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SOME APPLICATIONS OF OPTIMAL CONTROL TO INVERSE PROBLEMS IN ELASTOPLASTICITY

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The aim of this paper is to present the applications of the optimal control theory to solve several inverse problems for elastoplastic materials and structures. The optimal control theory permits to determine the internal state of a body from the knowledge both of the initial and the final, residual, geometry resulting from an unknown loading history.

1. Introduction

This article presents the applications of the optimal control theory to solve several inverse problems in non-linear mechanics, more precisely elastoplastic materials. Two decades ago Prof. H. D. Bui presented, in his book on inverse problems, some solution examples for the Cauchy problem based on control theory. This work is an extension of the ideas presented in that work [Bui 1993] for the case of structures with an elastoviscoplastic material behavior.

We shall start with the assumption of known constitutive equations and will only address the question of the determination of the internal state of the body resulting from an unknown loading history. In order to complement the missing loading data, we shall further suppose that both the initial and the final shape of the body are known. This problem setting has several applications ranging from the identification of causes of accidents to optimization of industrial processes and different solution methods have been proposed; see for example [Ballard and Constantinescu 1994; Constantinescu and Tardieu 2001; Gao and Mura 1989]. The solution proposed next is based on the optimal control theory, through the minimization of a suitably chosen functional cost among a class of admissible loading histories, as briefly discussed in [Stolz 2008]. It presents the additional advantage of proposing simultaneously an internal state and a possible loading history.

The paper starts with the introduction of the solution method based on optimal control theory to inverse problems in linear elasticity. Next a generalisation is presented for elastoplasticty and viscoplasticity. The final complexity is reached for elastoplastic problems under cyclic loading, where the optimal control method proved to be a powerfull tool to determine the limit cycle in elastoplasticity [Peigney and Stolz 2001; 2003] or the wear and the consequent the loss of material [Peigney 2004]. The approach is illustrated by a series of solutions of example problems.

2. Inverse problems in linear elasticity

2A. ***Setting of the problem.*** Let us consider a material body occupying in the reference configuration the volume Ω. The boundary of the body, denoted by $\partial\Omega$, is partitioned in two complementary parts Γ_o

Keywords: elastoplasticity, inverse problem, adjoint problem, optimal control.

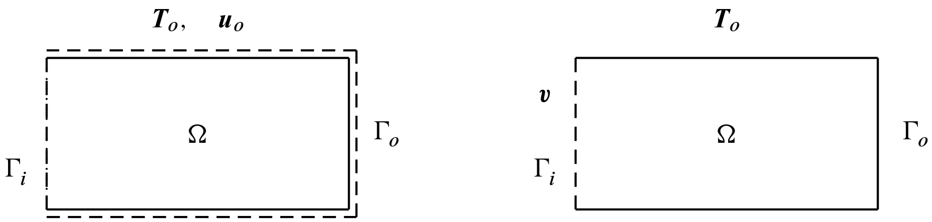

Figure 1. Left: a non-well posed problem: $\boldsymbol{T}_o$ and $\boldsymbol{u}_o$ are applied on Γ_o. Right: the well-posed problem: $\boldsymbol{v}$ on Γ_i (dashed) and $\boldsymbol{T}_o$ on Γ_o are prescribed.

and Γ_i, (see Figure 1). On the Γ_o part, both the displacement field $\boldsymbol{u}_o$ and the surface traction field $\boldsymbol{T}_o$ are known.

The inverse problem, considered next is the determination of both the displacement on the boundary part Γ_i and the complete displacement field $\boldsymbol{u}$ over Ω.

The unknown complete displacement field $\boldsymbol{u}$ should satisfy the following systems of partial differental equations and boundary conditions:

$$\begin{array}{lll}
\textit{Compatibility}: & \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^t \boldsymbol{u}), & \text{over } \Omega, \\
\textit{Constitutive law}: & \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}), & \text{over } \Omega, \\
\textit{Equilibrium}: & \operatorname{div} \boldsymbol{\sigma} = 0, & \text{over } \Omega, \\
\textit{Boundary conditions } \text{BCu}: & \boldsymbol{u} = \boldsymbol{u}_o, & \text{on } \Gamma_o, \\
\textit{Boundary conditions } \text{BCT}: & \boldsymbol{n}.\boldsymbol{\sigma} = \boldsymbol{T}_o, & \text{on } \Gamma_o.
\end{array}$$

This problem is not well posed in the Hadamard sense [Lions 1968], in the sense that existence and uniqueness of the solution is generally not insured and that small errors in the input data will conducts to large errors in the output data, the displacement field $\boldsymbol{u}$. Several solution methods have been dedicated to this problem setting: direct integration of Cauchy problem [Bui 1993; Bourgeois 1998], the quasi-reversibility method [Lions 1968; Bourgeois 1998], etc.

The solution proposed here is focused on the application of optimal control theory. An example of application is the steady-state heat conduction problem, where optimal control theory has been used to determine the history of heat sources [Delattre et al. 2002].

2B. ***A well posed problem.*** The ill posed character of the initial problem setting can be corrected, by relaxing the overdetermined data on Γ_o boundary and proposing data for Γ_i. For the overall probelm setting, this means changing the boundary conditions (BCu) with the conditions (BCi) applied on the complementary boundary Γ_i:

$$\text{BCi:} \quad \boldsymbol{u} = \boldsymbol{v}, \quad \text{on } \Gamma_i. \tag{2-1}$$

As a consequence, boundary condition are complete, defined on complementary parts and the corrected problem is know well posed. One can recognize, the form of a classical problem in small strain linear elasticity.

For the optimal control problem setting we shall denote this problem as the primal problem. Moreover, its solution $\boldsymbol{u}_{\text{sol}}(\boldsymbol{v}, \boldsymbol{T}_o)$ is then a linear function of the prescribed values on the surface tractions and displacements boundaries: $\boldsymbol{T}_o$ on Γ_o and $\boldsymbol{v}$ on Γ_i.

2C. ***The idea of control.*** The prescribed displacement $\boldsymbol{v}$ on the complementary boundary Γ_i, introduced before (2-1), will now play the role of a control variable which is optimized in order to satisfy the following condition: the solution of the primal problem $\boldsymbol{u}_{\text{sol}}(\boldsymbol{v}, \boldsymbol{T}_o)$ must match to the displacement field $\boldsymbol{u}_o$ on the boundary Γ_o. From a mathematical point of view, the optimal displacement $\boldsymbol{v}$ realizes the minimum of the functional J defined as

$$J(\boldsymbol{v}) = \int_{\Gamma_o} \tfrac{1}{2}\|\boldsymbol{u}(\boldsymbol{v}, \boldsymbol{T}_o) - \boldsymbol{u}_o\|^2 \,\mathrm{ds} + r \int_{\Gamma_i} \tfrac{1}{2}\|\boldsymbol{v}\|^2 \,\mathrm{ds}. \tag{2-2}$$

2D. ***The optimization method.*** A series of operations, adding the variational form of the primal problem to the functional J and using the boundary conditions, transforms the problem in a new optimization problem. The solution $\boldsymbol{u}$ of the inverse problem realizes an optimal point for the functional $\mathcal{J}$:

$$\mathcal{J}(\boldsymbol{u}, \boldsymbol{u}^*) = -\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}^*) \,\mathrm{d}\Omega + \int_{\Gamma_o} \boldsymbol{u}^*.\boldsymbol{T}_o \,\mathrm{ds} + h \int_{\Gamma_o} \tfrac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_o\|^2 \,\mathrm{ds} + r \int_{\Gamma_i} \tfrac{1}{2}\|\boldsymbol{u}\|^2 \,\mathrm{ds},$$

among the set of kinematically admissible fields $\boldsymbol{u}^*$ such that $\boldsymbol{u}^* = 0$ over Γ_i.

Outline of proof. The optimal point of the functional is characterized by the canceling of its first order variations. The variations of $\mathcal{J}$ are defined by

$$\delta\mathcal{J} = -\int_\Omega \Big(\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u}^*) + \boldsymbol{\sigma}^* : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\Big) \,\mathrm{d}\Omega + \int_{\Gamma_o} \delta\boldsymbol{u}^*.\boldsymbol{T}_o \,\mathrm{ds} \\ + h \int_{\Gamma_o} (\boldsymbol{u} - \boldsymbol{u}_o).\delta\boldsymbol{u} \,\mathrm{ds} + r \int_{\Gamma_i} \boldsymbol{u}.\delta\boldsymbol{u} \,\mathrm{ds} = \frac{\partial\mathcal{J}}{\partial\boldsymbol{u}^*}.\delta\boldsymbol{u}^* + \frac{\partial\mathcal{J}}{\partial\boldsymbol{u}}.\delta\boldsymbol{u}.$$

where we have set $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\boldsymbol{\sigma}^* = \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}^*)$, and $\delta\boldsymbol{u}$ and $\delta\boldsymbol{u}^*$ are free kinematically admissible with 0 virtual displacements field.

The optimization of the functional $\mathcal{J}$ leads to two sets of equations, representing respectively the primal and the adjoint problem.

The condition $\partial\mathcal{J}/\partial\boldsymbol{u}^* = 0$ corresponds to the equations of the already defined primal problem:

$$\operatorname{div}\boldsymbol{\sigma} = 0, \qquad \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{over } \Omega, \qquad \boldsymbol{\sigma}.\boldsymbol{n} = \boldsymbol{T}_o \quad \text{on } \Gamma_o. \tag{2-3}$$

The variations $\partial\mathcal{J}/\partial\boldsymbol{u} = 0$ corresponds to the equations of the so-called adjoint problem (2-4). This set is satisfied by the adjoint displacement field $\boldsymbol{u}^*$ and the condition of optimality (2-5):

$$0 = \operatorname{div}\boldsymbol{\sigma}^* \quad \text{over } \Omega, \qquad \boldsymbol{\sigma}^*.\boldsymbol{n} = h(\boldsymbol{u} - \boldsymbol{u}_0) \quad \text{on } \Gamma_o, \qquad \boldsymbol{u}^* = 0 \quad \text{on } \Gamma_i, \tag{2-4}$$

$$r\boldsymbol{u} = \boldsymbol{n}.\boldsymbol{\sigma}^* \quad \text{on } \Gamma_i. \tag{2-5}$$

One can further remark that the adjoint problem has the same structure as the primal problem, i.e. linear elasticity and small strain. However, the physical dimensions of the fields depend on the choice of the cost functional. This is a direct mathematical consequence of the fact that linearized elasticity is a self-adjoint problem.

In the next section, we now extend this method for solving inverse problems in the case of viscoplastic and elastoplastic materials.

3. Examples of inverse problem in elastoplasticity

Consider a body Ω with a known local material behaviour. Starting from the initial known shape of the body and an unknown loading history, the body occupies a different residual shape. The inverse problem studied next should estimate both the internal state governed by the plastic strain and a loading history compatible with the measured residual geometry.

The solution method of this inverse problem in elastoplasticity rests on several general assumptions:

- The loading history is given by the surface tractions: $\boldsymbol{T}(x, t_f) = \boldsymbol{T}_o$, $t \in (0, t_f)$ on Γ_o belonging to a class of possible loading histories.
- The elastoplastic evolution problem is solved and consequently the plastic strain $\boldsymbol{\varepsilon}_p(x, t)$ is an direct output of this evolution problem.
- The optimal loading history $\boldsymbol{T}^{\mathrm{op}}$ is chosen among the given class of admissible loading histories, such that the final shape, defined by the displacement field corresponding to the final time step, i.e. $\boldsymbol{u}(x, t_f)$ matches the measured residual displacement $\boldsymbol{u}_o$. This condition is expressed by the minimum of the cost functional J defined by

$$J(\boldsymbol{T}, \boldsymbol{\varepsilon}_p, t_f) = \int_{\Gamma_o} \frac{r}{2} \|\boldsymbol{u}(t_f) - \boldsymbol{u}_o\|^2 \,\mathrm{ds}, \tag{3-1}$$

which measures the mismatch between the two displacement fields.

These assumptions will permit to estimate the loading history and the internal state from the given data $\boldsymbol{u}_o$, $\boldsymbol{T}_o$. The natural control variables introduced in this problem are the histories of the surface traction $\boldsymbol{T}(x, t)$ and the plastic strain field $\boldsymbol{\varepsilon}_p(x, t)$.

To illustrate the solution of the elastoplastic inverse problem using the proposed method, let us consider the following examples: (i) a three-bar lattice under traction, (ii) the bending of a beam in plane strain and (iii) a hollow sphere under pressure.

In a first step, on these examples a particular solution of the direct problem is obtained in order to define a final residual shape. In the second step, starting from this final shape as given data, the optimal control theory provides the solution of the inverse elastoplastic problem.

3A. ***A three-bar lattice under traction.*** Let us consider the problem of an elastoplastic lattice, consisting of three bars of lengths $L_1 = L$, $L_2 = L_3 = L\sqrt{2}$, as depicted in Figure 2. The displacement of the point O is denoted by $\boldsymbol{u} = v\boldsymbol{e}_1 + h\boldsymbol{e}_2$.

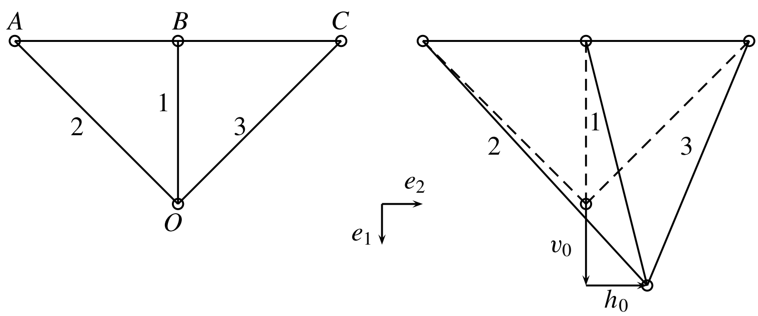

Figure 2. Three elastoplastic bars: initial (left) and residual configuration (right).

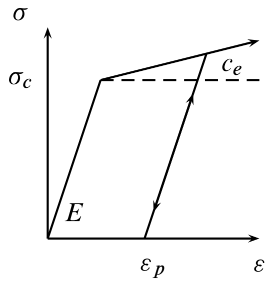

Figure 3. The constitutive law.

In the initial state the bars are stress free. After an unknown loading and unloading history, the lattice is in a final configuration, exhibiting a residual global displacement $\boldsymbol{u}_o = (v_o, h_o)$. This final shape is depending on the internal plastic strains δ_p^i.

Let us assume that the material behaves as an elastoplastic medium with linear hardening law as depicted in Figure 3. The material parameters of the model are: E is the Young's modulus, S is the section of the bar the hardening modulus c_o. One defines $c_e = Ec_o/(E + c_o)$.

Under an external loading, characterized by the force (V, H) applied in O with an increasing and decreasing amplitude the bars recover the static equilibrium in the absence of external forces and the final position of the point O is now (v_o, h_o). Let us assume that the distribution of plastic stretches is $(\delta_i^p,\ i = 1, 3)$ in the bars.

The equilibrium state of the system in terms of displacement, stretches and normal tractions must satisfy the set of equations:

- *Compatibility:*

$$\delta_1 = v, \quad \delta_2 = \frac{v+h}{\sqrt{2}}, \quad \delta_3 = \frac{v-h}{\sqrt{2}}. \tag{3-2}$$

- *Constitutive behaviour:*

$$\delta_1 = K\,N_1 + \delta_1^p, \quad \delta_2 = K\sqrt{2}\,N_2 + \delta_2^p, \quad \delta_3 = K\sqrt{2}\,N_3 + \delta_3^p. \tag{3-3}$$

- *Equilibrium:*

$$V = N_1 + \frac{N_2 + N_3}{\sqrt{2}}, \quad H = \frac{N_2 - N_3}{\sqrt{2}}. \tag{3-4}$$

- *Domain of reversibility:*

$$|N_i - c_i\delta_i^p| \le N_c. \tag{3-5}$$

Here $c_1 = S\dfrac{c_o}{L} = c$, $c_2 = c_3 = \dfrac{c}{\sqrt{2}}$ and $K = \dfrac{L}{ES}$, $N_i = S\sigma_i$, $\varepsilon_i L_i = \delta_i$, $\varepsilon_i^p L_i = \delta_i^p$.

3A1. *The direct problem.* For a given loading history $(V(t), H(t), t \in [0, t_f])$, one can compute the evolution of the system taking into account the elastoplastic constitutive law. The stresses N_i, $i = 1, 3$, lie in the convex set of reversibility, (3-5). The evolution of the plastic stretches is governed by the normality rule described as

$$|N_i - c_i\delta_i^p| \le N_c, \quad \lambda_i \ge 0, \quad \dot{\delta}_i^p = \lambda_i \frac{N_i - c_i\delta_i^p}{N_c} = \lambda_i n_i, \quad \lambda_i(|N_i - c_i\delta_i^p| - N_c) = 0. \tag{3-6}$$

In the initial configuration there is no external loads and no prestresses, therefore the lattice is considered stress free. As a consequence, the plastic stretches are identically to zero until one of the normal traction N_i reaches the critical yield value $\pm N_c$. The initial domain of reversibility, corresponding to the initial elastic domain of the lattice, in terms of external loads (V, H) is obtained by solving the system consisting of (3-2), (3-3), (3-4) with vanishing plastic stretches $\delta_i^p = 0$ and the imposing the inequations $|N_i| - N_c \le 0$:

$$|N_1| = \left|\frac{2V}{2+\sqrt{2}}\right| \le N_c, \quad |N_2| = \left|\frac{V}{2+\sqrt{2}} + \frac{H}{\sqrt{2}}\right| \le N_c, \quad |N_3| = \left|\frac{V}{2+\sqrt{2}} - \frac{H}{\sqrt{2}}\right| \le N_c.$$

If the applied load (V, H) leaves the reversibility domain, plasticity occurs. In this case, after a loading-unloading cycle a distribution of plastic stretches δ_i^p is obtained in the bars of the lattice. Therefore, when the applied stresses (V, H) return to $(0, 0)$, the lattice will not be stress free and will exhibit a residual shape (v^p, h^p). Let us denote, the residual stresses, i.e. tractions in the bars, N_i^r.

For the application of an external load, starting from this state, a new equilibrium of tractions in each bar N_i is obtained satisfying the equalities

$$N_1 = \frac{2V}{2+\sqrt{2}} + N_1^r, \quad N_2 = \frac{V}{2+\sqrt{2}} + \frac{H}{\sqrt{2}} + N_2^r, \quad N_3 = \frac{V}{2+\sqrt{2}} - \frac{H}{\sqrt{2}} + N_3^r. \tag{3-7}$$

Again this state has to be compatible with the actual domain of reversibility, which takes into account the existence of residual stresses: $|N_i - c_i\delta_i^p| - N_c \le 0$.

At the final unloaded state, when the external loads are $(H, V) = (0, 0)$, the global displacement is (v^p, h^p) and the local stretches $\delta_i = \delta_i^{ir}$ of the final residual configuration satisfy the equations (3-2), (3-3), (3-4). Let us now remark, that the internal residual stresses N_i^r depend only of the incompatibility of the plastic stretches $\delta_{\text{in}} = \sqrt{2}\delta_1^p - \delta_2^p - \delta_3^p$ and

$$N_1^r = -\frac{\delta_{\text{in}}}{K(2+\sqrt{2})}, \quad N_2^r\sqrt{2} = N_3^r\sqrt{2} = \frac{\delta_{\text{in}}}{K(2+\sqrt{2})}. \tag{3-8}$$

Moreover, the residual shape satisfies

$$v^p = \frac{2}{2+\sqrt{2}}(\delta_1^p + \tfrac{1}{2}(\delta_2^p + \delta_3^p)), \quad h^p\sqrt{2} = \delta_2^p - \delta_3^p. \tag{3-9}$$

3A2. *The solution for a loading history with $H = 0$.* Consider a loading-unloading process with $H = 0$. In this case, $N_2 = N_3$ and $h = 0$. The phase of increasing V will be decomposed in three steps as a function of the maximum value of V_m.

- *Elastic step.* For increasing V, the first part of the loading corresponds to a linear elastic response of the lattice
$$v = KN_1 = 2KN_2, \quad N_1 + \sqrt{2}N_2 = V. \tag{3-10}$$
This occurs under the condition $V_m \le V_1$:
$$v = \frac{K\sqrt{2}}{1+\sqrt{2}}V, \quad V \le V_1\frac{1+\sqrt{2}}{\sqrt{2}}N_c. \tag{3-11}$$

- *Plasticity of bar 1.* In the case when $V_m \geq V_1$, bar 1 is deformed plastically, starting with the assumption $\delta_2^p = \delta_3^p = 0$, one obtains

$$N_1 = N_c + c\delta_1^p, \quad K N_1 = v - \delta_1^p, \tag{3-12}$$

then

$$\delta_1^p = \frac{v - K N_c}{1 + Kc}, \quad V = \frac{N_c}{1 + Kc} + \Big(\frac{c}{1+Kc} + \frac{1}{K\sqrt{2}}\Big)v. \tag{3-13}$$

Let us consider a state δ_1^p, and a variation of the external loading $\dot{V} > 0$, then $\dot{\delta}_1^p \geq 0$ and the solution satisfies the normality rule. This solution is valid if and only if $|N_2| - N_c \leq 0$. The inequality implies that $V_m \leq V_2$ where V_2 is given by

$$V_2 = (1 + \sqrt{2})N_c + \frac{Kc}{1 + Kc} N_c. \tag{3-14}$$

- *Plasticity of the whole system.* When $V_m > V_2$ the three bars are deformed plastically. The system consisting of (3-2), (3-4), (3-3) is rewritten with $h = 0$, $\delta_2^p = \delta_3^p$, and the condition of plasticity:

$$N_1 = N_c + c\delta_1^p, \quad N_2 = N_c + \frac{c}{\sqrt{2}}\delta_2^p. \tag{3-15}$$

Using (3-3) and (3-4), we have

$$v = K N_c + (1 + Kc)\delta_1^p = 2K N_c + \sqrt{2}(1 + Kc)\delta_2^p, \tag{3-16}$$

$$V = N_c(1 + \sqrt{2}) + \frac{cK}{1 + Kc} N_c + \frac{c}{1 + Kc}\frac{1 + \sqrt{2}}{\sqrt{2}}(v - 2K N_c). \tag{3-17}$$

The residual displacement is obtained as

$$\frac{1 + \sqrt{2}}{\sqrt{2}} v^p = \delta_1^p + \delta_2^p. \tag{3-18}$$

3A3. *Inverse problems.* Let us assume that the residual shape is given by the displacement $\boldsymbol{u}_o = (v_o, u_o)$. We shall determine the best history of loading $(H(t), V(t), t \in [0, t_f])$ given by the problem of optimization based on the functional

$$J = \tfrac{1}{2}(v(t_f) - v_o)^2 + \tfrac{1}{2}(h(t_f) - h_o)^2 = \tfrac{1}{2}\big\|\boldsymbol{u}(\delta_p^i)(t_f) - \boldsymbol{u}_o\big\|^2, \tag{3-19}$$

According to this mathematical definition, the solution $(v(t_f), h(t_f))$ of the direct problem obtained from the optimal history is close to the measured displacement (v_o, h_o). Generally, the solution of this problem is not unique, there are several local minima and some restrictions on the history and on the plastic strain must be added to obtain a unique solution.

There are two classes of inverse problems which can be illustrated by the simple example. Class 1 consists in estimating the internal state without determining a loading history, taking as control variables the plastic stretches. Class 2 addresses the determination of a complete history of the loading and of the internal state.

Class 1: *Estimation of the plastic strain.* In this case, the plastic stretches are the control variables and we consider only the residual state. We seek to optimize the internal state such that the functional $J_o(\delta_i^p)$ is minimum, where J_o is

$$J_o = \tfrac{1}{2}r(v - v_o)^2 + \tfrac{1}{2}r(h - h_o)^2 = \frac{r}{2}\left\|\boldsymbol{u}(\delta_p^i) - \boldsymbol{u}_o\right\|^2. \tag{3-20}$$

In this functional, we must solve the system satisfied by the residual state only. Eliminating the equations of compatibility, we can introduce an adjoint state (h^*, v^*) to take the equilibrium into account simultaneously with the local constitutive law. Introducing the rigidities $C_1 = \dfrac{1}{K}$, $C_2 = C_3 = \dfrac{1}{K\sqrt{2}}$ of each bar and the functional $\mathscr{L}$, given by

$$\mathscr{L}(\boldsymbol{u}^*, \boldsymbol{u}, \delta_i^p) = -v^* C_1(v - \delta_1^p) - \frac{v^* + h^*}{\sqrt{2}} C_2\left(\frac{v+h}{\sqrt{2}} - \delta_2^p\right) - \frac{v^* - h^*}{\sqrt{2}} C_3\left(\frac{v-h}{\sqrt{2}} - \delta_3^p\right), \tag{3-21}$$

the equilibrium satisfies the variational formulation

$$\frac{\partial \mathscr{L}}{\partial \boldsymbol{u}^*} = 0, \quad \forall \boldsymbol{u}^* = (h^*, v^*). \tag{3-22}$$

Now the inverse problem is solved by minimization of the functional

$$\hat{J}(\boldsymbol{u}, \boldsymbol{u}^*, \delta_i^p) = J_o + \mathscr{L}. \tag{3-23}$$

In general the functional $\hat{J}$ posseses many minima and the addition of constraints is then necessary to ensure uniqueness.

For example we can minimize simultaneously a norm of the plastic stretches $D = N_c \sum_i \frac{1}{2} L_i (\delta_p^i)^2$, and the functional takes the form:

$$J(\delta_p^i) = \frac{r}{2}\|\boldsymbol{u}(\delta_p^i) - \boldsymbol{u}_r\|^2 - \mathscr{L} + q N_c \sum_i \tfrac{1}{2} L_i (\delta_p^i)^2.$$

Outline of proof. The variations with respect to $\boldsymbol{u}^*$ ensure the equilibrium equations

$$N_1 + \frac{1}{\sqrt{2}}(N_2 + N_3) = 0, \quad \frac{1}{\sqrt{2}}(N_2 - N_3) = 0, \tag{3-24}$$

and the constitutive law

$$N_1 = C_1(v - \delta_1^p), \quad N_2 = C_2\left(\frac{v+h}{\sqrt{2}} - \delta_2^p\right), \quad N_3 = C_3\left(\frac{v-h}{\sqrt{2}} - \delta_2^p\right). \tag{3-25}$$

Denoting the adjoint stresses by

$$N_1^* = C_1 v^*, \quad N_2^* = C_2 \frac{v^* + h^*}{\sqrt{2}}, \quad N_3^* = C_3 \frac{v^* - h^*}{\sqrt{2}}. \tag{3-26}$$

From the variations with respect to $\boldsymbol{u} = (h, v)$, we obtain the equations of the adjoint state:

$$N_1^* + \frac{1}{\sqrt{2}}(N_2^* + N_3^*) = r(v - v_o), \quad \frac{1}{\sqrt{2}}(N_2^* - N_3^*) = r(h - h_o). \tag{3-27}$$

Finally the variations with respect to δ_i^p give the conditions of optimality

$$N_i^* + q N_c L_i \delta_i^p = 0. \tag{3-28}$$

The system is complete and has a unique solution depending on the choice of r, q.

However, if the norm D is changed into the internal stored energy W in the residual tension N_i^r,

$$W = \tfrac{1}{2} L\big((N_1^r)^2 + \sqrt{2}(N_2^r)^2 + \sqrt{2}(N_3^r)^2\big), \tag{3-29}$$

this energy is convex in δ_{in} but not in δ_i^p, the resulting functional is not convex in δ_p^i and the uniqueness is not guaranteed.

Class 2: *Estimation of the loading history.*

In order to obtain information about the history of loading the direct problem of evolution must be solved for a class of loading.

For example, consider the family of radial loading given by $(H, V) = \mu(t)(H_o, V_o)$, if we assume that during the unloading step no plasticity occurs, then the loading is characterized by the maximum of μ and the direction (H_o, V_o).

From the given residual shape (v_o, h_o), we apply a loading (V, H) and assuming that the answer is purely elastic, we can define (v_m, h_m) by

$$v_m = v_o + \frac{K\sqrt{2}}{1+\sqrt{2}} V, \qquad h_m = h_o + K\sqrt{2} H. \tag{3-30}$$

For an estimation δ_i^p of the plastic stretches, the domain of reversibility is known and (V, H) must be inside this domain. We propose to find δ_i^p and (V, H) such that the displacement (v, h), satisfying the problem of equilibrium and the domain of reversibility, is close to the displacement (v_m, h_m). For that purpose we introduce the functional

$$\mathcal{J}(\boldsymbol{u}, \boldsymbol{u}^*, \delta_i^p, V, H) = \frac{r}{2}\|\boldsymbol{u} - \boldsymbol{u}_m\|^2 + \mathcal{L} + \sum_i \frac{\alpha_i L_i}{2}(|N_i - c_i \delta_i^p| - N_c)^2. \tag{3-31}$$

In this expression, the tractions N_i satisfy the constitutive law (3-3). The three constants α_i are chosen as $\alpha_i = \alpha > 0$ or $\alpha_i = 0$ depending or whether the bar i has been deformed plastically or not.

Denoting the adjoint traction by N_i^* as in (3-26), it is easy to prove that the optimality conditions on $\mathcal{J}$ with respect to $\boldsymbol{u}^*$, that is $\dfrac{\partial \mathcal{J}}{\partial \boldsymbol{u}^*} = 0$ give exactly the equations (3-4), (3-3). The conditions with respect to $\boldsymbol{u}$: $\dfrac{\partial \mathcal{J}}{\partial \boldsymbol{u}} = 0$ are equivalent to the adjoint problem (n_i being defined as in (3-6)):

$$r(v - v_m) = -\sum_{i=1}^{3} n_i \alpha_i L_i C_i (|N_i - c_i \delta_i^p| - N_c) + N_1^* + \frac{N_2^* + N_3^*}{\sqrt{2}}, \tag{3-32}$$

$$r(h - h_m) = -\sum_{i=2}^{3} n_i \alpha_i L_i C_i (|N_i - c_i \delta_i^p| - N_c) + \frac{N_2^* - N_3^*}{\sqrt{2}}. \tag{3-33}$$

The optimization with respect to δ_i^p gives

$$N_i^* - \alpha_i n_i (|N_i - c_i \delta_i^p| - N_c) \frac{1+Kc}{K} = 0, \tag{3-34}$$

and finally the optimization with respect to the loading (V, H) defines the adjoint displacement

$$v^* = r(v - v_m) \frac{K\sqrt{2}}{1+\sqrt{2}}, \quad u^* = r(h - h_m) K\sqrt{2}. \tag{3-35}$$

The system is complete. For a given residual shape, we must choose the α_i a priori. For a solution of the optimization, we must verify that the inequalities (3-5) are satisfied.

3A4. *Inverse problem when $h_o = 0$.* In this case, it is natural to consider that $\delta_2^{ir} = \delta_3^{ir}$ and then $H = 0$, $N_2 = N_3$ and $\delta_2 = \delta_3$.

We consider $V \geq 0$, then $n_i = 1$. For $v_o \neq 0$, we consider first that the plasticity occured only in bar 1, then $\alpha_2 = \alpha_3 = 0$ and $\delta_2^p = \delta_3^p = 0$. Then, $N_2^* = N_3^* = 0$, this implies that $v^* = 0$ and $|N_1 - c\delta_1^p| - N_c = 0$ simultaneously with $v = v_m$. We deduce immediatly that

$$v = \frac{2K}{2+\sqrt{2}} V + \frac{2}{2+\sqrt{2}} \delta_1^p, \quad \delta_1^p = \frac{2}{2+\sqrt{2}} v_o, \tag{3-36}$$

$$V = (1+\sqrt{2}) N_c + \frac{c}{1+Kc}(v - KN_c) + \frac{K}{\sqrt{2}} v. \tag{3-37}$$

This is the solution if the traction in the bars (2 and 3) are in the domain of reversibility ($N_2 \leq N_c$). This condition implies $v = 2KN_2 < 2KN_c$ that is equivalent to $V \leq V_2$. If this condition is not fulfilled, we consider that $\alpha_2 = \alpha_3 = \alpha$ and $u^* = 0$. The equilibrium implies that

$$V = \frac{v}{K} \frac{1+\sqrt{2}}{\sqrt{2}} - \frac{\delta_1^p + \delta_2^p}{K}, \quad v^p = \frac{\sqrt{2}}{1+\sqrt{2}} (\delta_1^p + \delta_2^p). \tag{3-38}$$

Using the definition of the adjoint stresses (3-26), the optimisation with respect to δ_i^p two relations and the optimisation with respect to v we obtain

$$v^* = \alpha (N_1 - c\delta_1^p - N_c)(1+Kc), \tag{3-39}$$

$$v^* = 2\alpha \left(N_2 - \frac{c}{\sqrt{2}} \delta_2^p - N_c \right)(1+Kc), \tag{3-40}$$

$$r(v^p - v_o) = -\frac{1+\sqrt{2}}{K\sqrt{2}} v^* + \frac{1+2\sqrt{2}}{K(1+Kc)} v^*. \tag{3-41}$$

The optimality condition (3-35), after the elimination of v, implies that $v^* = 0$ and $v^p = v_o$. The value of V is recovered.

For the given residual state obtained from a radial loading-unloading history, we recover the maximum loading and the internal state using the optimal control theory.

3B. ***Study of an elastoplastic beam in plane motion.*** The beam is initially rectilinear. After an unknown vertical distribution of loading, the beam takes a residual shape due to plastic strain.

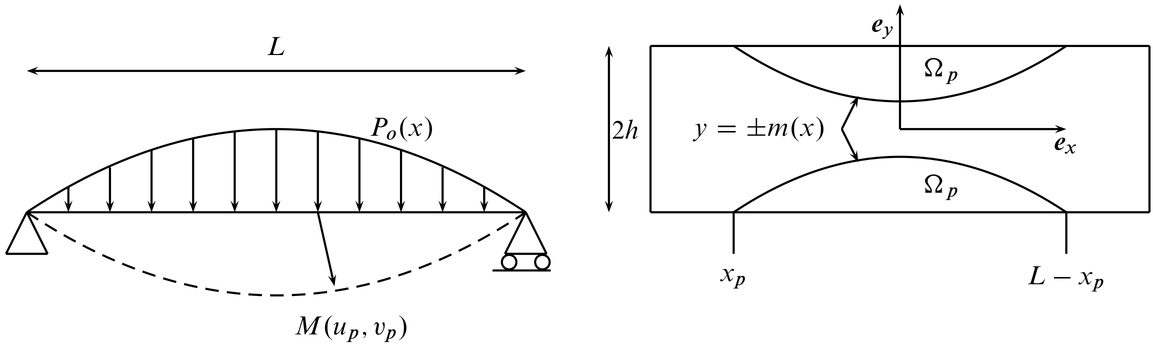

Figure 4. The initial state and final state (- -) of the beam. The geometry and the plastic zone Ω_p.

The goal of the inverse problem is to estimate a plastic strain distribution with an associated loading which is compatible with this residual shape.

The plane motion of the beam is characterized by the displacement u along $\boldsymbol{e}_x$ and a vertical displacement v in direction $\boldsymbol{e}_y$. We consider that $u(0,t) = v(0,t) = v(L,t) = 0$. We assume that the stress and the total strain are uniaxial:

$$\boldsymbol{\sigma} = \sigma \boldsymbol{e}_x \otimes \boldsymbol{e}_x, \quad \boldsymbol{\varepsilon} = (u' + y v'')\boldsymbol{e}_x \otimes \boldsymbol{e}_x, \tag{3-42}$$

and the plastic strain is described by the function $\alpha(x, y)$:

$$\boldsymbol{\varepsilon}_p = \alpha(x, y)\boldsymbol{e}_x \otimes \boldsymbol{e}_x. \tag{3-43}$$

Then the free energy is given by

$$w(u, v, \alpha) = \tfrac{1}{2}E(u' + y v'' - \alpha)^2 + \tfrac{1}{2}H\alpha^2. \tag{3-44}$$

The thermodynamical force A associated to the plastic strain must be within the domain of reversibility

$$A = \sigma - H\alpha, \quad f(A) = \|A\| - k \le 0, \tag{3-45}$$

and the evolution of the plastic strain inside the beam is given by the normality rule.

$$\dot{\alpha} = \lambda A/\|A\|, \quad \lambda f = 0, \quad \lambda \ge 0, \quad f \le 0. \tag{3-46}$$

For a given distribution of α and a given pressure on the beam, the displacement (u, v) minimizes the potential energy of the beam

$$E(u, v, \alpha) = \int_0^L \int_{-h}^{h} w \, dy \, dx + \int_0^L p(x) v(x) \, dx. \tag{3-47}$$

The traction N and the moment M are defined as

$$N = ES(u' - < \alpha >), \quad M = EIv'' - ES\langle y\alpha\rangle, \tag{3-48}$$

where $S = 2h$, $I = \frac{2}{3}h^3$, $\langle f \rangle = \dfrac{1}{2h}\displaystyle\int_{-h}^{h} f\,dy$. With these notations, the equilibrium of the beam satisfies

$$N_{,x} = 0, \quad M_{,xx} + p = 0, \quad N(L) = 0, \quad M(0) = M(L) = 0. \tag{3-49}$$

Then the thermodynamical force A is

$$A = \frac{N}{S} + E\langle\alpha\rangle + y\left(\frac{M}{I} + \frac{ES}{I}\langle y\alpha\rangle\right) - (E+H)\alpha. \tag{3-50}$$

The norm of A does not exceed k.

3B1. *The direct problem for a loading-unloading process.* For a distribution of pressure $p(x) = \mu(t)P_o(x)$ with $\mu(t) = qt$ for $t \in [0, t_m]$ and $\mu(t) = q(t_m - t)$ for $t \in [t_m, 2t_m]$, we solve the direct problem.

We consider that the plastic zone Ω_p is symmetric with respect to $\boldsymbol{e}_x$, then $\alpha(x, y, t) = \alpha(x, -y, t)$. The value of the thermodynamical force A determines the form of $\alpha(x, y)$:

$$\alpha(x, y, t) = \alpha_o(x, t) + y\alpha_1(x, t). \tag{3-51}$$

The evolution of the beam follows three phases: a response purely elastic, an evolution with plasticity, and a purely elastic unloading.

• *Elastic phase:* For the proposed loading, the response of the beam is linear elastic when $f(A) \leq 0, \forall(x, y)$ and

$$\alpha(x, y) = 0 \quad \text{for } t < t_c.$$

At $t = t_c$ some points of the beam are such that $f(A) = 0$. After this critical time ($t > t_c$), the plasticity develops inside the beam, the boundary of the plastic zone Ω_p is the plane curve $y = \pm m(x, t)$.

• *Phase of loading in plasticity:* In the plastic zone Ω_p

$$\Omega_p = \{(x, y) \in \Omega /\ y \leq -m(x, t),\ y \geq m(x, t),\ x \in]x_p(t), L - x_p(t)[\,\}, \tag{3-52}$$

the thermodynamical force $A(x, y)$ satisfies $A = k$, this implies that

$$\alpha_o = -\frac{k}{E+H}, \quad \langle y\alpha\rangle = \frac{1}{2h}\int_{-h}^{m} y\alpha\,dy. \tag{3-53}$$

And we obtain

$$\frac{M}{I} + \frac{kE}{I(E+H)}(h^2 - m^2) - \frac{2E}{3I}m^3\alpha_1 - H\alpha_1 = 0. \tag{3-54}$$

The plastic strain is then determined if we know the equation of the curve $y = \pm m(x, t)$.

In the complementary domain, namely, $-m(x, t) \geq y \geq m(x, t)$ for $x \in [x_p(t), L - x_p(t)]$, the local response is purely elastic:

$$A = \frac{My}{I} + \frac{2Ey}{I}\frac{k}{2(E+H)}(h^2 - m^2) + \frac{2Ey}{3I}\alpha_1(h^3 - m^3). \tag{3-55}$$

On the boundary of this domain, the continuity of A determines the equation of the boundary $y = -m(x, t)$:

$$-k = \frac{Mm}{I} + \frac{2Em}{I}\frac{k}{2(E+H)}(h^2 - m^2) + \frac{2Em}{3I}\alpha_1(h^3 - m^3). \tag{3-56}$$

In the complementary domain, namely for $x \in]0, x_p[\cup]x_p, L[$, we have

$$A = \frac{My}{I}. \tag{3-57}$$

For the sake of simplicity, we consider now the case of perfect plasticity, $H = 0$, for which an explicit equation for the boundary is obtained:

$$m(x,t) = \sqrt{3\Big(h^2 + \frac{M(x,t)}{k}\Big)}. \tag{3-58}$$

The local fields satisfy the equations

$$u(x,t) = 0, \tag{3-59}$$

$$v''(x,t) = \frac{k}{Em} \qquad \text{if } x \in [x_p(t), L - x_p(t)], \tag{3-60}$$

$$v''(x,t) = -\frac{M}{EI} \qquad \text{otherwise}, \tag{3-61}$$

and the plastic strain is

$$\alpha(x,y) = \frac{k}{E}\Big(\operatorname{sign}(y) - \frac{y}{m}\Big). \tag{3-62}$$

These equations are true for $t \in [0, t_m]$. At $t = t_m$ the moment M is maximum M_m.

• *Phase of unloading:* From $t \geq t_m$ the loading decreases. Let us assume that the loading rests within the domain of reversibility, then $\lambda(x, y) = 0$ and the unloading is purely elastic and that we obtain the value $M(x, 2t_m) = M_m(x)$ at $t = 2t_m$ then the solution satisfies:

- In the plastic zone, namely for $x \in [x_p(t_m), L - x_p(t_m)]$,

$$v_f''(x) = v''(x, 2t_m) = \frac{k}{E\sqrt{3\Big(h^2 + \frac{M_m(x)}{k}\Big)}} + \frac{M_m(x)}{EI} = \psi(M_m). \tag{3-63}$$

- In the complementary part,

$$v_f''(x) = 0 = \psi(M_m). \tag{3-64}$$

This solution determines the residual shape if the condition of elastic unloading is satisfied that is $M(x,t) \leq -\frac{2}{3}kh^2$.

The distribution of the moment M_m depends only on $P_o(x)$ and then the residual shape is governed by the equations

$$M_m'' = P_o, \quad v'' = \psi(M_m), \quad M_m(0) = M_m(L) = 0, \quad v(0) = v(L) = 0. \tag{3-65}$$

3B2. *The inverse problem.* We assume that the residual displacement v_o is given, and we try to determine the best process p of loading-unloading which gives a displacement $v(p)$ as close as possible to v_o by solving the direct problem. The loading p is controlled such that the functional

$$J(p) = \frac{1}{2}\int_0^L (v(p) - v_o)^2\,dx + \frac{1}{2}r\int_0^L p^2\,dx. \tag{3-66}$$

For a given family $p(x,t)$ we can solve the direct problem, and then optimize J. The equations of the direct problem are equivalent to the variational form by introducing the functional $\mathscr{L}$, which depends on adjoint fields:

$$\mathscr{L} = \int_0^L \big(M^*(p - M'') + v^*(\psi(M) - v'')\big)\,dx. \tag{3-67}$$

The solution of the inverse problem is obtained as optimization of $\mathscr{J} = J + \mathscr{L}$, the variations for the adjoint fields give rise to the equations of the direct problem, and the condition of optimality is obtained as a boundary condition:

$$0 = \int_0^L (M^* + rp)\,p^*\,dx. \tag{3-68}$$

Assume that $v_o(x)$ is the solution of the elastoplastic problem with $P_o(x) = p_o$. The goal of the inverse problem is to find the best $p(x,t)$ for which the displacement $v(p)$ is close to v_o.

The analysis with uniform $p(x,t)$ gives the exact solution. It is obvious because the solution of the direct problem with uniform p satisfies the equations of the inverse problem and the condition of optimality determines the value of p.

Other profiles for $P(x)$ can be used. For example, the choice of a triangular shape or a sinus shape for $P_o(x)$ gives a very close plastic zone. This can be shown numerically. The error on the shape does not exceed 1%.

4. Estimation of the internal state in elastoplasticity.

Consider now a body Ω. The body is submitted to an increasing loading, but the history of the loading is not known. Only the final shape $\boldsymbol{u}_o$ and the final loading $\boldsymbol{T}_o$ on the boundary Γ_o are known.

For an elastoplastic material, the stress satisfies the constitutive law

$$\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon}(u) - \boldsymbol{\varepsilon}_p), \quad f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_p) \le 0, \tag{4-1}$$

where the domain of reversibility $f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_p) \le 0$, is defined by the convex function f.

These stresses are in equilibrium with given boundary conditions ($\boldsymbol{T}_o$ on Γ_o) and are divergence free if there is no body force:

$$\operatorname{div} \boldsymbol{\sigma} = 0 \quad \text{over } \Omega, \qquad \boldsymbol{n}.\boldsymbol{\sigma} = \boldsymbol{T}_o \quad \text{over } \Gamma_o. \tag{4-2}$$

The plastic strain is isochoric:

$$\operatorname{Tr} \boldsymbol{\varepsilon}_p = 0. \tag{4-3}$$

To estimate the internal state, from the known residual shape, we can solve a problem of linear elasticity controlled by $\boldsymbol{\varepsilon}_p$: find the displacement $\boldsymbol{v}$ satisfying the following conditions:

- *Compatibility*: $2\boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla \boldsymbol{u} + \nabla^t \boldsymbol{u}$,
- *Constitutive law*: $\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p)$,
- *Boundary condition*: $\boldsymbol{n}.\boldsymbol{\sigma} = \boldsymbol{T}_o$ over Γ_o, $\boldsymbol{u} = 0$ over Γ_i.

The best estimation of $\boldsymbol{\varepsilon}_p$ must be such that on Γ_o the displacement solution of the direct problem $\boldsymbol{v}_{\text{sol}}(\boldsymbol{\varepsilon}_p)$ is close to $\boldsymbol{u}_o$ on Γ_o.

Previously, we proposed to solve the problem of evolution for an elastoplastic material for a given class of loading history, but this is a tedious task, because we must solve the direct problem many times to determine the optimal state. We propose to consider another way to estimate the elastoplastic state.

To obtain a solution of the optimization close to a problem of elastoplasticity, we decompose the body Ω into two domains, one where $\boldsymbol{\varepsilon}_p = 0$ and a plastic zone Ω_p where the stresses are close to the domain of reversibility.

These constraints are prescribed during the optimization process by introducing new terms in the functional to minimize:

$$J(\boldsymbol{u}, \boldsymbol{\varepsilon}_p, \Omega_p) = \frac{1}{2}\int_{\Gamma_o} h\|\boldsymbol{u}-\boldsymbol{u}_o\|^2 \,\mathrm{d}\Omega + \frac{\beta}{2}\int_{\Omega_p} \big(f(\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{\varepsilon}_p), \boldsymbol{\varepsilon}_p)\big)^2 \,\mathrm{d}\Omega. \tag{4-4}$$

The yielding function $f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}_p)$ is rewritten in terms of the local behaviour:

$$f(\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{\varepsilon}_p), \boldsymbol{\varepsilon}_p) = \mathrm{Y}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}_p). \tag{4-5}$$

To satisfy the problem of equilibrium, we introduce an adjoint state such that

$$\int_\Omega \nabla\boldsymbol{u}^* : \mathbb{C} : (\nabla\boldsymbol{u} - \boldsymbol{\varepsilon}_p)\,\mathrm{d}\Omega - \int_{\Gamma_o} \boldsymbol{u}^*.\boldsymbol{T}_o \,\mathrm{d}s = 0 \quad \text{for all } \boldsymbol{u}^* \text{ satisfying } \boldsymbol{u}^* = 0 \text{ over } \Gamma_i, \tag{4-6}$$

where the constitutive law has been taken into account. The displacement $\boldsymbol{u}$ and the adjoint displacement $\boldsymbol{u}^*$ are continuous along the boundary of the plastic zone.

Then, the functional to be optimized is reduced to

$$\begin{aligned}\mathcal{J}(\overset{\star}{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{\varepsilon}_p, \Omega_p) = \frac{1}{2}\int_{\Gamma_o} h\|\boldsymbol{u}-\boldsymbol{u}_o\|^2 \,\mathrm{d}\Omega + \frac{\beta}{2}\int_{\Omega_p} (\mathrm{Y}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}_p))^2 \,\mathrm{d}\Omega,\\ - \int_\Omega \nabla\overset{\star}{\boldsymbol{u}} : \mathbb{C} : (\nabla\boldsymbol{u} - \boldsymbol{\varepsilon}_p)\,\mathrm{d}\Omega + \int_{\Gamma_o} \overset{\star}{\boldsymbol{u}}.\boldsymbol{T}_o \,\mathrm{d}s.\end{aligned}$$

4A. ***Elements of proof.*** The displacements $\boldsymbol{u}$ and $\overset{\star}{\boldsymbol{u}}$ are continuous on the boundary of the plastic zone Γ_p, therefore their variations satisfy Hadamard conditions of continuity along Γ_p. When the position of Γ_p is moved by its normal velocity $\delta\phi$, the variations of the displacement satisfy

$$|[\delta\boldsymbol{u}]| + \delta\phi|[\nabla\boldsymbol{u}]|.\boldsymbol{n} = 0, \quad |[\delta\overset{\star}{\boldsymbol{u}}]| + \delta\phi|[\nabla\overset{\star}{\boldsymbol{u}}]|.\boldsymbol{n} = 0. \tag{4-7}$$

- *Variations with respect to* $\overset{\star}{\boldsymbol{u}}$ ($\partial\mathcal{J}/\partial\overset{\star}{\boldsymbol{u}} = 0$) are equivalent to the equations of the direct problem for a given distribution of $\boldsymbol{\varepsilon}_p$

$$\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p), \qquad \operatorname{div}\boldsymbol{\sigma} = 0, \qquad \boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{T}_o \quad \text{over } \Gamma_o, \qquad |[\boldsymbol{\sigma}]|.\boldsymbol{n} = 0 \quad \text{over } \Gamma_p. \tag{4-8}$$

- *The variations with respect to* $\boldsymbol{u}$ ($\partial\mathcal{J}/\partial\boldsymbol{u} = 0$) determine the adjoint state:

$$\begin{aligned}\overset{\star}{\boldsymbol{\sigma}} &= \mathbb{C} : \boldsymbol{\varepsilon}(\overset{\star}{\boldsymbol{u}}) - \beta\mathrm{Y}\frac{\partial\mathrm{Y}}{\partial\boldsymbol{\varepsilon}} && \text{over } \Omega_p,\\ \overset{\star}{\boldsymbol{\sigma}} &= \mathbb{C} : \boldsymbol{\varepsilon}(\overset{\star}{\boldsymbol{u}}) && \text{over } \Omega/\Omega_p,\\ \operatorname{div}\overset{\star}{\boldsymbol{\sigma}} &= 0, &&\\ \overset{\star}{\boldsymbol{\sigma}}.\boldsymbol{n} &= h(\boldsymbol{u}-\boldsymbol{u}_o) && \text{over } \Gamma_o,\end{aligned}$$

$$|[\overset{\star}{\boldsymbol{\sigma}}]|.\boldsymbol{n} = \beta \mathrm{Y} \frac{\partial \mathrm{Y}}{\partial \boldsymbol{\varepsilon}}.\boldsymbol{n} \qquad \text{over } \Gamma_p.$$

- *The condition of optimality* with respect to $\boldsymbol{\varepsilon}_p$ determines the plastic strain

$$\operatorname{dev}(\mathbb{C} : \nabla \overset{\star}{\boldsymbol{u}}) + \beta \mathrm{Y} \frac{\partial \mathrm{Y}}{\partial \boldsymbol{\varepsilon}_p} = 0, \tag{4-9}$$

where $\operatorname{dev}(\boldsymbol{a}) = \boldsymbol{a} - \operatorname{tr}(\boldsymbol{a})\boldsymbol{I}/3$.

- *Variation with respect to the domain* Ω_p*.* Finally, the condition of optimality on the plastic zone gives an equation of continuity of the energy

$$-\big|[\boldsymbol{\varepsilon}(\overset{\star}{\boldsymbol{u}}) : \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p)]\big| + \boldsymbol{\sigma} : \big|[\nabla \overset{\star}{\boldsymbol{u}}]\big| + \overset{\star}{\boldsymbol{\sigma}}_- : \big|[\nabla \boldsymbol{u}]\big| + \tfrac{1}{2}\beta \mathrm{Y}^2 = 0. \tag{4-10}$$

This optimization permits the determination of a plastic zone with an evaluation of the plastic strain.

The quality of the solution depends on the choice of the coefficients α, β, k. That can be investigated through analytical solutions for elastoplastic materials on simple geometries such as cylinders or spheres.

Consider the solution of a direct problem of an increasing loading process. The final state is a distribution of $\boldsymbol{\varepsilon}_p$, such that along the boundary of the plastic zone, the plastic strain is $\boldsymbol{\varepsilon}_p = 0$, and $f = 0$. Then it is obvious that the solution of the evolution problem satisfies the minimum of the proposed functional. It can be noticed that the condition of optimality (4-10) is verified, because the plastic strain vanishes on Γ_p and $\mathrm{Y} = 0$, the other contributions are identically zero because the elastic moduli are continuous, the displacements $\boldsymbol{u}$, the traction $\boldsymbol{\sigma}.\boldsymbol{n}$ are continuous and $\overset{\star}{\boldsymbol{\sigma}} = 0$, $\overset{\star}{\boldsymbol{u}} = 0$.

4B. ***Case of a hollow sphere.*** The solution of a hollow sphere under radial tension is well known for an elasto perfectly-plastic material. The external radius of the sphere is R_e, the radius of the void is R_i and the porosity is $c = R_i^3/R_e^3$ (see Figure 5). For an increasing loading the internal state is determined and is used for the boundary conditions applied to R_e for the inverse problem.

4B1. *Solution of the direct problem.* The plastic zone Ω_p is the spherical domain $r \in [R_i, R_p]$, where $p = R_p^3/R_e^3$. The solution is purely radial: $\boldsymbol{u} = u(r)\boldsymbol{e}_r$. The Cauchy stress is given by

$$\boldsymbol{\sigma} = \sigma_{rr}(r)\boldsymbol{e}_r \otimes \boldsymbol{e}_r + \sigma_{tt}(r)(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi), \tag{4-11}$$

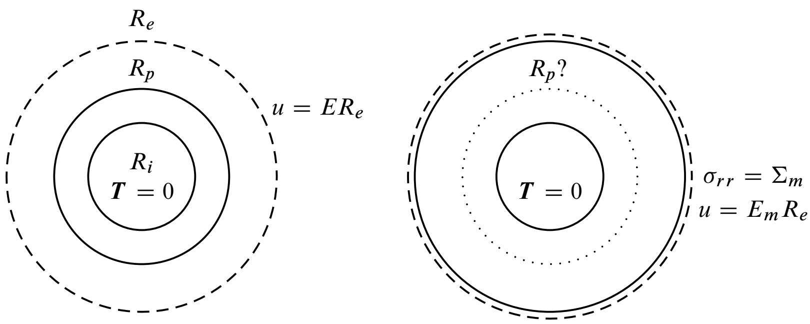

Figure 5. The hollow sphere. Left: the direct problem. Right; the inverse problem under given boundary conditions on R_e. On the dashed circle the displacement is prescribed, on the solid-line circle the traction is given.

and the domain of reversibility is determined by $\mathrm{Y} = \sigma_{tt} - \sigma_{rr} - \sigma_o \leq 0$ where σ_o is a constant. The plastic strain has the form

$$\boldsymbol{\varepsilon}_p = \varepsilon_p (2\boldsymbol{e}_r \otimes \boldsymbol{e}_r - \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi), \quad \varepsilon_p = \frac{3\kappa + 4\mu}{18\kappa\mu} \sigma_o \Big(1 - \frac{R_p^3}{r^3}\Big). \tag{4-12}$$

For an increasing imposed E the global response of the sphere is

$$\Sigma = 3K(c)(E - E_p) = \sigma_{rr}(R_e), \quad K(c) = (1-c)\frac{4\kappa\mu}{3\kappa c + 4\mu}, \tag{4-13}$$

$$E = \frac{u}{r}(R_e), \qquad E_p = \frac{3\kappa + 4\mu}{18\mu\kappa} \frac{\sigma_o}{1-c} \Big(p - c - c \ln \frac{p}{c}\Big). \tag{4-14}$$

During the loading process,

$$E = \frac{2\sigma_o}{3}\Big(\frac{p}{4\mu} + \frac{1}{3\kappa}\Big(1 + \ln\frac{p}{c}\Big)\Big), \quad \Sigma = \frac{2}{3}\sigma_o\Big(1 - p + \ln\frac{p}{c}\Big), \quad \sigma_{rr} = 2\sigma_o \ln\frac{r}{R_i} \text{ over } \Omega_p.$$

We remark that the radius R_p of the plastic zone is a increasing function with E.

4B2. *The inverse problem.* We assume that the measurement on the external boundary of the sphere is the strain E_m and the tension Σ_m. We consider that the fields depend only upon r and $\boldsymbol{\varepsilon}_p$ is isochoric, then the function $\mathcal{J}$ is

$$\mathcal{J}(\boldsymbol{u}, \overset{\star}{\boldsymbol{u}}, \varepsilon_p, R_p) = \frac{1}{2} h R_e^2 \Big(\frac{u(R_e)}{R_e} - E_m\Big)^2 + \frac{\beta}{2}\int_{R_i}^{R_p} \mathrm{Y}^2 r^2\, dr - \int_{R_i}^{R_e} \nabla\overset{\star}{\boldsymbol{u}} : \mathbb{C} : (\nabla\boldsymbol{u} - \boldsymbol{\varepsilon}_p)\, r^2\, dr + R_e^2\, \Sigma_m\, u^*(R_e), \tag{4-15}$$

where Y is evaluated in terms of $u(r)$ and ε_p:

$$\mathrm{Y} = 2\mu\Big(\frac{u}{r} - \frac{\mathrm{d}u}{\mathrm{d}r}\Big) + 6\mu\varepsilon_p - \sigma_o.$$

The plastic domain is controlled by R_p. The plastic zone must be optimized. The variations with respect to ε_p give

$$4\mu\Big(\frac{\mathrm{d}u^*}{\mathrm{d}r} - \frac{u^*}{r}\Big) - 6\beta\mu \mathrm{Y} = 0. \tag{4-16}$$

The variations with respect to $u(r)$ furnish the equations of the adjoint state:

$$0 = \frac{\mathrm{d}\sigma_{rr}^*}{\mathrm{d}r} + \frac{2}{r}(\sigma_{rr}^* - \sigma_{tt}^*) \qquad \text{if } r \geq R_p, \tag{4-17}$$

$$0 = \frac{\mathrm{d}\sigma_{rr}^*}{\mathrm{d}r} + \frac{2}{r}(\sigma_{rr}^* - \sigma_{tt}^*) - 2\mu\beta\Big(\frac{\mathrm{dY}}{\mathrm{d}r} + \frac{3}{r}\mathrm{Y}\Big) \quad \text{if } r \leq R_p, \tag{4-18}$$

where the local constitutive law $\overset{\star}{\boldsymbol{\sigma}} = \mathbb{C} : \boldsymbol{\varepsilon}(\overset{\star}{\boldsymbol{u}})$ has been taken into account. Then the adjoint displacement

$\overset{\star}{\boldsymbol{u}}$ satisfies

$$u^* = \begin{cases} a_e r + B_e/r^2 & \text{if } r \geq R_p, \\ u^* = a_p r + B_p/r^2 & \text{if } r \leq R_p, \end{cases} \tag{4-19}$$

and the local function Y is then deduced:

$$\mathrm{Y} = -\frac{2}{3\alpha} r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u^*}{r}\right) = \frac{2B_p}{\beta r^3}. \tag{4-20}$$

Having determined Y, the stresses in Ω_p are obtained by integration of local equations of equilibrium resulting from the variations of $\mathcal{J}$ with respect to $\overset{\star}{\boldsymbol{u}}$:

$$\sigma_{rr} = 2\sigma_o \ln \frac{r}{R_i} + \frac{4B_p}{3\beta}\left(\frac{1}{r^3} - \frac{1}{R_i^3}\right). \tag{4-21}$$

We also know σ_{tt} from Y and $3\mathrm{Tr}(\boldsymbol{\varepsilon}) = \sigma_{rr} + 2\sigma_{tt}$ gives the displacement in Ω_p:

$$3\kappa(r^2 u - R_i^2 u_i) = 2\sigma_o r^3 \log \frac{r}{R_i} + \frac{4B_p}{3R_i^3\beta}(r^3 - R_i^3). \tag{4-22}$$

Along $r = R_p$ we obtain

$$3\kappa\left(p\frac{u_p}{R_p} - c\frac{u_i}{R_i}\right) = \frac{2\sigma_o}{3} p \log \frac{p}{c} + (p-c)\frac{4B_p}{3\beta R_i^3}. \tag{4-23}$$

Finally the problem of optimization is reduced to the system ($B_p = b_p R_p^3$, $B_e = b_e R_p^3$)

$$\begin{aligned} \Sigma_m &= 3\kappa a - 4\mu b\, p, \\ \sigma_{rr}^*(R_e) &= 3\kappa a_e - 4\mu b_e p = h R_e(a + bp - E_m), \\ a_e + b_e &= a_p + b_p, \\ 3\kappa a_p - 4\mu b_p p/c &= 2\mu\alpha f(R_i) = -\frac{4\mu p}{c} b_p, \\ 3\kappa(a_e - a_p) - 4\mu(b_e - b_p) &= 4\mu b_p, \\ 3\kappa a - 4\mu b &= 2\sigma_o \ln \frac{p}{c} + \frac{4}{3\beta} b_p\left(1 - \frac{p}{c}\right). \end{aligned}$$

It is obvious that $a_p = 0$. For a given R_p this system has an unique solution.

To this system, we must add the condition (4-10), corresponding to the variation of the functional with respect to R_p.

Assuming that (E_m, Σ_m) is a solution of the direct problem (E, Σ), the solution of inverse problem is obtained taking $b_p = 0$ and the adjoint field $\overset{\star}{\boldsymbol{u}} = 0$. Along $r = R_p$, we have $\mathrm{Y} = 0$. As a consequence, we can now recover the solution of the direct problem.

However for boundary conditions, $\Sigma_m = \Sigma + \Delta\Sigma$ and $E_m = E + \Delta E$, where the Σ_m, E_m is not a solution of a direct problem under radial loading, the solution of the inverse problem is the sum of the solution of the direct problem and of that of the gap between it and the solution at R_p. It is easy to show that the solution is unique, especially the radius R_p is close to the radius obtained by the direct problem if the measured strain E_m is close to E.

This analysis can be extended to the case of elastoplasticity with linear hardening, and the final conclusions of such an extension are quite similar to the present case. The problem for elastoplastic material is not always regular, and the proposed variational approach is not general.

Next, the case of viscoplasticity is investigated as a regularisation of the elastoplastic problem.

5. Boundary control and extension in viscoplasticity

Let Ω be a domain with external boundary $\partial\Omega$. The body has an elastoviscoplastic behaviour. The state of the body is defined by the value of the strain $\boldsymbol{\varepsilon}$ and internal parameters α. The local behaviour is defined by a free energy $w(\boldsymbol{\varepsilon}, \alpha)$ and we assume that the internal state has an evolution satisfying the normality rule defined by a potential of dissipation Φ. The free energy is defined classically as a reversible part due to elasticity and energy embedded in the residual stresses and hardening

$$w(\boldsymbol{\varepsilon}, \alpha) = \tfrac{1}{2}(\boldsymbol{\varepsilon} - \alpha) : \mathbb{C} : (\boldsymbol{\varepsilon} - \alpha) + W(\alpha). \tag{5-1}$$

The state equations are given by

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : (\boldsymbol{\varepsilon} - \alpha), \quad A = -\frac{\partial w}{\partial \alpha}, \tag{5-2}$$

and the evolution of the internal state satisfies

$$\dot{\alpha} = \frac{\partial \Phi}{\partial A}. \tag{5-3}$$

We seek an estimation of the loading history along a part Γ_T knowing both the initial state of the body and the final position at final time t_f of Γ_T.

In linearized elasticity, the problem is easy to solve. Assuming that we know the displacement of the boundary, it is easy to determine the resulting traction on the boundary. Here the problem is more difficult because the final state depends in a fundamental the history of the loading, and this loading path is unknown.

The optimal control theory is used to give some possible answers to the problem of estimating the internal state, while simultaneously providing an optimal loading history compatible with the given residual shape.

Consider that the initial state is naturally $\boldsymbol{u}(x, t_o) = 0, \alpha(x, t_o) = 0, \forall x \in \Omega$. On Γ_u the displacement is prescribed $\boldsymbol{u}(x, t) = 0$. The final state $\boldsymbol{u}(x, t_f) = \boldsymbol{u}_o(x)$ is known along the complementary part Γ_T of the boundary. We seek the best history of the loading $\boldsymbol{T}_o(x, t)$ applied on Γ_T and the internal state $\alpha(x, t)$ such that the resulting displacement along Γ_T is close to the measured displacement $\boldsymbol{u}_o$ at time t_f.

For a given history $\boldsymbol{T}_o(x, t)$ imposed on Γ_T we determine the solution $\boldsymbol{u}(x, t), \alpha(x, t)$ satisfying the primal problem of evolution corresponding to the set of equations

- *Compatibility*: $2\boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla \boldsymbol{u} + \nabla^t \boldsymbol{u}$ over Ω, $\boldsymbol{u} = 0$ along Γ_u.
- *Equilibrium*: $\operatorname{div} \boldsymbol{\sigma} = 0$ ove rΩ, $\boldsymbol{n}.\boldsymbol{\sigma} = \boldsymbol{T}_o$ along Γ_T.
- *Constitutive law*: $\boldsymbol{\sigma} = \dfrac{\partial w}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : (\boldsymbol{\varepsilon} - \alpha), \quad A = -\dfrac{\partial w}{\partial \alpha}, \quad \dot{\alpha} = \dfrac{\partial \Phi}{\partial A}.$

For this behaviour, the displacement $\boldsymbol{u}$, the local state $\boldsymbol{\varepsilon}, \alpha$ and the stresses $\boldsymbol{\sigma}$ are functions of position and time.

In particular, the solution of the problem of evolution at t_f defines the displacement $\boldsymbol{u}(x, t_f)$ along Γ_T. The best history $\boldsymbol{T}_o(x, t)$ is determined by an optimality condition. The function to optimize is chosen as

$$J(\boldsymbol{u}, \boldsymbol{T}_o) = \int_{\Gamma_T} \tfrac{1}{2}k\|\boldsymbol{u}(x, t_f) - \boldsymbol{u}_o(x)\|^2 \,\mathrm{ds} + \int_0^{t_f}\int_{\Gamma_T} \tfrac{1}{2}\dot{\boldsymbol{T}}_o.H.\dot{\boldsymbol{T}}_o \,\mathrm{ds}\,\mathrm{dt}. \tag{5-4}$$

To solve the problem, we adopt a variational form of the primal problem of evolution by introducing the functional $\mathscr{L}$:

$$\mathscr{L} = -\int_0^{t_f}\int_\Omega (\dot{\boldsymbol{\varepsilon}}, \dot{\alpha})^t.\mathscr{W}.(\boldsymbol{\varepsilon}^*, \alpha^*)\,\mathrm{d}\Omega\,\mathrm{dt} + \int_0^{t_f}\int_{\Gamma_T} \dot{\boldsymbol{T}}_o.\boldsymbol{u}^* \,\mathrm{ds}\,\mathrm{dt} + \int_0^{t_f}\int_\Omega \Big(A^*.\Big(-\dot{\alpha} + \frac{\partial \Phi}{\partial A}\Big) - \alpha^*.\dot{A}\Big)\,\mathrm{d}\Omega\,\mathrm{dt}.$$

where $\mathscr{W}$ is related to the second derivative of the free energy

$$\begin{bmatrix} \dot{\boldsymbol{\sigma}} \\ -\dot{A} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\,\partial \boldsymbol{\varepsilon}} & \dfrac{\partial^2 w}{\partial \boldsymbol{\varepsilon}\,\partial \alpha} \\ \dfrac{\partial^2 w}{\partial \alpha\,\partial \boldsymbol{\varepsilon}} & \dfrac{\partial^2 w}{\partial \alpha\,\partial \alpha} \end{bmatrix} . \begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\alpha} \end{bmatrix} = \mathscr{W}(\boldsymbol{\varepsilon}, \alpha).\begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\alpha} \end{bmatrix}. \tag{5-5}$$

Let us introduce the notation

$$\begin{bmatrix} \dot{\boldsymbol{\sigma}} \\ \dot{B} \end{bmatrix} = \mathscr{W}(\boldsymbol{\varepsilon}, \alpha).\begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \dot{\alpha} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{\sigma}^* \\ B^* \end{bmatrix} = \mathscr{W}(\boldsymbol{\varepsilon}, \alpha).\begin{bmatrix} \boldsymbol{\varepsilon}^* \\ \alpha^* \end{bmatrix}. \tag{5-6}$$

The variations of $\mathscr{L}$ are

$$\begin{aligned} \delta\mathscr{L} = &-\int_0^{t_f}\int_\Omega \Big(\delta\dot{\boldsymbol{\varepsilon}} : \boldsymbol{\sigma}^* + \delta\dot{\alpha} B^* + \delta\boldsymbol{\varepsilon}^*\dot{\boldsymbol{\sigma}} + \delta\alpha^*\dot{B}\Big)\,\mathrm{d}\Omega\,\mathrm{dt} + \int_0^{t_f}\int_{\Gamma_T} (\delta\dot{\boldsymbol{T}}_o.\boldsymbol{u}^* + \dot{\boldsymbol{T}}_o.\delta\boldsymbol{u}^*)\,\mathrm{ds}\,\mathrm{dt} \\ &+ \int_0^{t_f}\int_\Omega \Big(\delta A^*.\Big(-\dot{\alpha} + \frac{\partial \Phi}{\partial A}\Big) + A^*\Big(-\delta\dot{\alpha} + \frac{\partial^2 \Phi}{\partial A\,\partial A}\delta A\Big) - \delta\alpha^*\dot{A} - \alpha^*\delta\dot{A}\Big)\,\mathrm{d}\Omega\,\mathrm{dt}. \end{aligned}$$

Due to the integration over the history $t \in [0, t_f]$, the variations of the functional give the rate equations of the quantities and conditions at final time t_f. It is obvious that the variation with respect to $\boldsymbol{u}^*$ gives directly that $\dot{\boldsymbol{\sigma}}$ is statically admissible with the prescribed history loading on Γ_T:

$$\mathrm{div}\,\dot{\boldsymbol{\sigma}} = 0, \quad \boldsymbol{n}.\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{T}}_o. \tag{5-7}$$

The variation with respect to A^* implies the evolution of the internal state α, and the variation with respect to α^* implies the constitutive law: $\dot{A} + \dot{B} = 0$. Conversely the variations with respect to $\boldsymbol{u}, \alpha$ lead to the adjoint problem:

- *Equilibrium*:

$$\mathrm{div}\,\dot{\boldsymbol{\sigma}}^* = 0, \qquad \boldsymbol{n}.\dot{\boldsymbol{\sigma}}^* = 0 \quad \text{on } \Gamma_T.$$

- *Boundary condition*:

$$\dot{\boldsymbol{u}}^*(x, t) = 0, \qquad \boldsymbol{u}^*(x, t_f) = 0 \quad \text{along } \Gamma_u.$$

- *Constitutive law*:

$$A^* + B^* = 0, \quad \dot{\alpha}^* = -\frac{\partial^2 \Phi}{\partial A\, \partial A} A^*, \quad \alpha^*(t_f) = 0,$$
$$\boldsymbol{\sigma}^*(x, t_f) = \mathbb{C} : \boldsymbol{\varepsilon}^*(x, t_f) \quad \text{over } \Omega.$$

To ensure the existence of a solution, the two potentials w, Φ must have regular second derivative.

To obtain an estimation of the loading history and of the internal state compatible with the measured displacement $\boldsymbol{u}_o$ over Γ_T at time t_f the functional to optimize is $\mathcal{J} = J + \mathcal{L}$ and the conditions of optimality are then

$$0 = \boldsymbol{u}^* + H.\dot{\boldsymbol{T}}_o \qquad \text{on } \Gamma_T, \tag{5-8}$$
$$0 = \boldsymbol{n}.\boldsymbol{\sigma}^*(t_f) + k(\boldsymbol{u}(t_f) - \boldsymbol{u}_o) \qquad \text{on } \Gamma_T, \tag{5-9}$$

where the first condition gives the boundary condition on Γ_T for the adjoint displacement and the second one is imposed by the variation with respect to $\boldsymbol{u}(t_f)$.

It can be noticed that the adjoint problem is a viscoelastic problem, whose modulus of viscosity depends on the primal solution. The viscoplastic potential of dissipation must be regular. Due to the dependance of time in viscoplasticity, the inverse problem must be solved numerically. The solution depends on the duration and the solution is obtained by a optimization with respect to t_f.

5A. ***Cyclic loading.*** The extension of the presented solution method for cyclic loading has been performed to determine the cyclic asymptotic answer of viscoplastic structures with applications to fatigue. In this case the functional $\mathcal{L}$ is not changed, and the cost function J is chosen as a measure of the gap on periodicity on the generalized stress space $\mathcal{A} = (\boldsymbol{\sigma}, A)$. Such a cost functional is, for example,

$$J(\mathcal{A}) = \int_\Omega \tfrac{1}{2}(\mathcal{A}(T) - \mathcal{A}(0)).\mathbb{H}.(\mathcal{A}(T) - \mathcal{A}(0))\, \mathrm{d}\Omega. \tag{5-10}$$

The solution is obtained by a resolution of a problem of minimization, the uniqueness is proved in [Peigney and Stolz 2003].

6. Conclusion

This article proposed a general method for resolution of inverse problems in elastoplasticity. The formulation is based on the definition of the control variables: the plastic strain and the history of loading. For each class of inverse problems, an appropriate functional is chosen. The solution stems from the introduction of an adjoint state solution of the adjoint problem. In elasticity, the adjoint problem has the same form than the primal problem. Applications on bars and on a beam have been presented with analytical resolutions. They give the main ideas developed in this article and are illustrations of the method based on optimal control. In elastoplasticity, for the estimation of the plastic strain and the plastic zone, the solution of the inverse problem for the given examples is the sum of the solution of the primal one and of an elastic one given by the adjoint state which measures the discrepancy with respect to the exact solution.

A general formulation is obtained in elastoviscoplasticity. In this case the solution is given by the introduction of an adjoint state, which is the solution of a viscoelastic problem whose characteristics are given by the primal solution. In this general case, the problem to solve is dependent upon the history of the local fields. Other applications and examples can be found in [Bourgeois 1998].

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In Memoriam: Huy Duong Bui