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The paper presents an application of the method of fundamental solutions to model auxetic materials. The utility of the method has been illustrated by solution of basic elastostatic problem of searching displacements distribution for materials with different auxetic properties. Some remarks on most often difficulties occurring during computational process are also included. The presented numerical examples show that the method of fundamental solutions could be effective numerical tool for researching various properties of auxetic materials.

1. Introduction

Solving many modern engineering problems it requires to use more and more advanced computational methods. Due to high progress in developing those methods in recent years, it was possible to take into account many important and complex problems especially in the field of mechanics, bioengineering or material science. In that domain, where the real problems could be described by differential equations, the main numerical tool to provide computer simulation is the finite element method (FEM) [Zienkiewicz and Taylor 2000]. That method is also most commonly used for describing materials of new properties and their nontrivial behaviors.

In this paper an alternative numerical method has been used to solve elastostatic problems for materials with negative Poisson's ratio as isotropic auxetic materials. The presented method, called Method of Fundamental Solutions (FMS), is a numerical method for solving boundary problems of differential equations, which belongs to the group of meshless methods. That method has been developed by Kupradze and Alexidze [Kupradze 1967; Kupradze and Aleksidze 1964]. Till now, MFS was successfully applied to solve mechanical problems [Bogomolny 1985; Fairweather et al. 2003; Goldberg 1995; Kita 2003; Mathon and Johnston 1977; Poullikkas et al. 2001], and also the elastic ones [Karageorghis and Fairweather 2000; de Medeiros et al. 2004; Poullikkas 1998; Poullikkas et al. 2002]. The main advantage of MFS is a possibility to obtain (with the help of numerical approximation) analytical solution of the problem in the entire considered region. One can obtain the solution by solving just one system of linear equations. That fact simplifies the numerical procedures and calculations and gives a possibility to control the approximation errors on the linear algebra level.

Some papers show that even in three-dimensional cases, the method could be effective and really fast [Maruszewski et al. 2014]. Despite advantages of presented method it is still not commonly used

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in material science field. In a recent authors' publication they proved, that it could be useful tool for searching and modeling of modern materials [Walczak et al. 2014].

In this paper authors focused on modeling the group of materials of unusual mechanical properties, which Poisson's ratio (PR) [Landau and Lifshitz 1986; Alderson et al. 2014] is negative. For common materials PR is positive and if PR is negative, the body shrinks/expands transversally when compressed/stretched. Such kind of materials we call auxetics and they were taken under considerations already in eighties last century [Almgren 1985; Lakes 1987], although their properties [Gilat and Aboudi 2013; Lim 2004; Yang et al. 2013] and structures [Gaspar et al. 2011; Wojciechowski 2003; Chen et al. 2009] are still under intensive studies. In this context, developing of new numerical methods to simulate them and to investigate various models seem to be important and are in particular interest.

The aim of the present paper is to show utility of described method as an excellent tool to investigate uncommon behavior of materials with auxetic properties. It is continuation of authors' study on developing of the MFS [Walczak et al. 2014] and to the best of their knowledge, the MFS method is used in this paper for the first time in such investigations.

The structure of the paper is as follows. Mathematical model with governing equations for the studied model is presented in Section 2. In Section 3 the MFS is briefly sketched, and the fundamental solutions for the basic differential equations of the given problems are presented. Computer simulations are discussed in Section 4, where the utility of the method is illustrated by solving the basic elastostatic problem of searching displacement distribution in materials with different auxetic properties. Moreover, a comparison to existing investigations and method used to consider the above problem has been also presented. Summary and conclusions are presented in Section 5.

2. Mathematical model of the problem

In the paper a three-dimensional problem of equilibrium of a solid has been solved numerically using the method of fundamental solutions (MFS). The body is assumed as homogeneous, isotropic and linearly elastic one. The last assumption, in practice, is satisfied only when small deformations are taken into account. Moreover, we assume that the material properties are constant (i.e., they do not depend on time for instance). The mechanical response of such a solid due to a given loading is unambiguously determined by two material constants. In continuum mechanics the Lamé constants λ and μ are usually used. In literature devoted to the various engineering applications in the role of the two necessary material constants appear rather Young's modulus E and Poisson's ratio (PR) ν. The relationship between those material constants are as follows so:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{1}$$

The modulus of elasticity E and the constant μ (i.e., the shear modulus) are always positive. However Poisson's ratio and the Lamé constant λ may be negative, although for most solids they are also positive. In the branch of physics dedicated to auxetic materials, which develops recently rapidly, more popular is associating special features of auxetic solids with the negative values of PR.

Let us denote by Ω the region in space occupied by the body. It is bounded by a surface $\partial\Omega$ which should be piecewise smooth. According to all previously mentioned assumptions, the equilibrium of the considered solid is described by the Cauchy–Navier equations. In the absence of external body forces, they take the homogeneous form [Poullikkas 1998; Poullikkas et al. 2002]:

$$\mu\nabla\cdot(\nabla\boldsymbol{u})+(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})=\mathbf{0}, \tag{2}$$

where $\boldsymbol{u}$ is the displacement vector at the point $\boldsymbol{x}=(x_1,x_2,x_3)$, and $\boldsymbol{x}\in\Omega$.

On the surface $\partial\Omega$ we postulate the boundary conditions of mixed type. Let $\partial\Omega_{\mathrm{I}}$ be a part of $\partial\Omega$, on which the Dirichlet boundary conditions are given. Therefore, we can write:

$$\boldsymbol{u}(\boldsymbol{x}^B)=\boldsymbol{f}(\boldsymbol{x}^B), \tag{3}$$

where $\boldsymbol{x}_B\in\partial\Omega_{\mathrm{I}}$, and $\boldsymbol{f}(\boldsymbol{x}^B)$ is a known vector function.

On the other part of $\partial\Omega$, which we denote by $\partial\Omega_{\mathrm{II}}$, the Neumann boundary conditions are postulated. We write them as follows:

$$\boldsymbol{t}(\boldsymbol{x}^B)=\boldsymbol{g}(\boldsymbol{x}^B), \tag{4}$$

where $\boldsymbol{x}^B\in\partial\Omega_{\mathrm{II}}$, $\boldsymbol{t}$ is the stress vector, and $\boldsymbol{g}(\boldsymbol{x}^B)$ is a known vector function describing external traction. It should be noted, that due to existence and uniqueness of the solution of the formulated boundary problem the following relation is required:

$$\partial\Omega_{\mathrm{I}}\cup\partial\Omega_{\mathrm{II}}=\partial\Omega,\qquad \partial\Omega_{\mathrm{I}}\cap\partial\Omega_{\mathrm{II}}=\varnothing. \tag{5}$$

3. Method of fundamental solutions

The equation of equilibrium (2) together with the boundary conditions (3) and (4) form the boundary value problem. The fundamental solutions of this system are in the form

$$G_{ij}(\boldsymbol{x},\boldsymbol{z})=\frac{(3-4\nu)|\boldsymbol{x}-\boldsymbol{z}|^2\delta_{ij}+(x_i-z_i)(x_j-z_j)}{16\pi\mu(1-\nu)|\boldsymbol{x}-\boldsymbol{z}|^3}, \tag{6}$$

where $|\boldsymbol{x}-\boldsymbol{z}|$ is the distance between the points $\boldsymbol{x}$ and $\boldsymbol{z}$. One can observe that

$$G_{ij}(\boldsymbol{x},\boldsymbol{z})=G_{ji}(\boldsymbol{x},\boldsymbol{z}). \tag{7}$$

The solutions given by (6) satisfy the differential equation (2) at each point of the space, except for one point at which it is not defined. Each fundamental solution represents the displacements at the point $\boldsymbol{x}$, that are caused by an unit force acting at the source point $\boldsymbol{z}$, namely the displacement at the point $\boldsymbol{x}$ in the direction of the axis x_j which has been caused by the i-th component of the force is equal to the fundamental solution given by (6) [Poullikkas 1998].

Let us consider N source points $\boldsymbol{z}_k$, $k=1,2,\ldots,N$ placed outside the considered body. Then, one can write the solution of the boundary value problem as the linear combination of the fundamental solutions

$$u_i(\boldsymbol{x})=\sum_{k=1}^{N}a_{kj}G_{ij}(\boldsymbol{x},\boldsymbol{z}_k),\qquad i=1,2,3, \tag{8}$$

where $\boldsymbol{x}\in\Omega\cup\partial\Omega$, $\boldsymbol{z}\in E^3-(\Omega\cup\partial\Omega)$, a_{kj} are unknown coefficients. One can easily see, that since the singularities of solutions are placed outside the body, any function of the form (8) satisfies the differential equation (2) in the region Ω. However, it should be noticed, that the boundary conditions (3) and (4) may be satisfied only approximately. To do that, one should define the set of collocation points placed on the surface $\partial\Omega$, which satisfy the boundary conditions. Let us choose M collocation points. Introducing the linear combinations (8) into the boundary conditions we obtain the system of linear equations with $3N$ unknown coefficients a_{kj}:

$$\boldsymbol{B} \cdot \boldsymbol{A} = \boldsymbol{F}, \tag{9}$$

where $\boldsymbol{A}$ is the $3N$-dimensional vector of unknown coefficients a_{kj}, $\boldsymbol{F}$ is the $3M$-dimensional vector containing the values of known functions f_i at each collocation point. $3M \times 3N$-dimensional matrix $\boldsymbol{B}$ consists of the terms which, after a suitable ordering each of the boundary conditions, are multipliers of the unknown coefficients a_{kj}.

Important advantage of the method of fundamental solutions is that the approximate solutions are differentiable. The accuracy of obtained solutions strongly depends on geometry of the problem and location of source points and collocation points. It is well known that in the case of elliptic differential equations, the biggest errors in approximate solutions of the given boundary value problem occur on the boundary of the considered region [Poullikkas et al. 2001; Karageorghis and Fairweather 2000; de Medeiros et al. 2004; Poullikkas 1998; Poullikkas et al. 2002]. It implies that to find the most accurate solution one should place the sets of the collocation points and the source points, respectively to minimize those errors. Also number of points in those sets is the important method parameter. To find some interesting remarks see [Poullikkas 1998; Poullikkas et al. 2002; Maruszewski et al. 2014; Walczak et al. 2014].

4. Numerical simulations

Let us consider a boundary value problem with the mixed boundary conditions. The solid in the shape of a cube with length of the edge 0.1 m, which is shown in Figure 1, is fixed on its one wall $x_1 = 0$. On the opposite wall there is applied the pressure $p = 100\,\text{MPa}$. The remaining four walls are free of loadings. It means that considered body is stretched in the direction of x_1 axis. Therefore the boundary value problem can be written as

$$\mu u_{i,kk} + (\lambda + \mu) u_{k,ik} = 0, \quad i = 1, 2, 3, \quad \boldsymbol{x} \in \Omega, \tag{10}$$

$$\boldsymbol{u}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega_1, \tag{11}$$

$$\boldsymbol{t}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega_3 \cup \Omega_4 \cup \Omega_5 \cup \Omega_6, \tag{12}$$

$$\boldsymbol{t}(\boldsymbol{x}) = p\boldsymbol{e}_1, \quad \boldsymbol{x} \in \Omega_2, \tag{13}$$

where $\boldsymbol{e}_1$ denotes the unit vector of the x_1 axis and $\Omega = (0, 0.1) \times (0, 0.1) \times (0, 0.1)$,

$$\begin{aligned}
\partial\Omega_1 &= \{\boldsymbol{x} : x_1 = 0,\ 0 \le x_2 \le 0.1, 0 \le x_3 \le 0.1\},\\
\partial\Omega_2 &= \{\boldsymbol{x} : x_1 = 0.1,\ 0 \le x_2 \le 0.1, 0 \le x_3 \le 0.1\},\\
\partial\Omega_3 &= \{\boldsymbol{x} : x_2 = 0,\ 0 \le x_1 \le 0.1, 0 \le x_3 \le 0.1\},\\
\partial\Omega_4 &= \{\boldsymbol{x} : x_2 = 0.1,\ 0 \le x_1 \le 0.1, 0 \le x_3 \le 0.1\},\\
\partial\Omega_5 &= \{\boldsymbol{x} : x_3 = 0,\ 0 \le x_1 \le 0.1, 0 \le x_2 \le 0.1\},\\
\partial\Omega_6 &= \{\boldsymbol{x} : x_3 = 0.1,\ 0 \le x_1 \le 0.1, 0 \le x_2 \le 0.1\}.
\end{aligned}$$

The subject of the numerical experiments is a set of the cubes, which are made of auxetic materials of which Poisson's ratio is from range $\nu = \langle -0.9, -0.3 \rangle$. The original software written in FORTRAN 95

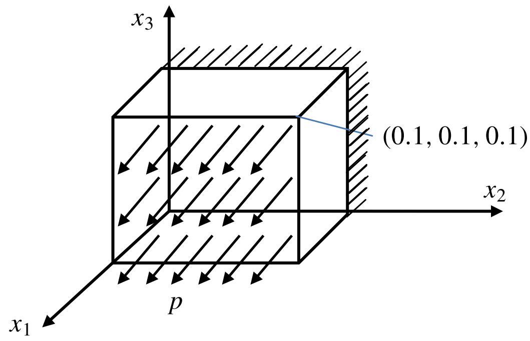

Figure 1. Loads and boundary conditions of considered body.

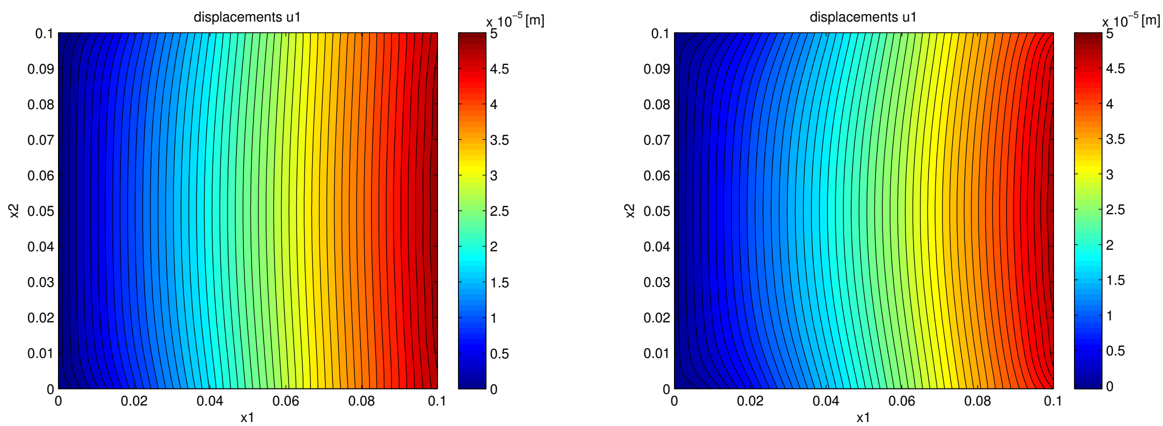

Figure 2. The component u_1 of the displacement vector on the $x_3 = 0.05$ cross-section for the material with $\nu = -0.3$ (left), $\nu = -0.5$ (right).

with implemented MFS was used to simulate behavior of considered group of materials. For each simulation exactly the same number of nodes (collocation points and source points) has been used. One should also notice, that number of applied source points was equal to number of collocation points and was equal to 864 in each considered case. That approach leads always to square main matrix of the system, and authors' experiences show that for mixed boundary conditions it gives the most accurate results.

Often considered problem where MFS is applied is a shape of the surface where the source points are placed. Generally one can find two approaches: one when those points are located on the sphere with the center placed in the middle of considered body having respectively big radius, and the other one when the source points are placed on the surface which shape and distance is very close to that of the considered body. Of course, it is placed outside of the body. In this paper the second way has been applied to all computer simulations (see [Maruszewski et al. 2014]).

In Figure 2 the component u_1 of the displacement vector on the plane $x_3 = 0.05$ is shown for two auxetic materials with $\nu = -0.3$ (left) and $\nu = -0.5$ (right). The observed symmetrical distribution of displacements is consistent with the symmetry of the boundary conditions as well as the load. Fixing only one wall results in the largest displacements close to the opposite one.

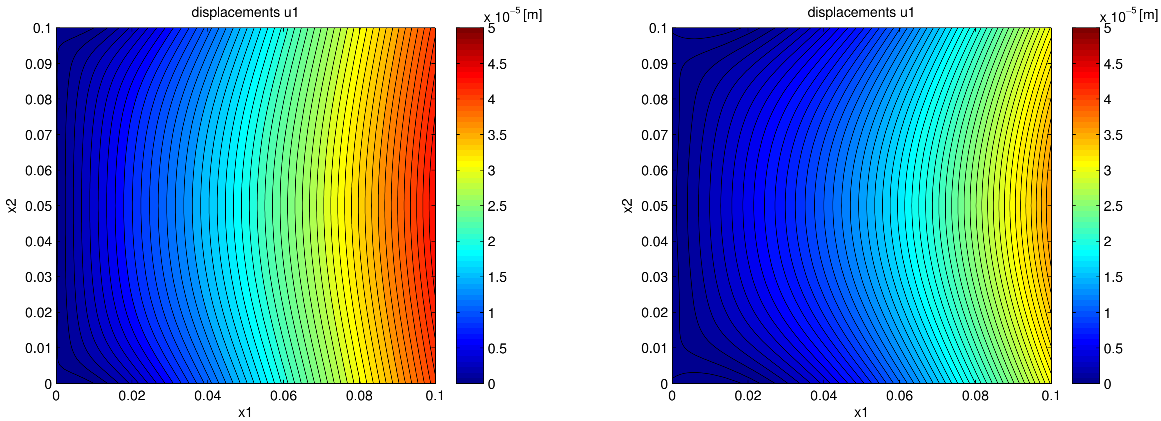

Figure 3. The component u_1 of the displacement vector on the $x_3 = 0.05$ cross-section for the material with $\nu = -0.7$ (left), $\nu = -0.9$ (right).

One can see that these two maps of displacements are similar. The values of the displacement u_1 along the left edge of each half of Figure 2 are zero, which is a consequence of fixing the wall. Positive values of displacements u_1 on the right hand side at both graphs are observed. Stretching state in the direction of x_1 axis and only little difference between values of displacement can also be observed. The situation changes, when we take into account the lower value of Poisson's ratio. In Figure 3 the component u_1 of the displacement vector on the plane $x_3 = 0.05$ is shown for another two auxetic materials with $\nu = -0.7$ (left) and $\nu = -0.9$ (right). One can observe, that there are some irregularities in those distributions occurring near the left corners (close to the fixed wall). Especially, that is visible in the Figure 3 (right), where the regions with different values of displacement component u_1 could be seen. That is not numerical effect, but it corresponds to auxetic nature of the considered body. In classical material this effect doesn't occur.

From the other hand, comparing the map included in Figure 2 to the map in Figure 3 we can observe, that maximal values of the displacement u_1 decrease with decreasing of PR value for considered materials. It should also be noticed, that the lines of the constant displacements are more curved with decreasing of PR value. Analyzing presented maps, one can see, that those lines are less perpendicular to x_1 axis in the case of material with the $\nu = -0.9$ than in the other cases. To visualize in better way the situation, the additional maps were drawn for all cases. In Figure 4 the component u_1 of the displacement vector is shown for materials with $\nu = -0.3$ (left), $\nu = -0.5$ (right), but this time cross-section of the cube was made by the plane $x_1 = x_3$.

Left edge of this map represents the edge of the cube, that is placed along x_2 axis and the right edge is placed along the straight $x_2 = x_3 = 0.1$. One can see that these two maps of displacements are similar to each other. The effect of bending of the lines of constant displacements is also visible what was mentioned earlier, when material has lower value of PR. As can be also seen, the values of displacements u_1 on the fixed walls, are close to zero. Stretching in the direction of x_1 axis, gives positive values of displacements u_1 in whole region of the both graphs. In Figure 5 the component u_1 of the displacement vector is presented for materials with $\nu - 0.7$ (left), $\nu - 0.9$ (right), in cross-section of the cube, that was made by the plane $x_1 = x_3$.

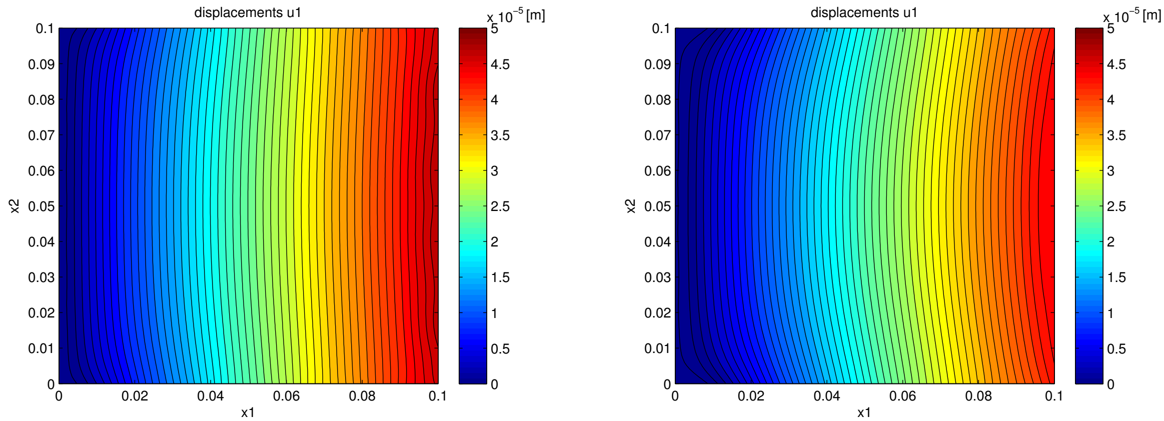

Figure 4. The component u_1 of the displacement vector on the $x_1 = x_3$ cross-section for the material with $\nu = -0.3$ (left), $\nu = -0.5$ (right).

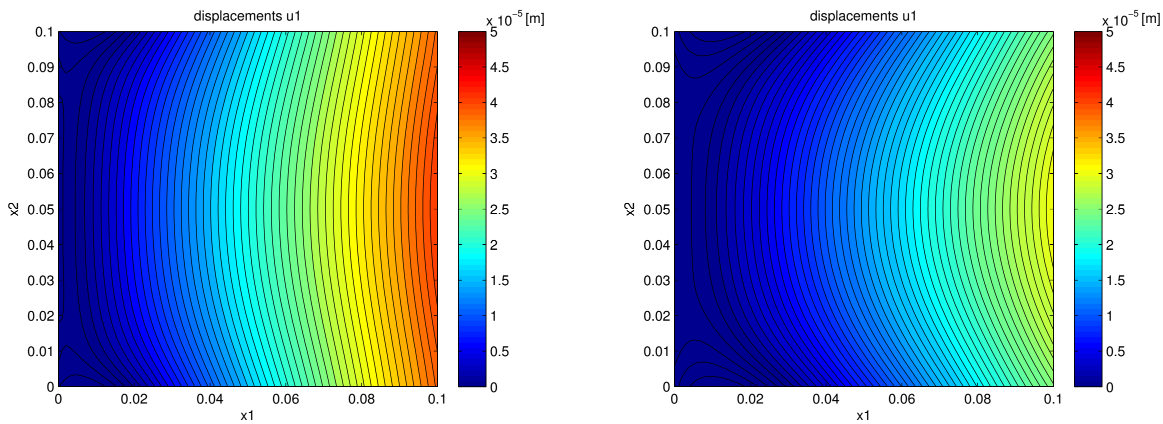

Figure 5. The component u_1 of the displacement vector on the $x_1 = x_3$ cross-section for the material with $\nu = -0.7$ (left), $\nu = -0.9$ (right).

One can see very close similarity between distributions presented in Figure 5 to distributions of displacements shown in Figure 3. For auxetic material with Poisson's ratio $\nu = -0.7$ and especially for material with $\nu = -0.9$, exactly the same irregularities occur near left boundary. Because cross-section used for creating maps presented in Figure 4 and Figure 5 is diagonal plane for considered cube, it could be seen that the strongest irregularities occur near the cube corners. In comparison to the auxetic materials the classical ones don't have irregularities in their corners, what one can observe in Figure 6 and Figure 7.

We can also see in Figure 6 and Figure 7 that the isolines representing displacements u_1 for both each of the classical materials are getting parallel with the growth of a distance from fixed walls. The auxetic materials, what is well visible, behave differently.

Generally we can conclude that some interesting effects in auxetics materials could be seen only for lower value of negative Poisson's ratio (close to -1).

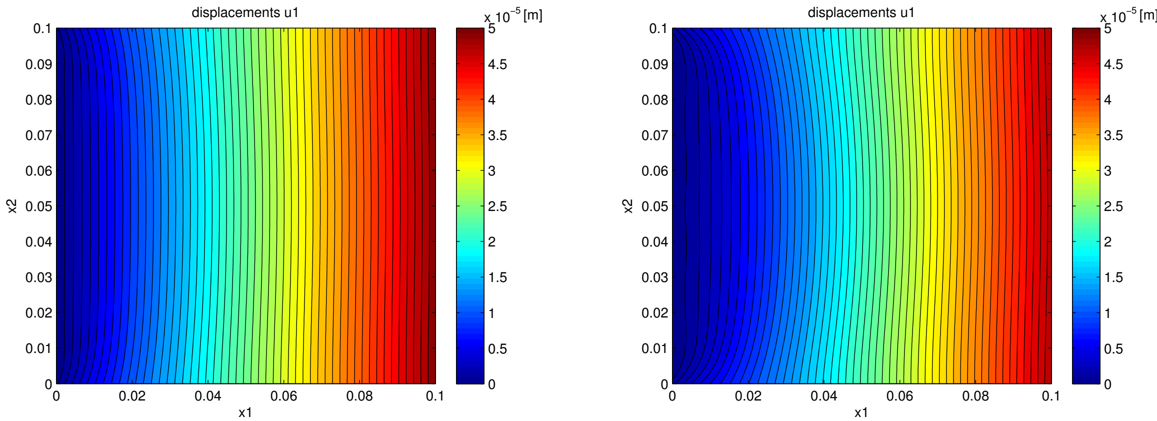

Figure 6. The component u_1 of the displacement vector on the $x_3 = 0.05$ cross-section for the material with $\nu = 0.2$ (left), $\nu = 0.4$ (right).

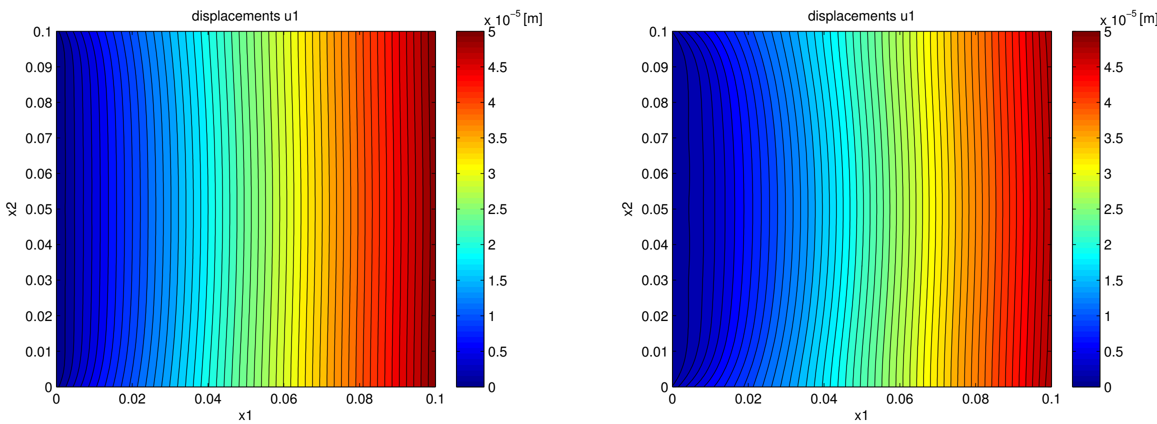

Figure 7. The component u_1 of the displacement vector on the $x_1 = x_3$ cross-section for the material with $\nu = 0.2$ (left), $\nu = 0.4$ (right).

An analysis of the results presented in Figures 2–7 allows us to draw additional important conclusions.

• The results obtained with the help of the fundamental solutions method applied to the basic equations of elasticity (11) of the auxetic continua are practically the same as those obtained by Lakes [1992] who has analyzed de Saint-Venant's principle applied to auxetic elastic materials. It proves that the both methods are equivalent each other. That equivalence results also from the fact that both models do not go beyond the irreversible thermodynamics of continua for normal and auxetic materials. That fact has been shown by Landau and Lifshitz [1986]. Considering the sufficient conditions for a minimum of the Helmholtz free energy in the state of thermodynamic equilibrium, they proved that the value of Poisson's ratio of isotropic materials belongs to the interval $(-1; 0, 5)$.

• From Figures 2–5 it can be seen that if the negative Poisson's ratio is lower, the values of the component $u_{1,1}$ of the displacement gradient are generally smaller.

• Comparing the displacement distribution in the auxetic material to that in the classical one (Figure 5 versus Figure 6 and Figure 7) we see that auxetic occurs faster stiff than normal material if Poisson's ratio decreases.

5. Conclusions

The presented method is effective and fast numerical tool for researching material behavior with different mechanical properties. The biggest advantage of presented method in comparison to FEM is that in the MFS there is no need to define classical mesh of considered body. Numerical procedures used for the mesh generation processes are usually time-consuming ones. In the meshless method one can define only the two sets of points that are not connected to each other. One may also treat it as two clouds of points where one of them is applied to define boundary conditions (collocation points) and the second one should be located just outside the considered body. What is also important (see Section 3) the obtained results have analytical character — of course they are obtained in numerical way, but after solution of just one linear system of equations (9), it is possible to find the displacements field in any point of considered region only using relation (8) and without any additional approximation. It implies the much more effective algorithm to providing numerical simulations especially in three-dimensional cases. Let us underline that all computer simulations have been done during a relative short time (few minutes).

The only problems occurring during the use of the method of fundamental solutions are connected with optimization of parameters like number of collocation points, number of source points or with the choice how source points should be located outside the considered body. It is nontrivial task and it is especially difficult, when one takes into account the boundary value problem with mixed boundary conditions (like in presented example in this paper). Those problems were considered in some publications [Poullikkas 1998; Poullikkas et al. 2002; Maruszewski et al. 2014; Walczak et al. 2014], where authors have tested MFS for such mechanical problems, for which the analytical solutions were known. However, one should notice that the MFS is still under development.

Appendix

To determine the solution of any boundary value problem defined by relations (2)–(4) with use of MFS one should write the system of linear algebraic equations (see (9)). We have assumed the displacement vector in each point $x \in \Omega \cup \partial\Omega$ as a linear combination defined by (8). To applied Dirichlet's boundary conditions we can rewrite them in the form

$$u_1(\boldsymbol{A}, \boldsymbol{Z}, \boldsymbol{x}) = \sum_{k=1}^{N} a_{k1} G_{11}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k2} G_{12}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k3} G_{13}(\boldsymbol{x}, \boldsymbol{z_k}),$$

$$u_2(\boldsymbol{A}, \boldsymbol{Z}, \boldsymbol{x}) = \sum_{k=1}^{N} a_{k1} G_{21}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k2} G_{22}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k3} G_{23}(\boldsymbol{x}, \boldsymbol{z_k}),$$

$$u_3(\boldsymbol{A}, \boldsymbol{Z}, \boldsymbol{x}) = \sum_{k=1}^{N} a_{k1} G_{31}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k2} G_{32}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k3} G_{33}(\boldsymbol{x}, \boldsymbol{z_k}),$$

where G are fundamental solutions (see (6)), the $3N$-dimensional vector $\boldsymbol{Z} = [\boldsymbol{z_1}, \boldsymbol{z_2}, \dots, \boldsymbol{z_N}]$ contains coordinates of N source points $\boldsymbol{z_j} = [x_1^j, x_2^j, x_3^j] \in \Omega \cup \partial\Omega$, $j = 1, \ldots, N$ and $3N$-dimensional vector $\boldsymbol{A} = [a_{11}, a_{21}, \ldots, a_{N1}, a_{12}, a_{22}, \ldots, a_{N2}, a_{13}, a_{23}, \ldots, a_{N3}]$ contains unknown coefficients. If Neumann's boundary conditions have been postulated we can write the stress vector components in the form

$$
\begin{aligned}
t_1(\boldsymbol{A}, \boldsymbol{Z}, \boldsymbol{x}) &= \sum_{k=1}^{N} a_{k1} T_{11}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k2} T_{12}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k3} T_{13}(\boldsymbol{x}, \boldsymbol{z_k}),\\
t_2(\boldsymbol{A}, \boldsymbol{Z}, \boldsymbol{x}) &= \sum_{k=1}^{N} a_{k1} T_{21}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k2} T_{22}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k3} T_{23}(\boldsymbol{x}, \boldsymbol{z_k}),\\
t_3(\boldsymbol{A}, \boldsymbol{Z}, \boldsymbol{x}) &= \sum_{k=1}^{N} a_{k1} T_{31}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k2} T_{32}(\boldsymbol{x}, \boldsymbol{z_k}) + \sum_{k=1}^{N} a_{k3} T_{33}(\boldsymbol{x}, \boldsymbol{z_k}),
\end{aligned}
$$

where $T(\boldsymbol{x}, z)$ are functions analogous to the functions $G(\boldsymbol{x}, z)$ and should be defined as

$$
T_{11}(\boldsymbol{x}, z) = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial G_{11}}{\partial x_1} + \nu\frac{\partial G_{21}}{\partial x_2} + \nu\frac{\partial G_{31}}{\partial x_3}\right]n_1 + \mu\left(\frac{\partial G_{11}}{\partial x_2} + \frac{\partial G_{21}}{\partial x_1}\right)n_2 + \mu\left(\frac{\partial G_{11}}{\partial x_3} + \frac{\partial G_{31}}{\partial x_1}\right)n_3,
$$

$$
\begin{aligned}
T_{12}(\boldsymbol{x}, z) &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial G_{12}}{\partial x_1} + \nu\frac{\partial G_{22}}{\partial x_2} + \nu\frac{\partial G_{32}}{\partial x_3}\right]n_1 + \mu\left(\frac{\partial G_{12}}{\partial x_2} + \frac{\partial G_{22}}{\partial x_1}\right)n_2 + \mu\left(\frac{\partial G_{12}}{\partial x_3} + \frac{\partial G_{32}}{\partial x_1}\right)n_3\\
&= T_{21}(\boldsymbol{x}, z),
\end{aligned}
$$

$$
\begin{aligned}
T_{13}(\boldsymbol{x}, z) &= \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial G_{13}}{\partial x_1} + \nu\frac{\partial G_{23}}{\partial x_2} + \nu\frac{\partial G_{33}}{\partial x_3}\right]n_1 + \mu\left(\frac{\partial G_{13}}{\partial x_2} + \frac{\partial G_{23}}{\partial x_1}\right)n_2 + \mu\left(\frac{\partial G_{13}}{\partial x_3} + \frac{\partial G_{33}}{\partial x_1}\right)n_3\\
&= T_{31}(\boldsymbol{x}, z),
\end{aligned}
$$

$$
T_{22}(\boldsymbol{x}, z) = \frac{2\mu}{1-2\nu}\left[\nu\frac{\partial G_{12}}{\partial x_1} + (1-\nu)\frac{\partial G_{22}}{\partial x_2} + \nu\frac{\partial G_{32}}{\partial x_3}\right]n_2 + \mu\left(\frac{\partial G_{22}}{\partial x_1} + \frac{\partial G_{12}}{\partial x_2}\right)n_1 + \mu\left(\frac{\partial G_{22}}{\partial x_3} + \frac{\partial G_{32}}{\partial x_2}\right)n_3,
$$

$$
\begin{aligned}
T_{23}(\boldsymbol{x}, z) &= \frac{2\mu}{1-2\nu}\left[\nu\frac{\partial G_{13}}{\partial x_1} + (1-\nu)\frac{\partial G_{23}}{\partial x_2} + \nu\frac{\partial G_{33}}{\partial x_3}\right]n_2 + \mu\left(\frac{\partial G_{23}}{\partial x_3} + \frac{\partial G_{33}}{\partial x_2}\right)n_3 + \mu\left(\frac{\partial G_{23}}{\partial x_1} + \frac{\partial G_{13}}{\partial x_2}\right)n_1\\
&= T_{32}(\boldsymbol{x}, z),
\end{aligned}
$$

$$
T_{33}(\boldsymbol{x}, z) = \frac{2\mu}{1-2\nu}\left[\nu\frac{\partial G_{13}}{\partial x_1} + \nu\frac{\partial G_{23}}{\partial x_2} + (1-\nu)\frac{\partial G_{33}}{\partial x_3}\right]n_3 + \mu\left(\frac{\partial G_{33}}{\partial x_2} + \frac{\partial G_{23}}{\partial x_3}\right)n_3 + \mu\left(\frac{\partial G_{33}}{\partial x_1} + \frac{\partial G_{13}}{\partial x_3}\right)n_1,
$$

where n_1, n_2, n_3 denote the components of the unit vector normal to the surface $\partial\Omega$.

Finally to find the solution of considered problem we should write the linear system of algebraic equations (9) in the following form:

$$\begin{bmatrix} G_{11}(\boldsymbol{x}^1, z_1) \cdots G_{11}(\boldsymbol{x}^1, z_N) & G_{12}(\boldsymbol{x}^1, z_1) \cdots G_{12}(\boldsymbol{x}^1, z_N) & G_{13}(\boldsymbol{x}^1, z_1) \cdots G_{13}(\boldsymbol{x}^1, z_N) \\ G_{21}(\boldsymbol{x}^1, z_1) \cdots G_{21}(\boldsymbol{x}^1, z_N) & G_{22}(\boldsymbol{x}^1, z_1) \cdots G_{22}(\boldsymbol{x}^1, z_N) & G_{23}(\boldsymbol{x}^1, z_1) \cdots G_{23}(\boldsymbol{x}^1, z_N) \\ G_{31}(\boldsymbol{x}^1, z_1) \cdots G_{31}(\boldsymbol{x}^1, z_N) & G_{32}(\boldsymbol{x}^1, z_1) \cdots G_{32}(\boldsymbol{x}^1, z_N) & G_{33}(\boldsymbol{x}^1, z_1) \cdots G_{33}(\boldsymbol{x}^1, z_N) \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ G_{11}(\boldsymbol{x}^M, z_1) \cdots G_{11}(\boldsymbol{x}^M, z_N) & G_{12}(\boldsymbol{x}^M, z_1) \cdots G_{12}(\boldsymbol{x}^M, z_N) & G_{13}(\boldsymbol{x}^M, z_1) \cdots G_{13}(\boldsymbol{x}^1, z_N) \\ G_{21}(\boldsymbol{x}^M, z_1) \cdots G_{21}(\boldsymbol{x}^M, z_N) & G_{22}(\boldsymbol{x}^M, z_1) \cdots G_{22}(\boldsymbol{x}^M, z_N) & G_{23}(\boldsymbol{x}^M, z_1) \cdots G_{23}(\boldsymbol{x}^M, z_N) \\ G_{31}(\boldsymbol{x}^M, z_1) \cdots G_{31}(\boldsymbol{x}^M, z_N) & G_{32}(\boldsymbol{x}^M, z_1) \cdots G_{32}(\boldsymbol{x}^M, z_N) & G_{33}(\boldsymbol{x}^M, z_1) \cdots G_{33}(\boldsymbol{x}^M, z_N) \end{bmatrix} \times \begin{bmatrix} a_{11} \\ \vdots \\ a_{N1} \\ \\ a_{12} \\ \vdots \\ a_{1N} \\ \\ a_{13} \\ \vdots \\ a_{N3} \end{bmatrix} = \begin{bmatrix} a_{11} \\ \vdots \\ a_{N1} \\ \\ a_{12} \\ \vdots \\ a_{1N} \\ \\ a_{13} \\ \vdots \\ a_{N3} \end{bmatrix},$$

where vector $\boldsymbol{F} = [f_1^1, f_2^1, f_3^1, \dots, f_1^M, f_2^M, f_3^M]$ contains the value of displacement vector or stress vector components applied in M collocation points. If in any point the Neumann boundary condition is applied, we only have to replace all functions G to the function T, respectively in three rows of the matrix which are written for considered point. If the number of collocation points M is equal to the number of source points N, we obtain square system of $3M = 3N$ equations. It is possible to define more collocation points than sources ($M > N$). In that case, one should find the approximate solution in the least square sense.

After solving of the system and obtaining the vector of coefficients A we can easily determine the displacement vector in each point of the considered body with the help of (8).

References

[Alderson et al. 2014] K. L. Alderson, A. Alderson, J. N. Grima, and K. W. Wojciechowski, "Auxetic materials and related systems", *Phys. Status Solidi B* **251** (2014), 263–266.

[Almgren 1985] R. F. Almgren, "An isotropic three-dimensional structure with Poisson's ratio $= -1$", *J. Elasticity* **15** (1985), 427–430.

[Bogomolny 1985] A. Bogomolny, "Fundamental solutions method for elliptic boundary value problems", *SIAM J. Numer. Anal.* **22**:4 (1985), 644–669.

[Chen et al. 2009] L. Chen, C. Liu, J. Wang, W. Zhang, C. Hu, and S. Fan, "Auxetic materials with large negative Poisson's ratios based on highly oriented carbon nanotube structures", *Appl. Phys. Lett.* **94** (2009), 253111.

[Fairweather et al. 2003] G. Fairweather, A. Karageorghis, and P. A. Martin, "The method of fundamental solutions for scattering and radiation problems", *Eng. Anal. Bound. Elem.* **27**:7 (2003), 759–769.

[Gaspar et al. 2011] N. Gaspar, C. W. Smith, A. Alderson, J. N. Grima, and K. E. Evans, "A generalised three-dimensional tethered-nodule model for auxetic materials", *Journal of Material Science* **46** (2011), 372–384.

[Gilat and Aboudi 2013] R. Gilat and J. Aboudi, "Behavior of elastoplastic auxetic microstructural arrays", *Materials* **6** (2013), 726–737.

[Goldberg 1995] M. A. Goldberg, "The method of fundamental solutions for Poisson's equation", *Eng. Anal. Bound. Elem.* **16** (1995), 205–213.

[Karageorghis and Fairweather 2000] A. Karageorghis and G. Fairweather, "The method of fundamental solutions for axisymmetric elasticity problems", *Comput. Mech.* **25** (2000), 524–532.

[Kita 2003] E. Kita, "Indirect Trefftz method for boundary value problem of Poisson equation", *Eng. Anal. Bound. Elem.* **27** (2003), 825–833.

[Kupradze 1967] V. Kupradze, "A method for the approximate solution of limiting problems in mathematical physics", *U.S.S.R. Comput. Math. Math. Phys.* **4**:6 (1967), 199–205.

[Kupradze and Aleksidze 1964] V. D. Kupradze and M. A. Aleksidze, "The method of functional equations for the approximate solution of certain boundary-value problems", *Ž. Vyčisl. Mat. i Mat. Fiz.* **4** (1964), 683–715. In Russian; translated in *U.S.S.R. Comput. Math. Math. Phys.* **4**:4 (1966), 82–126.

[Lakes 1987] R. Lakes, "Foam structures with a negative Poisson's ratio", *Science* **235** (1987), 1038–1040.

[Lakes 1992] R. S. Lakes, "Saint-Venant end effects for materials with negative Poisson's ratios", *J. Applied Mechanics* **59** (1992), 744–746.

[Landau and Lifshitz 1986] L. D. Landau and E. M. Lifshitz, *Course of theoretical physics, VII: Theory of elasticity*, 3rd ed., Pergamon Press, Oxford, 1986.

[Lim 2004] T. C. Lim, "Elastic properties of a Poisson–Shear material", *Journal of Material Science* **39** (2004), 4965–4969.

[Maruszewski et al. 2014] B. T. Maruszewski, G. Sypniewska-Kamińska, and T. Walczak, "Application of MFS for 3D elastostatic problems — location of the sources and condition number analysis", pp. 237–243 in *Recent advances in computational mechanics: Proceedings of the 20th conference on computer methods in mechanics (CMM2013)* (Poznań, Poland, 2013), edited by T. Łodygowski et al., CRC Press, Boca Raton, FL, 2014.

[Mathon and Johnston 1977] R. Mathon and R. L. Johnston, "The approximate solution of elliptic boundary-value problems by fundamental solutions", *SIAM J. Numer. Anal.* **14**:4 (1977), 638–650.

[de Medeiros et al. 2004] G. C. de Medeiros, P. W. Partridge, and J. O. Brandão, "The method of fundamental solutions with dual reciprocity for some problems in elasticity", *Eng. Anal. Bound. Elem.* **28** (2004), 453–461.

[Poullikkas 1998] A. Poullikkas, "The method of fundamental solutions for three-dimensional elastostatics problems", *Comput. Struct.* **80** (1998), 100–107.

[Poullikkas et al. 2001] A. Poullikkas, A. Karageorghis, and G. Georgiou, "The numerical solution of three-dimensional Signorini problems with the method of fundamental solutions", *Eng. Anal. Bound. Elem.* **25** (2001), 221–227.

[Poullikkas et al. 2002] A. Poullikkas, A. Karageorghis, and G. Georgiou, "The method of fundamental solutions for three-dimensional elastostatics problems", *Comput. Struct.* **80**:3-4 (2002), 365–370.

[Walczak et al. 2014] T. Walczak, G. Sypniewska-Kaminska, B. T. Maruszewski, and K. W. Wojciechowski, "Mesh versus meshless method of elastic displacement determination in a common and an auxetic material", *Phys. Status Solidi B* **251** (2014), 2225–2232.

[Wojciechowski 2003] K. W. Wojciechowski, "Non-chiral, molecular model of negative Poisson ratio in two dimensions", *J. Phys. A* **36**:47 (2003), 11765–11778.

[Yang et al. 2013] L. Yang, O. Harrysson, H. West, and D. Cormier, "Modeling of uniaxial compression in a 3D periodic re-entrant lattice structure", *Journal of Material Science* **48** (2013), 1413–1422.

[Zienkiewicz and Taylor 2000] O. C. Zienkiewicz and R. L. Taylor, *The finite element method, I: The basis*, 5th ed., Butterworth-Heinemann, Oxford, UK, 2000.

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