 0}\int_{S_d}\{(W+T)n_k v_k - \sigma_{ij}u_{j,k}n_i v_k\}\,dS, \tag{114}$$

where S_d is an arbitrary surface surrounds the inhomogeneity, moving with it and shrinking on it, and the n_k are the components of the unit outward normal $\boldsymbol{n}$ to the surface S_d. Furthermore, near the core of the moving inhomogeneity, leading-order terms of the fields satisfy the relation $\partial/\partial t = -v_k\partial/\partial x_k$ [Freund 1972], so we can write $u_{j,k}v_k = -\dot{u}_j$ in (114) to obtain

$$\delta\dot{\mathscr{E}}^{\text{tot}} = \lim_{S_d\to 0}\int_{S_d}\{(W+T)v_n + \sigma_{ij}\dot{u}_j n_i\}\,dS, \tag{115}$$

where v_n is the component of the velocity of the inhomogeneity in the direction of the outward normal $\boldsymbol{n}$ to the surface S_d. In agreement with the expression for the energy release rate into the core of the moving inhomogeneity as given by Eshelby [1970, Equation 78] we define the energy release rate $\mathscr{G}$ by

$$v\mathscr{G} \equiv \delta\dot{\mathscr{E}}^{\text{tot}} = \lim_{S_d\to 0}\int_{S_d}\{(W+T)v_n + \sigma_{ij}\dot{u}_j n_i\}\,dS, \tag{116}$$

which represents rate of energy loss of the system flowing into the inhomogeneity under translation. Equation (116) is in agreement with the energy release for a moving crack by [Atkinson and Eshelby 1968, Equation 9; Freund 1972, Equation 13; Freund 1990, p. 262], for dislocations [Clifton and Markenscoff 1981] and moving phase boundaries [Markenscoff and Ni 2010; Ni and Markenscoff 2015]. As proven in [Freund 1972], the above expression is path-independent for a crack, and will also be now for an inhomogeneity, since it is a weaker singularity.

5.2. ***Scaling of the inhomogeneity.*** In this case we use the transformation such that $\phi_i = lx_i$, $\phi_4 = lt$ and $\psi_j = \frac{1}{2}(1-n)lu_j$, i.e., scaling of the inhomogeneity. Inserting it in (106) gives

$$\begin{aligned}
\delta\Pi^{\mathscr{H}} &= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \Big\{ \frac{\partial}{\partial x_i}(Tlx_i) + \frac{\partial}{\partial t}(Tlt) + \rho\dot{u}_j \frac{\partial}{\partial t}\big(\tfrac{1}{2}(1-n)lu_j - u_{j,i}lx_i - \dot{u}_j lt\big)\Big\}\, dV\, dt \\
&= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \big\{ lx_i\rho\dot{u}_k\dot{u}_{k,i} + nlT + lt\rho\dot{u}_k\ddot{u}_k + Tl + \tfrac{1}{2}(1-n)l\rho\dot{u}_j\dot{u}_j - lx_i\rho\dot{u}_j\dot{u}_{j,i} \\
&\qquad\qquad - lt\rho\dot{u}_j\ddot{u}_j - l\rho\dot{u}_j\dot{u}_j\big\}\, dV\, dt \\
&= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \big\{ nTl + Tl + (1-n)Tl - 2Tl \big\}\, dV\, dt \\
&= -\delta\Pi^{\mathscr{L}},
\end{aligned} \tag{117}$$

where n is equal to number of spatial dimensions.

Thus, under an infinitesimal scaling of the inhomogeneity, the variation of the Lagrangian is equal to the negative variation of the Hamiltonian. Taking the time derivative of (117), we can write

$$\delta\dot{\Pi}^{\mathscr{H}} = -\delta\dot{\Pi}^{\mathscr{L}}, \tag{118}$$

which, using (90), can be written as

$$\delta\dot{\Pi}^{\mathscr{H}} = -\delta\dot{\Pi}^{\mathscr{L}} = -\epsilon \int_\Omega \left\{ \frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t} \right\} \overline{\psi}_j\, dV + \epsilon l M^{\mathrm{dyn}}, \tag{119}$$

where M^{dyn} is defined by (63). Now, from (99) and (119), we can define

$$\delta\mathscr{E}^{\mathrm{tot}} \equiv \delta\dot{\Pi}^{\mathscr{H}} = -\epsilon \int_\Omega \left\{ \frac{\partial\sigma_{ij}}{\partial x_i} - \frac{\partial(\rho\dot{u}_j)}{\partial t} \right\} \overline{\psi}_j\, dV + \epsilon l M^{\mathrm{dyn}}, \tag{120}$$

where $\delta\mathscr{E}^{\mathrm{tot}}$ is the change of the total energy in the volume Ω due to the scaling of the inhomogeneity evaluated at time t. In (120), the term in the curly brackets in the integrand is the linear momentum balance expression (Equation (17)). Therefore, (120) can be stated as the proposition that if linear momentum is conserved in the whole domain, then the change of the total energy of the system per unit infinitesimal scaling ϵl, under the scaling transformation (85), is equal to the dynamic M-integral.

5.3. ***Rotation of the inhomogeneity.*** In this case we use the transformation such that $\phi_i = \varepsilon_{ilm}\omega_m x_l$, $\phi_4 = 0$ and $\psi_j = \varepsilon_{jlm}\omega_m(u_l + x_l)$, i.e., rotation of the inhomogeneity. Inserting it in (106) gives

$$\begin{aligned}
\delta\Pi^{\mathscr{H}} &= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \Big\{ \frac{\partial}{\partial x_i}(T\varepsilon_{ilm}\omega_m x_l) + \rho\dot{u}_j \frac{\partial}{\partial t}(\varepsilon_{jlm}\omega_m(u_l + x_l) - u_{j,i}\varepsilon_{ilm}\omega_m x_l)\Big\}\, dV\, dt \\
&= -\delta\Pi^{\mathscr{L}} + 2\epsilon \int_0^t \int_\Omega \big\{ \varepsilon_{ilm}\omega_m x_l\rho\dot{u}_k\dot{u}_{k,i} + T\varepsilon_{ilm}\omega_m\delta_{il} \\
&\qquad\qquad + \rho\dot{u}_j\varepsilon_{jlm}\omega_m\dot{u}_l - \rho\dot{u}_j\dot{u}_{j,i}\varepsilon_{ilm}\omega_m x_l\big\}\, dV\, dt.
\end{aligned} \tag{121}$$

The first term of the integrand on the right-hand side cancels with the fourth term, the second term is zero because δ_{il} is symmetric in i and l but ε_{ilm} is skew-symmetric in i and l, and similarly the third term is also zero because $\dot{u}_j \dot{u}_l$ is symmetric in j and l but ε_{jlm} is skew-symmetric in j and l. Hence, we obtain

$$\delta \Pi^{\mathcal{H}} = -\delta \Pi^{\mathcal{L}}. \tag{122}$$

Thus, under an infinitesimal rotation of the inhomogeneity, for an isotropic material the variation of the Lagrangian is equal to the negative variation of the Hamiltonian. Taking the time derivative of (122), we can write

$$\delta \dot{\Pi}^{\mathcal{H}} = -\delta \dot{\Pi}^{\mathcal{L}}, \tag{123}$$

which, using (97), can be written as

$$\delta \dot{\Pi}^{\mathcal{H}} = -\delta \dot{\Pi}^{\mathcal{L}} = -\epsilon \int_{\Omega} \left\{ \frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial (\rho \dot{u}_j)}{\partial t} \right\} \overline{\psi}_j \, dV \\ + \epsilon \omega_m \int_{\Omega} \left[\frac{\partial}{\partial t} (\rho \varepsilon_{mlj} x_l \dot{u}_j) - \frac{\partial}{\partial x_i} (\varepsilon_{mlj} x_l \sigma_{ij}) \right] dV + \epsilon \omega_m L_m^{\text{dyn}}, \tag{124}$$

where L_m^{dyn} is defined by (75). Now, from (99) and (124), we can define

$$\delta \mathscr{E}^{\text{tot}} \equiv \delta \dot{\Pi}^{\mathcal{H}} = -\epsilon \int_{\Omega} \left\{ \frac{\partial \sigma_{ij}}{\partial x_i} - \frac{\partial (\rho \dot{u}_j)}{\partial t} \right\} \overline{\psi}_j \, dV \\ + \epsilon \omega_m \int_{\Omega} \left[\frac{\partial}{\partial t} (\rho \varepsilon_{mlj} x_l \dot{u}_j) - \frac{\partial}{\partial x_i} (\varepsilon_{mlj} x_l \sigma_{ij}) \right] dV + \epsilon \omega_m L_m^{\text{dyn}}, \tag{125}$$

where $\delta \mathscr{E}^{\text{tot}}$ is the change of the total energy in the volume Ω due to the rotation of the inhomogeneity evaluated at time t. In (125), the term in the curly brackets in the first integrand is the linear momentum balance expression (Equation (17)) and the second integrand on the right hand side is the angular momentum expression (Equation (69)). Therefore, (125) can be stated as the proposition that, for an isotropic material, if linear and angular momenta are conserved in the whole domain, then the change of the total energy of the system per unit infinitesimal rotation $\epsilon \omega_m$, under the rotation transformation (91) with "*lever arm* $x_i + u_i$" is equal to the dynamic L-integral.

6. Dissipative propositions

6.1. *Translation of the inhomogeneity.* From relation (111), we state the following proposition:

Proposition 1 [Gupta and Markenscoff 2012]. *Under the translation transformation of Equation (79), the total energy loss of the system per unit infinitesimal translation is equal to the dynamic J-integral if and only if linear momentum is conserved in the domain, provided that $u_{i,j}$ is invertible.*

This proposition extends to elastodynamics the earlier proposition for the static J-integral [Gupta and Markenscoff 2008].

6.2. *Scaling of the inhomogeneity.* From relation (120), we state the following proposition:

Proposition 2. *If linear momentum is conserved in the domain, under the scaling transformation of Equation (85) the total energy loss of the system per unit infinitesimal scaling parameter is equal to the dynamic M-integral.*

This proposition is immediately extended to elastostatics for the static M-integral.

6.3. ***Rotation of the inhomogeneity.*** From relation (125), we state the following proposition:

Proposition 3. *If linear and angular momenta are conserved in the domain, for an isotropic material under the rotation transformation of Equation (91) the total energy loss of the system per unit infinitesimal rotation is equal to the dynamic L-integral.*

This proposition is immediately extended to elastostatics for the static L-integral.

These propositions express the fact that, when analyticity is lost due to the inhomogeneity (inhomogeneities create discontinuities in the stress), the classical energy conservation of elasticity theory is not valid any longer. Extending his famous result (force on an elastic singularity) to the other transformations, we quote here Eshelby [1951, p. 108]: "When all sources of internal stress and inhomogeneity within Σ are given a small displacement $\delta\xi_l$, the energy $F_l\delta\xi_l$ is available for conversion into kinetic energy or dissipation by some process not considered in the elastic theory."

7. Conclusions

By applying Noether's theorem, we derived the group of infinitesimal transformations of translation, scaling and rotation in elastodynamics under which the Lagrangian functional remains invariant and obtained the corresponding conservation laws. For inhomogeneities, we demonstrated that, under these transformations, the variation of the Lagrangian is equal to the negative of the variation of the Hamiltonian, and this provide the relations between the conservation integrals and the total energy loss of the system due to these transformations. This leads to the propositions that, under scaling of the inhomogeneity, if linear momentum is conserved in the domain, then the total energy loss of the system per unit infinitesimal scaling is equal to the dynamic M-integral, and under rotation, if linear and angular momenta are conserved in the domain, then the total energy loss of the system per unit infinitesimal rotation is equal to the dynamic L-integral. Thus, the propositions are physically interpreted as dissipative mechanisms for the loss of the Hamiltonian energy due to translation, scaling or rotation of the inhomogeneity; these propositions extend the static counterparts [Budiansky and Rice 1973] to elastodynamics.

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