Journal of
Mechanics of
Materials and Structures

RELATION BETWEEN THE MAXWELL EQUATIONS AND BOUNDARY CONDITIONS IN PIEZOELECTRIC AND PIEZOMAGNETIC FRACTURE MECHANICS AND ITS APPLICATION

Hao Tian-hu

Volume 10, No. 4 July 2015

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 10, No. 4, 2015
dx.doi.org/10.2140/jomms.2015.10.447

RELATION BETWEEN THE MAXWELL EQUATIONS AND BOUNDARY CONDITIONS IN PIEZOELECTRIC AND PIEZOMAGNETIC FRACTURE MECHANICS AND ITS APPLICATION

HAO TIAN-HU

This paper presents the relation between the Maxwell equations and the boundary conditions in piezoelectric and piezomagnetic fracture mechanics. In addition, considering that the case after deformation (current configuration in nonlinear elasticity) is very important for these conditions, the significance of them has been studied for this case. The application of them has also been researched. Moreover, the stress field of the solid material caused by the electric field has been discussed. In the conclusion, it is briefly discussed how to determine the crack open or not, which is of vital importance for semipermeable and impermeable boundary conditions.

1. Introduction

In mechanics, along with rapid development of the computing technology, the methods of solution have been various. Accordingly, the equations of constitutive and the boundary conditions should be paid more attention. Consequently, in the research on the solid fracture mechanics of piezoelectric and piezomagnetic materials, the exploration of the relation between the Maxwell equations and the electromagnetic boundary conditions is necessary. Although many authors have researched on these boundary conditions such as Kumar and Singh [1997], yet the study of this relation is not enough. In this paper, firstly, the relation between the permeable electromagnetic boundary conditions and Maxwell equations has been studied. Then, the permeable, the semipermeable and the impermeable electromagnetic boundary conditions have been discussed [Zhang et al. 2002]. In particular, for the semipermeable electromagnetic boundary condition, the body after deformation must be dealt with. Therefore, we must cope with the nonlinear elasticity. We know that this theory is very complicated. In order to avoid this difficulty, we consider using the approximated direct method instead of the iteration method. Consequently, we don't need to carry out this repeat calculation.

Lastly, the problem of the stress field of the solid material caused by the electric field had been discussed. It is briefly discussed how to determine the crack open or not.

2. Maxwell equations and permeable conditions

It is known that the Maxwell equations can be written in two forms. They are differential form and integral form. The Maxwell equations in these forms are

$$\int_S \boldsymbol{D}\cdot d\boldsymbol{S} = q_0 \quad \text{or} \quad \mathrm{Div}\boldsymbol{D} = q_1, \quad (\partial D_1/\partial x_1 + \partial D_2/\partial x_2 + \partial D_3/\partial x_3 = q_1), \tag{1}$$

Keywords: Maxwell equations, electric-magnetic boundary condition.

where $\boldsymbol{D}$ is the electric displacement vector, S is the whole surface of a body, q_0 is the total charge in the body, q_1 is the charge density and $d\boldsymbol{S}$ is a vector as in the course of vector analysis.

$$\begin{gathered}\int_k \boldsymbol{E}\cdot d\boldsymbol{k} = -\int_{S_k}(\partial\boldsymbol{B}/\partial t)\cdot d\boldsymbol{S} \quad \text{or} \quad \mathrm{Curl}\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t,\\ \partial E_2/\partial x_1 - \partial E_1/\partial x_2 = -\partial B_3/\partial t,\\ \partial E_1/\partial x_3 - \partial E_3/\partial x_1 = -\partial B_2/\partial t,\\ \partial E_3/\partial x_2 - \partial E_2/\partial x_3 = -\partial B_1/\partial t.\end{gathered} \tag{2}$$

where $\boldsymbol{E}$ is electric field vector, $\boldsymbol{B}$ is magnetic induction vector, k is a closed curve, $d\boldsymbol{k}$ is the tangential vector of k and S_k is a surface whose boundary curve is k.

$$\int_S \boldsymbol{B}\cdot d\boldsymbol{S} = 0 \quad \text{or} \quad \mathrm{div}\boldsymbol{B} = 0, \quad (\partial B_1/\partial x_1 + \partial B_2/\partial x_2 + \partial B_3/\partial x_3 = 0), \tag{3}$$

$$\begin{gathered}\int_k \boldsymbol{H}\cdot d\boldsymbol{k} = \boldsymbol{J}_0 + \int_{S_k}(\partial\boldsymbol{D}/\partial t)\cdot d\boldsymbol{S} \quad \text{or} \quad \mathrm{Curl}\boldsymbol{H} = \boldsymbol{J}_0 + \partial\boldsymbol{D}/\partial t,\\ \partial H_2/\partial x_1 - \partial H_1/\partial x_2 = J_{03} + \partial D_3/\partial t,\\ \partial H_1/\partial x_3 - \partial H_3/\partial x_1 = J_{02} + \partial D_2/\partial t,\\ \partial H_3/\partial x_2 - \partial H_2/\partial x_3 = J_{01} + \partial D_1/\partial t,\end{gathered} \tag{4}$$

where $\boldsymbol{H}$ is magnetic field intensity vector and J_{01}, J_{02}, J_{03}, are the components of the current intensity vector $\boldsymbol{J}_0$.

Only the static condition and the cases $q_0 = 0$, $q_1 = 0$, $\boldsymbol{J}_0 = 0$ are dealt with.

The equations (1) and (3) become

$$\begin{gathered}\int_S \boldsymbol{D}\cdot d\boldsymbol{S} = 0 \quad \text{or} \quad \mathrm{Div}\boldsymbol{D} = 0 \quad (\partial D_1/\partial x_1 + \partial D_2/\partial x_2 + \partial D_3/\partial x_3 = 0), \quad \text{and}\\ \int_S \boldsymbol{B}\cdot d\boldsymbol{S} = 0 \quad \text{or} \quad \mathrm{Div}\boldsymbol{B} = 0 \quad (\partial B_1/\partial x_1 + \partial B_2/\partial x_2 + \partial B_3/\partial x_3 = 0).\end{gathered} \tag{5}$$

Considering $\partial\boldsymbol{D}/\partial t = 0$ and $\partial\boldsymbol{B}/\partial t = 0$ (static condition) and $\boldsymbol{J}_0 = 0$, the equations (2) and (4) become

$$\begin{gathered}\int_k \boldsymbol{E}\cdot d\boldsymbol{k} = 0 \quad \text{or} \quad \mathrm{Curl}\boldsymbol{E} = 0\\ (\partial E_2/\partial x_1 - \partial E_1/\partial x_2 = 0, \quad \partial E_1/\partial x_3 - \partial E_3/\partial x_1 = 0, \quad \partial E_3/\partial x_2 - \partial E_2/\partial x_3 = 0), \quad \text{and}\\ \int_k \boldsymbol{H}\cdot d\boldsymbol{k} = \boldsymbol{0} \quad \text{or} \quad \mathrm{Curl}\boldsymbol{H} = 0\\ (\partial H_2/\partial x_1 - \partial H_1/\partial x_2 = 0, \quad \partial H_1/\partial x_3 - \partial H_3/\partial x_1 = 0, \quad \partial H_3/\partial x_2 - \partial H_2/\partial x_3 = 0).\end{gathered} \tag{6}$$

On the basis of Equation (6), we have

$$E_i = \partial\phi/\partial x_i \text{ and } H_i = \partial\phi_1/\partial x_i \tag{6a}$$

where ϕ is the electric potential and ϕ_1 is the magnetic potential.

For plane case, on the Ox_1x_2 plane, two integrals in (5) are all computed along the surface curve p of the area and become

$$\int_p D_n\,dp = 0 \quad \text{and} \quad \int_p B_n\,dp = 0, \quad \text{or}$$
$$\partial D_1/\partial x_1 + \partial D_2/\partial x_2 = 0 \quad \text{and} \quad \partial B_1/\partial x_1 + \partial B_2/\partial x_2 = 0, \tag{7}$$

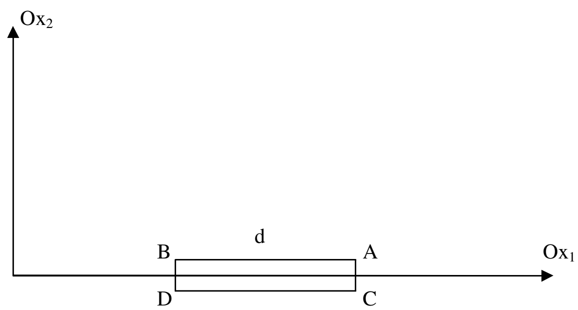
where p is the surface curve in the plane, D_n is the normal component of vector D and B_n is the normal component of vector B.

The two integrals in (6) become

$$\int_p E_t\,dp = 0 \quad \text{or} \quad \partial E_2/\partial x_1 - \partial E_1/\partial x_2 = 0, \quad \text{and}$$
$$\int_p H_t\,dp = 0 \quad \text{or} \quad \partial H_2/\partial x_1 - \partial H_1/\partial x_2 = 0, \tag{8}$$

where E_t is the tangential component of vector E and H_t is the tangential component of vector H.

Now, based on Maxwell equations, the permeable boundary conditions for the static electric and magnetic case are studied. One considers a surface, which can be the interface of two materials. A short segment of this surface is studied (we shall discuss it in detail in Appendix). For convenience, the segment in the studied plane is a part of Ox_1 axis. In order to study the conditions on the segment, a rectangle is taken, as shown in the figure:



The longer side is parallel to the segment, i.e., Ox_1 axis with width d. The shorter is perpendicular to the segment, i.e., Ox_2 axis and its length trends to zero. For the rectangle, the two integrals in (5) and (7) can be computed.

Considering the area is very small, one can be sure that the value of D, $E(\phi)$, B and $H(\phi_1)$ are constants on one side but can be different on other side. Therefore, the contribution on the shorter side tends to zero. For the longer side, they are parallel to Ox_1. The tangent component of vector $\boldsymbol{E}$ is E_1. Similarly, the normal component of vector $\boldsymbol{D}$ is D_2. The equation $\int_p D_n\,dp = 0$ and $\int_p E_t\,dp = 0$ becomes

$$(D_2^+ - D_2^-)d = 0, \quad \text{i.e.,} \quad D_2^+ = D_2^-, \quad \text{and} \quad (E_1^+ - E_1^-)d = 0, \quad \text{i.e.,} \quad E_1^+ = E_1^-(\phi^+ = \phi^-), \tag{9}$$

where D_2^+ is the D_2 on the upper surface of the interface and D_2^- is that on the lower surface. Similarly, E_1^+ and E_1^- can be understand as D_2^+ and D_2^-.

For the magnetic field, one has

$$B_2^+ = B_2^{--} \quad \text{and} \quad H_1^+ = H_1^-(\phi_1^+ = \phi_1^-). \tag{10}$$

It is the permeable boundary conditions for a surface in static electric and magnetic field. For the static electric field, in the piezoelectric fracture mechanics, it is the well known boundary condition of [Parton 1976; Mikhailov and Parton 1990].

As a matter of fact, that is an old boundary condition. In any textbook of the theory of electromagnetism, for the interface between two materials, one can find this boundary condition.

3. Semipermeable conditions and impermeable condition

Although the permeable boundary conditions are deduced from Maxwell equations, yet they have not considered the existence of crack and only for a surface or an interface in the materials. When one directly uses them for the piezoelectric and piezomagnetic fracture mechanics, they may result in larger deviations sometime. However, the importance of these conditions is that they can be the basis of the further discussion on the boundary conditions for the piezoelectric and piezomagnetic fracture mechanics.

Now, the semipermeable boundary conditions and the impermeable boundary conditions are considered. Firstly, the cracks can be divided into two kinds. The first is for the cracks with the opening voids full of fluid (air) after deformation. We always dealt with this kind. The second has not the opening voids such as the antiplane case (as $u_1 = u_2 = 0$, the crack can not be opening), the crack subjected to compression stress, etc. For the second, as the void does not exist after deformation, the permeable equations $D_2^+ = D_2^-$ and $E_1^+ = E_1^-(\phi^+ = \phi^-)$, $B_2^+ = B_2^{--}$ and $H_1^+ = H_1^-(\phi_1^+ = \phi_1^-)$ can be accepted.

Then, for the first, when studying the crack full of air, it is improper to write the boundary condition before deformation as the classical theory; otherwise the crack is only a slit without void and no air can be exist in it. Therefore, we must consider the boundary condition after deformation (the current configuration in nonlinear elasticity). In the meantime, the opening crack becomes a void. At the surface of the void, on the interface between the fluid (in void) and the solid (outside void), we have the interface boundary conditions

$$D_2^+ = D_2^- \quad \text{and} \quad E_1^+ = E_1^-(\phi^+ = \phi^-), \quad B_2^+ = B_2^{--} \quad \text{and} \quad H_1^+ = H_1^-(\phi_1^+ = \phi_1^-), \tag{11}$$

where D_2^+, D_2^-, E_1^+, $E_1^-(\phi^+, \phi^-)$, B_2^+, B_2^-, H_1^+, $H_1^-(\phi_1^+, \phi_1^-)$ belong to the fluid (in void) and the solid (outside void).

In the fluid (in void), there are the various basic equations of the fluid (air), such as

$$\partial^2\phi_i/\partial x_1^2 + \partial^2\phi_i/\partial x_2^2 + \partial^2\phi_i/\partial x_3^2 = 0 \quad \text{and} \quad \partial^2\phi_{1i}/\partial x_1^2 + \partial^2\phi_{1i}/\partial x_2^2 + \partial^2\phi_{1i}/\partial x_3^2 = 0, \tag{12}$$

where ϕ_i and ϕ_{1i} are the electric and magnetic potentials of the fluid components in the void. For the solid (outside void), the basic equations are well known and we shall not discuss them here. It is the all conditions satisfied by the body with void including air. To solve it is a complicated problem. Generally, it is better to use the nonlinear elasticity to solve this problem but the nonlinear elasticity is too tough to study. Now, a simpler method is accepted in study. This method is as follows. For convenience, when studying the crack void full of air, the boundary after deformation can be accepted as the boundary before deformation (a closed slit) adding the evaluated boundary displacement. Naturally, we know that this displacement is also found by classical theory. In this theory, the displacement field is calculated from the boundary before deformation, i.e., on the nondeformation body. This result is not precise but results in the little deviation. Nevertheless, to the case of crack void full of air, the deviation is not negligible.

As mentioned above, the approximate boundary after deformation is determined. Now we consider the solid (outside void) and the fluid (in void). As mentioned above, the constitutive equation of the solid is well known and that of the fluid are various and complicated. In order to avoid that of fluid, considering the void is very small after deformation, Hao [1993] and Hao and Shen [1994] have used the linear change of ϕ and ϕ_1 along the surface normal to replace the rigorous solution of the complicated equation (that is one of the basic assumption of this boundary condition). In the void, E_2 and H_2 (for small deformation case, E_n and H_n are replaced by E_2 and H_2) become $-(\phi^+ - \phi^-)/(u_2^+ - u_2^-)$ and $-(\phi_1^+ - \phi_1^-)/(u_2^+ - u_2^-)$, and u_2 is the evaluated boundary displacement component as above mentioned.

Considering $D_2 = \epsilon_a E_2$ and $B_2 = \epsilon_{a1} H_2$ in air, one obtains

$$(u_2^+ - u_2^-)D_2 = -\epsilon_a(\phi^+ - \phi^-), \quad (u_2^+ - u_2^-)B_2 = -\epsilon_{a1}(\phi_1^+ - \phi_1^-) \tag{13}$$

where ϵ_a and ϵ_{a1} are the electric and magnetic permitivities of air.

Since in the void, E_2 and H_2 become the constants along the normal, D_2 and B_2 are also the constants along the normal. Then, we obtain

$$D_2^+ = D_2^- = -\epsilon_a \frac{\phi^+ - \phi^-}{u_2^+ - u_2^-}, \quad B_2^+ = B_2^- = -\epsilon_{a1} \frac{\phi_1^+ - \phi_1^-}{u_2^+ - u_2^-}. \tag{14}$$

In fact, this boundary condition is obtained from the conception of current configuration in finite deformation theory and the linear change of potential as [Hao 2004]. In short, the conception of current configuration is that we must deal with the crack boundary after deformation when studying a crack.

The equation (14) is the semipermeable boundary condition. For the piezoelectric case, it is suggested by [Hao 1993; Hao and Shen 1994].

It is approximate to use an average rate of change of potential ϕ to take the place of the actual rate. However, as it is only an approximate boundary condition rather than an exact result, I can be sure that for disagreeing it we must be based on some contrary examples, not one exact example.

It is apparent that Equation (13) will be reduced to $\phi^+ = \phi^-$ or $\phi_1^+ = \phi_1^-$ (one of the permeable boundary conditions) when $u_2^+ - u_2^- = 0$ (closed), and to the following equation under the condition $\epsilon_a = 0$ and $\epsilon_{a1} = 0$:

$$D_2^+ = D_2^- = 0, \; B_2^+ = B_2^- = 0. \tag{15}$$

The Equation (15) is the impermeable boundary conditions.

4. Some problems about the application of these boundary conditions

In order to avoid the irrationality in the result, we must decide to suitably use these boundary conditions. The permeable boundary condition is obtained from the Maxwell equations exactly. Therefore, we must determine in what situation this boundary may be accepted. If we can be sure that the crack is closed, the permeable boundary condition should be accepted. However, it is not easy to determine the crack being closed. In order to do it, from [Hao 2001], we know

$$u_2^+ - u_2^- = 2\mathrm{Re}\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})[f_j'(x_1)^+ - f_j'(x_1)^-]/\mu, \tag{16a}$$

where the constants can be found in [Hao 2001].

It is an exact formula to decide whether the crack is closed or not but it is too complicated to be used. We shall discuss it in detail later.

When we know that the crack is closed, the permeable boundary condition can be accepted.

From the Equation (16), we can also decide that the crack is open. At this time, the semipermeable or the impermeable boundary condition can be considered.

About the semipermeable boundary condition, although it has considered the electric field in the air, yet it seems to be too complicated to deal with. However, many results can be accepted by us to study this problem without difficulty. These results tell us that D_2^+ can be determined directly without complicated computing. For an example, to the common multiple collinear cracks (naturally, single crack) under the simple remote load case, we have following result.

In general case, there are four functions of complex variables $f_j''(z_j)$ and the displacements and potential can be

$$\begin{aligned}\phi^+ - \phi^- &= 2\mathrm{Re}\sum_{j=1}^{4}(h_{1r}k_{rj} - \xi_{1\alpha}d_{\alpha j})[f_j'(x_1)^+ - f_j'(x_1)^-],\\ u_2^+ - u_2^- &= 2\mathrm{Re}\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})[f_j'(x_1)^+ - f_j'(x_1)^-]/\mu_j\end{aligned} \tag{16b}$$

where the constants can be found in [Hao 2001].

The functions $f_j''(z_j)$ can be obtained from

$$\begin{gathered}\sum_{j=1}^{4} l_{ij} f_j''(z) = e_i + i f_i + e_i[Q(z) - 1], \quad i = 1, \dots, 4,\\ Q(z) = \frac{z^n + c_1 z^{n-1} + \cdots + c_{n-1}z + c_n}{\prod_{k=1}^{n}[(z-a_k)(z-b_k)]^{1/2}},\end{gathered} \tag{17}$$

where $c_1, c_2, \dots, c_{n-1}, c_n$ are defined by single value requirements of displacements and potential and a_k and b_k are the two tips of the k-th crack but no relation with material constants.

Then, using linear algebra method, one can find functions $f_j''(z_j)$ as [Hao 2001]

$$f_i''(z_i) = \sum_{j=1}^{4} x_{ij}\{D_j + e_j Q(z_i)\}, \quad D_j = e_j + i f_j - e_j, \tag{18}$$

where x_{ij} is determined by linear algebra method as [Hao 2001].

We introduce

$$P'(x_1) = -Q(x_1). \tag{19}$$

Then, there is

$$\begin{aligned}
f_i''(x_1)^+ - f_i''(x_1)^- &= -\Big[\sum_{j=1}^{4} x_{ij}e_j Q(x_1)\Big]^- + \Big[\sum_{j=1}^{4} x_{ij}e_j Q(x_1)\Big]^+ \\
&= \Big[\sum_{j=1}^{4} x_{ij}e_j P'(x_1)\Big]^- - \Big[\sum_{j=1}^{4} x_{ij}e_j P'(x_1)\Big]^+ \\
&= \sum_{j=1}^{4} x_{ij}e_j[P'(x_1)^- - P'(x_1)^+].
\end{aligned} \tag{20}$$

From Equation (20), we obtain

$$\begin{aligned}
f_i'(x_1)^+ - f_i'(x_1)^- &= \sum_{j=1}^{4} x_{ij}e_j[P(x_1)^- - P(x_1)^+] \\
&= [P(x_1)^- - P(x_1)^+]\sum_{j=1}^{4} x_{ij}e_j \\
&= A_i[P(x_1)^- - P(x_1)^+],
\end{aligned} \tag{21}$$

$$A_i = \sum_{j=1}^{4} x_{ij}e_j. \tag{22}$$

Then, we have

$$\begin{aligned}
\phi^+ - \phi^- &= 2\mathrm{Re}\sum_{j=1}^{4}(h_{1r}k_{rj} - \xi_{1\alpha}d_{\alpha j})A_j[P(x_1)^- - P(x_1)^+] \\
&= 2\mathrm{Re}[P(x_1)^- - P(x_1)^+]\sum_{j=1}^{4}(h_{1r}k_{rj} - \xi_{1\alpha}d_{\alpha j})A_j, \\
u_2^+ - u_2^- &= 2\mathrm{Re}\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})A_j[P(x_1)^- - P(x_1)^+]/\mu_j \\
&= 2\mathrm{Re}[P(x_1)^- - P(x_1)^+]\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})A_j/\mu_j.
\end{aligned} \tag{23}$$

One can find that the function $[P(x_1)^+ - P(x_1)^-]$ is imaginary. Therefore, we have

$$\begin{aligned}
\phi^+ - \phi^- &= 2\mathrm{i}[P(x_1)^- - P(x_1)^+]\mathrm{Im}\sum_{j=1}^{4}(h_{1r}k_{rj} - \xi_{1\alpha}d_{\alpha j})A_j, \\
u_2^+ - u_2^- &= 2\mathrm{i}[P(x_1)^- - P(x_1)^+]\mathrm{Im}\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})A_j/\mu_j.
\end{aligned} \tag{24}$$

Therefore, D_2^+ can be approximately expressed as

$$\begin{aligned} D_2^+ &= -\epsilon_a \frac{[P(x_1)^- - P(x_1)^+]\operatorname{Im}\sum_{j=1}^4 (h_{1r}k_{rj} - \xi_{1\alpha}d_{\alpha j})A_j}{[P(x_1)^- - P(x_1)^+]\operatorname{Im}\sum_{j=1}^4 (\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})A_j/\mu_j} \\ &= -\epsilon_a \frac{\operatorname{Im}\sum_{j=1}^4 (h_{1r}k_{rj} - \xi_{1\alpha}d_{\alpha j})A_j}{\operatorname{Im}\sum_{j=1}^4 (\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})A_j/\mu_j}, \end{aligned} \tag{25}$$

D_2^+ is no relation with these coordinates x_i and can be determined directly without the iteration method.

We must pay attention to that although the expressions of $\phi^+ - \phi^-$ and $u_2^+ - u_2^-$ are exact, yet for D_2^+ it is approximate. For an example, we consider a crack. In general case, its $\phi^+ - \phi^-$ and $u_2^+ - u_2^-$ may be proportional to $(a^2 - z^2)^{1/2}$. When the load leads $u_2^+ - u_2^-$ tending to zero, the crack should be closed. Therefore, we must accept the permeable condition. As $\phi^+ - \phi^-$ can also tend to zero, the value $(\phi^+ - \phi^-)/(u_2^+ - u_2^-)$ may tend to a constant. However, because the crack is closed, there is no air in the crack void and $(\phi^+ - \phi^-)/(u_2^+ - u_2^-)$ is a constant without significance.

5. Stress field caused by the electric field

Now, we study the stress field caused by the electric field. Essentially it is the acting force of electric field on the solid element. The acting force caused by the electric field is a body force. It is known that the stress field caused by the electric field is a square but that by the piezoelectric field is linear [Fang and Yin 1989, 4.7.1, p. 209]. Therefore, the stress field caused by the electric field is always smaller than that by the piezoelectric field [ibid., 4.7.2, p. 210] and always can be neglected.

Due to the complexity of this problem, the stress distribution caused by the electric field will be discussed in detail in another paper.

6. Conclusions

For the electric-magnetic fracture mechanics, the relation between the Maxwell equations and the permeable, semipermeable and impermeable electromagnetic boundary conditions has been studied. Then, the application of these boundary conditions has been discussed. Lastly, the stress field caused by the electric field also has been discussed. It is known that permeable electromagnetic boundary conditions are exact for the closed crack. When we can be sure that the crack is not closed, the semipermeable or the impermeable electromagnetic boundary condition is accepted. Naturally, it has some trouble to use formula (16a) to decide whether the crack is open or not. However, for the cracks on a straight line (for example, the cracks on Ox_1) and $D_2^\infty = 0$, it is easy to deal with. From the equation (24), we know

$$u_2^+ - u_2^- = 2\mathrm{i}[P(x_1)^- - P(x_1)^+]\operatorname{Im}\sum_{j=1}^4 (\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})A_j/\mu_j \tag{26}$$

where $A_j = m_j\sigma_2^\infty$ and m_j is a constant and no use for our discussion.

Substituting them into (26), we have

$$u_2^+ - u_2^- = 2\mathrm{i}[P(x_1)^- - P(x_1)^+]\sigma_2^\infty \operatorname{Im}\sum_{j=1}^4 (\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})m_j/\mu_j. \tag{27}$$

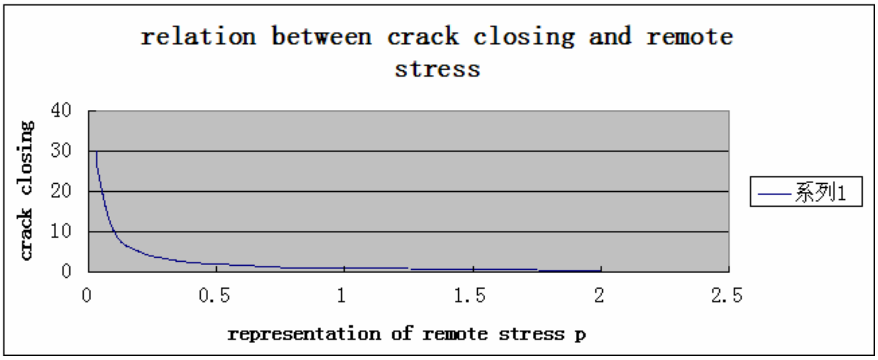


Figure 1. The relation between the crack closing and the representation of remote stress p.

It is clear that the value $u_2^+ - u_2^-$ is proportional to the value σ_2^∞ which is the same with the theory of elasticity.

For one crack case, we know

$$u_2^+ - u_2^- = -2\mathrm{i}[P(x_1)^+ - P(x_1)^-](k_2/\pi^{1/2}a^{1/2})\,\mathrm{Im}\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})m_j/\mu_j, \tag{28}$$

where k_2 is the stress intensity factor and a is the half length of the crack.

In order to clarify the crack closing, it is explained in Figure 1.

For convenience, the term $4a\sigma_2^\infty\,\mathrm{Im}\sum_{j=1}^{4}(\beta_{2r}k_{rj} + \eta_{2\alpha}d_{\alpha j})m_j/\mu_j$ is replaced by $10^{11}p$, where $2a$ is the crack length.

From Figure 1 we know that when the representation of remote stress p tends to 0, the crack closing to infinite.

Previously, we only consider the crack being traction free at its surface. When there is homogeneous load σ_0 on crack surface and $\sigma_2^\infty = 0$, we resolve it into two cases. The one is homogeneous stress σ_0 on the whole solid and the another is $\sigma_2^\infty = -\sigma_0$. On the basis of the sum of the two cases, all boundary conditions are satisfied. The case of homogeneous stress σ_0 on the whole solid is a homogeneous field. It is easy to deal with. The case of $-\sigma_0$ at infinite is that of $\sigma_2^\infty = -\sigma_0$. It has been discussed in the equation (27).

Appendix: About the boundary condition

In order to discuss the boundary condition easily, we consider the plane potential fluid mechanics. Firstly, we introduce the conception of source, sink and vortex point. Naturally, they are the plane potential fluid field which has two components parallel to Ox_1 and Ox_2. The velocity fields of source and sink are radius. At every point, the velocity is parallel to the radius. The vortex point is tangential velocity field. At every point, the velocity is perpendicular to the radius. Therefore, the vortex point velocity field is a circular ring field. In fluid mechanics, we call this field in rotational field. We know that the conception of source seems to be the water spring. The conception of sink is contrary to that of source. For the vortex point, we always seem to observe it at the water surface. Sometimes, the sources (naturally, sinks and vortex points) can be considered a line source. That seems to be observed at the water surface. For convenience, we deal with the plane problem on the plane with axis Ox_1 and Ox_2. Now, we consider a slit on the interface of two materials. The slit is on the axis Ox_1. We must be sure that because there are two materials, the velocity must be different in the two materials. Letting the velocities be V_1 and V_2 in the two materials. When there is no source and sink in the slit, letting the values V_{12} and V_{22} are components of V_1 and V_2 paralleling to Ox_2. $V_{12} = V_{22}$. When there is no vortex point in the slit, letting the values V_{11} and V_{21} are components of V_1 and V_2 paralleling to Ox_1. $V_{11} = V_{21}$. When there are some sources or sinks in the slit, $V_{12} \neq V_{22}$ across the slit. When there are some vortex points in the slit, $V_{11} \neq V_{21}$ across the slit.

On the basis of above discussed, the static electric field is easy to deal with. The positive and negative charge is corresponding to the source and sink. Naturally, it seems to be difficult to find anything corresponding to the vortex point. However, when we study the magnetic field around the wire, we seem to meet the point corresponding to it. Therefore, for the slit of the interface of two materials on Ox_1 in the electric field, we can be sure that $D_{12} = D_{22}$ (corresponding to $V_{12} = V_{22}$ in fluid). Now, we must pay attention to the condition $V_{11} = V_{21}$ in fluid. We know that in fluid mechanics, we prove the condition $V_{11} = V_{21}$ based on the condition no vortex point in the slit. In fluid mechanics, as above mentioned, the condition no vortex point is corresponding to that of nonrotation. In Maxwell equations, as above mentioned, the condition nonrotation is discussed in equations (2) and (6). Here, the physical quantity E pays a leading role. Therefore, the condition $V_{11} = V_{21}$ in fluid mechanics, is corresponding to the condition $E_{11} = E_{21}$ in Maxwell equations.

Therefore, we obtain the boundary condition on the interface

$$D_{12} = D_{22}, \quad E_{11} = E_{21}, \tag{29}$$

where D_{1j} is the j-th of D_1, D_{2j} is the j-th of D_2, E_{1j} is the j-th of E_1 and E_{2j} is the j-th of E_2.

Naturally, these boundary conditions above mentioned, are obtained based on the analogy method. Now, we shall prove it by integral form of Maxwell equations exactly.

Letting BA and CD being the two longer sides of the rectangle (their length equals d) and AC, BD being the two shorter sides (their length tends to 0) and the segment of Ox_1 in the rectangle being the interface, we have $\int_p D_n\,dp = 0$ and $\int_p E_t\,dp = 0$. We consider the right spiral rule. The direction of four tops of the rectangle is $BACDB$. As BA and CD are parallel to Ox_1, their normal is parallel to Ox_2. The vector D_n becomes D_2. As their tangent is parallel to Ox_1, the vector E_t becomes E_1. The equation $\int_p D_n\,dp = 0$ becomes

$$-D_{12BA}AB + D_{22CD}CD + \text{ small contribution of the sides } AC \text{ and } DB = 0, \tag{30}$$

where D_{12} is the second component of the electric displacement D_1 in upper half plane $x_2 > 0$, D_{22} is the second component of the electric displacement D_2 in lower half plane $x_2 < 0$, D_{12BA} is D_{12} on BA and D_{22CD} is D_{22} on CD.

Considering the contribution of the shorter sides AC and DB tending to zero, we neglect it and consider the two sides AC and DB being the upper and lower surfaces of the interface. Therefore,

$$-D_{2BA}AB + D_{2CD}CD = -D_2^+AB + D_2^-CD = 0, \tag{31}$$

where $D_2^+ = D_{2BA}$ is the D_2 on the upper surface of the interface and $D_2^- = D_{2CD}$ is the D_2 on the lower surface of the interface.

Substituting $BA = CD = d$, we obtain

$$d(-D_2^+ + D_2^-) = 0. \tag{32}$$

Therefore, we have

$$D_2^+ = D_2^-. \tag{33}$$

Considering the E_t becomes E_1, the equation $\int_p E_t\, dp = 0$ becomes

$$-E_{11BA}BA + E_{21CD}CD + \text{ small contribution of the sides } AC \text{ and } DB = 0,$$

where E_{11} is the first component of the electric field E_1 in upper half plane $x_2 > 0$, E_{21} is the first component of the electric field E_1 in lower half plane $x_2 < 0$, E_{11BA} is E_1 on BA and E_{21CD} is E_1 on CD.

Considering the contribution of the shorter sides AC and DB being very little, we neglect it and consider the two sides AC and DB being the upper and lower surfaces of the interface. Therefore,

$$-E_{11BA}AB + E_{21CD}CD = -E_1^+ AB + E_2^- CD = 0, \tag{34}$$

where $E_1^+ = E_{11BA}$ is the E_1 on the upper surface of the interface and $E_2^- = E_{21CD}$ is the E_2 on the lower surface of the interface.

Substituting $BA = CD = d$, we obtain

$$d(-E_1^+ + E_1^-) = 0. \tag{35}$$

And so we have

$$E_1^+ = E_1^-. \tag{36}$$

Therefore, on the upper and lower surface of the interface, the components of vectors E_1 and D_2 on both surfaces are equal. When on the upper and lower surface of the interface, the components of vector E_1 on both surfaces are equal, on the upper and lower surface, the function ϕ ($\phi = \partial E_1/\partial x_1$) on both surfaces is equal (when the interface is $-\infty$–$+\infty$, on the upper and lower surface, the difference between the two functions ϕ^+ and ϕ^- may be a constant).

This result is the famous permeable condition in piezoelectric fracture mechanics [Parton 1976; Parton and Kudryavtsev 1988; Mikhailov and Parton 1990].

In fact, this is an old result in electrodynamics: $\int_q D_n\, dq = 0$. We can find it in any textbook such as [Coelho 1979].

Acknowledgment

Thanks for Professor Huang K. C. thanks to whom I have overcome many difficulties. Thanks for Professor Shen Z. Y. for his valuable help on English.

References

[Coelho 1979] R. Coelho, *Physics of dielectrics for the engineer*, Fundamental studies in engineering **1**, Elsevier, Amsterdam, 1979.

[Fang and Yin 1989] J. X. Fang and Z. W. Yin, 电介质物理学 , Science Press, Beijing, 1989.

[Hao 1993] T. H. Hao, "A boundary condition in electromagnetic fracture mechanics", *J. China Text. Univ.* **19**:1 (1993), 53. In Chinese.

[Hao 2001] T. H. Hao, "Multiple collinear cracks in a piezoelectric material", *Int. J. Solids Struct.* **38**:50–51 (2001), 9201–9208.

[Hao 2004] T. H. Hao, "Perturbation method of the piezoelectric fracture mechanics considering the permittivity of the medium in the crack gap", *Int. J. Fract.* **126**:1 (2004), 57–69.

[Hao and Shen 1994] T.-H. Hao and Z.-Y. Shen, "A new electric boundary condition of electric fracture mechanics and its applications", *Eng. Fract. Mech.* **47**:6 (1994), 793–802.

[Kumar and Singh 1997] S. Kumar and R. N. Singh, "Influence of applied electric field and mechanical boundary condition on the stress distribution at the crack tip in piezoelectric materials", *Mater. Sci. Eng. A* **231**:1–2 (1997), 1–9.

[Mikhailov and Parton 1990] G. K. Mikhailov and V. Z. Parton, *Electromagnetoelasticity*, Hemisphere, New York, 1990.

[Parton 1976] V. Z. Parton, "Fracture mechanics of piezoelectric materials", *Acta Astronaut.* **3**:9–10 (1976), 671–683.

[Parton and Kudryavtsev 1988] V. Z. Parton and B. A. Kudryavtsev, *Electromagnetoelasticity: piezoelectrics and electrically conductive solids*, Gordon and Breach, New York, 1988.

[Zhang et al. 2002] T.-Y. Zhang, M. Zhao, and P. Tong, "Fracture of piezoelectric ceramics", *Adv. Appl. Mech.* **38** (2002), 147–289.

Received 8 Mar 2014. Revised 4 Oct 2014. Accepted 27 Apr 2015.

HAO TIAN-HU: haoth0000@aliyun.com
State Key Lab for Modification of Polymer Materials and Chemical Fibers, Donghua University, P.O. Box 220, Shanghai, 200051, China

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

Cover photo: Mando Gomez, www.mandolux.com

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2015 is US $565/year for the electronic version, and $725/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 10, No. 4 July 2015

1559-3959(2015)10:4;1-9