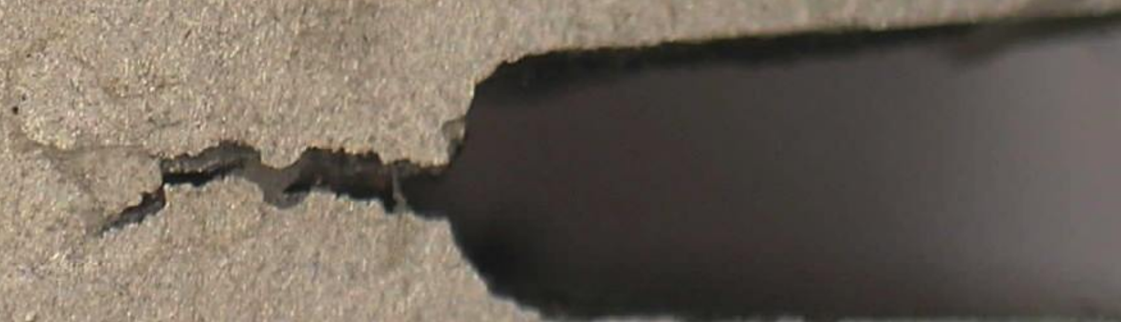
Journal of

Mechanics of Materials and Structures



HARMONIC SHAPES IN ISOTROPIC LAMINATED PLATES

Xu Wang and Peter Schiavone

Volume 10, No. 3

May 2015

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 10, No. 3, 2015
dx.doi.org/10.2140/jomms.2015.10.433

HARMONIC SHAPES IN ISOTROPIC LAMINATED PLATES

XU WANG AND PETER SCHIAVONE

Harmonic shapes are known to minimize stress disturbance when introduced into an elastic body as either holes or inclusions. This paper is concerned with the design of harmonic shapes in an isotropic laminated plate. Specifically, we require that the harmonic shape does not disturb the sum of the two normal membrane stress resultants and that of the two normal bending moments when inserted into a uniformly loaded laminated plate. Using complex variable methods, we demonstrate how a single harmonic shape (hole or rigid inclusion) and two interacting harmonic shapes can be successfully designed to meet our requirements. In our discussion, the two interacting harmonic shapes include (i) two interacting harmonic holes, (ii) two interacting harmonic rigid inclusions, and (iii) one harmonic hole interacting with another harmonic rigid inclusion.

1. Introduction

The minimization of stress concentrations in composite materials remains a priority among researchers and practitioners alike. To date, various criteria have been proposed and successfully applied to the design of the shape of holes or inclusions which produce minimum stress concentrations when inserted into an elastic body (see for example [Mansfield 1953; Cherepanov 1974; Bjorkman and Richards 1976]). The design of such optimal structural shapes inevitably leads to the solution of an inverse problem in elasticity [Bui 1993; Bonnet and Constantinescu 2005]. The "neutral condition" proposed by Mansfield [1953] and further developed by Ru [1998] and Milton and Serkov [2001] is the most stringent yet most difficult to realize since it requires that the introduction of the corresponding neutral hole or inclusion leaves the stress distribution in the original uncut body completely undisturbed. The "equal strength condition" introduced by Cherepanov [1974] requires that the hoop stress be constant along the boundary of the hole or inclusion. The "harmonic field condition" advanced by Bjorkman and Richards [1976; 1979a] and further developed by Ru [1999a; 1999b] and Wang et al. [2005] requires that the introduction of the harmonic hole or inclusion does not alter the first invariant of the stress field anywhere in the surrounding elastic body. This design condition has many implications: (i) the Laplacian component of the stress field remains unchanged; (ii) there is no change in volume energy; (iii) there is no elastic rotation; (iv) harmonic holes or rigid inclusions produce minimum stress concentrations in constant fields. Interestingly, Bjorkman and Richards [1976; 1979b] observed that, under constant applied fields, the harmonic field condition and the equal strength condition are essentially equivalent in that they produce the same result.

Even though the analysis of bending and stretching deformations of thin plates in the presence of various defects such as dislocations, holes, cracks, anti-cracks and inhomogeneities has received considerable attention (see, for example, [Sih and Rice 1964; Zakharov and Becker 2000; Hasebe and Wang

Keywords: harmonic shape, stress concentration, inverse problem, isotropic laminated plate, conformal mapping.

2000; 2002; Wang and Hasebe 2000a; 2000b; 2001; Cheng and Reddy 2002, 2003; 2004b; 2004a; Yin 2005a; 2005b; Wang and Schiavone 2013; Wang and Zhou 2013], the design of optimal hole shapes to minimize stress concentrations in laminated plates has, in contrast, been investigated by only a few researchers (see, for example, [Vellaichamy et al. 1990; Budiansky et al. 1993; Senocak and Waas 1995]. In this paper, we continue the work in this area by extending Bjorkman and Richards' harmonic field condition for linear plane elasticity to isotropic laminated plates. To achieve our design objective, we make use of the complex variable formulation proposed by Beom and Earmme [1998] and developed recently by Wang and Zhou [2014] for the analysis of isotropic laminated plates.

2. Formulation

Consider an undeformed plate of uniform thickness h in a Cartesian coordinate system $\{x_i\}$ $(i = 1, 2, 3)$ in which $x_3 = 0$ is on the main plane. The plate is composed of an isotropic, linearly elastic material that can be inhomogeneous and/or laminated in the thickness direction. Repeated Greek indices imply summation over the range of the index from 1 to 2.

The displacement field in the Kirchhoff plate theory takes the form

$$\tilde{u}_\alpha = u_\alpha + x_3 \vartheta_\alpha, \quad \tilde{u}_3 = w, \tag{1}$$

where the in-plane displacements u_α, deflection w, and the slopes $\vartheta_\alpha = -w_{,\alpha}$ on the main plane are all independent of x_3.

An integral operator is first introduced as follows: $Q(\,\cdot\,) = \int_{-h_0}^{h-h_0} (\,\cdot\,)\, dx_3$, with h_0 being the distance between the main plane and the lower surface of the plate. Then the constitutive equations of the laminated isotropic plate are [Beom and Earmme 1998]

$$\begin{aligned} N_{\alpha\beta} &= A_{\alpha\beta\omega\rho}\varepsilon_{\omega\rho} + B_{\alpha\beta\omega\rho}\kappa_{\omega\rho}, \\ M_{\alpha\beta} &= B_{\alpha\beta\omega\rho}\varepsilon_{\omega\rho} + D_{\alpha\beta\omega\rho}\kappa_{\omega\rho}, \end{aligned} \tag{2}$$

where $N_{\alpha\beta}$ and $M_{\alpha\beta}$ are, respectively, the membrane stress resultants and bending moments defined by $N_{\alpha\beta} = Q\sigma_{\alpha\beta}$ and $M_{\alpha\beta} = Qx_3\sigma_{\alpha\beta}$, with $\sigma_{\alpha\beta}$ being the stresses; $\varepsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are the main plane strains and curvatures, defined as $\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$ and $\kappa_{\alpha\beta} = -w_{,\alpha\beta}$; $A_{\alpha\beta\omega\rho}$, $B_{\alpha\beta\omega\rho}$ and $D_{\alpha\beta\omega\rho}$ are the extensional, coupling and bending stiffness tensors, given by

$$\begin{aligned} A_{\alpha\beta\omega\rho} &= A_{12}\delta_{\alpha\beta}\delta_{\omega\rho} + \tfrac{1}{2}(A_{11} - A_{12})(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}), \\ B_{\alpha\beta\omega\rho} &= B_{12}\delta_{\alpha\beta}\delta_{\omega\rho} - \tfrac{1}{2}B_{12}(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}), \\ D_{\alpha\beta\omega\rho} &= D_{12}\delta_{\alpha\beta}\delta_{\omega\rho} + \tfrac{1}{2}(D_{11} - D_{12})(\delta_{\alpha\omega}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\omega}), \end{aligned} \tag{3}$$

with $\delta_{\alpha\beta}$ being the Kronecker delta, $A_{ij} = QC_{ij}$, $B_{ij} = Qx_3C_{ij}$, and $D_{ij} = Qx_3^2C_{ij}$ $(ij = 11, 12)$. In addition, C_{11} and C_{12} can be expressed in terms of the Young's modulus $E = E(x_3)$ and Poisson's ratio $\nu = \nu(x_3)$ of the plate as $C_{11} = E/(1-\nu^2)$ and $C_{12} = \nu E/(1-\nu^2)$. Expression (3) implies that the main plane is chosen such that $B_{11} = 0$. Consequently, it is found that

$$h_0 = \frac{\int_0^h X_3 C_{11}\, dX_3}{\int_0^h C_{11}\, dX_3},$$

with $X_3 = x_3 + h_0$ being the vertical coordinate of the given point from the lowest surface of the plate.

In the absence of external loads on the top and bottom surfaces of the plate, the equilibrium equations are given by

$$N_{\alpha\beta,\beta} = 0, \quad R_{\beta,\beta} = 0, \tag{4}$$

where $R_\beta = M_{\alpha\beta,\alpha}$ are the transverse shearing forces.

Substitution of (2) into (4) yields the decoupled equations

$$(A_{11} + A_{12})u_{\beta,\beta\alpha} + (A_{11} - A_{12})u_{\alpha,\beta\beta} = 0, \quad w_{,\alpha\alpha\beta\beta} = 0. \tag{5}$$

By considering (5), the membrane stress resultants, bending moments, transverse shearing forces, in-plane displacements, deflection and slopes on the main plane of the plate, and the four stress functions φ_α and η_α can be expressed in terms of four analytic functions $\phi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as [Beom and Earmme 1998; Cheng and Reddy 2002; Wang and Zhou 2014]

$$\begin{aligned} &N_{11} + N_{22} = 4\,\mathrm{Re}\{\phi'(z) + B\Phi'(z)\}, \\ &N_{22} - N_{11} + 2\mathrm{i}N_{12} = 2\big(\bar{z}\phi''(z) + \psi'(z) + B\bar{z}\Phi''(z) + B\Psi'(z)\big), \end{aligned} \tag{6}$$

$$\begin{aligned} &M_{11} + M_{22} = 4D(1+\nu^D)\,\mathrm{Re}\{\Phi'(z)\} + \frac{B(\kappa^A - 1)}{\mu}\,\mathrm{Re}\{\phi'(z)\}, \\ &M_{22} - M_{11} + 2\mathrm{i}M_{12} = -2D(1-\nu^D)\big(\bar{z}\Phi''(z) + \Psi'(z)\big) - \frac{B}{\mu}\big(\bar{z}\phi''(z) + \psi'(z)\big), \\ &R_1 - \mathrm{i}R_2 = 4D\Phi''(z) + \frac{B(\kappa^A + 1)}{2\mu}\phi''(z), \end{aligned} \tag{7}$$

$$\begin{aligned} &2\mu(u_1 + \mathrm{i}u_2) = \kappa^A\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \\ &\vartheta_1 + \mathrm{i}\vartheta_2 = \Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)}, \quad w = -\,\mathrm{Re}\{\bar{z}\Phi(z) + \chi(z)\}, \\ &\varphi_1 + \mathrm{i}\varphi_2 = \mathrm{i}\big(\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\big) + \mathrm{i}B\big(\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi(z)}\big), \\ &\eta_1 + \mathrm{i}\eta_2 = \mathrm{i}D(1-\nu^D)\big(\kappa^D\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)}\big) + \mathrm{i}\frac{B}{2\mu}\big(\kappa^A\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\big), \end{aligned} \tag{8}$$

where $\Psi(z) = \chi'(z)$, and

$$\begin{aligned} &\mu = \tfrac{1}{2}(A_{11} - A_{12}), \quad B = B_{12}, \quad D = D_{11}, \quad \nu^A = \frac{A_{12}}{A_{11}}, \quad \nu^D = \frac{D_{12}}{D_{11}}, \\ &\kappa^A = \frac{3A_{11} - A_{12}}{A_{11} + A_{12}} = \frac{3 - \nu^A}{1 + \nu^A}, \quad \kappa^D = \frac{3D_{11} + D_{12}}{D_{11} - D_{12}} = \frac{3 + \nu^D}{1 - \nu^D}. \end{aligned} \tag{9}$$

Moreover, the membrane stress resultants, bending moments, transverse shearing forces, and modified Kirchhoff transverse shearing forces $V_1 = R_1 + M_{12,2}$ and $V_2 = R_2 + M_{21,1}$ can be expressed in terms of the four stress functions φ_α and η_α as [Cheng and Reddy 2002]

$$N_{\alpha\beta} = -\epsilon_{\beta\omega}\varphi_{\alpha,\omega}, \quad M_{\alpha\beta} = -\epsilon_{\beta\omega}\eta_{\alpha,\omega} - \tfrac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega}, \quad R_\alpha = -\tfrac{1}{2}\epsilon_{\alpha\beta}\eta_{\omega,\omega\beta}, \quad V_\alpha = -\epsilon_{\alpha\omega}\eta_{\omega,\omega\omega}, \tag{10}$$

where $\epsilon_{\alpha\beta}$ are the components of the two-dimensional permutation tensor.

Now we consider a laminated plate subjected to remote uniform membrane stress resultants $N_{\alpha\beta}^\infty$ and bending moments $M_{\alpha\beta}^\infty$. The asymptotic behaviors of $\phi(z)$, $\psi(z)$, $\Phi(z)$ and $\Psi(z)$ at infinity can then be simply derived as

$$\phi(z) \cong \delta_1 z + O(1), \quad \psi(z) \cong \delta_2 z + O(1), \quad \Phi(z) \cong \gamma_1 z + O(1), \quad \Psi(z) \cong \gamma_2 z + O(1), \tag{11}$$

where the two real constants δ_1, γ_1 and the two complex constants δ_2, γ_2 are related to the remote uniform loading through

$$\begin{aligned}
\delta_1 &= \frac{\mu D(1+\nu^D)(N_{11}^\infty + N_{22}^\infty) - B\mu(M_{11}^\infty + M_{22}^\infty)}{4\mu D(1+\nu^D) - B^2(\kappa^A - 1)},\\
\gamma_1 &= \frac{4\mu(M_{11}^\infty + M_{22}^\infty) - B(\kappa^A - 1)(N_{11}^\infty + N_{22}^\infty)}{16\mu D(1+\nu^D) - 4B^2(\kappa^A - 1)},\\
\delta_2 &= \frac{\mu D(1-\nu^D)(N_{22}^\infty - N_{11}^\infty + 2\mathrm{i}N_{12}^\infty) + B\mu(M_{22}^\infty - M_{11}^\infty + 2\mathrm{i}M_{12}^\infty)}{2\mu D(1-\nu^D) - B^2},\\
\gamma_2 &= \frac{-2\mu(M_{22}^\infty - M_{11}^\infty + 2\mathrm{i}M_{12}^\infty) - B(N_{22}^\infty - N_{11}^\infty + 2\mathrm{i}N_{12}^\infty)}{4\mu D(1-\nu^D) - 2B^2}.
\end{aligned} \tag{12}$$

In the context of an isotropic laminated plate, Bjorkman and Richards' harmonic field conditions now become that the two sums $N_{11} + N_{22}$ and $M_{11} + M_{22}$ remain unchanged everywhere in the surrounding laminated plate after the introduction of the harmonic hole or inclusion. It is further deduced from $(6)_1$ and $(7)_1$ that $\phi(z)$ and $\Phi(z)$ must take the following form in order to ensure that the shape is harmonic:

$$\phi(z) = \delta_1 z, \quad \Phi(z) = \gamma_1 z. \tag{13}$$

In the next two sections we will address in detail a single harmonic hole or rigid inclusion and two interacting harmonic shapes.

3. A single harmonic hole or rigid inclusion

The single harmonic hole or rigid inclusion forms a simply connected bounded domain with Lipschitz boundary. As such, we consider the conformal mapping function [Kantorovich and Krylov 1950]

$$z = \omega(\xi) = R\Big(\xi + \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \cdots\Big), \quad \xi(z) = \omega^{-1}(z), \quad |\xi| \ge 1, \tag{14}$$

where R is a real scaling constant and a_i $(i = 1, 2, \dots)$ are complex constants. This function conformally maps the exterior of the hole or the rigid inclusion in the z-plane onto the exterior of the unit circle $|\xi| = 1$ in the ξ-plane. For convenience, we write $\psi(z) = \psi(\omega(\xi)) = \psi(\xi)$ and $\Psi(z) = \Psi(\omega(\xi)) = \Psi(\xi)$.

3.1. ***A single harmonic hole.*** In this case, $\varphi_1 = \varphi_2 = \eta_1 = \eta_2 = 0$ along the edge of the hole. By enforcing this free edge boundary condition on $|\xi| = 1$, we arrive at these expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$\begin{aligned}
\psi(\xi) &= \frac{-\big(4\mu D(1-\nu^D) + B^2(\kappa^A - 1)\big)\delta_1 - 8B\mu D\gamma_1}{2\mu D(1-\nu^D) - B^2} R\Big(\frac{1}{\xi} + \bar{a}_1\xi + \bar{a}_2\xi^2 + \cdots\Big),\\
\Psi(\xi) &= \frac{B(\kappa^A + 1)\delta_1 + \big(4\mu D(1+\nu^D) + 2B^2\big)\gamma_1}{2\mu D(1-\nu^D) - B^2} R\Big(\frac{1}{\xi} + \bar{a}_1\xi + \bar{a}_2\xi^2 + \cdots\Big).
\end{aligned} \tag{15}$$

By satisfying the asymptotic conditions for $\psi(z)$ and $\Psi(z)$ in the second and fourth expressions of (11), we obtain

$$a_2 = a_3 = \cdots = 0, \tag{16}$$

which implies that the harmonic hole must be of elliptical shape, and

$$\begin{aligned}\delta_2 &= \frac{-\big(4\mu D(1-\nu^D)+B^2(\kappa^A-1)\big)\delta_1 - 8B\mu D\gamma_1}{2\mu D(1-\nu^D)-B^2}\bar{a}_1,\\ \gamma_2 &= \frac{B(\kappa^A+1)\delta_1+\big(4\mu D(1+\nu^D)+2B^2\big)\gamma_1}{2\mu D(1-\nu^D)-B^2}\bar{a}_1.\end{aligned} \tag{17}$$

Consequently, the remote uniform loading should satisfy the restrictions

$$\frac{N_{11}^\infty - N_{22}^\infty - 2\mathrm{i}N_{12}^\infty}{N_{11}^\infty + N_{22}^\infty} = \frac{M_{11}^\infty - M_{22}^\infty - 2\mathrm{i}M_{12}^\infty}{M_{11}^\infty + M_{22}^\infty} = \bar{a}_1. \tag{18}$$

The hoop membrane stress resultant $N_{\theta\theta}$ and hoop bending moment $M_{\theta\theta}$ are both constant along the boundary of the elliptic hole:

$$N_{\theta\theta} = N_{11}^\infty + N_{22}^\infty, \quad M_{\theta\theta} = M_{11}^\infty + M_{22}^\infty. \tag{19}$$

Similar to the argument by Bjorkman and Richards [1976], the harmonic hole simultaneously produces minimum values of $N_{\theta\theta}$ and $M_{\theta\theta}$, and thus is optimal. We note that (18) and (19) are in agreement with the corresponding results in [ibid.] for a harmonic hole in an isotropic and homogeneous plate subjected to in-plane loading.

3.2. ***A single harmonic rigid inclusion.*** In this case, $u_1 = u_2 = \vartheta_1 = \vartheta_2 = 0$ along the edge of the rigid inclusion. By enforcing this boundary condition for a rigidly clamped edge on $|\xi| = 1$, we arrive at these expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$\begin{aligned}\psi(\xi) &= (\kappa^A-1)\delta_1 R\Big(\frac{1}{\xi}+\bar{a}_1\xi+\bar{a}_2\xi^2+\cdots\Big),\\ \Psi(\xi) &= -2\gamma_1 R\Big(\frac{1}{\xi}+\bar{a}_1\xi+\bar{a}_2\xi^2+\cdots\Big).\end{aligned} \tag{20}$$

By satisfying the asymptotic conditions on $\psi(z)$ and $\Psi(z)$ in the second and fourth expressions of (11), we obtain

$$a_2 = a_3 = \cdots = 0, \tag{21}$$

which implies that the harmonic rigid inclusion must be elliptical and

$$\delta_2 = (\kappa^A-1)\delta_1\bar{a}_1, \quad \gamma_2 = -2\gamma_1\bar{a}_1. \tag{22}$$

Thus, the remote uniform loading should satisfy the restrictions

$$\begin{aligned}&\frac{\big(4\mu D(1+\nu^D)-B^2(\kappa^A-1)\big)(N_{22}^\infty - N_{11}^\infty + 2\mathrm{i}N_{12}^\infty)}{\big(2\mu D(1+\nu^D)+B^2\big)(\kappa^A-1)(N_{11}^\infty+N_{22}^\infty)-2B\mu(\kappa^A+1)(M_{11}^\infty+M_{22}^\infty)}\\ &= \frac{\big(4\mu D(1+\nu^D)-B^2(\kappa^A-1)\big)(M_{22}^\infty - M_{11}^\infty + 2\mathrm{i}M_{12}^\infty)}{\big(4\mu D(1-\nu^D)+B^2(\kappa^A-1)\big)(M_{11}^\infty+M_{22}^\infty)-2BD(\kappa^A-1)(N_{11}^\infty+N_{22}^\infty)} = \bar{a}_1.\end{aligned} \tag{23}$$

The interfacial normal membrane stress resultant $N_{\rho\rho}$ and interfacial normal bending moment $M_{\rho\rho}$ are uniformly distributed along the inclusion/matrix interface as

$$N_{\rho\rho} = (\kappa^A + 1)\delta_1, \quad M_{\rho\rho} = 4D\gamma_1, \tag{24}$$

where δ_1, γ_1 are as in (12).

The hoop membrane stress resultant and hoop bending moment are both constant along the inclusion/matrix interface on the matrix side and are given by

$$\begin{aligned} N_{\theta\theta} &= \frac{\big(\mu D(1+\nu^D)(3-\kappa^A) - B^2(\kappa^A-1)\big)(N_{11}^\infty + N_{22}^\infty) + B\mu(\kappa^A+1)(M_{11}^\infty + M_{22}^\infty)}{4\mu D(1+\nu^D) - B^2(\kappa^A-1)}, \\ M_{\theta\theta} &= \frac{\big(4\mu D\nu^D - B^2(\kappa^A-1)\big)(M_{11}^\infty + M_{22}^\infty) + BD(\kappa^A-1)(N_{11}^\infty + N_{22}^\infty)}{4\mu D(1+\nu^D) - B^2(\kappa^A-1)}. \end{aligned} \tag{25}$$

When $B = 0$, (23)–(25) reduce to

$$\frac{2(N_{22}^\infty - N_{11}^\infty + 2\mathrm{i}N_{12}^\infty)}{(\kappa^A-1)(N_{11}^\infty + N_{22}^\infty)} = \bar{a}_1, \quad N_{\rho\rho} = \frac{\kappa^A+1}{4}(N_{11}^\infty + N_{22}^\infty), \quad N_{\theta\theta} = \frac{3-\kappa^A}{4}(N_{11}^\infty + N_{22}^\infty), \tag{26}$$

$$\frac{(1+\nu^D)(M_{22}^\infty - M_{11}^\infty + 2\mathrm{i}M_{12}^\infty)}{(1-\nu^D)(M_{11}^\infty + M_{22}^\infty)} = \bar{a}_1, \quad M_{\rho\rho} = \frac{M_{11}^\infty + M_{22}^\infty}{1+\nu^D}, \quad M_{\theta\theta} = \frac{\nu^D(M_{11}^\infty + M_{22}^\infty)}{1+\nu^D}. \tag{27}$$

The results in (26) agree with those in [Bjorkman and Richards 1979b] for a harmonic rigid inclusion in an isotropic and homogeneous plate under in-plane loads.

4. Two interacting harmonic shapes

In order to obtain two interacting harmonic shapes (which include three typical cases: (i) two interacting harmonic holes, (ii) two interacting harmonic rigid inclusions, and (iii) one harmonic hole interacting with another harmonic rigid inclusion), we first introduce the conformal mapping function [Wang 2012]

$$z = \omega(\xi) = R\left(\frac{1}{\xi-\lambda} + \frac{a}{\xi-\lambda^{-1}} + \frac{\Lambda^{-1}a}{\rho\xi-\lambda^{-1}} + \sum_{n=1}^{\infty}(a_n\xi^n + a_{-n}\xi^{-n})\right),$$
$$\xi(z) = \omega^{-1}(z), \quad 1 \le |\xi| \le \rho^{-1/2}, \tag{28}$$

where R is a real scaling constant, λ is a real constant with $1 < |\lambda| < \rho^{-1/2}$, a and Λ are complex constants, and a_n, a_{-n} are complex constants to be determined. In (28), the first-order pole at $\xi = \lambda$ is located within the annulus $1 \le |\xi| \le \rho^{-1/2}$, whereas the two first-order poles at $\xi = \lambda^{-1}$ and $\xi = (\rho\lambda)^{-1}$ are both located outside the annulus. The function (28) will conformally map the matrix region (excluding the two harmonic shapes) in the z-plane onto an annulus $1 \le |\xi| \le \rho^{-1/2}$ in the ξ-plane, and the left and right interfaces L_1 and L_2 formed between the two harmonic shapes and the matrix are mapped onto two coaxial circles with radii 1 and $\rho^{-1/2}$, respectively, in the ξ-plane. It is also apparent that each of the two interacting shapes will be nonelliptical [Wang 2012].

4.1. ***Two interacting harmonic holes.*** By enforcing the free edge boundary condition on the inner circle $|\xi| = 1$, we obtain these expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$\begin{aligned}\psi(\xi) &= \frac{-\big(4\mu D(1-\nu^D) + B^2(\kappa^A - 1)\big)\delta_1 - 8B\mu D\gamma_1}{2\mu D(1-\nu^D) - B^2} \\ &\qquad \times R\bigg(\frac{1}{\xi^{-1}-\lambda} - \frac{\bar{a}\lambda^2}{\xi-\lambda} + \frac{\bar{\Lambda}^{-1}\bar{a}}{\rho\xi^{-1}-\lambda^{-1}} + \sum_{n=1}^{\infty}(\bar{a}_n\xi^{-n} + \bar{a}_{-n}\xi^n)\bigg), \\ \Psi(\xi) &= \frac{B(\kappa^A+1)\delta_1 + \big(4\mu D(1+\nu^D) + 2B^2\big)\gamma_1}{2\mu D(1-\nu^D) - B^2} \\ &\qquad \times R\bigg(\frac{1}{\xi^{-1}-\lambda} - \frac{\bar{a}\lambda^2}{\xi-\lambda} + \frac{\bar{\Lambda}^{-1}\bar{a}}{\rho\xi^{-1}-\lambda^{-1}} + \sum_{n=1}^{\infty}(\bar{a}_n\xi^{-n} + \bar{a}_{-n}\xi^n)\bigg).\end{aligned} \tag{29}$$

Similarly, by enforcing the free-edge boundary condition on the outer circle $|\xi| = \rho^{-1/2}$, we obtain another set of expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$\begin{aligned}\psi(\xi) &= \frac{-\big(4\mu D(1-\nu^D) + B^2(\kappa^A - 1)\big)\delta_1 - 8B\mu D\gamma_1}{2\mu D(1-\nu^D) - B^2} \\ &\qquad \times R\bigg(\frac{\lambda^{-1}}{1-\rho\lambda\xi} + \frac{\bar{a}}{\rho^{-1}\xi^{-1}-\lambda^{-1}} - \frac{\bar{\Lambda}^{-1}\bar{a}\lambda^2}{\xi-\lambda} + \sum_{n=1}^{\infty}(\bar{a}_n\rho^{-n}\xi^{-n} + \bar{a}_{-n}\rho^n\xi^n)\bigg), \\ \Psi(\xi) &= \frac{B(\kappa^A+1)\delta_1 + \big(4\mu D(1+\nu^D) + 2B^2\big)\gamma_1}{2\mu D(1-\nu^D) - B^2} \\ &\qquad \times R\bigg(\frac{\lambda^{-1}}{1-\rho\lambda\xi} + \frac{\bar{a}}{\rho^{-1}\xi^{-1}-\lambda^{-1}} - \frac{\bar{\Lambda}^{-1}\bar{a}\lambda^2}{\xi-\lambda} + \sum_{n=1}^{\infty}(\bar{a}_n\rho^{-n}\xi^{-n} + \bar{a}_{-n}\rho^n\xi^n)\bigg).\end{aligned} \tag{30}$$

The two expressions for $\psi(\xi)$ and $\Psi(\xi)$ obtained in (29) and (30) must coincide. As a result, the unknown parameters in (28) are given by

$$\Lambda = \bar{\Lambda} = 1, \quad a_n = \frac{\lambda^{-n-1} + a\rho^n\lambda^{n+1}}{1-\rho^{-n}}, \quad a_{-n} = \frac{\lambda^{n-1} + a\lambda^{-n+1}}{\rho^{-n}-1}, \quad n = 1, 2, \ldots. \tag{31}$$

Remark. The interacting harmonic shapes can now be uniquely determined using (28) for given real numbers Λ, ρ, λ and a given complex number a.

In addition, the satisfaction of the remote boundary conditions on $\psi(z)$ and $\Psi(z)$ in the second and fourth expressions of (11) will yield

$$\begin{aligned}\delta_2 &= -\frac{-\big(4\mu D(1-\nu^D) + B^2(\kappa^A - 1)\big)\delta_1 - 8B\mu D\gamma_1}{2\mu D(1-\nu^D) - B^2}\bar{a}\lambda^2, \\ \gamma_2 &= -\frac{B(\kappa^A+1)\delta_1 + \big(4\mu D(1+\nu^D) + 2B^2\big)\gamma_1}{2\mu D(1-\nu^D) - B^2}\bar{a}\lambda^2.\end{aligned} \tag{32}$$

Thus, the remote uniform loading should satisfy the restrictions

$$\frac{N_{11}^{\infty}-N_{22}^{\infty}-2\mathrm{i}N_{12}^{\infty}}{N_{11}^{\infty}+N_{22}^{\infty}}=\frac{M_{11}^{\infty}-M_{22}^{\infty}-2\mathrm{i}M_{12}^{\infty}}{M_{11}^{\infty}+M_{22}^{\infty}}=-\bar{a}\lambda^2. \tag{33}$$

The hoop membrane stress resultant and hoop bending moment are both uniformly distributed along the boundary of the two harmonic holes as

$$N_{\theta\theta}=N_{11}^{\infty}+N_{22}^{\infty},\quad M_{\theta\theta}=M_{11}^{\infty}+M_{22}^{\infty},\quad z\in L_1\cup L_2. \tag{34}$$

4.2. *Two interacting harmonic rigid inclusions.* By enforcing the boundary condition for a rigidly clamped edge on the inner circle $|\xi|=1$, we obtain these expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$\begin{aligned}\psi(\xi)&=(\kappa^A-1)\delta_1 R\left(\frac{1}{\xi^{-1}-\lambda}-\frac{\bar{a}\lambda^2}{\xi-\lambda}+\frac{\bar{\Lambda}^{-1}\bar{a}}{\rho\xi^{-1}-\lambda^{-1}}+\sum_{n=1}^{\infty}(\bar{a}_n\xi^{-n}+\bar{a}_{-n}\xi^n)\right),\\ \Psi(\xi)&=-2\gamma_1 R\left(\frac{1}{\xi^{-1}-\lambda}-\frac{\bar{a}\lambda^2}{\xi-\lambda}+\frac{\bar{\Lambda}^{-1}\bar{a}}{\rho\xi^{-1}-\lambda^{-1}}+\sum_{n=1}^{\infty}(\bar{a}_n\xi^{-n}+\bar{a}_{-n}\xi^n)\right).\end{aligned} \tag{35}$$

Similarly, by enforcing the boundary condition for a rigidly clamped edge on the outer circle $|\xi|=\rho^{-1/2}$, we again obtain a second set of expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$\begin{aligned}\psi(\xi)&=(\kappa^A-1)\delta_1 R\left(\frac{\lambda^{-1}}{1-\rho\lambda\xi}+\frac{\bar{a}}{\rho^{-1}\xi^{-1}-\lambda^{-1}}-\frac{\bar{\Lambda}^{-1}\bar{a}\lambda^2}{\xi-\lambda}+\sum_{n=1}^{\infty}(\bar{a}_n\rho^{-n}\xi^{-n}+\bar{a}_{-n}\rho^n\xi^n)\right),\\ \Psi(\xi)&=-2\gamma_1 R\left(\frac{\lambda^{-1}}{1-\rho\lambda\xi}+\frac{\bar{a}}{\rho^{-1}\xi^{-1}-\lambda^{-1}}-\frac{\bar{\Lambda}^{-1}\bar{a}\lambda^2}{\xi-\lambda}+\sum_{n=1}^{\infty}(\bar{a}_n\rho^{-n}\xi^{-n}+\bar{a}_{-n}\rho^n\xi^n)\right).\end{aligned} \tag{36}$$

Equating the two expression for $\psi(\xi)$ and $\Psi(\xi)$, we obtain $\Lambda=1$; a_n and a_{-n} in (28) can also be determined from (31).

In addition, the satisfaction of the remote boundary conditions on $\psi(z)$ and $\Psi(z)$ in the second and fourth expressions of (11) yields

$$\delta_2=-(\kappa^A-1)\delta_1\bar{a}\lambda^2,\quad \gamma_2=2\gamma_1\bar{a}\lambda^2. \tag{37}$$

Thus, the remote uniform loading is constrained by the equations

$$\begin{aligned}&\frac{\big(4\mu D(1+\nu^D)-B^2(\kappa^A-1)\big)(N_{22}^{\infty}-N_{11}^{\infty}+2\mathrm{i}N_{12}^{\infty})}{\big(2\mu D(1+\nu^D)+B^2\big)(\kappa^A-1)(N_{11}^{\infty}+N_{22}^{\infty})-2B\mu(\kappa^A+1)(M_{11}^{\infty}+M_{22}^{\infty})}\\ &=\frac{\big(4\mu D(1+\nu^D)-B^2(\kappa^A-1)\big)(M_{22}^{\infty}-M_{11}^{\infty}+2\mathrm{i}M_{12}^{\infty})}{\big(4\mu D(1-\nu^D)+B^2(\kappa^A-1)\big)(M_{11}^{\infty}+M_{22}^{\infty})-2BD(\kappa^A-1)(N_{11}^{\infty}+N_{22}^{\infty})}=-\bar{a}\lambda^2.\end{aligned} \tag{38}$$

The interfacial normal membrane stress resultant and interfacial normal bending moment are uniformly distributed along the two inclusion/matrix interfaces as

$$N_{\rho\rho}=(\kappa^A+1)\delta_1,\quad M_{\rho\rho}=4D\gamma_1,\quad z\in L_1\cup L_2. \tag{39}$$

The hoop membrane stress resultant and hoop bending moment are both constant along the two inclusion/matrix interfaces on the matrix side and are given by

$$
\begin{aligned}
N_{\theta\theta} &= \frac{\big(\mu D(1+\nu^D)(3-\kappa^A) - B^2(\kappa^A-1)\big)(N_{11}^\infty + N_{22}^\infty) + B\mu(\kappa^A+1)(M_{11}^\infty + M_{22}^\infty)}{4\mu D(1+\nu^D) - B^2(\kappa^A-1)},\\
M_{\theta\theta} &= \frac{\big(4\mu D\nu^D - B^2(\kappa^A-1)\big)(M_{11}^\infty + M_{22}^\infty) + BD(\kappa^A-1)(N_{11}^\infty + N_{22}^\infty)}{4\mu D(1+\nu^D) - B^2(\kappa^A-1)}, \quad z \in L_1 \cup L_2.
\end{aligned} \tag{40}
$$

4.3. ***A harmonic hole interacting with a harmonic rigid inclusion.*** Without loss of generality, we assume that the left-hand shape is a hole, whilst the right-hand shape is a rigid inclusion. By enforcing the free edge boundary condition on the inner circle $|\xi| = 1$, we obtain these expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$
\begin{aligned}
\psi(\xi) &= \frac{-\big(4\mu D(1-\nu^D) + B^2(\kappa^A-1)\big)\delta_1 - 8B\mu D\gamma_1}{2\mu D(1-\nu^D) - B^2}\\
&\qquad \times R\left(\frac{1}{\xi^{-1}-\lambda} - \frac{\bar{a}\lambda^2}{\xi-\lambda} + \frac{\bar{\Lambda}^{-1}\bar{a}}{\rho\xi^{-1}-\lambda^{-1}} + \sum_{n=1}^{\infty}(\bar{a}_n\xi^{-n} + \bar{a}_{-n}\xi^n)\right),\\
\Psi(\xi) &= \frac{B(\kappa^A+1)\delta_1 + \big(4\mu D(1+\nu^D) + 2B^2\big)\gamma_1}{2\mu D(1-\nu^D) - B^2}\\
&\qquad \times R\left(\frac{1}{\xi^{-1}-\lambda} - \frac{\bar{a}\lambda^2}{\xi-\lambda} + \frac{\bar{\Lambda}^{-1}\bar{a}}{\rho\xi^{-1}-\lambda^{-1}} + \sum_{n=1}^{\infty}(\bar{a}_n\xi^{-n} + \bar{a}_{-n}\xi^n)\right).
\end{aligned} \tag{41}
$$

By enforcing the boundary condition for a rigidly clamped edge on the outer circle $|\xi| = \rho^{-1/2}$, we obtain a second set of expressions for $\psi(\xi)$ and $\Psi(\xi)$:

$$
\begin{aligned}
\psi(\xi) &= (\kappa^A-1)\delta_1 R\left(\frac{\lambda^{-1}}{1-\rho\lambda\xi} + \frac{\bar{a}}{\rho^{-1}\xi^{-1}-\lambda^{-1}} - \frac{\bar{\Lambda}^{-1}\bar{a}\lambda^2}{\xi-\lambda} + \sum_{n=1}^{\infty}(\bar{a}_n\rho^{-n}\xi^{-n} + \bar{a}_{-n}\rho^n\xi^n)\right),\\
\Psi(\xi) &= -2\gamma_1 R\left(\frac{\lambda^{-1}}{1-\rho\lambda\xi} + \frac{\bar{a}}{\rho^{-1}\xi^{-1}-\lambda^{-1}} - \frac{\bar{\Lambda}^{-1}\bar{a}\lambda^2}{\xi-\lambda} + \sum_{n=1}^{\infty}(\bar{a}_n\rho^{-n}\xi^{-n} + \bar{a}_{-n}\rho^n\xi^n)\right).
\end{aligned} \tag{42}
$$

Equating the two expressions for each of $\psi(\xi)$ and $\Psi(\xi)$, we obtain the expressions

$$
\begin{aligned}
\Lambda = \bar{\Lambda} &= -\frac{(\kappa^A-1)\big(2\mu D(1-\nu^D) - B^2\big)\delta_1}{\big(4\mu D(1-\nu^D) + B^2(\kappa^A-1)\big)\delta_1 + 8B\mu D\gamma_1}\\
&= -\frac{2\big(2\mu D(1-\nu^D) - B^2\big)\gamma_1}{B(\kappa^A+1)\delta_1 + \big(4\mu D(1+\nu^D) + 2B^2\big)\gamma_1}
\end{aligned} \tag{43}
$$

for the unknown parameters in (28), and

$$
a_n = \frac{\lambda^{-n-1} + a\Lambda^{-1}\rho^n\lambda^{n+1}}{1-\Lambda\rho^{-n}}, \quad a_{-n} = \frac{\lambda^{n-1} + a\lambda^{-n+1}}{\Lambda^{-1}\rho^{-n} - 1}, \quad n = 1, 2, \ldots. \tag{44}
$$

It can be further deduced from (43) that the two sums $N_{11}^{\infty}+N_{22}^{\infty}$ and $M_{11}^{\infty}+M_{22}^{\infty}$ cannot be set arbitrarily, and should satisfy the restriction

$$\frac{B(N_{11}^{\infty}+N_{22}^{\infty})}{\mu(M_{11}^{\infty}+M_{22}^{\infty})}=\alpha, \tag{45}$$

where

$$\alpha=\frac{\nu^A-\nu^D\pm\sqrt{(\nu^A-\nu^D)^2+2\beta(1-\nu^A)}}{1-\nu^A},\qquad \beta=\frac{B^2}{\mu D}>0. \tag{46}$$

As a result, the parameter Λ in (43) can be more explicitly determined as

$$\Lambda=-\frac{(\kappa^A-1)\big(2(1-\nu^D)-\beta\big)\big(2+\nu^A+\nu^D-\sqrt{(\nu^A-\nu^D)^2+2\beta(1-\nu^A)}\big)}{\big(4(1+\nu^D)-\beta(\kappa^A-1)\big)\big(2-\nu^A-\nu^D+\sqrt{(\nu^A-\nu^D)^2+2\beta(1-\nu^A)}\big)} \tag{47}$$

if the plus sign is chosen in (46), and

$$\Lambda=-\frac{(\kappa^A-1)\big(2(1-\nu^D)-\beta\big)\big(2+\nu^A+\nu^D+\sqrt{(\nu^A-\nu^D)^2+2\beta(1-\nu^A)}\big)}{\big(4(1+\nu^D)-\beta(\kappa^A-1)\big)\big(2-\nu^A-\nu^D-\sqrt{(\nu^A-\nu^D)^2+2\beta(1-\nu^A)}\big)} \tag{48}$$

if the minus sign is chosen in (46). Equations (47) and (48) suggest that there are two distinct values of Λ for given material parameters ν^A, ν^D and β.

For example, if we set $\nu^A=\frac{1}{4}$, $\nu^D=0.3$, $\beta=0.2$, then $\alpha=0.6667$ and $\Lambda=-0.2903$, or $\alpha=-0.8$ and $\Lambda=-1$.

In addition, the satisfaction of the remote boundary conditions for $\psi(z)$ and $\Psi(z)$ in the second and fourth expressions of (11) yields

$$\begin{aligned}\delta_2&=-\frac{-\big(4\mu D(1-\nu^D)+B^2(\kappa^A-1)\big)\delta_1-8B\mu D\gamma_1}{2\mu D(1-\nu^D)-B^2}\bar{a}\lambda^2,\\ \gamma_2&=-\frac{B(\kappa^A+1)\delta_1+\big(4\mu D(1+\nu^D)+2B^2\big)\gamma_1}{2\mu D(1-\nu^D)-B^2}\bar{a}\lambda^2.\end{aligned} \tag{49}$$

Thus, in addition to condition (45), the remote uniform loading should also satisfy the restrictions

$$\frac{N_{11}^{\infty}-N_{22}^{\infty}-2\mathrm{i}N_{12}^{\infty}}{N_{11}^{\infty}+N_{22}^{\infty}}=\frac{M_{11}^{\infty}-M_{22}^{\infty}-2\mathrm{i}M_{12}^{\infty}}{M_{11}^{\infty}+M_{22}^{\infty}}=-\bar{a}\lambda^2. \tag{50}$$

The hoop membrane stress resultant and hoop bending moment are both uniformly distributed along the boundary of the left harmonic hole as

$$N_{\theta\theta}=N_{11}^{\infty}+N_{22}^{\infty},\quad M_{\theta\theta}=M_{11}^{\infty}+M_{22}^{\infty},\quad z\in L_1. \tag{51}$$

Meanwhile, the interfacial normal membrane stress resultant and interfacial normal bending moment are uniformly distributed along the right inclusion/matrix interface as

$$\begin{aligned}N_{\rho\rho}&=\frac{(\kappa^A+1)(1+\nu^D-\frac{\beta}{\alpha})}{4(1+\nu^D)-\beta(\kappa^A-1)}(N_{11}^{\infty}+N_{22}^{\infty}),\\ M_{\rho\rho}&=\frac{4-\alpha(\kappa^A-1)}{4(1+\nu^D)-\beta(\kappa^A-1)}(M_{11}^{\infty}+M_{22}^{\infty}),\quad z\in L_2.\end{aligned} \tag{52}$$

The hoop membrane stress resultant and hoop bending moment are both constant along the right inclusion/matrix interface on the matrix side, and are given by

$$\begin{aligned} N_{\theta\theta} &= \frac{(1+\nu^D)(3-\kappa^A)+\frac{\beta}{\alpha}(\kappa^A+1)-\beta(\kappa^A-1)}{4(1+\nu^D)-\beta(\kappa^A-1)}(N_{11}^{\infty}+N_{22}^{\infty}), \\ M_{\theta\theta} &= \frac{4\nu^D+(\alpha-\beta)(\kappa^A-1)}{4(1+\nu^D)-\beta(\kappa^A-1)}(M_{11}^{\infty}+M_{22}^{\infty}), \quad z\in L_2. \end{aligned} \tag{53}$$

5. Conclusions

We have proposed a new harmonic condition in the context of isotropic laminated plates. By imposing this condition and by using the complex variable formulation for a laminated plate [Beom and Earmme 1998; Wang and Zhou 2014], we have successfully obtained (i) a single harmonic hole, (ii) a single harmonic rigid inclusion, (iii) two interacting harmonic holes, (iv) two interacting harmonic rigid inclusions, and (v) a harmonic hole interacting with another harmonic rigid inclusion when the laminated plate is subjected to remote uniform membrane stress resultants and bending moments. It is shown that a single harmonic hole or rigid inclusion must be elliptical in shape and that the remote uniform loading should satisfy (18) for a harmonic hole, or (23) for a harmonic rigid inclusion. Our results in Section 4 show that it is permissible to obtain two interacting nonelliptical harmonic shapes composed of (i) two holes, (ii) two rigid inclusions, or (iii) one hole near another rigid inclusion. For the case of two interacting harmonic holes or two interacting harmonic rigid inclusions, $\Lambda \equiv 1$ in (28); for the case of one harmonic hole interacting with another harmonic rigid inclusion, Λ is determined by Equations (47) and (48). The remote loading condition (33) for two interacting harmonic holes is identical in form to (18) for a single harmonic hole, the remote loading condition (38) for two interacting harmonic rigid inclusions is identical in form to (23) for a single harmonic rigid inclusion, whereas the remote loading conditions (45) and (50) for a harmonic hole interacting with another harmonic rigid inclusion are more stringent. It is observed that for all the cases discussed, the hoop membrane stress resultant and hoop bending moments are always constant along any existing boundary and interface (see (19), (25), (34), (40), (51) and (53)). Thus, the obtained harmonic shapes also satisfy the equal strength condition.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (grant number 11272121), and by the Natural Sciences and Engineering Research Council of Canada through a Discovery Grant.

References

[Beom and Earmme 1998] H. G. Beom and Y. Y. Earmme, "Complex variable method for problems of a laminate composed of multiple isotropic layers", *Int. J. Fract.* **92**:4 (1998), 305–324.

[Bjorkman and Richards 1976] G. S. Bjorkman, Jr. and R. Richards, Jr., "Harmonic holes: an inverse problem in elasticity", *J. Appl. Mech.* (*ASME*) **43** (1976), 414–418.

[Bjorkman and Richards 1979a] G. S. Bjorkman, Jr. and R. Richards, Jr., "Harmonic holes for nonconstant fields", *J. Appl. Mech.* (*ASME*) **46**:3 (1979), 573–576.

[Bjorkman and Richards 1979b] G. S. Bjorkman, Jr. and R. Richards, Jr., "The rigid harmonic inclusion", pp. 781–784 in *Proceedings of the Third Engineering Mechanics Division Specialty Conference* (Austin, TX, 1979), edited by C. P. Johnson, ASCE, New York, 1979.

[Bonnet and Constantinescu 2005] M. Bonnet and A. Constantinescu, "Inverse problems in elasticity", *Inverse Probl.* **21**:2 (2005), R1–R50.

[Budiansky et al. 1993] B. Budiansky, J. W. Hutchinson, and A. G. Evans, "On neutral holes in tailored, layered sheets", *J. Appl. Mech.* (*ASME*) **60**:4 (1993), 1056–1058.

[Bui 1993] H. D. Bui, *Introduction aux problèmes inverses en mécanique des máteriaux*, Eyrolles, Paris, 1993. Translated as *Inverse problems in the mechanics of materials: an introduction*, CRC Press, Boca Raton, FL, 1994.

[Cheng and Reddy 2002] Z.-Q. Cheng and J. N. Reddy, "Octet formalism for Kirchhoff anisotropic plates", *Phil. Trans. R. Soc. A* **458**:2022 (2002), 1499–1517.

[Cheng and Reddy 2003] Z.-Q. Cheng and J. N. Reddy, "Green's functions for infinite and semi-infinite anisotropic thin plates", *J. Appl. Mech.* (*ASME*) **70**:2 (2003), 260–267.

[Cheng and Reddy 2004a] Z.-Q. Cheng and J. N. Reddy, "Green's functions for an anisotropic thin plate with a crack or an anticrack", *Int. J. Eng. Sci.* **42**:3–4 (2004), 271–289.

[Cheng and Reddy 2004b] Z.-Q. Cheng and J. N. Reddy, "Laminated anisotropic thin plate with an elliptic inhomogeneity", *Mech. Mater.* **36**:7 (2004), 647–657.

[Cherepanov 1974] G. P. Cherepanov, "Обратные задачи теории упругости", *Prikl. Mat. Mekh.* **38**:6 (1974), 963–979. Translated as "Inverse problems of the plane theory of elasticity"*J. Appl. Math. Mech.* **38**:6 (1974), 915–931.

[Hasebe and Wang 2000] N. Hasebe and X.-F. Wang, "Green's functions of thin plate bending problem under fixed boundary", *J. Eng. Mech.* (*ASCE*) **126**:2 (2000), 206–213.

[Hasebe and Wang 2002] N. Hasebe and X.-F. Wang, "Green's functions for the bending of thin plates under various boundary conditions and applications: a review", *Thin-Walled Struct.* **40**:7–8 (2002), 611–624.

[Kantorovich and Krylov 1950] L. V. Kantorovich and V. I. Krylov, Приближенные методы высшего анализа, 3rd ed., State Publishing House of Technical Literature, Moscow, 1950. Translated as *Approximate methods of higher analysis*, Wiley, New York, 1958.

[Mansfield 1953] E. H. Mansfield, "Neutral holes in plane sheet: reinforced holes which are elastically equivalent to the uncut sheet", *Quart. J. Mech. Appl. Math.* **6** (1953), 370–378.

[Milton and Serkov 2001] G. W. Milton and S. K. Serkov, "Neutral coated inclusions in conductivity and anti-plane elasticity", *Phil. Trans. R. Soc. A* **457**:2012 (2001), 1973–1997.

[Ru 1998] C.-Q. Ru, "Interface design of neutral elastic inclusions", *Int. J. Solids Struct.* **35**:7–8 (1998), 559–572.

[Ru 1999a] C.-Q. Ru, "Three-phase elliptical inclusions with internal uniform hydrostatic stresses", *J. Mech. Phys. Solids* **47**:2 (1999), 259–273.

[Ru 1999b] C.-Q. Ru, "A new method for an inhomogeneity with stepwise graded interphase under thermomechanical loadings", *J. Elasticity* **56**:2 (1999), 107–127.

[Senocak and Waas 1995] E. Senocak and A. M. Waas, "Neutral cutouts in laminated plates", *Mech. Compos. Mater. Struct.* **2**:1 (1995), 71–89.

[Sih and Rice 1964] G. C. Sih and J. R. Rice, "The bending of plates of dissimilar materials with cracks", *J. Appl. Mech.* (*ASME*) **31**:3 (1964), 477–482.

[Vellaichamy et al. 1990] S. Vellaichamy, B. G. Prakash, and S. Brun, "Optimum design of cutouts in laminated composite structures", *Comput. Struct.* **37**:3 (1990), 241–246.

[Wang 2012] X. Wang, "Uniform fields inside two non-elliptical inclusions", *Math. Mech. Solids* **17**:7 (2012), 736–761.

[Wang and Hasebe 2000a] X.-F. Wang and N. Hasebe, "Green's function for mixed boundary value problem of thin plate", *J. Eng. Mech.* (*ASCE*) **126**:8 (2000), 787–794.

[Wang and Hasebe 2000b] X.-F. Wang and N. Hasebe, "Bending of a thin plate containing a rigid inclusion and a crack", *Eng. Anal. Bound. Elem.* **24**:2 (2000), 145–153.

[Wang and Hasebe 2001] X.-F. Wang and N. Hasebe, "Green's function of a point dislocation for the bending of a composite infinite plate with an elliptical hole at interface", *Arch. Appl. Mech.* **71**:4-5 (2001), 233–248.

[Wang and Schiavone 2013] X. Wang and P. Schiavone, "Interface cracks in Kirchhoff anisotropic thin plates of dissimilar materials", *J. Appl. Mech.* (*ASME*) **80**:4 (2013), Article ID #041025.

[Wang and Zhou 2013] X. Wang and K. Zhou, "Twelve-dimensional Stroh-like formalism for Kirchhoff anisotropic piezoelectric thin plates", *Int. J. Eng. Sci.* **71** (2013), 111–136.

[Wang and Zhou 2014] X. Wang and K. Zhou, "An inclusion of arbitrary shape in an infinite or semi-infinite isotropic multilayered plate", *Int. J. Appl. Mech.* **06**:01 (2014), Article ID #1450001.

[Wang et al. 2005] G. F. Wang, P. Schiavone, and C.-Q. Ru, "Harmonic shapes in finite elasticity under nonuniform loading", *J. Appl. Mech.* (*ASME*) **72**:5 (2005), 691–694.

[Yin 2005a] W.-L. Yin, "Green's function of anisotropic plates with unrestricted coupling and degeneracy, part 1: The infinite plate", *Compos. Struct.* **69**:3 (2005), 360–375.

[Yin 2005b] W.-L. Yin, "Green's function of anisotropic plates with unrestricted coupling and degeneracy, part 2: Other domains and special laminates", *Compos. Struct.* **69**:3 (2005), 376–386.

[Zakharov and Becker 2000] D. D. Zakharov and W. Becker, "Unsymmetric composite laminate with a discontinuity of the in-plane displacement or of the slope", *Acta Mech.* **144**:3-4 (2000), 127–135.

Received 13 Mar 2014. Accepted 17 Jul 2014.

XU WANG: xuwang_sun@hotmail.com
School of Mechanical and Power Engineering, East China University of Science and Technology, 130 Meilong Road, Shanghai, 200237, China

PETER SCHIAVONE: p.schiavone@ualberta.ca
Department of Mechanical Engineering, University of Alberta, 4-9 Mechanical Engineering Building, Edmonton, AB T6G 2G8, Canada

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2015 is US $565/year for the electronic version, and $725/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 10, No. 3 May 2015

Special issue
In Memoriam: Huy Duong Bui

1559-3959(2015)10:3;1-A