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INTEGRAL EQUATIONS FOR 2D AND 3D PROBLEMS OF THE SLIDING INTERFACE CRACK BETWEEN ELASTIC AND RIGID BODIES

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This paper revisits the sliding interface crack problem between elastic and rigid half-planes studied by Bui and Oueslati and provides an alternative method of derivation of the solution, which will then be extended to three-dimensional (3D) crack problems. Based upon the displacement continuation technique of complex potentials, an appropriate Green function for the isolated edge dislocation dipole at the interface is given. Then by considering the sliding condition along the interface crack, the field equations can be obtained for the two-dimensional (2D) problem. Furthermore, it is shown that the edge dislocation dipole in 2D appears to be a particular form of the fundamental Kupradze–Basheleishvili tensor in 3D, which provides a method for deriving the coupled nonlinear integral equations for the same frictional interface plane crack of an arbitrary shape. The present work describes how the 3D sliding interface crack is related to the same problem in 2D.

1. Introduction

Cracks lying along the bonds between dissimilar materials or between elastic and rigid bodies are called interface cracks. They are encountered in delamination of composites, film-substrate combinations, fiber-reinforced materials, etc. Consequently, they are of great practical and theoretical importance and have been widely investigated in the literature.

It is well known that, in opening mode under tensile remote stress, there are overlaps of the interface crack faces due to the oscillatory behavior for the stress ahead of the crack tips [Williams 1959; England 1965; Rice 1988; Willis 1971]. To overcome this physical inconsistency, Comninou [1977; Comninou and Dundurs 1980] introduced a contact region behind the crack tip. If the compressive normal load is high enough, then the contact zone is the entire crack faces. Thus, the closed interface crack model combined with unilateral contact with friction must be considered. In the literature, many works are devoted to asymptotic solutions, including the effect of Coulomb's law of dry friction [Deng 1994; Audoly 2000; Bui and Oueslati 2005; Bui 1975; Bui and Oueslati 2004]. It is worth noting that analysis of frictional cracks is rather complex because firstly the problem is nonlinear and secondly the solution is not generally unique when no information is available on the crack history.

Crack problems have been often analyzed on the basis of Green's function or distribution of dislocations [Weertman 1996; Hills et al. 1996]. For interface cracks in bimaterials, the Green function is based on the solution of an edge dislocation at the interface, provided by [Nakahara et al. 1972]. Interface edge dislocation between two elastic layers was also studied by Suo and Hutchinson [1990]. In these solutions, the singularities of displacement and stresses are merely of the same nature as in the homogeneous case

Keywords: interface crack, edge dislocation dipole, Kupradze–Basheleishvili tensor, singular integral equation.

with a little change in some coefficients: with displacement $\boldsymbol{u} \cong O(\log r)$ and u_1 discontinuous (term in $\operatorname{atan}(x_2/x_1)$) and with stress $\boldsymbol{\sigma} \cong O(1/r)$.

When the crack problems become 3D, the conventional methods such as Muskhelishvili's complex potentials can no longer be applied and appropriate methods have been developed. Without being exhaustive, it can be mentioned that the Neuber–Papkovich representation has been used [Kassir and Sih 1973]. An elegant and powerful tool is provided by the potential theory introduced by Kupradze [1963]. This method was used by Bui [1975] and Kossecka [1971] for deriving the integral equation for the crack opening displacement. Later Bui [1977] and Weaver [1977] proposed an integral equations method for the plane crack problem with an arbitrary shape. Furthermore, the weight functions for cracked bodies, introduced by Bueckner [1973], are often used for obtaining the stress intensity factors (SIFs) for 3D cracks. Fukuama and Madariaga [1995] derived a boundary integral equation for plane cracks in the static and dynamic cases. It is worth noting that few 3D crack problems have been solved analytically and the trend is to use numerical computations.

In the present work, we revisit the problem of the interface crack between an elastic half-plane and a rigid one in the presence of Coulomb's law of dry friction. The study is limited to the sliding branch of the friction law. In [Bui and Oueslati 2005], the solutions in complex form for nonhomogeneous loading at infinity were worked out. Here we derive this solution in a different manner by considering distribution of edge dislocation dipoles that correspond to the 2D expression of the fundamental Kupradze–Basheleishvili tensor in 3D. This remark allows us to extend the approach of solution to the 3D interface crack. A set of coupled integral equations for the crack displacement discontinuities is obtained.

2. Basic equations for the 2D sliding interface crack

Consider an elastic solid, with Lamé coefficient μ and λ, occupying the upper half-space Ω^+ and bonded to the lower rigid half-plane Ω^- along the x_1 axis, except on the crack $[b, a]$. We do not know the location of the crack tips b and a, which have to be determined by the solution of the equations.

Following [Bui and Oueslati 2005], let the uncracked system be subjected to some remote, not necessarily homogeneous, stress fields $\sigma_{ij}^{\infty}(x)$. Let the resulting stress field near the interface, prior to delamination, such that $u_1 = u_2 = 0$ at the interface be noted $\sigma_{ij}^0(\boldsymbol{x})$. We consider that $\sigma_{ij}^0(\boldsymbol{x})$ is prescribed load at the interface level, prior to delamination.

Localized shear stress and normal stress at the interface will likely cause delamination if the magnitude of the shear stress is high enough. It should be noted that the inhomogeneous applied stress field has been considered by Simonov [1990] for a frictionless interface crack. Note that, prior to delamination, no slip occurs if the excess stress $Q := f\sigma_{22}^0(x_1, 0) - \sigma_{12}^0(x_1, 0)$ is negative. This is the no-slip condition.

Consider the equations in quasistatic plane strain elasticity with the complex variable $z = x_1 + ix_2$. Displacements and stresses are expressed in terms of analytical functions $\phi(z)$ and $\psi(z)$ in Ω^+ with possible poles at infinity and singularities at the crack tips [Muskhelishvili 1977]:

$$2\mu(u_1 - iu_2) = \{\kappa\overline{\phi(z)} - \bar{z}\phi'(z) - \psi(z)\}, \tag{1}$$

$$(\sigma_{11} + \sigma_{22}) = 4\operatorname{Re}\{\phi'(z)\}, \tag{2}$$

$$(\sigma_{22} + i\sigma_{12}) = \overline{\phi'(z)} + \phi'(z) + \bar{z}\phi''(z) + \psi'(z). \tag{3}$$

The boundary conditions are

$$u_1 = u_2 = 0 \qquad \text{on the bonded zone,} \tag{4}$$

$$u_2 = 0, \quad |\sigma_{12}| = -f\sigma_{22} \quad (\sigma_{22} \leq 0) \qquad \text{on the sliding crack} \tag{5}$$

with the following conditions at infinity:

$$\sigma(\boldsymbol{x}) \to \sigma^{\infty}(\boldsymbol{x}) \quad \text{for } |\boldsymbol{x}| \to \infty \text{ in } \Omega^+, \tag{6}$$

$$\phi'(z \to \infty) = \tfrac{1}{4}(\sigma_{11}^{\infty} + \sigma_{22}^{\infty}) - \frac{i}{1+\kappa}\sigma_{12}^{\infty}, \tag{7}$$

$$\psi'(z \to \infty) = \tfrac{1}{2}(\sigma_{22}^{\infty} - \sigma_{11}^{\infty}) + i\sigma_{12}^{\infty}, \tag{8}$$

where $\kappa = 3 - 4\nu$ in plane strain (ν is the Poisson ratio) and f is the friction coefficient. In (5), the inequality $\sigma_{22} \leq 0$ is not prescribed but has to be checked a posteriori. The crack is defined by the cut along $[b, a]$. Particles of Ω^+along the cut can move horizontally.

Equations (1)–(8) are strictly defined in the upper plane. However, we assume that $\phi(z)$ and $\psi(z)$ can be extended to the whole plane with a cut along the crack by analytical continuations across the bonded zones. Of course, expressions in the right-hand sides of (1)–(3) have no physical meaning in the lower half-plane for the problem considered. They are introduced for mathematical purposes in the derivation of the solution. Nevertheless, we shall see that the formal solution obtained in the lower half-plane corresponds to a similar problem of the elastic half-plane Ω^- adhering on the rigid upper body Ω^+. These solutions for the displacement fields in both problems are linked together by antisymmetry.

From condition (4) on the bonded zone, we obtain

$$\begin{aligned} 2\mu(u_1 - iu_2) &= \{\kappa\overline{\Phi(z)} - \bar{z}\phi'(z) - \psi(z)\} \\ &= \{\kappa\bar{\Phi}(z) - z\phi'(z) - \psi(z)\} = 0 \quad \text{(on the unbroken zone).} \end{aligned}$$

This equation suggests the following definition for the function $\psi(z)$ not only in Ω^+ but also in Ω^-:

$$\psi(z) := \kappa\bar{\phi}(z) - z\phi'(z). \tag{9}$$

The continuation used in (9) is normally referred to as a displacement continuation (rather than the more common stress continuation). Using (9) for (3) and (1), we find on the real axis $z = \bar{z}$

$$\sigma_{22} - i\sigma_{12} = \phi'(z) + \kappa\phi'(\bar{z}), \tag{10}$$

$$2\mu(u_1 - iu_2) = \{\kappa\overline{\phi(z)} - \bar{z}\phi'(z) - \psi(z)\} = \kappa\{\overline{\phi(z)} - \bar{\phi}(z)\}, \tag{11}$$

$$2\mu(u_1 + iu_2) = \{\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\} = \kappa\{\phi(z) - \phi(\bar{z})\}. \tag{12}$$

We recover the adherence condition $u_1 - iu_2 = 0$ on the bonded zone where the function $\phi(z)$ is continuous $\phi(z) = \phi(\bar{z})$. The condition that $u_2 = 0$, on the cut, corresponds to the imaginary part of the right-hand side of (12) being zero on the cut.

We introduce the auxiliary stress field $\theta_{ij}(\boldsymbol{x})$, the corresponding auxiliary displacement field $\boldsymbol{q}(\boldsymbol{x})$, and the associate auxiliary complex functions $\Theta(z)$ and $\Xi(z)$ by putting

$$\boldsymbol{\theta} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^0. \tag{13}$$

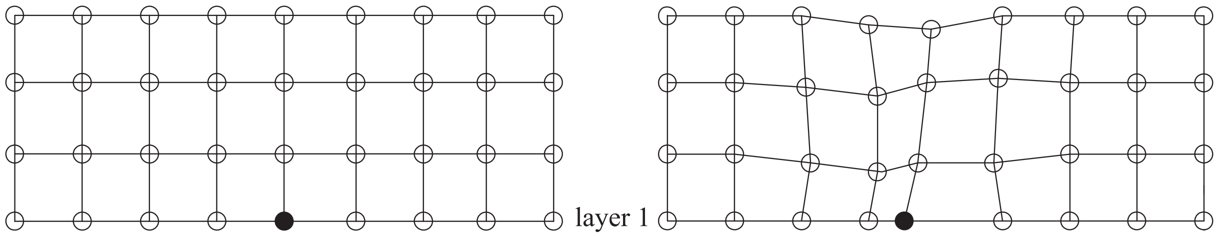

Figure 1. Atoms of layer 1 stuck on the rigid interface, except the black atom.

On the crack, since the displacement components associated to $\boldsymbol{\sigma}^0(\boldsymbol{x})$ vanish, the auxiliary displacement fields $q_1(x_1, 0)$ and $q_2(x_1, 0)$ coincide with the current displacement fields $u_1(x_1, 0)$ and $u_2(x_1, 0)$. At infinity, the stress field $\theta_{ij}(\boldsymbol{x})$ vanishes; hence, the function $\Theta'(z)$ vanishes at infinity:

$$\Theta'(z) \cong O(1/z) \quad \text{at infinity.} \tag{14}$$

Moreover, the continuity condition (or the compatibility condition) on the displacement (12) holds for the auxiliary fields $\boldsymbol{q}$ and gives

$$2\mu(q_1 + iq_2) = \kappa\{\Theta(z) - \Theta(\bar{z})\} \quad \text{for } z = \bar{z}. \tag{15}$$

3. The isolated edge dislocation dipole

Let us consider an atomic model of defect at the interface between an elastic medium and a rigid body shown in Figure 1. Suppose that all atoms of layer 1 are fixed, being stuck on the rigid interface, except the black atom, which is debonded from the substratum and can glide to the left.

When we move around the black atom along the lattice, from one atom of layer 1 to another one in the same layer 1, we recover the same unchanged positions of atoms. It is expected that displacement and stress fields in the upper half-plane are more singular than in the case of edge dislocation in homogeneous medium. To investigate the mathematical nature of such an isolated "defect" at the origin of the real axis as shown in Figure 1, where $q_2 = 0$ on the real axis, $q_1 = 0$ for ($x_1 \neq 0$, $x_2 = 0$), and $q_1 \neq 0$ at ($x_1 = 0$, $x_2 = 0$), we consider the displacement on the real axis from the Ω^+ side, given as

$$2\mu(q_1 + iq_2) = k\{\Theta(z) - \Theta(\bar{z})\} \quad (z = \bar{z}). \tag{16}$$

In the continuation of functions in Ω^- through the bonded zone, we may consider the lower half-plane as an elastic medium stuck on the rigid upper half-plane, except along the crack; in Ω^-, the displacement field is antisymmetric. Therefore, the defect shown in Figure 1 corresponds to the solution in Ω^+ of an edge dislocation dipole in shear mode as in Figure 2 (left). The edge dislocation dipole in the opening mode is depicted in Figure 2 (right).

Consider the function defined in the whole space (upper and lower complex plane Ω^+ and Ω^-) as

$$\Theta^{(h)}(z) = \chi \frac{1}{2ih\pi}\{\log(z+h) - \log(z)\} \tag{17}$$

with complex $\chi = c_1 + ic_2$, $c_1 > 0$ and $c_2 < 0$, and the cut along $(-h, 0)$.

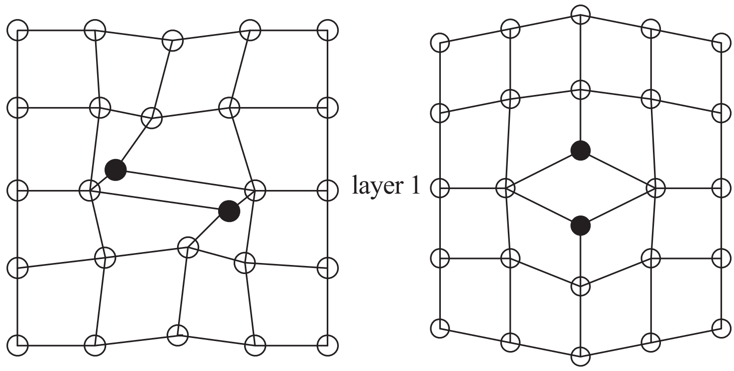

Figure 2. Edge dislocation dipoles in shear mode (left) and opening mode (right). Atoms of layer 1 are fixed, except the (double) black atoms moving in opposite directions in Ω^+ and Ω^-.

Outside the cut, $\Theta^{(h)}(z)$ is continuous so that $q_1^{(h)} + iq_2^{(h)} = 0$ on the part of the real axis where $x_2 = 0$, $x_1 < -h$, and $x_1 > 0$.

On the upper face of the cut, we find

$$2\mu(q_1^{(h)} + iq_2^{(h)}) = k(\Theta^{(h)}(z^+) - \Theta^{(h)}(z^-)) = -\frac{k}{h}(c_1 + ic_2). \tag{18}$$

Now taking the limit of (17) as $h \to 0$, we obtain firstly the derivative of the logarithm function:

$$\Theta^{(0)}(z) = \chi \frac{1}{2i\pi} \frac{1}{z}. \tag{19}$$

Secondly, the limit of the constant function (18) over the interval $[-h, 0]$ of vanishing length $h \to 0$, which is nothing but the Dirac delta function:

$$2\mu(q_1^{(0)} + iq_2^{(0)}) = -k(c_1 + ic_2)\delta(x_1). \tag{20}$$

The displacement singularity given in Ω^+ by the function (19) is higher than that of an isolated edge dislocation in homogeneous medium. At the defect itself, the displacement (20) can be considered as the Green function. Note that, for an edge dislocation in a bimaterial, the Dirac delta function appears in the stresses at the interface: $\sigma_{22} + i\sigma_{12} = 2\bar{B}(1/x_1 + \pi i\beta'\mu\delta(x_1))$ (for $x_2 = 0$), where B and β' are some bimaterial constants [Suo and Hutchinson 1990]. Here the Dirac delta function is found in the displacement at the interface. The components of the displacement at $\boldsymbol{x}$ in the upper half-plane, $x_2 > 0$, due to an isolated edge dislocation dipole at the origin $\boldsymbol{y} = \boldsymbol{0}$, in the pure shear mode $\chi = c_1$ $(c_2 = 0)$, are given as (with $\Theta(z) = \Theta^0(z) = c_1/2i\pi z$ and $\overline{\Theta}'(\bar{z}) = -c_1/2i\pi\bar{z}^2$)

$$2\mu(q_1 + iq_2) = k\{\Theta(z) - \Theta(\bar{z})\} + (\bar{z} - z)\overline{\Theta}'(\bar{z}) \quad \text{for } x_2 > 0, \tag{21}$$

$$2\mu q_1(x_1 + i0) = -kc_1\delta(x_1) \quad \text{and} \quad 2\mu q_2 = 0 \quad \text{for } x_2 = 0^+. \tag{22}$$

Thus, we obtain for $x_2 > 0$

$$2\mu(q_1 + iq_2) = -c_1\frac{k}{\pi}\frac{x_2}{x_1^2 + x_2^2} - c_1\frac{x_2(x_1^2 - x_2^2)}{\pi(x_1^2 + x_2^2)^2} - 2ic_1\frac{x_1x_2^2}{\pi(x_1^2 + x_2^2)^2}. \tag{23}$$

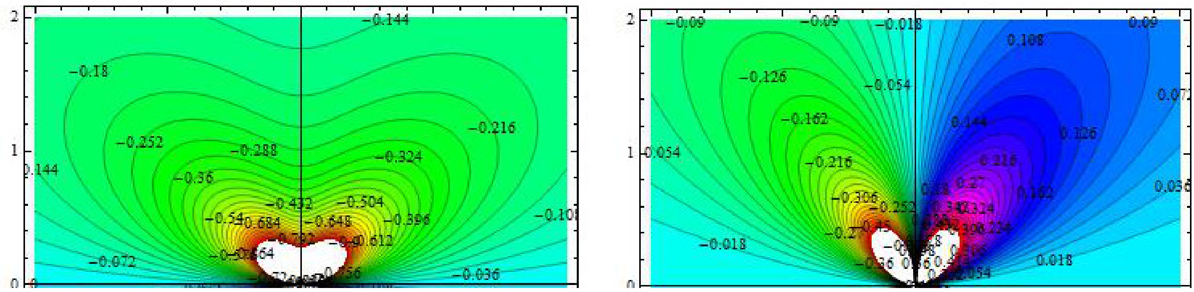

Figure 3. Contours of the displacement field given by (23) ($c_1 = 1$): on the left $2\mu q_1$ and on the right $2\mu q_2$.

The contours of the displacement field in the upper half-plane due to an isolated dipole dislocation given by (23) are depicted in Figure 3.

The displacement at $\boldsymbol{x} = (x_1, x_2)$ in the entire plane due to an edge dislocation dipole at $\boldsymbol{y} = (y_1, 0)$ on the real axis (of normal $\boldsymbol{e}^2$), not necessarily at the origin, satisfies the elastic equilibrium equation and the following boundary condition on the real axis ($\boldsymbol{x}^{\pm} = (x_1, \pm 0)$):

$$-\frac{2\mu}{kc_1}\boldsymbol{q}^{(0)}(\boldsymbol{x}^{\pm}, \boldsymbol{y}, \boldsymbol{e}^2) = \pm\delta(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{e}^1 \quad (x_2 = y_2 = 0). \tag{24}$$

We recover in (20), (21), (22), and (24) the particular 2D form of the Kupradze–Basheleishvili fundamental tensor $B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{n}_P)$ given for 3D solids [Kupradze 1963] with components (see Appendix A)

$$B_i^j(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{n}(\boldsymbol{y})) = \frac{1}{2\pi(\lambda+3\mu)}\{2\mu\delta_{ij} + 3(\lambda+\mu)(x_i - y_j)(x_j - y_j)\rho^{-2}\}\frac{\partial}{\partial n_y}\frac{1}{\rho}, \tag{25}$$

where $\rho = \sqrt{(x_i - y_i)(x_i - y_i)}$. The elastic displacement vector $\boldsymbol{B}^j(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{n}_P)$ at point $\boldsymbol{x}$ is discontinuous at $\boldsymbol{x} = \boldsymbol{y}$ on the plane P of normal $\boldsymbol{n}_P$. The discontinuity along the unit vector $\boldsymbol{e}^j$ is given by

$$\boldsymbol{B}^j(\boldsymbol{x}^{\pm}, \boldsymbol{y}, \boldsymbol{n}_P) = \pm\delta_P(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{e}^j, \tag{26}$$

where $\delta_P(\boldsymbol{x}-\boldsymbol{y})$ is the Dirac delta function on the plane P. Therefore, the vector $\boldsymbol{q}^{(0)}(\boldsymbol{x}^{\pm}, \boldsymbol{y}, \boldsymbol{e}^2)$ is the particular form in 2D of the vector $\boldsymbol{B}^1(\boldsymbol{x}^{\pm}, \boldsymbol{y}, \boldsymbol{e}^2)$ in 3D with $\boldsymbol{x}^{\pm} = (x_1, x_2, \pm 0)$.

By integrating (25) with respect to y_3, $-\infty < y_3 < \infty$, we obtain the 2D singular defect $\boldsymbol{q}^{(0)}(\boldsymbol{x}^{\pm}, \boldsymbol{y}, \boldsymbol{e}^2)$ for P normal to $\boldsymbol{e}^2$. Point defects with properties (24) or (26) are purely mathematical. They are boundary Green functions for determining the displacement field in 2D and 3D.

4. Solution of the Dirichlet boundary value problem and the singular integral equation for the 2D sliding crack

Consider the following boundary value problem of the elastic half-plane $x_2 > 0$, where both components of displacement $q_1(t)$ and $q_2(t) = 0$ are prescribed on the interface ($x_2 = 0$) with $q_1(\pm\infty) = 0$. Find the stresses on $x_2 = 0$.

Using results of the previous section, in the pure shear mode and from (19) and (20), the solution is

$$\Theta(z) = \frac{1}{2i\pi}\int_{-\infty}^{+\infty} \frac{-2\mu q_1(t)}{\kappa}\frac{dt}{z-t}. \tag{27}$$

Remark. It is possible to derive the potential $\Theta(z)$ given by (27) by another method. To this end, we search for a solution of (15) using the Cauchy integral kernel $1/(z-t)$ and by considering $\chi(t)$ as unknown density:

$$\Theta(z) = \frac{1}{2i\pi}\int_{-\infty}^{+\infty} \chi(t)\frac{dt}{z-t} = -\frac{1}{2i\pi}\int_{-\infty}^{+\infty} \chi(t)\frac{dt}{t-z}. \tag{28}$$

Thus, using $2\mu(q_1+iq_2) = \kappa\{\Theta(z)-\Theta(\bar{z})\}$ $(z=\bar{z})$, we obtain the density as $\chi(t) = -2\mu q_1(t)/\kappa$.

The stress in $x_2 > 0$ is then given by (10) as

$$\begin{aligned}\theta_{22}(x_1) - i\theta_{12}(x_1) &= \Theta'(z) + \kappa\Theta'(\bar{z}) \\ &= -\frac{1}{2i\pi}\int_{-\infty}^{+\infty} \chi'(t)\frac{dt}{t-z} - \frac{\kappa}{2i\pi}\int_{-\infty}^{+\infty} \chi'(t)\frac{dt}{t-\bar{z}} \\ &= \frac{2\mu}{\kappa}\left\{\frac{1}{2i\pi}\int_{-\infty}^{+\infty} q_1'(t)\frac{dt}{t-z} + \kappa\frac{1}{2i\pi}\int_{-\infty}^{+\infty} q_1'(t)\frac{dt}{t-\bar{z}}\right\}.\end{aligned}$$

Using Plemelj formulas ($z = x_1 + i0$ and $\bar{z} = x_1 - i0$), we then obtain the solution (pv = principal value) for the stress at the interface as

$$\theta_{12}(x_1) = \frac{\mu}{\kappa\pi}(1+\kappa)(\text{pv})\int_{-\infty}^{+\infty} q_1'(t)\frac{dt}{t-x_1}, \tag{29}$$

$$\theta_{22}(x_1) = \frac{\mu}{\kappa}(1-\kappa)q_1'(x_1). \tag{30}$$

Equations (29) and (30) are particular forms (for $q_2(t) = 0$) of the general solution derived in [Bui 1968], where general boundary conditions $q_1(t) \neq 0$ and $q_2(t) \neq 0$ are considered and where the Kupradze–Basheleishvili tensor $B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{n})$ has been used.

Equations (5), (29), and (30) allow us to establish the integral equation for the unknown $q_1'(x_1)$ in the frictional sliding interface crack problem considered in this paper with data σ_{12}^0 and σ_{22}^0. To this end, we have to express the friction condition on the current stresses $\sigma_{12} = f\sigma_{22}$, with $\sigma = \sigma^0 + \theta$, as

$$\theta_{12} - f\theta_{22} = f\sigma_{22}^0 - \sigma_{12}^0$$

and then obtain the integral equation

$$-f\frac{\mu}{\kappa}(1-\kappa)q_1'(x_1) + \frac{\mu}{\kappa\pi}(1+\kappa)(\text{pv})\int_b^a q_1'(t)\frac{dt}{t-x_1} = f\sigma_{22}^0(x_1) - \sigma_{12}^0(x_1). \tag{31}$$

It is a Carleman integral equation with the unknown $q_1'(x)$, the gradient of the horizontal displacement on the sliding crack lip. The general theory of such integral equations is provided in [Muskhelishvili 1977; Tricomi 1985] for example.

Let us introduce the coefficients $A = -f(\mu/\kappa)(1-\kappa)$ and $B = i(\mu/\kappa)(1+\kappa)$. The index of this integral equation is $\tau = (1/2i\pi)\log G$, where $G = (A-B)/(A+B) = -((\kappa+1)+if(\kappa-1))/((\kappa+1)-if(\kappa-1))$.

According to [Muskhelishvili 1977], the solution of the integral equation (31) vanishing at infinity is

$$q_1'(x_1) = \frac{X^+(x_1)+X^-(x_1)}{2(A+B)X^+(x_1)}(f\sigma_{22}^0(x_1)-\sigma_{12}^0(x_1)) + \frac{X^+(x_1)-X^-(x_1)}{2i\pi}(\mathrm{pv})\int_b^a \frac{f\sigma_{22}^0(t)-\sigma_{12}^0(t)}{(A+B)X^+(t)}\frac{dt}{(t-x_1)}, \tag{32}$$

where $X(z)=(z-b)^{n/2-\tau}(z-a)^{m/2+\tau}$ is the fundamental solution of the homogenous Riemann–Hilbert problem $X^+(z)=GX^-(z)$.

The closed-form expression of (32) gives

$$q_1'(x_1) = \frac{f\kappa(\kappa-1)}{\mu((\kappa+1)^2+f^2(\kappa-1)^2)}(f\sigma_{22}^0(x_1)-\sigma_{12}^0(x_1)) - \frac{\kappa(\kappa-1)X^+(x_1)}{\mu\pi\big((\kappa+1)^2+f^2(\kappa-1)^2\big)}(\mathrm{pv})\int_b^a \frac{f\sigma_{22}^0(t)-\sigma_{12}^0(t)}{X^+(t)}\frac{dt}{(t-x_1)}. \tag{33}$$

Some noticeable features are that G is equal to g, the coefficient of the Hilbert problem established in [Bui and Oueslati 2005, (22)], the index τ coincides with the coefficient α, and

$$\frac{f\sigma_{22}^0(x_1)-\sigma_{12}^0(x_1)}{A+B} = \frac{k}{\mu}E_0(x),$$

where $E_0(x)$ is the second member of the Hilbert problem derived in [Bui and Oueslati 2005].

The closed form and physical solution (square-integrable solution) of the integral equation (33) for a polynomial remote loading $(\sigma_{22}^0, \sigma_{12}^0)$ for example can be found in [Bui and Oueslati 2005].

5. Integral equations for the sliding planar crack of an arbitrary shape

Analysis of the 3D problem of sliding delamination between an elastic half-space Ω^+ $(x_3>0)$ and a rigid body under Coulomb's friction is very difficult. We do not attempt to solve this problem and restrict ourselves simply to the following one: given the auxiliary shear stresses $\theta_{31}(x_1,x_2,0)$ and $\theta_{32}(x_1,x_2,0)$ and normal stress $\theta_{33}(x_1,x_2,0)$ on the interface plane P $(x_3=0)$, prior to delamination, find the integral equations for tangential auxiliary displacements $\varphi_1(x_1,x_2):=q_1(x_1,x_2,0)$ and $\varphi_2(x_1,x_2):=q_2(x_1,x_2,0)$ in the planar crack S.

First of all, we generalize (29) and (30) to 3D problems. Let $\boldsymbol{x}=(x_1,x_2,x_3)$, $\boldsymbol{y}=(y_1,y_2,0)$, and $\boldsymbol{z}=(z_1,z_2,0)$. The fundamental Kupradze–Basheleishvili tensor (25) gives the displacement field in Ω^+:

$$q_i(\boldsymbol{x}) = \int_S B_i^j(\boldsymbol{x},\boldsymbol{y},\boldsymbol{e}^3)\varphi_j(\boldsymbol{y})\,dS_y. \tag{34}$$

To verify that, we take the limit $\boldsymbol{x}\to\boldsymbol{y}^+=(y_1,y_2,0^+)$ and using (26) obtain $q_j(\boldsymbol{y}^+)=\varphi_j(\boldsymbol{y})$. Note that the displacement of points in P lying outside S vanishes. This means that (34) corresponds exactly to the displacement field of the elastic body Ω^+ stuck on the rigid surface P, except on the crack S.

The density $\varphi_j(\boldsymbol{y})$ of the double layer potential (34) (see Appendix C) is exactly the displacement in S. For convenience, the displacement representation in terms of the single layer potential is outlined in Appendix B.

From (34), we can calculate the stress field in Ω^+ and also on P. Such a calculation is somewhat delicate because of strongly singular kernels. However, explicit formulas for the stresses on P have been given in [Bui 1968; 1977]:

- tangential stresses

$$\theta_{3\alpha}(\boldsymbol{y}) = \frac{\mu}{2\pi}(\mathrm{pv})\int_S \frac{\partial\varphi_\alpha}{\partial z_\beta}\frac{\partial}{\partial y_\beta}\frac{1}{\rho(\boldsymbol{y},\boldsymbol{z})}\,dS_z + \frac{\mu(\lambda+\mu)}{2\pi(\lambda+3\mu)}(\mathrm{pv})\int_S\left(\frac{\partial\varphi_1}{\partial z_1}+\frac{\partial\varphi_2}{\partial z_2}\right)\frac{\partial}{\partial y_\alpha}\frac{1}{\rho(\boldsymbol{y},\boldsymbol{z})}\,dS_z, \quad (35)$$

- normal stress

$$\theta_{33}(\boldsymbol{y}) = \frac{2\mu^2}{\lambda+3\mu}\left(\frac{\partial\varphi_1}{\partial y_1}+\frac{\partial\varphi_2}{\partial y_2}\right). \quad (36)$$

We are now in a position to derive the integral equations for the frictional interface crack S with prescribed stresses $(\sigma^0_{31},\sigma^0_{32},\sigma^0_{33})$ on the plane P. Assume that $\sigma^0_{31}<0$, $\sigma^0_{32}<0$, and $\sigma^0_{33}<0$. Such an assumption can only be verified a posteriori so that the friction law can be written as $\sigma_{31}-f\sigma_{33}\cos(\psi)=0$ and $\sigma_{32}-f\sigma_{33}\sin(\psi)=0$, where $\psi=\operatorname{atan}(\varphi_2/\varphi_1)$. We then obtain the coupled *nonlinear* integral equations for φ_α, which are the generalizations of (31) to 3D problems:

$$\frac{\mu}{2\pi}(\mathrm{pv})\int_S \frac{\partial\varphi_1}{\partial z_\beta}\frac{\partial}{\partial y_\beta}\frac{1}{\rho(\boldsymbol{y},\boldsymbol{z})}\,dS_z + \frac{\mu(\lambda+\mu)}{2\pi(\lambda+3\mu)}(\mathrm{pv})\int_S\left(\frac{\partial\varphi_1}{\partial z_1}+\frac{\partial\varphi_2}{\partial z_2}\right)+\frac{\partial}{\partial y_1}\frac{1}{\rho(\boldsymbol{y},\boldsymbol{z})}\,dS_z$$
$$-f\frac{\varphi_1}{\sqrt{\varphi_1^2+\varphi_2^2}}\frac{2\mu^2}{(\lambda+3\mu)}\left(\frac{\partial\varphi_1}{\partial y_1}+\frac{\partial\varphi_2}{\partial y_2}\right) = f\sigma^0_{33}(\boldsymbol{y})-\sigma^0_{31}(\boldsymbol{y}) \quad (37)$$

and

$$\frac{\mu}{2\pi}(\mathrm{pv})\int_S \frac{\partial\varphi_2}{\partial z_\beta}\frac{\partial}{\partial y_\beta}\frac{1}{\rho(\boldsymbol{y},\boldsymbol{z})}\,dS_z + \frac{\mu(\lambda+\mu)}{2\pi(\lambda+3\mu)}(\mathrm{pv})\int_S\left(\frac{\partial\varphi_1}{\partial z_1}+\frac{\partial\varphi_2}{\partial z_2}\right)\frac{\partial}{\partial y_2}\frac{1}{\rho(\boldsymbol{y},\boldsymbol{z})}\,dS_z$$
$$-f\frac{\varphi_2}{\sqrt{\varphi_1^2+\varphi_2^2}}\frac{2\mu^2}{(\lambda+3\mu)}\left(\frac{\partial\varphi_1}{\partial y_1}+\frac{\partial\varphi_2}{\partial y_2}\right) = f\sigma^0_{33}(\boldsymbol{y})-\sigma^0_{32}(\boldsymbol{y}). \quad (38)$$

It is out of the scope of this paper to derive the solution to (37)–(38), even by a numerical method. However, it is worth noting that these integral equations extend over the crack surface only, which has many advantages for numerical analysis.

Finally, if we interpret the SIFs K_{I} and K_{II} along the crack front, as factors of the tangential displacement discontinuity [Bui 2006], then the two sliding modes are nonlinearly coupled in the frictional 3D crack problems.

6. Conclusion

In this paper, the problem of the frictional sliding interface crack between an elastic half-plane and a rigid one has been analyzed by considering distribution of edge dislocation dipoles along the crack faces. We recover the field equations obtained in [Bui and Oueslati 2005] by means of complex potential representation. Then this approach has been extended to the planar interface crack with an arbitrary shape thanks to the fundamental Kupradze–Basheleishvili tensor. A couple of nonlinear integral equations for the crack displacement discontinuities have been derived. It is worth noting that these integrals are taken only over the crack face, which will be interesting in numerical computations. This is a topic for future work.

Appendix A: The Kupradze–Basheleishvili tensor

Let Ω be an elastic solid in 3D space $\mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$ and $\boldsymbol{x}$ be the Cartesian coordinates. The elastic operator L is defined such that the Lamé–Navier equation reads

$$L\boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \boldsymbol{f} = 0 \quad \text{in } \Omega,$$

where $\boldsymbol{u}$ is the displacement field in the solid Ω and $\boldsymbol{f}$ denotes for the body force.

The corresponding traction acting on the boundary of $\partial\Omega$ with the outward unit normal $\boldsymbol{n}$ is obtained from the displacement $\boldsymbol{u}$ by applying the traction operator

$$T^n := 2\mu\frac{\partial}{\partial n} + \lambda\boldsymbol{n}\,\mathrm{div} + \mu\boldsymbol{n}\wedge\mathrm{rot}\,.$$

The fundamental Kupradze–Basheleishvili operator associated to a plane P_y, with normal $\boldsymbol{n}_y$ at the point $\boldsymbol{y}$, is defined by the tensor $\boldsymbol{B}$ satisfying

$$L\boldsymbol{B}^k(\boldsymbol{x},\boldsymbol{y};\boldsymbol{n}_y) = 0, \quad k\in\{1,2,3\},\ \boldsymbol{x}\notin P_y \text{ for fixed } \boldsymbol{y}\in P_y,$$

and the boundary conditions

$$\boldsymbol{B}^k(\boldsymbol{x},\boldsymbol{y};\boldsymbol{n}_y) = \boldsymbol{0}, \quad \boldsymbol{x},\boldsymbol{y}\in P_y \text{ and } \boldsymbol{x}\neq\boldsymbol{y}.$$

The components of $\boldsymbol{B}^k(\boldsymbol{x},\boldsymbol{y};\boldsymbol{n}_y)$ read

$$B_i^k(\boldsymbol{x},\boldsymbol{y},\boldsymbol{n}_y) = \frac{1}{2\pi(\lambda+3\mu)}\{2\mu\delta_{ik} + 3(\lambda+\mu)(x_i-y_i)(x_k-y_k)\rho^{-2}\}\frac{\partial}{\partial n_y}\frac{1}{\rho},$$

where $\rho = \sqrt{(x_i-y_i)(x_i-y_i)}$ is the Euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. It should be noted that $\boldsymbol{B}^k(\boldsymbol{x},\boldsymbol{y};\boldsymbol{n}_y) = -\boldsymbol{B}^k(\boldsymbol{y},\boldsymbol{x};\boldsymbol{n}_y)$.

Following [Bui 2006], it can be shown, for a point $\boldsymbol{x}$ on the plane P_y, that the vectors defined by

$$\boldsymbol{B}^k(\boldsymbol{x}^{\pm},\boldsymbol{y};\boldsymbol{n}_y) = \pm\boldsymbol{e}^k\delta_{P_y}(\boldsymbol{y}-\boldsymbol{x})$$

are opposite Dirac delta distributions. Physically, the k-component of the Kupradze–Basheleishvili tensor is a dislocation dipole in the $\boldsymbol{e}^k$ direction. The previous relation is often used to describe the displacement jump on the crack faces.

Appendix B: Single layer potential

The displacement field solution of the elastic problem in the absence of body force can be obtained in terms of an integral surface $S_i(\boldsymbol{x})$ with a continuous density ψ_i and a kernel $V_i^k(\boldsymbol{x},\boldsymbol{z})$:

$$S_i(\boldsymbol{x}) = 2\int_{\partial\Omega} V_i^k(\boldsymbol{x},\boldsymbol{z})\psi_k(\boldsymbol{z})\,dS_z.$$

By analogy with Newton's potential, this integral is called the single layer potential. It is continuous across the surface $\partial\Omega$, but the corresponding stresses with a component along the normal should be discontinuous.

According to [Kupradze 1963], the stress vector exerted on the boundary $\partial\Omega$ is obtained by using the traction operator T^n to both sides of the previous equation:

$$T^n \boldsymbol{S}(\boldsymbol{y}^{\pm}) = \pm\boldsymbol{\psi}(\boldsymbol{y}) + 2(\text{pv}) \int_{\partial\Omega} V_i^k(\boldsymbol{x}, z)\psi_k(z)\, dS_z.$$

Appendix C: Double layer potential

The displacement field solution of the elastic problem in the absence of body force can be represented by an elastic potential $D_i(\boldsymbol{x})$ with a continuous density ϕ_i:

$$D_i(\boldsymbol{x}) = \int_{\partial\Omega} \boldsymbol{B}_i^k(\boldsymbol{x}, z; \boldsymbol{n}_z)\phi_k(z)\, dS_z.$$

It can be established that $\boldsymbol{D}(\boldsymbol{x})$ is discontinuous across the surface $\partial\Omega$ [Kupradze 1963]:

$$\boldsymbol{D}(\boldsymbol{x}^+) - \boldsymbol{D}(\boldsymbol{x}^-) = 2\boldsymbol{\phi}(\boldsymbol{x}).$$

The limit of $\boldsymbol{D}(\boldsymbol{x})$ when $\boldsymbol{x}$ tends to $\boldsymbol{x}^-$ belonging to the surface $\partial\Omega$ gives

$$D_i(\boldsymbol{x}^-) = \phi_i(\boldsymbol{x}^-) + \int_{\partial\Omega} \boldsymbol{B}_i^k(\boldsymbol{x}, z; \boldsymbol{n}_z)\phi_k(z)\, dS_z.$$

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