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ON CESÀRO MEANS OF ENERGY IN MICROPOLAR THERMOELASTIC DIFFUSION THEORY

MARIN MARIN AND SAMY REFAHY MAHMOUD

This paper is dedicated to the theory of thermoelasticity of micropolar diffusion. For the mixed initial boundary value problem defined in this context, we prove that the Cesàro means of the kinetic and strain energies of a solution with finite energy become asymptotically equal as time tends to infinity.

1. Introduction

Eringen [2003] has developed a continuum theory for a mixture of a micropolar elastic solid and a micropolar viscous fluid. All materials, whether natural or synthetic, possess microstructures.

In the micropolar continuum theory, the rotational degrees of freedom play a central role. The material points of porous solids and dirty fluids undergo translation and rotations. Thus, we have six degrees of freedom, instead of the three degrees of freedom considered in classical elasticity and fluid mechanics (see [Eringen 1999; 2001]). A large class of engineering materials, as well as soils, rocks, granular materials, sand and underground water mixtures may be modeled more realistically by means the theory proposed in [Eringen 2003]. Consolidation problems in the building industry and oil exploration problems fall into the domain of this theory.

In the last decade of time, the micropolar theory was extended to include thermal effects in many studies. One can refer to [Ieşan 2004; Marin and Florea 2014; Marin et al. 2013a; 2013b; Sharma and Marin 2014; Dhaliwal and Singh 1987] for a review on the micropolar thermoelasticity and a historical survey of the subject, as well as to [Eringen 1999] in the Continuum Physics series, in which the general theory of micromorphic media has been summed up.

Aouadi [2008] extended the micropolar theory to include thermal and diffusion effects. In fact, the development of high technologies in the years before, during, and after the second world war pronouncedly affected the investigations in which the fields of temperature and diffusion in solids cannot be neglected. At elevated and low temperatures, the processes of heat and mass transfer play the decisive role in many problems of satellites, returning space vehicles, and landing on water or land. These days, oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits. Diffusion can be defined as the random walk of an ensemble of particles from regions of high concentration to regions of lower concentration. Thermodiffusion in an elastic solid is due to coupling of the fields of temperature, mass diffusion and that of strain.

The earliest results concerning energy equipartition were dedicated to the abstract differential equations and to the abstract wave equation. The result established in [Levine 1977] using the Lagrange identity, represents a simplified proof that asymptotic equipartition occurs between the Cesàro means of

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the kinetic and potential energies. The asymptotic equipartition between the mean kinetic and strain energies in the context of linear elastodynamics was studied in [Day 1980]. Also, we can refer to [Dassios and Galanis 1980; Goldstein and Sandefur 1976] and, in specific cases, [Marin and Stan 2013; Teodorescu-Draghicescu and Vlase 2011; Vlase et al. 2012].

In the present paper we consider the linear theory of micropolar thermoelastic diffusion and we formulate the basic initial-boundary value problem in the framework of the linearized theory developed in [Aouadi 2008]. Then we study the asymptotic partition of the energy associated with the solution of this problem. In this aim we introduce the Cesàro mean of various parts of the total energy and use the methods deduced in [Levine 1977; Day 1980; Rionero and Chirita 1987; Marin 2009] to establish the relations that describe the asymptotic behavior of the mean energies. Thus, we use some Lagrange–Brun identities to prove that the mean thermal energy tends to zero as time goes to infinity and the asymptotic equipartition occurs between the Cesàro means of the kinetic and internal energies.

The asymptotic equipartition property is a familiar notion in differential equations field. This means that the kinetic and potential energy of a classical solution with finite energy become asymptotic equal in means as time tends to infinite. Such a property is presented in various papers for physical systems governed by nondissipative hyperbolic partial differential equations or systems of such equations.

But the system of equations governing our mixed initial boundary value problem consists of hyperbolic equations with dissipation and, therefore, does not belong to one of the categories considered previously in literature of subject. By using the dissipation mechanism of the system, we can prove that equipartition occurs between the mean kinetic and strain energies. Instead of abstracted version of this question, we prefer to emphasize the technique itself in the context of micropolar thermoelastic diffusion.

We want to outline that there are many papers which employ the various refinements of the Lagrange identity. One can refer to [Levine 1977; Day 1980; Rionero and Chirita 1987; Gurtin 1993].

The plan of our study is the following one. We first write down the mixed initial boundary value problem defined in the above context. Then we shall establish some Lagrange type identities and, also, we introduce the Cesàro means of various parts of the total energy associated to the solutions. Based on these estimations, at last, we establish the relations that describe the asymptotic behavior of the mean energies.

2. Basic equations and conditions

We assume that a bounded region B of three-dimensional Euclidean space $\mathbb{R}^3$ is occupied by a microstretch thermoelastic body, referred to the reference configuration and a fixed system of rectangular Cartesian axes. Let $\bar{B}$ denote the closure of B and call ∂B the boundary of the domain B. We consider ∂B be a piecewise smooth surface and designate by n_i the components of the outward unit normal to the surface ∂B. Letters in boldface stand for vector fields. We use the notation v_i to designate the components of the vector $\boldsymbol{v}$ in the underlying rectangular Cartesian coordinates frame. Superposed dots stand for the material time derivative. We shall employ the usual summation and differentiation conventions: the subscripts are understood to range over integers $(1, 2, 3)$. Summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

The spatial argument and time argument of a function will be omitted when there is no likelihood of confusion. We refer the motion of the body to a fixed system of rectangular Cartesian axes Ox_i, $i = 1, 2, 3$.

Let us denote by u_i the components of the displacement vector and by ϕ_i the components of the microrotation vector. Also, we denote by C the concentration of the diffusive material in the micropolar body and by T the temperature measured from the constant absolute temperature T_0 of the body in its reference state.

As usual, we denote by σ_{ij} the components of the stress tensor and by μ_{ij} the components of the couple stress tensor over B.

In the absence of body force, body couple force and heat supply fields, the field of basic equations for micropolar thermoelastic diffusion are the following (see [Aouadi 2008]):

- the equation of motion

$$\begin{aligned} \rho\ddot{u}_i &= \sigma_{ji,\,j}, \\ \epsilon_{ijk}\sigma_{jk} + \mu_{ji,\,j} &= \rho J_{ij}\ddot{\phi}_j; \end{aligned} \tag{1}$$

- the equation of energy

$$q_{i,\,i} = \rho T_0 \dot{S}; \tag{2}$$

- the equation of conservation of mass

$$\eta_{i,\,i} = \dot{C}. \tag{3}$$

In these equations we have used the following notation:

- ρ is the reference constant mass density;
- $J_{ij} = J_{ji}$ are the coefficients of microinertia;
- $\sigma_{ij},\ \mu_{ij}$ are the components of the stress;
- S is the entropy per unit mass;
- q_i are the components of heat flux vector;
- η_i are the components of the flow of diffusing mass vector.

Let us denote by θ the temperature, where $\theta = T - T_0$. Here T_0 is the temperature of the medium in its natural state.

If we suppose that the micropolar body, in the reference state, has a center of symmetry at each point, but is otherwise nonisotropic, we have the following constitutive equations:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\varepsilon_{kl} + p_{ijkl}\phi_{kl} + a_{ij}\theta + b_{ij}C, \\ \mu_{ij} &= p_{ijkl}\varepsilon_{kl} + d_{ijkl}\phi_{kl} + p_{ij}\theta + q_{ij}C, \\ \rho S &= -a_{ij}\varepsilon_{ij} - p_{ij}\phi_{ij} + \frac{\rho c_E}{T_0}\theta + \varpi C, \\ P &= b_{ij}\varepsilon_{ij} + q_{ij}\phi_{ij} - \varpi\theta + \varrho C, \\ q_i &= \kappa_{ij}\theta_{,\,j} \\ \eta_i &= d_{ij}P_{,\,j} \end{aligned} \tag{4}$$

In the above relations P is the chemical potential per unit mass, c_E is the specific heat at constant strain, c_{ijkl} is the tensor of elastic constants. Also, the equations $(4)_5$ and $(4)_6$ are know as Fourier's law and Fick's law, respectively. The constants ϖ and ϱ are measures of thermodiffusion effects and diffusive effects, respectively. The rest of parameters are material constants.

The characteristic quantities of the strain ε_{ij} and ϕ_{ij} used in the above equations, are defined by means of the geometric equations

$$\varepsilon_{ji} = u_{i,\,j} - \epsilon_{kji}\phi_k, \quad \phi_{ji} = \phi_{i,\,j}, \tag{5}$$

where ϵ_{ijk} is the alternating tensor.

The functions c_{ijkl}, p_{ijkl}, d_{ijmn}, a_{ij}, b_{ij}, p_{ij}, q_{ij}, κ_{ij}, d_{ij} and ϱ are the characteristic constitutive coefficients. Regarding these coefficients, the conductivity tensor κ_{ij} and the diffusion tensor d_{ij} we have the symmetry relations

$$c_{ijkl} = c_{klij}, \quad d_{ijkl} = d_{klij}, \quad p_{ijkl} = p_{klij}, \quad \kappa_{ij} = \kappa_{ji}, \quad d_{ij} = d_{ji}. \tag{6}$$

The density ρ, the coefficients of inertia J_{ij} and the temperature θ_0 are given constants which satisfy the conditions

$$\rho > 0, \quad J_{ij} > 0, \quad \theta_0 > 0. \tag{7}$$

In accordance with entropy production inequality we must assume that c_{ijmn}, p_{ijmn}, d_{ijmn}, a_{ij}, b_{ij} and κ_{ij} are positive definite tensors, i.e.,

$$\begin{aligned}
c_{ijkl}\xi_{ij}\xi_{mn} &\ge k_0\xi_{ij}\xi_{ij}, & k_0 &> 0, & &\text{for all } \xi_{ij} = \xi_{ji},\\
p_{ijkl}\xi_{ij}\xi_{mn} &\ge k_1\xi_{ij}\xi_{ij}, & k_1 &> 0, & &\text{for all } \xi_{ij},\\
d_{ijkl}\xi_{ij}\xi_{mn} &\ge k_2\xi_{ij}\xi_{ij}, & k_2 &> 0, & &\text{for all } \xi_{ij} = \xi_{ji},\\
a_{ij}\xi_i\xi_j &\ge k_3\xi_i\xi_i, & k_3 &> 0, & &\text{for all } \xi_i,\\
b_{ij}\xi_i\xi_j &\ge k_4\xi_i\xi_i, & k_4 &> 0, & &\text{for all } \xi_i,\\
\kappa_{ij}\xi_i\xi_j &\ge k_5\xi_i\xi_i, & k_5 &> 0, & &\text{for all } \xi_i.
\end{aligned} \tag{8}$$

The components of the surface traction t_i, the surface couple m_i, the heat flux q and the diffusion flux η, at regular points of ∂B, are given by

$$t_i = \sigma_{ji}n_j, \quad m_i = \mu_{ji}n_j, \quad q = q_i n_i, \quad \eta = \eta_i n_i,$$

respectively. By n_i we denoted the components of the outward unit normal of surface ∂B. Now, we admit the following prescribed boundary conditions:

$$\begin{aligned}
u_i &= 0 \quad \text{on } \partial B_1 \times [0,\infty), & t_i &= 0 \quad \text{on } \partial B_1^c \times [0,\infty),\\
\phi_i &= 0 \quad \text{on } \partial B_2 \times [0,\infty), & m_i &= 0 \quad \text{on } \partial B_2^c \times [0,\infty),\\
\theta &= 0 \quad \text{on } \partial B_3 \times [0,\infty), & q &= 0 \quad \text{on } \partial B_3^c \times [0,\infty),\\
P &= 0 \quad \text{on } \partial B_4 \times [0,\infty), & \eta &= 0 \quad \text{on } \partial B_4^c \times [0,\infty).
\end{aligned} \tag{9}$$

Here ∂B_1, ∂B_2, ∂B_3 and ∂B_4 with respective complements ∂B_1^c, ∂B_2^c, ∂B_3^c and ∂B_4^c are subsets of the surface ∂B such that

$$\begin{aligned}
&\partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \partial B_3 \cap \partial B_3^c = \partial B_4 \cap \partial B_4^c = \varnothing,\\
&\partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B_3 \cup \partial B_3^c = \partial B_4 \cup \partial B_4^c = \partial B.
\end{aligned}$$

Introducing the constitutive equations (4) into equations (1)–(3) we obtain the system of equations

$$
\begin{aligned}
\rho \ddot{u}_i &= c_{ijkl}\varepsilon_{kl,\,i} + p_{ijkl}\phi_{kl,\,j} + a_{ij}\theta_{,\,j} + b_{ij}C_{,\,j},\\
\rho J_{ij}\ddot{\phi}_j &= p_{ijkl}\varepsilon_{kl,\,j} + d_{ijkl}\phi_{kl,\,j} + p_{ij}\theta_{,\,j} + q_{ij}C_{,\,j}\\
&\qquad + \epsilon_{ijk}[c_{jkml}\varepsilon_{ml} + p_{jkml}\phi_{ml} + a_{jk}\theta + \beta_{jk}C],\\
\kappa_{ij}\theta_{,\,ij} &= \varrho c_E\dot{\theta} - T_0 a_{ij}\dot{\varepsilon}_{ij} - T_0 p_{ij}\dot{\phi}_{ij} + T_0\varpi\dot{C}\\
\dot{C} &= d_{ij}(\varrho C + b_{ij}\varepsilon_{ij} + q_{ij}\phi_{ij} - \varpi\theta)_{,\,ij},
\end{aligned}
\tag{10}
$$

To this system of equations we adjoin the initial conditions

$$
\begin{aligned}
&u_i(x,0) = u_i^0(x), && \dot{u}_i(x,0) = u_i^1(x), && \phi_i(x,0) = \phi_i^0(x), && \dot{\phi}_i(x,0) = \phi_i^1(x),\\
&\theta(x,0) = \theta^0(x), && \dot{\theta}(x,0) = \theta^1(x), && C(x,0) = C^0(x), && x \in B.
\end{aligned}
\tag{11}
$$

By a solution of the mixed initial boundary value problem of micropolar thermoelastic diffusion in the cylinder $\Omega_0 = B \times [0,\infty)$ we mean an ordered array $(u_i,\ \phi_i,\ \theta,\ C)$ which satisfies the system of equations (10) for all $(x,t) \in \Omega_0$, the boundary conditions (9) and the initial conditions (11).

Let us observe that if $\operatorname{meas}(\partial B_1) = 0$ and $\operatorname{meas}(\partial B_2) = 0$ then there exists a family of rigid motions and null temperature and null diffusion which satisfy the equations (10) and null boundary conditions. In this way we can decompose the initial data (u_i^0, u_i^1) and (ϕ_i^0, ϕ_i^1) as

$$
u_i^0 = u_i^* + U_i^0, \quad u_i^1 = \dot{u}_i^* + U_i^1, \quad \phi_i^0 = \phi_i^* + \Phi_i^0, \quad \phi_i^1 = \dot{\phi}_i^* + \Phi_i^1,
\tag{12}
$$

where $(u_i^*, \dot{u}_i^*)$ are rigid displacements and $(\phi_i^*, \dot{\phi}_i^*)$ are rigid microrotations determined such that (U_i^0, U_i^1) and (Φ_i^0, Φ_i^1) satisfy the restrictions

$$
\begin{aligned}
&\int_B (\rho U_i^0 + \rho J_{ij}\Phi_j^0)\,dV = 0, && \int_B \rho\epsilon_{ijk}x_j(U_k^0 + J_{lk}\Phi_l^0)\,dV = 0,\\
&\int_B (\rho U_i^1 + \rho J_{ij}\Phi_j^1)\,dV = 0, && \int_B \rho\epsilon_{ijk}x_j(U_k^1 + J_{lk}\Phi_l^1)\,dV = 0,
\end{aligned}
\tag{13}
$$

where ϵ_{ijk} is Ricci's tensor.

Similarly, if $\operatorname{meas}(\partial B_3) = 0$ then there exists a family of constant temperatures, null displacements, null microrotations and null diffusion which satisfy the equations (10) and null boundary conditions. Thus, we can decompose the initial data θ^0 and θ^1 in the form

$$
\theta^0 = \theta^* + T^0, \quad \theta^1 = \dot{\theta}^* + T^1,
\tag{14}
$$

where θ^* and $\dot{\theta}^*$ are constants temperatures determined such that T^0 and T^1 satisfy the restrictions

$$
\int_B T^0 dV = 0, \quad \int_B T^1 dV = 0.
\tag{15}
$$

3. Specific notations

We denote by $C^m(B)$ the class of scalar functions possessing derivatives up to the m-th order in the domain B which are continuous on B.

For $f \in C^m(B)$ we define the norm

$$\|f\|_{C^m(B)} = \sum_{k=1}^{m} \sum_{i_1,i_2,\ldots,i_k} \max |f_{,\,i_1 i_2 \ldots i_k}|.$$

By $\boldsymbol{C}^m(B)$ we denote the class of vector fields with six components $C^m(B)$.

For $\boldsymbol{w} \in \boldsymbol{C}^m(B)$ we define the norm

$$\|\boldsymbol{w}\|_{\boldsymbol{C}^m(B)} = \sum_{i=1}^{6} \|w_i\|_{C^m(B)}.$$

By $W_m(B)$ we denote the Hilbert space obtained as the completion of the space $C^m(B)$ by means of the norm $\|\cdot\|_{W_m(B)}$ induced by the inner product

$$(f, g)_{W_m(B)} = \sum_{k=1}^{m} \sum_{i_1,i_2,\ldots,i_k} \int_B f_{,\,i_1 i_2 \ldots i_k} g_{,\,i_1 i_2 \ldots i_k}\, dV.$$

Finally, we will denote by $\boldsymbol{W}_m(B)$ the space obtained as the completion of the space $\boldsymbol{C}^m(B)$ by means of the norm $\|\cdot\|_{\boldsymbol{W}_m(B)}$ induced by the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{W}_m(B)} = \sum_{i=1}^{6} (u_i, v_i)_{W_m(B)}.$$

We will use as norm in Cartesian product of the normed spaces the sum of the norms of the factor spaces. Let us introduce the notation

$$\begin{aligned}
\widehat{C}^1(B) &= \big\{\chi \in C^1(B) : \chi = 0 \text{ on } \partial B_3 \text{ or } \chi = 0 \text{ on } \partial B_4;\\
&\qquad\qquad \text{if } \operatorname{meas}(\partial B_3) = 0 \text{ or } \operatorname{meas}(\partial B_4) = 0 \text{ then } \textstyle\int_B \chi\, dV = 0\big\},\\
\widehat{\boldsymbol{C}}^1(B) &= \big\{(v_i, \psi_i) \in \boldsymbol{C}^1(B) : v_i = 0 \text{ on } \partial B_1,\ \psi_i = 0 \text{ on } \partial B_2;\\
&\qquad\qquad \text{if } \operatorname{meas}(\partial B_1) = \operatorname{meas}(\partial B_2) = 0 \text{ then } \textstyle\int_B (\rho v_i + \rho J_{ij}\psi_j)\, dV = 0\big\},\\
\widehat{W}_1(B) &= \text{the completion of } \widehat{C}^1(B) \text{ by means of the norm } \|\cdot\|_{W_1(B)},\\
\widehat{\boldsymbol{W}}_1(B) &= \text{the completion of } \widehat{\boldsymbol{C}}^1(B) \text{ by means of the norm } \|\cdot\|_{\boldsymbol{W}_1(B)}.
\end{aligned}$$

In this notation $W_m(B)$ represents the familiar Sobolev space (see [Adams 1975]) and $\boldsymbol{W}_m(B)$ is the Cartesian product $\boldsymbol{W}_m(B) = [W_m(B)]^6$.

The hypothesis (11) assures that the following (Korn's inequality) holds (see [Hlaváček and Nečas 1970a; 1970b]):

$$\int_B [c_{ijkl}\varepsilon_{kl}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u}) + 2p_{klij}\varepsilon_{kl}(\boldsymbol{u})\phi_{ij}(\boldsymbol{u}) + d_{klij}\phi_{kl}(\boldsymbol{u})\phi_{ij}(\boldsymbol{u})] \geq m_1 \int_B (u_i u_i + u_{i,j}u_{i,j} + \phi_i\phi_i + \phi_{i,j}\phi_{i,j})\, dV, \quad (16)$$

for all $\boldsymbol{u} = (u_i, \phi_i) \in \boldsymbol{W}_1(B)$, where m_1 is a constant, $m_1 > 0$, and

$$\varepsilon_{ji}(\boldsymbol{u}) = u_{i,\,j} - \varepsilon_{kji}\phi_k, \qquad \phi_{ji}(\boldsymbol{u}) = \phi_{i,\,j}.$$

Also, using the hypothesis (8), the following (Poincaré's inequality) holds for all $(\chi, \pi) \in \widehat{W}_1(B) \times \widehat{W}_1(B)$:

$$\int_B (\kappa_{ij}\chi_{,i}\chi_{,j} + d_{ij}\pi_{,i}\pi_{,j})\, dV \geq m_2 \int_B (\chi^2 + \pi^2)\, dV, \tag{17}$$

where m_2 is a constant, $m_2 > 0$.

If $\operatorname{meas}(\partial B_1) = \operatorname{meas}(\partial B_2) = 0$ then we can decompose the solution $((u_i, \phi_i), \theta, C)$ in the form

$$u_i = u_i^* + t\dot{u}_i^* + v_i, \quad \phi_i = \phi_i^* + t\dot{\phi}_i^* + \psi_i, \quad \theta = \chi, \quad C = \pi, \tag{18}$$

where $((v_i, \psi_i), \chi, \pi) \in \widehat{\mathbf{W}}_1(B) \times \widehat{W}_1(B) \times \widehat{W}_1(B)$ represents the solution of the system of equations (10) with the boundary conditions (9) and the initial conditions

$$\begin{aligned} &v_i(x,0) = U_i^0(x), \quad \dot{v}_i(x,0) = U_i^1(x), \quad \psi_i(x,0) = \Phi_i^0(x), \quad \dot{\psi}_i(x,0) = \Phi_i^1(x), \\ &\chi(x,0) = \theta^0(x), \quad \dot{\chi}(x,0) = \theta^1(x), \quad P(x,0) = \mathcal{P}^0(x), \quad \text{for all } x \in B. \end{aligned}$$

Now, we consider that $\operatorname{meas}(\partial B_4) = 0$.

Then we can decompose the solution $((u_i, \phi_i), \theta, C)$ in the form

$$u_i = v_i, \phi_i = \psi_i, \theta = \theta^* + \chi, C = \pi \tag{19}$$

where $((v_i, \psi_i), \chi, \pi) \in \widehat{\mathbf{W}}_1(B) \times \widehat{W}_1(B) \times \widehat{W}_1(B)$ represents the solution of the system of equations (10) with the boundary conditions initial conditions

$$\begin{aligned} &v_i(x,0) = u_i^0(x), \quad \dot{v}_i(x,0) = u_i^1(x), \quad \psi_i(x,0) = \varphi_i^0(x), \quad \dot{\psi}_i(x,0) = \varphi_i^1(x), \\ &\chi(x,0) = T^0(x), \quad \dot{\chi}(x,0) = T^1(x), \quad \pi(x,0) = C^0(x), \quad \text{for all } x \in B. \end{aligned} \tag{20}$$

4. Preliminary results

In this section we shall establish some evolutionary integral identities which form the basis in proving the relations that express the asymptotic partition of energy. In the first next theorem we prove a conservation law of total energy.

Theorem 1. *Let $((u_i, \phi_i), \theta, C)$ be a solution of the mixed initial boundary value problem defined by the equations (10), the boundary conditions (9) and the initial conditions (11). If we suppose that*

$$(u_i^0, \phi_i^0) \in \mathbf{W}_1(B), \quad (u_i^1, \phi_i^1) \in \mathbf{W}_0(B), \quad (\theta^0, C^0) \in W_1(B) \times W_1(B), \quad \theta^1 \in W_0(B),$$

then the following energy conservation law holds:

$$\mathscr{E}(t) + \frac{1}{T_0}\int_0^t \int_B \kappa_{ij}\theta_{,i}(s)\theta_{,j}(s)\, dV\, ds + \int_0^t \int_B C(s)\dot{P}(s)\, dV\, ds = \mathscr{E}(0) \tag{21}$$

for any $t \in [0, \infty)$, where

$$\begin{aligned} \mathscr{E}(t) = \frac{1}{2}\int_B [\rho\dot{u}_i(t)\dot{u}_i(t) + \rho J_{ij}\dot{\phi}_i(t)\dot{\phi}_j(t) + \frac{\rho c_E}{T_0}\theta^2(t) + 2\varpi C(t)\theta(t) - \varrho C^2(t)]\, dV \\ + \frac{1}{2}\int_B [c_{ijkl}\varepsilon_{ij}(t)\varepsilon_{kl}(t) + 2p_{ijkl}\varepsilon_{ij}(t)\phi_{kl}(t) + d_{ijkl}\phi_{ij}(t)\phi_{kl}(t)]\, dV. \end{aligned}$$

Proof. In view of equations of motion (10)$_1$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}[\rho\dot{u}_i(s)\dot{u}_i(s)] &= \rho\dot{u}_i(s)\ddot{u}_i(s) = \dot{u}_i(s)\sigma_{ij,\,j}(s)\\ &= \dot{u}_i(s)[c_{ijkl}\varepsilon_{kl}(s) + p_{ijkl}\phi_{kl}(s) + a_{ij}\theta(s) + b_{ij}C(s)]_{,\,j}\\ &= \{\dot{u}_i(s)[c_{ijkl}\varepsilon_{kl}(s) + p_{ijkl}\phi_{kl}(s) + a_{ij}\theta(s) + b_{ij}C(s)]\}_{,\,j}\\ &\quad - [c_{ijkl}\varepsilon_{kl}(s) + p_{ijkl}\phi_{kl}(s) + a_{ij}\theta(s) + b_{ij}C(s)]\dot{u}_{i,\,j}(s).\end{aligned} \tag{22}$$

Taking into account the equation (10)$_2$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}[\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)]\\ &= \rho J_{ij}\dot{\phi}_i(s)\ddot{\phi}_j(s) = \dot{\phi}_i(s)[\mu_{ij,\,j}(s) + \epsilon_{ijk}\sigma_{jk}(s)]\\ &= \dot{\phi}_i(s)\{[p_{ijkl}\varepsilon_{kl}(s) + d_{ijkl}\phi_{kl}(s) + p_{ij}\theta(s) + q_{ij}C(s)]_{,\,j} + \epsilon_{ijk}\sigma_{jk}(s)\}\\ &= \{\dot{\phi}_i(s)[p_{ijkl}\varepsilon_{kl}(s) + d_{ijkl}\phi_{kl}(s) + p_{ij}\theta(s) + q_{ij}C(s)]\}_{,\,j}\\ &\quad - [p_{ijkl}\varepsilon_{kl}(s) + d_{ijkl}\phi_{kl}(s) + p_{ij}\theta(s) + q_{ij}C(s)]\dot{\phi}_{ij}(s)\\ &\quad + \epsilon_{ijk}[c_{jkmn}\varepsilon_{mn}(s) + d_{jkmn}\phi_{mn}(s) + a_{jk}\theta(s) + b_{jk}C(s)]\dot{\phi}_i(s).\end{aligned} \tag{23}$$

Now we are adding equalities (22) and (23) member by member:

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}[\rho\dot{u}_i(s)\dot{u}_i(s) + \rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)]\\ &= \{\dot{u}_i(s)[c_{ijkl}\varepsilon_{kl}(s) + p_{ijkl}\phi_{kl}(s) + a_{ij}\theta(s) + b_{ij}C(s)]\}_{,\,j}\\ &\quad + \{\dot{\phi}_i(s)[p_{ijkl}\varepsilon_{kl}(s) + d_{ijkl}\phi_{kl}(s) + p_{ij}\theta(s) + q_{ij}C(s)]\}_{,\,j}\\ &\quad - c_{ijkl}\varepsilon_{kl}(s)\dot{\varepsilon}_{ij}(s) - p_{ijkl}(\phi_{kl}(s)\dot{\varepsilon}_{ij}(s) + \dot{\phi}_{kl}(s)\varepsilon_{ij}(s)) - d_{ijkl}\phi_{kl}(s)\dot{\phi}_{ij}(s)\\ &\quad - \theta(s)[a_{ij}\dot{\varepsilon}_{ij}(s) + p_{ij}\dot{\phi}_{ij}(s)] - C(s)[b_{ij}\dot{\varepsilon}_{ij}(s) + q_{ij}\dot{\phi}_{ij}(s)].\end{aligned} \tag{24}$$

On the other hand, by using the equations (10)$_4$ and (10)$_5$, we get

$$\begin{aligned}&\theta(s)[a_{ij}\dot{\varepsilon}_{ij}(s) + p_{ij}\dot{\phi}_{ij}(s)] + C(s)[b_{ij}\dot{\varepsilon}_{ij}(s) + q_{ij}\dot{\phi}_{ij}(s)]\\ &\qquad = \frac{1}{2}\frac{d}{ds}\Big[\frac{\rho c_E}{T_0}\theta^2(s) + 2\varpi C(s)\theta(s) - \varrho C^2(s)\Big] + C(s)\dot{P}(s) - \frac{1}{T_0}\kappa_{ij}\theta_{,\,ij}(s)\theta(s).\end{aligned} \tag{25}$$

From (24) and (25) we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\Big[\rho\dot{u}_i(s)\dot{u}_i(s) + \rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s) + \frac{\rho c_E}{T_0}\theta^2(s) + 2\varpi C(s)\theta(s) - \varrho C^2(s)\Big]\\ &\quad = \{\dot{u}_i(s)[c_{ijkl}\varepsilon_{kl}(s) + p_{ijkl}\phi_{kl}(s) + a_{ij}\theta(s) + b_{ij}C(s)]\}_{,\,j}\\ &\qquad + \{\dot{\phi}_i(s)[p_{ijkl}\varepsilon_{kl}(s) + d_{ijkl}\phi_{kl}(s) + p_{ij}\theta(s) + q_{ij}C(s)]\}_{,\,j}\\ &\qquad - c_{ijkl}\varepsilon_{kl}(s)\dot{\varepsilon}_{ij}(s) - p_{ijkl}(\phi_{kl}(s)\dot{\varepsilon}_{ij}(s) + \dot{\phi}_{kl}(s)\varepsilon_{ij}(s))\\ &\qquad - d_{ijkl}\phi_{kl}(s)\dot{\phi}_{ij}(s) - C(s)\dot{P}(s) + \frac{1}{T_0}\kappa_{ij}\theta_{,\,ij}(s)\theta(s),\end{aligned}$$

and this equality can be restated in the form

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\Big[\rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)+\frac{\rho c_E}{T_0}\theta^2(s)+2\varpi C(s)\theta(s)-\varrho C^2(s)\Big]\\
&\quad+\frac{1}{2}\frac{d}{ds}[\cdot c_{ijkl}\varepsilon_{kl}(s)\varepsilon_{ij}(s)+2p_{ijkl}\varepsilon_{ij}(s)\phi_{kl}(s)+d_{ijkl}\phi_{kl}(s)\phi_{ij}(s)]\\
&\quad+\frac{1}{T_0}\kappa_{ij}\theta_{,\,i}(s)\theta_{,\,j}(s)-C(s)\dot{P}(s)\\
&=\{\dot{u}_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,\,j}\\
&\quad+\{\dot{\phi}_i(s)[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+p_{ij}\theta(s)+q_{ij}C(s)]\}_{,\,j}+\Big(\frac{1}{T_0}\kappa_{ij}\theta_{,\,i}(s)\theta(s)\Big)_{,\,j}.
\end{aligned}\tag{26}$$

Now, we integrate the relation (26) over $B\times(0,t)$ and use the divergence theorem, the symmetry relations (6), the boundary conditions (9) and the initial conditions (11) so that we arrive at the desired result (21) and Theorem 1 is concluded. □

Theorem 2. *Consider $((u_i,\phi_i),\theta,C)$ a solution of the mixed initial boundary value problem defined by the equations (10), the boundary conditions (9) and the initial conditions (11). If we suppose that*

$$(u_i^0,\phi_i^0)\in \mathbf{W}_1(B),\ (u_i^1,\phi_i^1)\in \mathbf{W}_0(B),\ (\theta^0,C^0)\in W_1(B)\times W_1(B),\ \theta^1\in W_0(B),$$

then the following identity holds:

$$\begin{aligned}
&2\int_B[\rho u_i(t)\dot{u}_i(t)+\rho J_{ij}\phi_i(t)\dot{\phi}_j(t)]\,dV\\
&\quad+\int_B\frac{1}{T_0}k_{ij}\Big(\int_0^t\theta_{,i}(s)\,ds\Big)\Big(\int_0^t\theta_{,\,j}(s)\,ds\Big)\,dV+\int_B d_{ij}\Big(\int_0^t P_{,i}(s)\,ds\Big)\Big(\int_0^t P_{,j}(s)\,ds\Big)\,dV\\
&=2\int_0^t\int_B[\rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)]\,dV\,ds\\
&\quad-\int_0^t\int_B[c_{ijkl}\varepsilon_{kl}(s)\varepsilon_{ij}(s)+2p_{ijkl}\phi_{kl}(s)\varepsilon_{ij}(s)+d_{ijkl}\phi_{kl}(s)\phi_{ij}(s)]\,dV\,ds\\
&\quad-\int_0^t\int_B\Big[\frac{\rho c_E}{T_0}\theta^2(s)+2\varpi C(s)\theta(s)-\varrho C^2(s)\Big]\,dV\,ds+2\int_0^t\int_B[\rho S^0\theta(s)-P^0C(s)]\,dV\,ds\\
&\quad+2\int_B[\rho u_i^0\dot{u}_i^1+\rho J_{ij}\phi_i^0\dot{\phi}_j^1]\,dV,
\end{aligned}\tag{27}$$

where

$$\begin{aligned}
\rho S^0&=\frac{\rho c_E}{T_0}\theta^0-a_{ij}\varepsilon_{ij}^0-p_{ij}\phi_{ij}^0+\varpi C^0,\\
P^0&=b_{ij}\varepsilon_{ij}^0+q_{ij}\phi_{ij}^0-\varpi\theta^0+\varrho C^0\\
\varepsilon_{ji}^0&=u_{i,\,j}^0-\epsilon_{kji}\phi_k^0,\\
\phi_{ji}^0&=\phi_{i,\,j}^0.
\end{aligned}\tag{28}$$

Proof. In view of equations of motion (10)$_1$, we get

$$\begin{aligned}\frac{d}{ds}[\rho u_i(s)\dot{u}_i(s)] &= \rho\dot{u}_i(s)\dot{u}_i(s)+\rho u_i(s)\ddot{u}_i(s) = \rho\dot{u}_i(s)\dot{u}_i(s)+u_i(s)\sigma_{ij,\,j}(s)\\ &= \rho\dot{u}_i(s)\dot{u}_i(s)+u_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]_{,\,j}\\ &= \rho\dot{u}_i(s)\dot{u}_i(s)+\{u_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,\,j}\\ &\qquad -u_{i,\,j}(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\\ &= \rho\dot{u}_i(s)\dot{u}_i(s)-[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]u_{i,\,j}(s)\\ &\qquad +\{u_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,\,j}.\end{aligned}\tag{29}$$

Taking into account the equations of motion (10)$_2$, we obtain

$$\begin{aligned}\frac{d}{ds}[\rho J_{ij}\phi_i(s)\dot{\phi}_j(s)] &= \rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)+\rho J_{ij}\phi_i(s)\ddot{\phi}_j(s)\\ &= \rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)+\phi_i(s)[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+p_{ij}\theta(s)+q_{ij}C(s)]_{,\,j}\\ &\qquad +\epsilon_{ijk}[c_{jkmn}\varepsilon_{mn}(s)+d_{jkmn}\phi_{mn}(s)+a_{jk}\theta(s)+b_{jk}C(s)]\phi_i\\ &= \rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)+\{\phi_i(s)[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,\,j}\\ &\qquad -[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+p_{ij}\theta(s)+q_{ij}C(s)]\phi_{i,\,j}(s)\\ &\qquad +\epsilon_{ijk}[c_{jkmn}\varepsilon_{mn}(s)+p_{jkmn}\phi_{mn}(s)+a_{jk}\theta(s)+b_{jk}C(s)]\phi_i.\end{aligned}\tag{30}$$

Now we are adding equalities (29) and (30) member by member:

$$\begin{aligned}\frac{d}{ds}[\rho u_i(s)\dot{u}_i(s)+\rho J_{ij}\phi_i(s)\dot{\phi}_j(s)] &= \rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)\\ &\quad +\{u_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,\,j}\\ &\quad +\{\phi_i(s)[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+p_{ij}\theta(s)+q_{ij}C(s)]\}_{,\,j}\\ &\quad -c_{ijkl}\varepsilon_{kl}(s)\varepsilon_{ij}(s)-2p_{ijkl}\varepsilon_{ij}(s)\phi_{kl}(s)-d_{ijkl}\phi_{kl}(s)\phi_{ij}(s)\\ &\quad -[a_{ij}\varepsilon_{ij}(s)+p_{ij}\phi_{ij}(s)]\theta(s)-[b_{ij}\varepsilon_{ij}(s)+q_{ij}\phi_{ij}(s)]C(s).\end{aligned}\tag{31}$$

By an integration with respect to time variable in the equation (10)$_3$ and then by using the initial conditions (11) we deduce

$$\begin{aligned}&-[a_{ij}\varepsilon_{ij}(s)+p_{ij}\phi_{ij}(s)]\\ &\qquad = \frac{1}{T_0}\left(\int_0^s \kappa_{ij}\theta_{,\,i}(z)\,dz\right)_{,\,j}-\frac{\rho c_E}{T_0}\theta(s)-\varpi C(s)+\frac{\rho c_E}{T_0}\theta^0+\varpi C^0-a_{ij}\varepsilon^0_{ij}-p_{ij}\phi^0_{ij}.\end{aligned}$$

Here we multiply by $\theta(s)$ and obtain

$$\begin{aligned}-[a_{ij}\varepsilon_{ij}(s)+p_{ij}\phi_{ij}(s)]\theta(s) &= \frac{1}{T_0}\left(\int_0^s \kappa_{ij}\theta_{,\,i}(z)\,dz\right)_{,\,j}\theta(s)\\ &\quad -\frac{\rho c_E}{T_0}\theta^2(s)-\varpi C(s)\theta(s)+\left(\frac{\rho c_E}{T_0}\theta^0-a_{ij}\varepsilon^0_{ij}-p_{ij}\phi^0_{ij}+\varpi C^0\right)\theta(s).\end{aligned}\tag{32}$$

Also, by integrating with respect to the time variable, from Fick's law and the initial conditions (11) we deduce

$$b_{ij}\varepsilon_{ij}(s)+q_{ij}\phi_{ij}(s)=\left(\int_0^s d_{ij}P_{,i}(z)\,dz\right)_{,j}+\varpi\theta(s)-\varrho C(s)+b_{ij}\varepsilon_{ij}^0+q_{ij}\phi_{ij}^0-\varpi\theta^0+\varrho C^0.$$

Here we multiply by $C(s)$ so that we are led to

$$\begin{aligned}-[b_{ij}\varepsilon_{ij}(s)+q_{ij}\phi_{ij}(s)]C(s)=-\left(\int_0^s d_{ij}\;P_{,i}(z)\,dz\right)_{,j}C(s)\\ -\varpi\theta(s)C(s)+\varrho C^2(s)-(b_{ij}\varepsilon_{ij}^0+q_{ij}\phi_{ij}^0-\varpi\theta^0+\varrho C^0)C(s).\end{aligned}\tag{33}$$

By adding relations (32) and (33) member by member one obtains

$$\begin{aligned}&-[a_{ij}\varepsilon_{ij}(s)+p_{ij}\phi_{ij}(s)]\theta(s)-[b_{ij}\varepsilon_{ij}(s)+q_{ij}\phi_{ij}(s)]C(s)\\ &\quad=\frac{1}{T_0}\left(\int_0^s \kappa_{ij}\theta_{,i}(z)\,dz\right)_{,j}\theta(s)-\frac{\rho c_E}{T_0}\theta^2(s)-\varpi C(s)\theta(s)\\ &\qquad+\left(\frac{\rho c_E}{T_0}\theta^0-a_{ij}\varepsilon_{ij}^0-p_{ij}\phi_{ij}^0+\varpi C^0\right)\theta(s)\\ &\qquad-\left(\int_0^s d_{ij}P_{,i}(z)\,dz\right)_{,j}C(s)-\varpi\theta(s)C(s)+\varrho C^2(s)\\ &\qquad-(b_{ij}\varepsilon_{ij}^0+q_{ij}\phi_{ij}^0-\varpi\theta^0+\varrho C^0)C(s).\end{aligned}\tag{34}$$

We introduce in (34) into (31) so that we get the equality

$$\begin{aligned}&\frac{d}{ds}[\rho u_i(s)\dot{u}_i(s)+\rho J_{ij}\phi_i(s)\dot{\phi}_j(s)]\\ &\quad=\rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)\\ &\qquad+\{u_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,j}\\ &\qquad+\{\phi_i(s)[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+p_{ij}\theta(s)+q_{ij}C(s)]\}_{,j}\\ &\qquad-c_{ijkl}\varepsilon_{kl}(s)\varepsilon_{ij}(s)-2p_{ijkl}\phi_{kl}(s)\varepsilon_{ij}(s)-d_{ijkl}\phi_{kl}(s)\phi_{ij}(s)\\ &\qquad+\frac{1}{T_0}\left(\int_0^s \kappa_{ij}\theta_{,i}(z)\,dz\right)_{,j}\theta(s)-\frac{\rho c_E}{T_0}\theta^2(s)-\varpi C(s)\theta(s)\\ &\qquad+\left(\frac{\rho c_E}{T_0}\theta^0-a_{ij}\varepsilon_{ij}^0-p_{ij}\phi_{ij}^0+\varpi C^0\right)\theta(s)\\ &\qquad-\left(\int_0^s d_{ij}P_{,i}(z)\,dz\right)_{,j}C(s)-\varpi\theta(s)C(s)+\varrho C^2(s)\\ &\qquad-(b_{ij}\varepsilon_{ij}^0+q_{ij}\phi_{ij}^0-\varpi\theta^0+\varrho C^0)C(s).\end{aligned}$$

This relation can be restated in the form

$$
\begin{aligned}
\frac{d}{ds}&[\rho u_i(s)\dot{u}_i(s)+\rho J_{ij}\phi_i(s)\dot{\phi}_j(s)]\\
&=\rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)\\
&\quad+\{u_i(s)[c_{ijkl}\varepsilon_{kl}(s)+p_{ijkl}\phi_{kl}(s)+a_{ij}\theta(s)+b_{ij}C(s)]\}_{,\,j}\\
&\quad+\{\phi_i(s)[p_{ijkl}\varepsilon_{kl}(s)+d_{ijkl}\phi_{kl}(s)+p_{ij}\theta(s)+q_{ij}C(s)]\}_{,\,j}\\
&\quad-c_{ijkl}\varepsilon_{kl}(s)\varepsilon_{ij}(s)-2p_{ijkl}\phi_{kl}(s)\varepsilon_{ij}(s)-d_{ijkl}\phi_{kl}(s)\phi_{ij}(s)\\
&\quad+\rho S^0\theta(s)-P^0C(s)-\frac{\rho c_E}{T_0}\theta^2(s)-2\varpi C(s)\theta(s)+\varrho C^2\\
&\quad+\frac{1}{T_0}\Big(\kappa_{ij}\theta(s)\int_0^s\theta_{,\,i}(z)\,dz\Big)_{,\,j}-\frac{1}{T_0}\kappa_{ij}\theta_{,\,j}(s)\int_0^s\theta_{,\,i}(z)\,dz\\
&\quad-\Big(d_{ij}P(s)\int_0^s P_{,\,i}(z)\,dz\Big)_{,\,j}-d_{ij}P_{,\,j}(s)\int_0^s P_{,\,i}(z)\,dz,
\end{aligned}
\tag{35}
$$

where ρS^0 and P^0 are given in (28).

Integrating by parts, it is easy to obtain

$$
\kappa_{ij}\int_0^t\theta_{,\,i}(s)\Big(\int_0^s\theta_{,\,j}(z)\,dz\Big)\,ds=k_{ij}\int_0^s\theta_{,\,j}(z)\,dz\int_0^s\theta_{,\,i}(z)\,dz\Big|_{s=0}^{s=t}-k_{ij}\int_0^t\theta_{,\,j}(s)\Big(\int_0^s\theta_{,\,i}(z)\,dz\Big)\,ds.
$$

On the basis of symmetry of tensor κ_{ij}, from the above equality we deduce that

$$
2\kappa_{ij}\int_0^t\theta_{,\,i}(s)\Big(\int_0^s\theta_{,\,j}(z)\,dz\Big)\,ds=\kappa_{ij}\Big(\int_0^t\theta_{,\,i}(s)\,ds\Big)\Big(\int_0^t\theta_{,\,j}(s)\,ds\Big).\tag{36}
$$

Analogous, on the basis of symmetry of tensor d_{ij}, we obtain the equality

$$
2d_{ij}\int_0^t P_{,\,i}(s)\Big(\int_0^s P_{,\,j}(z)\,dz\Big)\,ds=d_{ij}\Big(\int_0^t P_{,\,i}(s)\,ds\Big)\Big(\int_0^t P_{,\,j}(s)\,ds\Big).\tag{37}
$$

Now we integrate the identity (35) over $B\times(0,t)$, then we employ the divergence theorem, the boundary conditions (9), the initial conditions (11), the symmetry relations (6) and relations (36)–(37) so that we are led to the desired identity (27).

This concludes Theorem 2. □

Theorem 3. *Consider $((u_i,\phi_i),\theta,C)$ a solution of the mixed initial boundary value problem defined by the equations (10), the boundary conditions (9) and the initial conditions (11). If we suppose that*

$$
(u_i^0,\varphi_i^0)\in\boldsymbol{W}_1(B),\quad(u_i^1,\varphi_i^1)\in\boldsymbol{W}_0(B),\quad(\theta^0,C^0)\in W_1(B)\times W_1(B),\quad\theta^1\in W_0(B),
$$

then we have the identity

$$
\begin{aligned}
2\int_B [\rho u_i(t)\dot{u}_i(t)+\rho J_{ij}\phi_i(t)\dot{\phi}_j(t)]\,dV+\int_B \kappa_{ij}\Big[\Big(\int_0^t \theta_{,i}(z)\,dz\Big)\Big(\int_0^t \theta_{,j}(z)\,dz\Big)\Big]\,dV \\
+\int_B d_{ij}\Big[\Big(\int_0^t P_{,i}(z)\,dz\Big)\Big(\int_0^t P_{,j}(z)\,dz\Big)\Big]\,dV \\
=\int_B \{\varrho[u_i^0\dot{u}_i(2t)+u_i^1 u_i(2t)]+\rho J_{ij}[\phi_i^0\dot{\phi}_j(2t)+\phi_i^1\phi_j(2t)]\}\,dV \\
+\int_0^t\int_B \{\rho S^0[\theta(t+s)-\theta(t-s)]+P^0[C(t+s)-C(t-s)]\}\,dV\,ds, \quad (38)
\end{aligned}
$$

where ϱS^0 *and* P^0 *are defined by relations* (28).

Proof. It is not difficult to prove the identity

$$
\frac{d}{ds}\{\rho[f_i(s)\dot{g}_i(s)-\dot{f}_i(s)g_i(s)]\}=\rho[f_i(s)\ddot{g}_i(s)-\ddot{f}_i(s)g_i(s)],
$$

where $f_i(x,s)$ and $g_i(x,s)$ are twice continuously differentiable functions with respect to time variable s. By integrating the above identity over $B\times(0,t)$ one obtains

$$
\begin{aligned}
\int_B \varrho[f_i(s)\dot{g}_i(s)-\dot{f}_i(s)g_i(s)]\,dV \\
=\int_0^t\int_B \varrho[f_i(s)\ddot{g}_i(s)-\ddot{f}_i(s)g_i(s)]\,dV\,ds+\int_B \varrho[f_i(0)\dot{g}_i(0)-\dot{f}_i(0)g_i(0)]\,dV. \quad (39)
\end{aligned}
$$

Now, we set in (46) the functions f_i and g_i as follows:

$$
f_i(x,s)=u_i(x,t-s),\quad g_i(x,s)=u_i(x,t+s)\quad \text{for } s\in[0,t],\ t\in(0,\infty),
$$

so that one obtains the identity

$$
\begin{aligned}
2\int_B \rho u_i(t)\dot{u}_i(t)\,dV=\int_B \varrho[u_i^0\dot{u}_i(2t)+u_i^1u_i(2t)]\,dV \\
+\int_0^t\int_B \rho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]\,dV\,ds,\quad t\in[0,\infty). \quad (40)
\end{aligned}
$$

Similarly, if we substitute in (46) the functions f_i and g_i defined by

$$
f_i(x,s)=\phi_i(x,t-s),\quad g_i(x,s)=\phi_i(x,t+s)\quad \text{for } s\in[0,t],\ t\in(0,\infty),
$$

then we are led to the identity

$$
\begin{aligned}
2\int_B \rho J_{ij}\phi_i(t)\dot{\phi}_j(t)\,dV=\int_B \rho J_{ij}[\phi_i^0\dot{\phi}_j(2t)+\phi_i^1\phi_j(2t)]\,dV \\
+\int_0^t\int_B \rho J_{ij}[\phi_i(t+s)\ddot{\phi}_j(t-s)-\phi_i(t-s)\ddot{\phi}_j(t+s)]\,dV\,ds,\quad t\in[0,\infty). \quad (41)
\end{aligned}
$$

We add the equalities (40) and (41) member by member and obtain

$$
\begin{aligned}
2\int_B &[\rho u_i(t)\dot{u}_i(t)+\rho J_{ij}\phi_i(t)\dot{\phi}_j(t)]\,dV\\
&=\int_B \varrho[u_i^0\dot{u}_i(2t)+u_i^1u_i(2t)]\,dV+\int_B \rho J_{ij}[\phi_i^0\dot{\phi}_j(2t)+\phi_i^1\phi_j(2t)]\,dV\\
&\quad+\int_0^t\int_B \rho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]\,dV\,ds\\
&\quad+\int_0^t\int_B \rho J_{ij}[\phi_i(t+s)\ddot{\phi}_j(t-s)-\phi_i(t-s)\ddot{\phi}_j(t+s)]\,dV\,ds
\end{aligned}
\tag{42}
$$

for any $t\in[0,\infty)$.

The last two integrals in (42) contain some inertial terms which will be eliminated in the following. First, taking into account the equations $(10)_1$ one obtains

$$
\begin{aligned}
\rho[&u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]\\
&=u_i(t+s)\rho\ddot{u}_i(t-s)-u_i(t-s)\rho\ddot{u}_i(t+s)\\
&=u_i(t+s)[c_{ijkl}\varepsilon_{kl}(t-s)+p_{ijkl}\phi_{kl}(t-s)+a_{ij}\theta(t-s)+b_{ij}\,C(t-s)]_{,\,j}\\
&\qquad-u_i(t-s)[c_{ijkl}\varepsilon_{kl}(t+s)+p_{ijkl}\phi_{kl}(t+s)+a_{ij}\theta(t+s)+b_{ij}\,C(t+s)]_{,\,j}\\
&=\{u_i(t+s)[c_{ijkl}\varepsilon_{kl}(t-s)+p_{ijkl}\phi_{kl}(t-s)+a_{ij}\theta(t-s)+b_{ij}\,C(t-s)]\}_{,\,j}\\
&\qquad-u_{i,\,j}(t+s)[c_{ijkl}\varepsilon_{kl}(t-s)+p_{ijkl}\phi_{kl}(t-s)+a_{ij}\theta(t-s)+b_{ij}\,C(t-s)]\\
&\qquad-\{u_i(t-s)[c_{ijkl}\varepsilon_{kl}(t+s)+p_{ijkl}\phi_{kl}(t+s)+a_{ij}\theta(t+s)+b_{ij}\,C(t+s)]\}_{,\,j}\\
&\qquad+u_{i,\,j}(t-s)[c_{ijkl}\varepsilon_{kl}(t+s)+p_{ijkl}\phi_{kl}(t+s)+a_{ij}\theta(t+s)+b_{ij}\,C(t+s)].
\end{aligned}
\tag{43}
$$

In view of equations $(10)_2$ we get

$$
\begin{aligned}
\rho J_{ij}&[\phi_i(t+s)\ddot{\phi}_j(t-s)-\phi_i(t-s)\ddot{\phi}_j(t+s)]\\
&=\phi_i(t+s)\rho J_{ij}\ddot{\phi}_j(t-s)-\phi_i(t-s)\rho J_{ij}\ddot{\phi}_i(t+s)\\
&=\phi_i(t+s)[p_{ijkl}\varepsilon_{kl}(t-s)+d_{ijkl}\phi_{kl}(t-s)+p_{ij}\theta(t-s)+q_{ij}\,C(t-s)]_{,\,j}\\
&\qquad+\epsilon_{ijk}\phi_i(t+s)[a_{jkmn}\varepsilon_{mn}(t-s)+p_{jkmn}\phi_{mn}(t-s)+a_{jk}\theta(t-s)+b_{jk}C(t-s)]\\
&\qquad-\phi_i(t-s)[p_{ijkl}\varepsilon_{kl}(t+s)+d_{ijkl}\phi_{kl}(t+s)+p_{ij}\theta(t+s)+q_{ij}\,C(t+s)]_{,\,j}\\
&\qquad-\epsilon_{ijk}\phi_i(t-s)[a_{jkmn}\varepsilon_{mn}(t+s)+p_{jkmn}\phi_{kl}(t+s)+a_{jk}\theta(t+s)+b_{jk}C(t+s)]\\
&=\{\phi_i(t+s)[p_{ijkl}\varepsilon_{kl}(t-s)+d_{ijkl}\phi_{kl}(t-s)+p_{ij}\theta(t-s)+q_{ij}\,C(t-s)]\}_{,\,j}\\
&\qquad-\phi_{i,\,j}(t+s)[p_{ijkl}\varepsilon_{kl}(t-s)+d_{ijkl}\phi_{kl}(t-s)+p_{ij}\theta(t-s)+q_{ij}\,C(t-s)]\\
&\qquad+\epsilon_{ijk}\phi_i(t+s)[a_{jkmn}\varepsilon_{mn}(t-s)+p_{jkmn}\phi_{mn}(t-s)+a_{jk}\theta(t-s)+b_{jk}C(t-s)]\\
&\qquad-\{\phi_i(t-s)[p_{ijkl}\varepsilon_{kl}(t+s)+d_{ijkl}\phi_{kl}(t+s)+p_{ij}\theta(t+s)+q_{ij}\,C(t+s)]\}_{,\,j}\\
&\qquad+\phi_{i,\,j}(t-s)[p_{ijkl}\varepsilon_{kl}(t+s)+d_{ijkl}\phi_{kl}(t+s)+p_{ij}\theta(t+s)+q_{ij}\,C(t+s)]\\
&\qquad-\epsilon_{ijk}\phi_i(t-s)[a_{jkmn}\varepsilon_{mn}(t+s)+p_{jkmn}\phi_{kl}(t+s)+a_{jk}\theta(t+s)+b_{jk}\,C(t+s)].
\end{aligned}
\tag{44}
$$

If we add the equalities (43) and (44) together, then we are led to

$$\begin{aligned}
\rho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]&+\rho J_{ij}[\phi_i(t+s)\ddot{\phi}_j(t-s)-\phi_i(t-s)\ddot{\phi}_j(t+s)]\\
=\{u_i(t+s)&[c_{ijkl}\varepsilon_{kl}(t-s)+p_{ijkl}\phi_{kl}(t-s)+a_{ij}\theta(t-s)+b_{ij}C(t-s)]\}_{,j}\\
&-\{u_i(t-s)[c_{ijkl}\varepsilon_{kl}(t+s)+p_{ijkl}\phi_{kl}(t+s)+a_{ij}\theta(t+s)+b_{ij}C(t+s)]\}_{,j}\\
&+\{\phi_i(t+s)[p_{ijkl}\varepsilon_{kl}(t-s)+d_{ijkl}\phi_{kl}(t-s)+p_{ij}\theta(t-s)+q_{ij}C(t-s)]\}_{,j}\\
&-\{\phi_i(t-s)[p_{ijkl}\varepsilon_{kl}(t+s)+d_{ijkl}\phi_{kl}(t+s)+p_{ij}\theta(t+s)+q_{ij}C(t+s)]\}_{,j}\\
&-[a_{ij}\varepsilon_{ij}(t+s)+p_{ij}\phi_{ij}(t+s)]\theta(t-s)+[a_{ij}\varepsilon_{ij}(t-s)+p_{ij}\phi_{ij}(t-s)]\theta(t+s)\\
&-[b_{ij}\varepsilon_{ij}(t+s)+q_{ij}\phi_{ij}(t+s)]C(t-s)+[b_{ij}\varepsilon_{ij}(t-s)+q_{ij}\phi_{ij}(t-s)]C(t+s).
\end{aligned} \tag{45}$$

For the last two rows of right-hand side of equality (45) we get the equivalent expressions using the energy equation and the equation of mass diffusion, respectively.

So, by an integration with respect to time variable, from the energy equation $(10)_3$ and the initial conditions (11) we deduce

$$\begin{aligned}
-[a_{ij}\varepsilon_{ij}(t+s)+p_{ij}\phi_{ij}(t+s)]\theta(t-s)=\frac{1}{T_0}\left(\int_0^{t+s}\kappa_{ij}\theta_{,i}(z)\,dz\right)_{,j}\theta(t-s)\\
-\frac{\rho c_E}{T_0}\theta(t-s)\theta(t+s)-\varpi\theta(t-s)C(t-s)+\rho S^0\theta(t-s),
\end{aligned} \tag{46}$$

where ρS^0 is defined in (28).

Similarly, we have

$$\begin{aligned}
[a_{ij}\varepsilon_{ij}(t-s)+p_{ij}\phi_{ij}(t-s)]\theta(t+s)=-\frac{1}{T_0}\left(\int_0^{t-s}\kappa_{ij}\theta_{,i}(z)\,dz\right)_{,j}\theta(t+s)\\
+\frac{\rho c_E}{T_0}\theta(t+s)\theta(t-s)+\varpi\theta(t-s)C(t+s)-\rho S^0\theta(t+s).
\end{aligned} \tag{47}$$

From (46) and (47) we deduce

$$\begin{aligned}
-[a_{ij}\varepsilon_{ij}(t+s)+p_{ij}\phi_{ij}(t+s)]\theta(t-s)&+[a_{ij}\varepsilon_{ij}(t-s)+p_{ij}\phi_{ij}(t-s)]\theta(t+s)\\
&=\varpi[\theta(t+s)C(t-s)-\theta(t-s)C(t+s)]+\rho S^0[\theta(t+s)-\theta(t-s)]\\
&\quad+\frac{1}{T_0}\kappa_{ij}\left[\theta(t-s)\left(\int_0^{t+s}\theta_{,i}(z)\,dz\right)_{,j}-\theta(t+s)\left(\int_0^{t-s}\theta_{,i}(z)\,dz\right)_{,j}\right].
\end{aligned} \tag{48}$$

The equality (48) can be restated in the form

$$\begin{aligned}
-[a_{ij}\varepsilon_{ij}(t+s)&+p_{ij}\phi_{ij}(t+s)]\theta(t-s)+[a_{ij}\varepsilon_{ij}(t-s)+p_{ij}\phi_{ij}(t-s)]\theta(t+s)\\
&=\varpi[\theta(t+s)C(t-s)-\theta(t-s)C(t+s)]+\rho S^0[\theta(t+s)-\theta(t-s)]\\
&\quad+\left[\frac{1}{T_0}\kappa_{ij}\theta(t-s)\int_0^{t+s}\theta_{,i}(z)\,dz-\frac{1}{T_0}\kappa_{ij}\theta(t+s)\int_0^{t-s}\theta_{,i}(z)\,dz\right]_{,j}\\
&\quad+\frac{1}{T_0}\kappa_{ij}\left[\theta_{,j}(t+s)\int_0^{t-s}\theta_{,i}(z)\,dz-\theta_{,i}(t-s)\int_0^{t+s}\theta_{,i}(z)\,dz\right].
\end{aligned} \tag{49}$$

It is easy to deduce that the last row in (49) can be written in the form

$$\frac{1}{T_0}\kappa_{ij}\left[\theta_{,j}(t+s)\int_0^{t-s}\theta_{,i}(z)\,dz-\theta_{,i}(t-s)\int_0^{t+s}\theta_{,i}(z)\,dz\right]\\ =\frac{1}{T_0}\kappa_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}\theta_{,i}(z)\,dz\right)\left(\int_0^{t-s}\theta_{,j}(z)\,dz\right)\right],$$

so that (49) receives the form

$$\begin{aligned}-[a_{ij}\varepsilon_{ij}(t+s)+p_{ij}\phi_{ij}(t+s)]\theta(t-s)+[a_{ij}\varepsilon_{ij}(t-s)+p_{ij}\phi_{ij}(t-s)]\theta(t+s)\\ =\varpi[\theta(t+s)C(t-s)-\theta(t-s)C(t+s)]+\rho S^0[\theta(t+s)-\theta(t-s)]\\ +\left[\frac{1}{T_0}\kappa_{ij}\theta(t-s)\int_0^{t+s}\theta_{,i}(z)\,dz-\frac{1}{T_0}\kappa_{ij}\theta(t+s)\int_0^{t-s}\theta_{,i}(z)\,dz\right]_{,j}\\ +\frac{1}{T_0}\kappa_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}\theta_{,i}(z)\,dz\right)\left(\int_0^{t-s}\theta_{,j}(z)\,dz\right)\right].\end{aligned}\tag{50}$$

Now, by an integration with respect to time variable, from the equation (3), Fick's law and the initial conditions (11) we deduce

$$\begin{aligned}&-[b_{ij}\varepsilon_{ij}(t+s)+q_{ij}\phi_{ij}(t+s)]C(t-s)\\ &=-\left(\int_0^{t+s}d_{ij}P_{,i}(z)\,dz\right)_{,j}C(t-s)-\varpi\theta(t+s)C(t-s)+\varrho C(t+s)C(t-s)-P^0C(t-s),\end{aligned}\tag{51}$$

where P^0 is defined in (28).

Similarly, we have

$$\begin{aligned}&[b_{ij}\varepsilon_{ij}(t-s)+q_{ij}\phi_{ij}(t-s)]C(t+s)\\ &=\left(\int_0^{t-s}d_{ij}P_{,i}(z)\,dz\right)_{,j}C(t+s)+\varpi\theta(t-s)C(t+s)-\varrho C(t-s)C(t+s)+P^0C(t+s),\end{aligned}\tag{52}$$

so that from (51) and (52) we are led to

$$\begin{aligned}-[b_{ij}\varepsilon_{ij}(t+s)+q_{ij}\phi_{ij}(t+s)]C(t-s)+[b_{ij}\varepsilon_{ij}(t-s)+q_{ij}\phi_{ij}(t-s)]C(t+s)\\ =\varpi[\theta(t-s)C(t+s)-\theta(t+s)C(t-s)]+P^0[C(t+s)-C(t-s)]\\ -\left(\int_0^{t-s}d_{ij}P_{,i}(z)\,dz\right)_{,j}C(t+s)+\left(\int_0^{t+s}d_{ij}P_{,i}(z)\,dz\right)_{,j}C(t-s).\end{aligned}\tag{53}$$

The equality (53) can be restated in the form

$$\begin{aligned}-[b_{ij}\varepsilon_{ij}(t+s)+q_{ij}\phi_{ij}(t+s)]C(t-s)+[b_{ij}\varepsilon_{ij}(t-s)+q_{ij}\phi_{ij}(t-s)]C(t+s)\\ =\varpi[\theta(t-s)C(t+s)-\theta(t+s)C(t-s)]+P^0[C(t+s)-C(t-s)]\\ +\left[d_{ij}(P(t+s)\int_0^{t-s}P_{,i}(z)\,dz-P(t-s)\int_0^{t+s}P_{,i}(z)\,dz)\right]_{,j}\\ -d_{ij}\left[P_{,j}(t+s)\int_0^{t-s}P_{,i}(z)\,dz-P_{,j}(t-s)\int_0^{t+s}P_{,i}(z)\,dz\right].\end{aligned}\tag{54}$$

It is easy to deduce that the last row in (54) can be written in the form

$$d_{ij}\left[P_{,j}(t+s)\int_0^{t-s}P_{,i}(z)\,dz-P_{,j}(t-s)\int_0^{t+s}P_{,i}(z)\,dz\right]=d_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}P_{,i}(z)\,dz\right)\left(\int_0^{t-s}P_{,j}(z)\,dz\right)\right],$$

so that (54) receives the form

$$\begin{aligned}-[b_{ij}\varepsilon_{ij}(t+s)+q_{ij}\phi_{ij}(t+s)]C(t-s)+[b_{ij}\varepsilon_{ij}(t-s)+q_{ij}\phi_{ij}(t-s)]C(t+s)\\
=\varpi[\theta(t-s)\,C(t+s)-\theta(t+s)\,C(t-s)]+P^0[C(t+s)-P(t-s)]\\
+\left[d_{ij}(P(t+s)\int_0^{t-s}P_{,i}(z)\,dz-P(t-s)\int_0^{t+s}P_{,i}(z)\,dz)\right]_{,j}\\
+d_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}P_{,i}(z)\,dz\right)\left(\int_0^{t-s}P_{,j}(z)\,dz\right)\right].\end{aligned} \tag{55}$$

We now introduce the expressions (50) and (55) into equality (45) and we are led to

$$\begin{aligned}&\rho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]+\rho J_{ij}[\phi_i(t+s)\ddot{\phi}_j(t-s)-\phi_i(t-s)\ddot{\phi}_j(t+s)]\\
&=\{u_i(t+s)[c_{ijkl}\varepsilon_{kl}(t-s)+p_{ijkl}\phi_{kl}(t-s)+a_{ij}(\theta(t-s)+\alpha\dot{\theta}(t-s))+b_{ij}\;P(t-s)]\}_{,j}\\
&-\{u_i(t-s)[c_{ijkl}\varepsilon_{kl}(t+s)+p_{ijkl}\phi_{kl}(t+s)+a_{ij}(\theta(t+s)+\alpha\dot{\theta}(t+s))+b_{ij}\;P(t+s)]\}_{,j}\\
&+\{\phi_i(t+s)[p_{ijkl}\varepsilon_{kl}(t-s)+d_{ijkl}\phi_{kl}(t-s)+\kappa_{ij}(\theta(t-s)+\alpha\dot{\theta}(t-s))+q_{ij}P(t-s)]\}_{,j}\\
&-\{\phi_i(t-s)[p_{ijkl}\varepsilon_{kl}(t+s)+d_{ijkl}\phi_{kl}(t+s)+\kappa_{ij}(\theta(t+s)+\alpha\dot{\theta}(t+s))+q_{ij}P(t+s)]\}_{,j}\\
&+\rho S^0[\theta(t+s)-\theta(t-s)]+P^0[C(t+s)-C(t-s)]\\
&+\left[\frac{1}{T_0}\kappa_{ij}\theta(t-s)\int_0^{t+s}\theta_{,i}(z)\,dz-\frac{1}{T_0}\kappa_{ij}\theta(t+s)\int_0^{t-s}\theta_{,i}(z)\,dz\right]_{,j}\\
&+\left[d_{ij}\left(P(t+s)\int_0^{t-s}P_{,i}(z)\,dz-P(t-s)\int_0^{t+s}P_{,i}(z)\,dz\right)\right]_{,j}\\
&+\frac{1}{T_0}\kappa_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}\theta_{,i}(z)\,dz\right)\left(\int_0^{t-s}\theta_{,j}(z)\,dz\right)\right]\\
&+d_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}P_{,i}(z)\,dz\right)\left(\int_0^{t-s}P_{,j}(z)\,dz\right)\right].\end{aligned} \tag{56}$$

We now substitute the relation (56) into (42) and we use the divergence theorem and the boundary conditions (9) in order to obtain

$$\begin{aligned}&2\int_B[\rho u_i(t)\dot{u}_i(t)+\rho J_{ij}\phi_i(t)\dot{\phi}_j(t)]\,dV\\
&=\int_B\varrho[u_i^0\dot{u}_i(2t)+u_i^1u_i(2t)]\,dV+\int_B\rho J_{ij}[\phi_i^0\dot{\phi}_j(2t)+\phi_i^1\phi_j(2t)]\,dV\\
&+\int_0^t\int_B\{\rho S^0[\theta(t+s)-\theta(t-s)]+P^0[C(t+s)-C(t-s)]\}\,dV\,ds\\
&+\int_0^t\int_B\frac{1}{T_0}\kappa_{ij}\frac{d}{ds}\left[\left(\int_0^{t+s}\theta_{,i}(z)\,dz\right)\left(\int_0^{t-s}\theta_{,j}(z)\,dz\right)\right]dV\,ds\end{aligned}$$

$$+\int_0^t\int_B d_{ij}\frac{d}{ds}\bigg[\bigg(\int_0^{t+s}P_{,i}(z)\,dz\bigg)\bigg(\int_0^{t-s}P_{,j}(z)\,dz\bigg)\bigg]dV\,ds. \tag{57}$$

It is not difficult to observe that

$$\int_0^t\frac{d}{ds}\bigg[\bigg(\int_0^{t+s}\theta_{,i}(z)\,dz\bigg)\bigg(\int_0^{t-s}\theta_{,j}(z)\,dz\bigg)\bigg]ds=-\bigg(\int_0^t\theta_{,i}(z)\,dz\bigg)\bigg(\int_0^t\theta_{,j}(z)\,dz\bigg),$$

and, analogously,

$$\int_0^t\frac{d}{ds}\bigg[\bigg(\int_0^{t+s}P_{,i}(z)\,dz\bigg)\bigg(\int_0^{t-s}P_{,j}(z)\,dz\bigg)\bigg]ds=-\bigg(\int_0^t P_{,i}(z)\,dz\bigg)\bigg(\int_0^t P_{,j}(z)\,dz\bigg).$$

Finally, by using the last two relations in (57), one obtains the desired equality (38) so that Theorem 3 is proved. □

5. Equipartition of energy

In this section we shall use the identities (21), (27) and (38) such that by using the hypotheses made in Section 2 we establish the asymptotic partition of total energy.

Let us introduce the Cesàro means of all energies contained in the identity (21). So, we have Cesàro means of kinetic energy, strain energy, thermal energy and energy of dissipation, respectively:

$$\begin{aligned}
\mathcal{K}_C(t)&\equiv\frac{1}{2t}\int_0^t\int_B[\rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)]\,dV\,ds,\\
\mathcal{S}_C(t)&\equiv\frac{1}{2t}\int_0^t\int_B[c_{ijkl}\varepsilon_{ij}(t)\varepsilon_{kl}(t)+2p_{ijkl}(t)\varepsilon_{ij}(t)\phi_{kl}+d_{ijkl}\phi_{ij}(t)\phi_{kl}]\,dV\,ds,\\
\mathcal{T}_C(t)&\equiv\frac{1}{2t}\int_0^t\int_B\frac{\rho c_E}{T_0}\theta^2(s)\,dV\,ds,\\
\Gamma_C(t)&\equiv\frac{1}{t}\int_0^t\int_0^s\int_B[\frac{1}{T_0}\kappa_{ij}\theta_{,i}(\xi)\theta_{,j}(\xi)+C(\xi)\dot{P}(\xi)]\,dV\,d\xi\,ds.
\end{aligned} \tag{58}$$

In the following theorem we state and prove the main result of our study.

Theorem 4. *Consider* $((u_i,\phi_i),\theta,C)$ *a solution of the mixed initial boundary value problem defined by equations* (10), *the boundary conditions* (9) *and the initial conditions* (11). *We assume that the hypotheses from* Section 2 *are satisfied. Then, for all initial data*

$$(u_i^0,\phi_i^0)\in\mathbf{W}_1(B),\quad (u_i^1,\phi_i^1)\in\mathbf{W}_0(B),\quad (\theta^0,C^0)\in W_1(B)\times W_1(B),\quad \theta^1\in W_0(B),$$

we have the following relations:

(i) *If* $\operatorname{meas}(\partial B_3)\neq 0$, *then*

$$\lim_{t\to\infty}\mathcal{T}_C(t)=0. \tag{59}$$

(ii) *If* $\operatorname{meas}(\partial B_1)\neq 0$, $\operatorname{meas}(\partial B_2)\neq 0$ *and* $\operatorname{meas}(\partial B_4)\neq 0$, *then*

$$\lim_{t\to\infty}\mathcal{K}_C(t)=\lim_{t\to\infty}\mathcal{S}_C(t), \tag{60}$$

$$\lim_{t\to\infty}\Gamma_C(t)=\mathcal{E}(0)-2\lim_{t\to\infty}\mathcal{K}_C(t)=\mathcal{E}(0)-2\lim_{t\to\infty}\mathcal{S}_C(t). \tag{61}$$

(iii) *If* $\operatorname{meas}(\partial B_1) = 0$ *or* $\operatorname{meas}(\partial B_2) = 0$ *or* $\operatorname{meas}(\partial B_3) = 0$ *or* $\operatorname{meas}(\partial B_4) = 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{S}_C(t) + \frac{1}{2}\int_B [\rho \dot{u}_i^* \dot{u}_i^* + \rho J_{ij} \dot{\phi}_i^* \dot{\phi}_j^*]\, dV, \tag{62}$$

$$\begin{aligned} \lim_{t\to\infty} \Gamma_C(t) &= \mathcal{E}(0) - 2\lim_{t\to\infty} \mathcal{K}_C(t) + \frac{1}{2}\int_B [\rho \dot{u}_i^* \dot{u}_i^* + \rho J_{ij} \dot{\phi}_i^* \dot{\phi}_j^*]\, dV \\ &= \mathcal{E}(0) - 2\lim_{t\to\infty} \mathcal{S}_C(t) - \frac{1}{2}\int_B [\rho \dot{u}_i^* \dot{u}_i^* + \rho J_{ij} \dot{\phi}_i^* \dot{\phi}_j^*]\, dV. \end{aligned} \tag{63}$$

Proof. (i) Suppose that $\operatorname{meas}(\partial B_3) \neq 0$. It is easy to prove that $\theta \in \widehat{W}_1(B)$. Therefore, we can apply the Poincaré inequality (17) so that from the identity (21) one obtains

$$\int_0^t \int_B \frac{\rho c_E}{T_0} \theta^2(s)\, dV\, ds \le \frac{1}{m_2} \int_0^t \int_B \kappa_{ij} \theta_{,i}(s) \theta_{,j}(s)\, dV\, ds \le \frac{1}{m_2} \mathcal{E}(0). \tag{64}$$

From relation (64), taking into account (58), we obtain the conclusion (59).

(ii) We first use the energy conservation law (21) and the hypotheses of Section 2 in order to obtain the following estimates:

$$\int_B \frac{\rho c_E}{T_0} \theta^2(t)\, dV \le \mathcal{E}(0), \qquad t \in [0, \infty), \tag{65}$$

$$\int_B [\rho \dot{u}_i(t) \dot{u}_i(t) + \rho J_{ij} \dot{\phi}_i(t) \dot{\phi}_j(t)]\, dV \le 2\mathcal{E}(0), \qquad t \in [0, \infty), \tag{66}$$

$$\int_0^t \int_B \left[\frac{\rho c_E}{T_0} \kappa_{ij} \theta_{,i}(s) \theta_{,j}(s) + C(s) \dot{P}(s) \right] dV\, ds \le \mathcal{E}(0), \quad t \in [0, \infty). \tag{67}$$

On the basis of identities (27) and (38) we will find some relationships between the types of energy. So, from (27) we find

$$\begin{aligned} &\frac{1}{2t} \int_0^t \int_B [\rho \dot{u}_i(s) \dot{u}_i(s) + \rho J_{ij} \dot{\phi}_i(s) \dot{\phi}_j(s)]\, dV\, ds \\ &\quad - \frac{1}{2t} \int_0^t \int_B [c_{ijkl} \varepsilon_{ij}(s) \varepsilon_{kl}(s) + 2 p_{ijkl} \phi_{ij}(s) \varepsilon_{kl}(s) + d_{ijkl} \phi_{ij}(s) \phi_{kl}(s)]\, dV\, ds \\ &= \frac{1}{2t} \int_0^t \int_B \left[\frac{\rho c_E}{T_0} \theta^2(s) + 2\varpi C(s) \theta(s) - \varrho C^2(s) \right] dV\, ds - \frac{1}{2t} \int_B [\rho u_i^0 u_i^1 + \rho J_{ij} \phi_i^0 \phi_i^1]\, dV \\ &\quad - \frac{1}{4t} \int_0^t \int_B [\rho S^0 \theta(s) - P^0 C(s)]\, dV\, ds + \frac{1}{2t} \int_B [\rho u_i(t) \dot{u}_i(t) + \rho J_{ij} \phi_i(t) \dot{\phi}_j(t)]\, dV \\ &\quad + \frac{1}{4t} \int_B \frac{1}{T_0} \kappa_{ij} \left(\int_0^t \theta_{,i}(\xi) d\xi \right) \left(\int_0^t \theta_{,j}(\xi) d\xi \right) dV \\ &\quad + \frac{1}{4t} \int_B d_{ij} \left(\int_0^t P_{,i}(\xi) d\xi \right) \left(\int_0^t P_{,j}(\xi) d\xi \right) dV. \end{aligned} \tag{68}$$

By using (38), from (68) one obtains

$$\begin{aligned}
&\frac{1}{2t}\int_0^t\int_B[\rho\dot{u}_i(s)\dot{u}_i(s)+\rho J_{ij}\dot{\phi}_i(s)\dot{\phi}_j(s)]\,dV\,ds\\
&\qquad-\frac{1}{2t}\int_0^t\int_B[c_{ijkl}\varepsilon_{ij}(s)\varepsilon_{kl}(s)+2p_{ijkl}\phi_{ij}(s)\varepsilon_{kl}(s)+d_{ijkl}\phi_{ij}(s)\phi_{kl}(s)]\,dV\,ds\\
&=\frac{1}{2t}\int_0^t\int_B\left[\frac{\rho c_E}{T_0}\theta^2(s)+2\varpi C(s)\theta(s)-\varrho C^2(s)\right]dV\,ds\\
&\qquad-\frac{1}{2t}\int_B[\rho u_i^0u_i^1+\rho J_{ij}\phi_i^0\phi_i^1]\,dV-\frac{1}{4t}\int_0^t\int_B[\rho S^0\theta(s)-P^0C(s)]\,dV\,ds\\
&\qquad+\frac{1}{4t}\int_B\{\varrho[u_i^0\dot{u}_i(2t)+u_i^1u_i(2t)]+\rho J_{ij}[\phi_i^0\dot{\phi}_j(2t)+\phi_i^1\phi_j(2t)]\}\,dV\\
&\qquad+\frac{1}{4t}\int_0^t\int_B\{\rho S^0[\theta(t+s)-\theta(t-s)]+P^0[C(t+s)-C(t-s)]\}\,dV\,ds.
\end{aligned}\tag{69}$$

Taking into account the notations (58) and using the initial conditions (11), from the identity (69) we deduce

$$\begin{aligned}
&\mathcal{K}_C(t)-\mathcal{S}_C(t)\\
&\quad=-\frac{1}{2t}\int_B[\rho u_i^0u_i^1+\rho J_{ij}\phi_i^0\phi_i^1]\,dV\\
&\qquad+\frac{1}{2t}\int_B[\rho u_i^0\dot{u}_i(2t)+\rho J_{ij}\phi_i^0\dot{\phi}_j(2t)]\,dV+\frac{1}{2t}\int_B[\rho u_i^1u_i(2t)+\rho J_{ij}\phi_i^1\phi_j(2t)]\,dV\\
&\qquad+\frac{1}{4t}\int_0^t\int_B[2\varpi C(s)\theta(s)-\varrho C(s)]\,dV\,ds-\frac{1}{4t}\int_0^t\int_B[\rho S^0\theta(s)-P^0C(s)]\,dV\,ds\\
&\qquad+\frac{1}{4t}\int_0^t\int_B\{\rho S^0[\theta(t+s)-\theta(t-s)]+P^0[C(t+s)-C(t-s)]\}\,dV\,ds+\mathcal{T}_C(t).
\end{aligned}\tag{70}$$

Now we will use the Schwarz and Cauchy inequalities on the right-hand side of (70). Then, by using the relations (64)–(67) we get

$$\begin{aligned}
\left|-\frac{1}{2t}\int_B[\rho u_i^0u_i^1+\rho J_{ij}\phi_i^0\phi_j^1]\,dV\right|&\le\frac{1}{4t}\int_B[\rho(u_i^0u_i^0+u_i^1u_i^1)+\rho J_{ij}(\phi_i^0\phi_j^0+\phi_i^1\phi_j^1)]\,dV;\\
\left|\frac{1}{4t}\int_B[\rho u_i^0\dot{u}_i(2t)+\rho J_{ij}\phi_i^0\dot{\phi}_j(2t)]\,dV\right|&\le\frac{1}{8t}\int_B[\rho u_i^0u_i^0+\rho J_{ij}\phi_i^0\phi_j^0]\,dV+\frac{1}{4t}\,\mathscr{E}(0).
\end{aligned}\tag{71}$$

Since $(u_i,\phi_i)\in\widehat{\boldsymbol{W}}_1(B)$, and $P\in\widehat{W}_1(B)$ by using the conditions (7), the Korn's inequality (16), the identity (21) and the inequalities (8) one obtains

$$\begin{aligned}
&\int_B[\rho u_i(s)u_i(s)+\rho J_{ij}\phi_i(s)\phi_j(s)]\,dV\\
&\le\frac{\rho}{m_1}\int_B[c_{ijkl}\varepsilon_{ij}(s)\varepsilon_{kl}(s)+2p_{ijkl}\phi_{ij}(s)\varepsilon_{kl}(s)+d_{ijkl}\phi_{ij}(s)\phi_{kl}(s)]\,dV\le\frac{2\rho}{m_1}\,\mathscr{E}(0),\qquad s\in[0,\infty).
\end{aligned}\tag{72}$$

Thus, by using (72) we are led to

$$\left|\frac{1}{4t}\int_B[\rho u_i^1u_i(2t)+\rho J_{ij}\phi_i^1\phi_j(2t)]\,dV\right|\le\frac{1}{8t}\int_B[\rho u_i^1u_i^1+\rho J_{ij}\phi_i^1\phi_j^1]\,dV+\frac{\rho}{4tm_1}\,\mathscr{E}(0).\tag{73}$$

Taking into account the estimates (71) and (73) and the relation (59), we pass to the limit in (70) as $t \to \infty$ and conclude that the relation (60) holds.

Also, it is not difficult to observe that the relation (61) is obtained from (21) by taking the Cesàro mean and by using the relations (59) and (60).

(iii) We use the decomposition (18) from Section 3, the relation (12), (13) and the fact that $(u_i, \phi_i) \in \widehat{W}_1(B)$ and $P \in \widehat{W}_1(B)$ in order to obtain the identity

$$\frac{1}{4t}\int_B [\rho u_i^0 \dot{u}_i(2t) + \rho J_{ij}\phi_i^0\dot{\phi}_j(2t)]\,dV = \frac{1}{4t}\int_B [\rho u_i^* \dot{u}_i^* + \rho J_{ij}\phi_i^*\dot{\phi}_j^*]\,dV + \frac{1}{2}\int_B [\rho \dot{u}_i^* \dot{u}_i^* + \rho J_{ij}\dot{\phi}_i^*\dot{\phi}_j^*]\,dV + \frac{1}{4t}\int_B [\rho U_i^0 v_i(2t) + \rho J_{ij}\Phi_i^0\psi_j(2t)]\,dV. \quad (74)$$

Also, since $(v_i, \psi_i) \in \widehat{W}_1(B)$, Korn's inequality (16) leads to the relation

$$\begin{aligned}\frac{1}{4t}\int_B &[\rho v_i(s)v_i(s) + \rho J_{ij}\psi_i(s)\psi_j(s)]\,dV \\ &\le \frac{\rho}{m_1}\int_B [c_{ijkl}\bar{\varepsilon}_{ij}(s)\bar{\varepsilon}_{kl}(s) + 2p_{ijkl}\bar{\phi}_{ij}(s)\bar{\varepsilon}_{kl}(s) + d_{ijkl}\bar{\phi}_{ij}(s)\bar{\phi}_{mn}(s)]\,dV \\ &= \frac{\rho}{m_1}\int_B [c_{ijkl}\varepsilon_{ij}(s)\varepsilon_{kl}(s) + 2p_{ijkl}\phi_{ij}(s)\varepsilon_{kl}(s) \\ &\qquad + d_{ijkl}\phi_{ij}(s)\phi_{mn}(s)]\,dV \le \frac{2\rho}{m_1}\mathscr{E}(0), \quad s \in [0, \infty), \end{aligned} \quad (75)$$

where $\bar{\varepsilon}_{ji} = v_{i,j} - \epsilon_{kji}\psi_k$, $\bar{\phi}_{ij} = \psi_{j,i}$.

Passing to the limit in (70) as $t \to \infty$ and taking into account the relations (71), (74) and (75) one obtains the conclusion (63). Finally, the relation (63) is obtained on the basis of (21) by taking the Cesàro mean and by using the relations (59), (62) and (71). Thus, the proof of Theorem 4 is complete. □

6. Conclusion

As a concluding remark, we must outline that the relations (60) and (62), restricted to the class of initial data for which $\dot{u}_i^* = \dot{\phi}_i^* = 0$, prove the asymptotic equipartition in mean of the kinetic and strain energies. The presence of other components of total energy does not influence this behavior.

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MARIN MARIN: m.marin@unitbv.ro
Department of Mathematics and Computer Sciences, Transilvania University of Braşov, Strada Universităţii 1, 500068 Braşov, Romania

SAMY REFAHY MAHMOUD: srhassan@kau.edu.sa
Department of Mathematics, Faculty of Science, King Abdul Azis University, 21589 Jeddah, Saudi Arabia

and

Department of Mathematics, Sohag University, Nasr City Eastern avenue, Sohag, 82524, Egypt

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