RELATION BETWEEN THE MAXWELL EQUATIONS AND BOUNDARY CONDITIONS IN PIEZOELECTRIC AND PIEZOMAGNETIC FRACTURE MECHANICS AND ITS APPLICATION

Hao Tian-hu

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This paper presents the relation between the Maxwell equations and the boundary conditions in piezoelectric and piezomagnetic fracture mechanics. In addition, considering that the case after deformation (current configuration in nonlinear elasticity) is very important for these conditions, the significance of them has been studied for this case. The application of them has also been researched. Moreover, the stress field of the solid material caused by the electric field has been discussed. In the conclusion, it is briefly discussed how to determine the crack open or not, which is of vital importance for semipermeable and impermeable boundary conditions.

1. Introduction

In mechanics, along with rapid development of the computing technology, the methods of solution have been various. Accordingly, the equations of constitutive and the boundary conditions should be paid more attention. Consequently, in the research on the solid fracture mechanics of piezoelectric and piezomagnetic materials, the exploration of the relation between the Maxwell equations and the electromagnetic boundary conditions is necessary. Although many authors have researched on these boundary conditions such as Kumar and Singh [1997], yet the study of this relation is not enough. In this paper, firstly, the relation between the permeable electromagnetic boundary conditions and Maxwell equations has been studied. Then, the permeable, the semipermeable and the impermeable electromagnetic boundary conditions have been discussed [Zhang et al. 2002]. In particular, for the semipermeable electromagnetic boundary condition, the body after deformation must be dealt with. Therefore, we must cope with the nonlinear elasticity. We know that this theory is very complicated. In order to avoid this difficulty, we consider using the approximated direct method instead of the iteration method. Consequently, we don’t need to carry out this repeat calculation.

Lastly, the problem of the stress field of the solid material caused by the electric field had been discussed. It is briefly discussed how to determine the crack open or not.

2. Maxwell equations and permeable conditions

It is known that the Maxwell equations can be written in two forms. They are differential form and integral form. The Maxwell equations in these forms are

\[ \int_S D \cdot dS = q_0 \quad \text{or} \quad \text{Div} D = q_1, \quad (\partial D_1/\partial x_1 + \partial D_2/\partial x_2 + \partial D_3/\partial x_3 = q_1), \]  

(1)

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where $D$ is the electric displacement vector, $S$ is the whole surface of a body, $q_0$ is the total charge in the body, $q_1$ is the charge density and $dS$ is a vector as in the course of vector analysis.

\[
\int_k E \cdot dk = - \int_{S_k} (\partial B / \partial t) \cdot dS \quad \text{or} \quad \text{Curl}E = - \partial B / \partial t,
\]

\[
\begin{align*}
\partial E_2 / \partial x_1 - \partial E_1 / \partial x_2 &= - \partial B_3 / \partial t, \\
\partial E_1 / \partial x_3 - \partial E_3 / \partial x_1 &= - \partial B_2 / \partial t, \\
\partial E_3 / \partial x_2 - \partial E_2 / \partial x_3 &= - \partial B_1 / \partial t.
\end{align*}
\]

(2)

where $E$ is electric field vector, $B$ is magnetic induction vector, $k$ is a closed curve, $dk$ is the tangential vector of $k$ and $S_k$ is a surface whose boundary curve is $k$.

\[
\int_S B \cdot dS = 0 \quad \text{or} \quad \text{div}B = 0, \quad (\partial B_1 / \partial x_1 + \partial B_2 / \partial x_2 + \partial B_3 / \partial x_3 = 0),
\]

(3)

\[
\int_k H \cdot dk = J_0 + \int_{S_k} (\partial D / \partial t) \cdot dS \quad \text{or} \quad \text{Curl}H = J_0 + \partial D / \partial t,
\]

\[
\begin{align*}
\partial H_2 / \partial x_1 - \partial H_1 / \partial x_2 &= J_{03} + \partial D_3 / \partial t, \\
\partial H_1 / \partial x_3 - \partial H_3 / \partial x_1 &= J_{02} + \partial D_2 / \partial t, \\
\partial H_3 / \partial x_2 - \partial H_2 / \partial x_3 &= J_{01} + \partial D_1 / \partial t.
\end{align*}
\]

(4)

where $H$ is magnetic field intensity vector and $J_{01}$, $J_{02}$, $J_{03}$, are the components of the current intensity vector $J_0$.

Only the static condition and the cases $q_0 = 0$, $q_1 = 0$, $J_0 = 0$ are dealt with.

The equations (1) and (3) become

\[
\int_S D \cdot dS = 0 \quad \text{or} \quad \text{Div} D = 0 \quad (\partial D_1 / \partial x_1 + \partial D_2 / \partial x_2 + \partial D_3 / \partial x_3 = 0), \quad \text{and}
\]

\[
\int_S B \cdot dS = 0 \quad \text{or} \quad \text{Div} B = 0 \quad (\partial B_1 / \partial x_1 + \partial B_2 / \partial x_2 + \partial B_3 / \partial x_3 = 0).
\]

(5)

Considering $\partial D / \partial t = 0$ and $\partial B / \partial t = 0$ (static condition) and $J_0 = 0$, the equations (2) and (4) become

\[
\begin{align*}
\int_k E \cdot dk &= 0 \quad \text{or} \quad \text{Curl}E = 0 \\
(\partial E_2 / \partial x_1 - \partial E_1 / \partial x_2 = 0, \quad \partial E_1 / \partial x_3 - \partial E_3 / \partial x_1 = 0, \quad \partial E_3 / \partial x_2 - \partial E_2 / \partial x_3 = 0), \quad \text{and}
\end{align*}
\]

\[
\int_k H \cdot dk = 0 \quad \text{or} \quad \text{Curl}H = 0
\]

\[
(\partial H_2 / \partial x_1 - \partial H_1 / \partial x_2 = 0, \quad \partial H_1 / \partial x_3 - \partial H_3 / \partial x_1 = 0, \quad \partial H_3 / \partial x_2 - \partial H_2 / \partial x_3 = 0).
\]

(6)

On the basis of Equation (6), we have

\[
E_i = \partial \phi / \partial x_i \quad \text{and} \quad H_i = \partial \phi_1 / \partial x_i
\]

(6a)

where $\phi$ is the electric potential and $\phi_1$ is the magnetic potential.
For plane case, on the $Ox_1x_2$ plane, two integrals in (5) are all computed along the surface curve $p$ of the area and become

$$\int_p D_n \, dp = 0 \quad \text{and} \quad \int_p B_n \, dp = 0, \quad \text{or} \quad \partial D_1/\partial x_1 + \partial D_2/\partial x_2 = 0 \quad \text{and} \quad \partial B_1/\partial x_1 + \partial B_2/\partial x_2 = 0,$$

where $p$ is the surface curve in the plane, $D_n$ is the normal component of vector $D$ and $B_n$ is the normal component of vector $B$.

The two integrals in (6) become

$$\int_p E_t \, dp = 0 \quad \text{or} \quad \partial E_2/\partial x_1 - \partial E_1/\partial x_2 = 0, \quad \text{and} \quad \int_p H_t \, dp = 0 \quad \text{or} \quad \partial H_2/\partial x_1 - \partial H_1/\partial x_2 = 0,$$

where $E_t$ is the tangential component of vector $E$ and $H_t$ is the tangential component of vector $H$.

Now, based on Maxwell equations, the permeable boundary conditions for the static electric and magnetic case are studied. One considers a surface, which can be the interface of two materials. A short segment of this surface is studied (we shall discuss it in detail in Appendix). For convenience, the segment in the studied plane is a part of $Ox_1$ axis. In order to study the conditions on the segment, a rectangle is taken, as shown in the figure:

The longer side is parallel to the segment, i.e., $Ox_1$ axis with width $d$. The shorter is perpendicular to the segment, i.e., $Ox_2$ axis and its length trends to zero. For the rectangle, the two integrals in (5) and (7) can be computed.

Considering the area is very small, one can be sure that the value of $D$, $E(\phi)$, $B$ and $H(\phi_1)$ are constants on one side but can be different on other side. Therefore, the contribution on the shorter side tends to zero. For the longer side, they are parallel to $Ox_1$. The tangent component of vector $E$ is $E_1$. Similarly, the normal component of vector $D$ is $D_2$. The equation $\int_p D_n \, dp = 0$ and $\int_p E_t \, dp = 0$ becomes

$$(D_2^+ - D_2^-)d = 0, \quad \text{i.e.,} \quad D_2^+ = D_2^-, \quad \text{and} \quad (E_1^+ - E_1^-)d = 0, \quad \text{i.e.,} \quad E_1^+ = E_1^- (\phi^+ = \phi^-),$$

where $D_2^+$ is the $D_2$ on the upper surface of the interface and $D_2^-$ is that on the lower surface. Similarly, $E_1^+$ and $E_1^-$ can be understand as $D_2^+$ and $D_2^-$. 
For the magnetic field, one has
\[ B_2^+ = B_2^- \quad \text{and} \quad H_1^+ = H_1^- (\phi_1^+ = \phi_1^-). \] (10)

It is the permeable boundary conditions for a surface in static electric and magnetic field. For the static electric field, in the piezoelectric fracture mechanics, it is the well known boundary condition of [Parton 1976; Mikhailov and Parton 1990].

As a matter of fact, that is an old boundary condition. In any textbook of the theory of electromagnetism, for the interface between two materials, one can find this boundary condition.

3. Semipermeable conditions and impermeable condition

Although the permeable boundary conditions are deduced from Maxwell equations, yet they have not considered the existence of crack and only for a surface or an interface in the materials. When one directly uses them for the piezoelectric and piezomagnetic fracture mechanics, they may result in larger deviations sometime. However, the importance of these conditions is that they can be the basis of the further discussion on the boundary conditions for the piezoelectric and piezomagnetic fracture mechanics.

Now, the semipermeable boundary conditions and the impermeable boundary conditions are considered. Firstly, the cracks can be divided into two kinds. The first is for the cracks with the opening voids full of fluid (air) after deformation. We always dealt with this kind. The second has not the opening voids such as the antiplane case (as \( u_1 = u_2 = 0 \), the crack can not be opening), the crack subjected to compression stress, etc. For the second, as the void does not exist after deformation, the permeable equations
\[ D_2^+ = D_2^- \quad \text{and} \quad E_1^+ = E_1^- (\phi_1^+ = \phi_1^-), \]
\[ B_2^+ = B_2^- \quad \text{and} \quad H_1^+ = H_1^- (\phi_1^+ = \phi_1^-), \] (11)
can be accepted.

Then, for the first, when studying the crack full of air, it is improper to write the boundary condition before deformation as the classical theory; otherwise the crack is only a slit without void and no air can be exist in it. Therefore, we must consider the boundary condition after deformation (the current configuration in nonlinear elasticity). In the meantime, the opening crack becomes a void. At the surface of the void, on the interface between the fluid (in void) and the solid (outside void), we have the interface boundary conditions
\[ D_2^+ = D_2^- \quad \text{and} \quad E_1^+ = E_1^- (\phi_1^+ = \phi_1^-), \]
\[ B_2^+ = B_2^- \quad \text{and} \quad H_1^+ = H_1^- (\phi_1^+ = \phi_1^-), \] (11)
where \( D_2^+, D_2^-, E_1^+, E_1^- (\phi_1^+, \phi_1^-), B_2^+, B_2^-, H_1^+, H_1^- (\phi_1^+, \phi_1^-) \) belong to the fluid (in void) and the solid (outside void).

In the fluid (in void), there are the various basic equations of the fluid (air), such as
\[ \partial^2 \phi_i / \partial x_1^2 + \partial^2 \phi_i / \partial x_2^2 + \partial^2 \phi_i / \partial x_3^2 = 0 \quad \text{and} \quad \partial^2 \phi_{ii} / \partial x_1^2 + \partial^2 \phi_{ii} / \partial x_2^2 + \partial^2 \phi_{ii} / \partial x_3^2 = 0, \] (12)
where \( \phi_i \) and \( \phi_{ii} \) are the electric and magnetic potentials of the fluid components in the void. For the solid (outside void), the basic equations are well known and we shall not discuss them here. It is the all conditions satisfied by the body with void including air. To solve it is a complicated problem. Generally, it is better to use the nonlinear elasticity to solve this problem but the nonlinear elasticity is too tough to study. Now, a simpler method is accepted in study. This method is as follows. For convenience, when studying the crack void full of air, the boundary after deformation can be accepted as the boundary before deformation (a closed slit) adding the evaluated boundary displacement. Naturally, we know that this
displacement is also found by classical theory. In this theory, the displacement field is calculated from the boundary before deformation, i.e., on the nondeformation body. This result is not precise but results in the little deviation. Nevertheless, to the case of crack void full of air, the deviation is not negligible.

As mentioned above, the approximate boundary after deformation is determined. Now we consider the solid (outside void) and the fluid (in void). As mentioned above, the constitutive equation of the solid is well known and that of the fluid are various and complicated. In order to avoid that of fluid, considering the void is very small after deformation, Hao [1993] and Hao and Shen [1994] have used the linear change of φ and φ₁ along the surface normal to replace the rigorous solution of the complicated equation (that is one of the basic assumption of this boundary condition). In the void, $E_2$ and $H_2$ (for small deformation case, $E_n$ and $H_n$ are replaced by $E_2$ and $H_2$) become $-(φ^+ - φ^-)/(u_2^+ - u_2^-)$ and $-(φ_1^+ - φ_1^-)/(u_2^+ - u_2^-)$, and $u_2$ is the evaluated boundary displacement component as above mentioned.

Considering $D_2 = ε_a E_2$ and $B_2 = ε_{a1} H_2$ in air, one obtains

$$D_2^+ = D_2^- = -ε_a(φ^+ - φ^-), \quad B_2^+ = B_2^- = -ε_{a1}(φ_1^+ - φ_1^-) \quad (13)$$

where $ε_a$ and $ε_{a1}$ are the electric and magnetic permitivities of air.

Since in the void, $E_2$ and $H_2$ become the constants along the normal, $D_2$ and $B_2$ are also the constants along the normal. Then, we obtain

$$D_2^+ = D_2^- = 0, \quad B_2^+ = B_2^- = 0. \quad (14)$$

In fact, this boundary condition is obtained from the conception of current configuration in finite deformation theory and the linear change of potential as [Hao 2004]. In short, the conception of current configuration is that we must deal with the crack boundary after deformation when studying a crack.

The equation (14) is the semipermeable boundary condition. For the piezoelectric case, it is suggested by [Hao 1993; Hao and Shen 1994].

It is approximate to use an average rate of change of potential $φ$ to take the place of the actual rate. However, as it is only an approximate boundary condition rather than an exact result, I can be sure that for disagreeing it we must be based on some contrary examples, not one exact example.

It is apparent that Equation (13) will be reduced to $φ^+ = φ^-$ or $φ_1^+ = φ_1^-$ (one of the permeable boundary conditions) when $u_2^+ - u_2^- = 0$ (closed), and to the following equation under the condition $ε_a = 0$ and $ε_{a1} = 0$:

$$D_2^+ = D_2^- = 0, \quad B_2^+ = B_2^- = 0. \quad (15)$$

The Equation (15) is the impermeable boundary conditions.

### 4. Some problems about the application of these boundary conditions

In order to avoid the irrationality in the result, we must decide to suitably use these boundary conditions. The permeable boundary condition is obtained from the Maxwell equations exactly. Therefore, we must determine in what situation this boundary may be accepted. If we can be sure that the crack is closed, the permeable boundary condition should be accepted. However, it is not easy to determine the crack
being closed. In order to do it, from [Hao 2001], we know
\[ u_2^+ - u_2^- = 2\text{Re} \sum_{j=1}^{4} (\beta_{2r}k_{rj} + \eta_{2a}d_{aj})[f_j'(x_1)^+ - f_j'(x_1)^-]/\mu, \] (16a)

where the constants can be found in [Hao 2001].

It is an exact formula to decide whether the crack is closed or not but it is too complicated to be used. We shall discuss it in detail later.

When we know that the crack is closed, the permeable boundary condition can be accepted.

From the Equation (16), we can also decide that the crack is open. At this time, the semipermeable or the impermeable boundary condition can be considered.

About the semipermeable boundary condition, although it has considered the electric field in the air, yet it seems to be too complicated to deal with. However, many results can be accepted by us to study this problem without difficulty. These results tell us that \( D_j^+ \) can be determined directly without complicated computing. For an example, to the common multiple collinear cracks (naturally, single crack) under the simple remote load case, we have following result.

In general case, there are four functions of complex variables \( f_j''(z_j) \) and the displacements and potential can be
\[ \phi^+ - \phi^- = 2\text{Re} \sum_{j=1}^{4} (h_{1r}k_{rj} - \xi_{1a}d_{aj})[f_j'(x_1)^+ - f_j'(x_1)^-], \] (16b)
\[ u_2^+ - u_2^- = 2\text{Re} \sum_{j=1}^{4} (\beta_{2r}k_{rj} + \eta_{2a}d_{aj})[f_j'(x_1)^+ - f_j'(x_1)^-]/\mu_j \]

where the constants can be found in [Hao 2001].

The functions \( f_j''(z_j) \) can be obtained from
\[ \sum_{j=1}^{4} l_{ij}f_j''(z) = e_i + if_i + e_i[Q(z) - 1], \quad i = 1, \ldots, 4, \] (17)
\[ Q(z) = z^n + c_1z^{n-1} + \cdots + c_{n-1}z + c_n \prod_{k=1}^{n}(z - a_k)(z - b_k)^{1/2}, \]
where \( c_1, c_2, \ldots, c_{n-1}, c_n \) are defined by single value requirements of displacements and potential and \( a_k \) and \( b_k \) are the two tips of the \( k \)-th crack but no relation with material constants.

Then, using linear algebra method, one can find functions \( f_j''(z_j) \) as [Hao 2001]
\[ f_i''(z_i) = \sum_{j=1}^{4} x_{ij}[D_j + e_jQ(z_i)], \quad D_j = e_j + if_j - e_j, \] (18)

where \( x_{ij} \) is determined by linear algebra method as [Hao 2001].

We introduce
\[ P'(x_1) = -Q(x_1). \] (19)
Then, there is
\[
f''_i(x_1)^+ - f''_i(x_1)^- = -\left[ \sum_{j=1}^{4} x_{ij} e_j Q(x_1) \right]^+ + \left[ \sum_{j=1}^{4} x_{ij} e_j Q(x_1) \right]^-
\]
\[
= \left[ \sum_{j=1}^{4} x_{ij} e_j P'(x_1) \right]^- - \left[ \sum_{j=1}^{4} x_{ij} e_j P'(x_1) \right]^+
\]
\[
= \sum_{j=1}^{4} x_{ij} e_j [P'(x_1)^- - P'(x_1)^+].
\]

From Equation (20), we obtain
\[
f'_i(x_1)^+ - f'_i(x_1)^- = \sum_{j=1}^{4} x_{ij} e_j [P(x_1)^- - P(x_1)^+]
\]
\[
= [P(x_1)^- - P(x_1)^+] \sum_{j=1}^{4} x_{ij} e_j
\]
\[
= A_i [P(x_1)^- - P(x_1)^+],
\]
\[\tag{21}\]
\[A_i = \sum_{j=1}^{4} x_{ij} e_j. \tag{22}\]

Then, we have
\[
\phi^+ - \phi^- = 2\text{Re} \sum_{j=1}^{4} (h_{1r} k_{rj} - \xi_{1ad}) A_j [P(x_1)^- - P(x_1)^+]
\]
\[
= 2\text{Re} [P(x_1)^- - P(x_1)^+] \sum_{j=1}^{4} (h_{1r} k_{rj} - \xi_{1ad}) A_j,
\]
\[\tag{23}\]
\[
u_2^+ - \nu_2^- = 2\text{Re} \sum_{j=1}^{4} (\beta_{2r} k_{rj} + \eta_{2ad}) A_j [P(x_1)^- - P(x_1)^+] / \mu_j
\]
\[
= 2\text{Re} [P(x_1)^- - P(x_1)^+] \sum_{j=1}^{4} (\beta_{2r} k_{rj} + \eta_{2ad}) A_j / \mu_j.
\]

One can find that the function \([P(x_1)^+ - P(x_1)^-]\) is imaginary. Therefore, we have
\[
\phi^+ - \phi^- = 2i [P(x_1)^- - P(x_1)^+] \text{Im} \sum_{j=1}^{4} (h_{1r} k_{rj} - \xi_{1ad}) A_j,
\]
\[\tag{24}\]
\[
u_2^+ - \nu_2^- = 2i [P(x_1)^- - P(x_1)^+] \text{Im} \sum_{j=1}^{4} (\beta_{2r} k_{rj} + \eta_{2ad}) A_j / \mu_j.
\]
Therefore, $D_2^+$ can be approximately expressed as

$$D_2^+ = -\varepsilon_a \frac{[P(x_1)^- - P(x_1)^+] \text{Im} \sum_{j=1}^{4} (h_{1r}k_{rj} - \xi_{1a}d_{aj}) A_j}{[P(x_1)^- - P(x_1)^+] \text{Im} \sum_{j=1}^{4} (\beta_{2r}k_{rj} + \eta_{2a}d_{aj}) A_j / \mu_j}$$

$$= -\varepsilon_a \frac{\text{Im} \sum_{j=1}^{4} (h_{1r}k_{rj} - \xi_{1a}d_{aj}) A_j}{\text{Im} \sum_{j=1}^{4} (\beta_{2r}k_{rj} + \eta_{2a}d_{aj}) A_j / \mu_j}.$$  

(25)

$D_2^+$ is no relation with these coordinates $x_i$ and can be determined directly without the iteration method.

We must pay attention to that although the expressions of $\phi^+-\phi^-$ and $u_2^+ - u_2^-$ are exact, yet for $D_2^+$ it is approximate. For an example, we consider a crack. In general case, its $\phi^+-\phi^-$ and $u_2^+ - u_2^-$ may be proportional to $(a^2 - z^2)^{1/2}$. When the load leads $u_2^+ - u_2^-$ tending to zero, the crack should be closed. Therefore, we must accept the permeable condition. As $\phi^+-\phi^-$ can also tend to zero, the value $(\phi^+-\phi^-)/(u_2^+ - u_2^-)$ may tend to a constant. However, because the crack is closed, there is no air in the crack void and $(\phi^+-\phi^-)/(u_2^+ - u_2^-)$ is a constant without significance.

5. Stress field caused by the electric field

Now, we study the stress field caused by the electric field. Essentially it is the acting force of electric field on the solid element. The acting force caused by the electric field is a body force. It is known that the stress field caused by the electric field is a square but that by the piezoelectric field is linear [Fang and Yin 1989, 4.7.1, p. 209]. Therefore, the stress field caused by the electric field is always smaller than that by the piezoelectric field [ibid., 4.7.2, p. 210] and always can be neglected.

Due to the complexity of this problem, the stress distribution caused by the electric field will be discussed in detail in another paper.

6. Conclusions

For the electric-magnetic fracture mechanics, the relation between the Maxwell equations and the permeable, semipermeable and impermeable electromagnetic boundary conditions has been studied. Then, the application of these boundary conditions has been discussed. Lastly, the stress field caused by the electric field also has been discussed. It is known that permeable electromagnetic boundary conditions are exact for the closed crack. When we can be sure that the crack is not closed, the semipermeable or the impermeable electromagnetic boundary condition is accepted. Naturally, it has some trouble to use formula (16a) to decide whether the crack is open or not. However, for the cracks on a straight line (for example, the cracks on $Ox_1$) and $D_2^\infty = 0$, it is easy to deal with. From the equation (24), we know

$$u_2^+ - u_2^- = 2i[P(x_1)^- - P(x_1)^+] \text{Im} \sum_{j=1}^{4} (\beta_{2r}k_{rj} + \eta_{2a}d_{aj}) A_j / \mu_j.$$  

(26)

where $A_j = m_j \sigma_2^\infty$ and $m_j$ is a constant and no use for our discussion.

Substituting them into (26), we have

$$u_2^+ - u_2^- = 2i[P(x_1)^- - P(x_1)^+] \sigma_2^\infty \text{Im} \sum_{j=1}^{4} (\beta_{2r}k_{rj} + \eta_{2a}d_{aj}) m_j / \mu_j.$$  

(27)
Figure 1. The relation between the crack closing and the representation of remote stress $p$.

It is clear that the value $u_2^+ - u_2^-$ is proportional to the value $\sigma_2^\infty$ which is the same with the theory of elasticity.

For one crack case, we know

$$u_2^+ - u_2^- = -2i[P(x_1)^+ - P(x_1)^-](k_2/\pi^{1/2}a^{1/2}) \text{Im} \sum_{j=1}^{4} (\beta_{2rj} + \eta_{2a}d_{aj}) m_j/\mu_j,$$  \hspace{1cm} (28)

where $k_2$ is the stress intensity factor and $a$ is the half length of the crack.

In order to clarify the crack closing, it is explained in Figure 1.

For convenience, the term $4a\sigma_2^\infty \text{Im} \sum_{j=1}^{4} (\beta_{2rj} + \eta_{2a}d_{aj}) m_j/\mu_j$ is replaced by $10^{11}p$, where $2a$ is the crack length.

From Figure 1 we know that when the representation of remote stress $p$ tends to 0, the crack closing to infinite.

Previously, we only consider the crack being traction free at its surface. When there is homogeneous load $\sigma_0$ on crack surface and $\sigma_2^\infty = 0$, we resolve it into two cases. The one is homogeneous stress $\sigma_0$ on the whole solid and the another is $\sigma_2^\infty = -\sigma_0$. On the basis of the sum of the two cases, all boundary conditions are satisfied. The case of homogeneous stress $\sigma_0$ on the whole solid is a homogeneous field. It is easy to deal with. The case of $-\sigma_0$ at infinite is that of $\sigma_2^\infty = -\sigma_0$. It has been discussed in the equation (27).

Appendix: About the boundary condition

In order to discuss the boundary condition easily, we consider the plane potential fluid mechanics. Firstly, we introduce the conception of source, sink and vortex point. Naturally, they are the plane potential fluid field which has two components parallel to $Ox_1$ and $Ox_2$. The velocity fields of source and sink are radius. At every point, the velocity is parallel to the radius. The vortex point is tangential velocity field. At every point, the velocity is perpendicular to the radius. Therefore, the vortex point velocity field is a circular ring field. In fluid mechanics, we call this field in rotational field. We know that the conception of source seems to be the water spring. The conception of sink is contrary to that of source. For the vortex point, we always seem to observe it at the water surface. Sometimes, the sources (naturally, sinks
and vortex points) can be considered a line source. That seems to be observed at the water surface. For convenience, we deal with the plane problem on the plane with axis \( Ox_1 \) and \( Ox_2 \). Now, we consider a slit on the interface of two materials. The slit is on the axis \( Ox_1 \). We must be sure that because there are two materials, the velocity must be different in the two materials. Letting the velocities be \( V_1 \) and \( V_2 \) in the two materials. When there is no source and sink in the slit, letting the values \( V_{12} \) and \( V_{22} \) are components of \( V_1 \) and \( V_2 \) paralleling to \( Ox_2 \). \( V_{12} = V_{22} \). When there is no vortex point in the slit, letting the values \( V_{11} \) and \( V_{21} \) are components of \( V_1 \) and \( V_2 \) paralleling to \( Ox_1 \). \( V_{11} = V_{21} \). When there are some sources or sinks in the slit, \( V_{12} \neq V_{22} \) across the slit. When there are some vortex points in the slit, \( V_{11} \neq V_{21} \) across the slit.

On the basis of above discussed, the static electric field is easy to deal with. The positive and negative charge is corresponding to the source and sink. Naturally, it seems to be difficult to find anything corresponding to the vortex point. However, when we study the magnetic field around the wire, we seem to meet the point corresponding to it. Therefore, for the slit of the interface of two materials on \( Ox_1 \) in the electric field, we can be sure that \( D_{12} = D_{22} \) (corresponding to \( V_{12} = V_{22} \) in fluid). Now, we must pay attention to the condition \( V_{11} = V_{21} \) in fluid. We know that in fluid mechanics, we prove the condition \( V_{11} = V_{21} \) based on the condition no vortex point in the slit. In fluid mechanics, as above mentioned, the condition no vortex point is corresponding to that of nonrotation. In Maxwell equations, as above mentioned, the condition nonrotation is discussed in equations (2) and (6). Here, the physical quantity \( E \) pays a leading role. Therefore, the condition \( V_{11} = V_{21} \) in fluid mechanics, is corresponding to the condition \( E_{11} = E_{21} \) in Maxwell equations.

Therefore, we obtain the boundary condition on the interface

\[
D_{12} = D_{22}, \quad E_{11} = E_{21}, \quad (29)
\]

where \( D_{1j} \) is the \( j \)-th of \( D_1 \), \( D_{2j} \) is the \( j \)-th of \( D_2 \), \( E_{1j} \) is the \( j \)-th of \( E_1 \) and \( E_{2j} \) is the \( j \)-th of \( E_2 \).

Naturally, these boundary conditions above mentioned, are obtained based on the analogy method. Now, we shall prove it by integral form of Maxwell equations exactly.

Letting \( BA \) and \( CD \) being the two longer sides of the rectangle (their length equals \( d \)) and \( AC, BD \) being the two shorter sides (their length tends to 0) and the segment of \( Ox_1 \) in the rectangle being the interface, we have \( \int_{p} D_n \, dp = 0 \) and \( \int_{p} E_i \, dp = 0 \). We consider the right spiral rule. The direction of four tops of the rectangle is \( BACDB \). As \( BA \) and \( CD \) are parallel to \( Ox_1 \), their normal is parallel to \( Ox_2 \). The vector \( D_n \) becomes \( D_2 \). As their tangent is parallel to \( Ox_1 \), the vector \( E_i \) becomes \( E_1 \). The equation \( \int_{p} D_n \, dp = 0 \) becomes

\[
-D_{12BA}AB + D_{22CD}CD + \text{ small contribution of the sides } AC \text{ and } DB = 0, \quad (30)
\]

where \( D_{12} \) is the second component of the electric displacement \( D_1 \) in upper half plane \( x_2 > 0 \), \( D_{22} \) is the second component of the electric displacement \( D_2 \) in lower half plane \( x_2 < 0 \), \( D_{12BA} \) is \( D_{12} \) on \( BA \) and \( D_{22CD} \) is \( D_{22} \) on \( CD \).

Considering the contribution of the shorter sides \( AC \) and \( DB \) tending to zero, we neglect it and consider the two sides \( AC \) and \( DB \) being the upper and lower surfaces of the interface. Therefore,

\[
-D_{2BA}AB + D_{2CD}CD = -D_2^+ AB + D_2^- CD = 0, \quad (31)
\]
where $D_2^+ = D_{2BA}$ is the $D_2$ on the upper surface of the interface and $D_2^- = D_{2CD}$ is the $D_2$ on the lower surface of the interface.

Substituting $BA = CD = d$, we obtain

$$d(-D_2^+ + D_2^-) = 0. \quad (32)$$

Therefore, we have

$$D_2^+ = D_2^-. \quad (33)$$

Considering the $E_i$ becomes $E_1$, the equation $\int_p E_i \, dp = 0$ becomes

$$-E_{11BA}BA + E_{21CD}CD + \text{small contribution of the sides } AC \text{ and } DB = 0,$$

where $E_{11}$ is the first component of the electric field $E_1$ in upper half plane $x_2 > 0$, $E_{21}$ is the first component of the electric field $E_1$ in lower half plane $x_2 < 0$. $E_{11BA}$ is $E_1$ on $BA$ and $E_{21CD}$ is $E_1$ on $CD$.

Considering the contribution of the shorter sides $AC$ and $DB$ being very little, we neglect it and consider the two sides $AC$ and $DB$ being the upper and lower surfaces of the interface. Therefore,

$$-E_{11BA}AB + E_{21CD}CD = -E_1^+AB + E_2^-CD = 0, \quad (34)$$

where $E_1^+ = E_{11BA}$ is the $E_1$ on the upper surface of the interface and $E_2^- = E_{21CD}$ is the $E_2$ on the lower surface of the interface.

Substituting $BA = CD = d$, we obtain

$$d(-E_1^+ + E_1^-) = 0. \quad (35)$$

And so we have

$$E_1^+ = E_1^- \quad (36)$$

Therefore, on the upper and lower surface of the interface, the components of vectors $E_1$ and $D_2$ on both surfaces are equal. When on the upper and lower surface of the interface, the components of vector $E_1$ on both surfaces are equal, on the upper and lower surface, the function $\phi (\phi = \partial E_1/\partial x_1)$ on both surfaces is equal (when the interface is $-\infty--\infty$, on the upper and lower surface, the difference between the two functions $\phi^+$ and $\phi^-$ may be a constant).

This result is the famous permeable condition in piezoelectric fracture mechanics [Parton 1976; Parton and Kudryavtsev 1988; Mikhailov and Parton 1990].

In fact, this is an old result in electrodynamics: $\int q \, D_n \, dq = 0$. We can find it in any textbook such as [Coelho 1979].

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References


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HAO TIAN-HU: haoth0000@aliyun.com
State Key Lab for Modification of Polymer Materials and Chemical Fibers, Donghua University, P.O. Box 220, Shanghai, 200051, China
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