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AN EULERIAN FORMULATION FOR LARGE DEFORMATIONS OF ELASTICALLY ISOTROPIC ELASTIC-VISCOPLASTIC MEMBRANES

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Typical models of membrane-like structures use a Lagrangian formulation of a hyperelastic membrane with a specified reference configuration. Here, an Eulerian formulation is proposed for modeling elastically isotropic, elastic-viscoplastic membranes. The membrane is modeled as a composite of an elastic and an inelastic component with evolution equations for elastic deformation tensors for each component. The model includes hyperelastic response as a special case and has a smooth elastic-inelastic transition capable of modeling both rate-independent and rate-dependent inelastic response. Strongly objective numerical algorithms are developed for integrating the proposed evolution equations. Also, an example of an initially flat circular membrane loaded by a follower pressure is considered to examine: rate-independent elastic and elastic-plastic responses, as well as rate-dependent inelastic relaxation effects.

1. Introduction

The Cosserat surface model of an elastic membrane can be developed as a restricted theory of a shell with no bending stiffness [Naghdi 1972, Section 14]. In this formulation the membrane is modeled as a deformable two-dimensional surface in three-dimensional space that can change area and distort. More specifically, the membrane is modeled as a hyperelastic solid using a Lagrangian formulation with director vectors that depend on convected coordinates related to a reference configuration.

Standard formulations of plasticity theory [Hill 1950; Green and Naghdi 1965; Cristescu 1967; Lee 1969; Lubliner 1990; Bertram 2005] are Lagrangian, with constitutive equations that depend on: a total deformation measure from a reference configuration, an inelastic deformation measure from the reference configuration and an elastic deformation measure from an intermediate configuration. In particular, for finite deformations it is common to use a multiplicative form relating total, elastic and plastic deformations [Bilby 1960; Kröner 1960; Lee 1969]. Moreover, overstress models of viscoplasticity for rate-dependent plasticity were introduced by Malvern [1951] and Perzyna [1963].

Eckart [1948] seems to be the first to emphasize that the stress tensor is determined by a constitutive equation that depends on an elastic deformation tensor which is determined by integrating an evolution equation. This evolution equation for elastic deformation is Eulerian in nature since it depends only on quantities that characterize the present state of the material, which can be measured in principle. More specifically, this evolution equation does not depend on tensor quantities, like total and plastic deformation, that depend on arbitrariness of reference and intermediate configurations. Eckart's formulation [1948] is limited to elastically isotropic material response and is essentially the same as the one proposed later by Leonov [1976] for polymeric media. Rubin [1994] generalized this approach

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for elastically anisotropy materials. An important feature of this formulation [Rubin 1994] is that it removes unphysical arbitrariness of the specification of a reference configuration and an intermediate configuration from which elastic deformation is measured [Rubin 2012].

The main objective of this paper is to present a new formulation of an isotropic elastic-viscoplastic membrane for large deformations based on recent developments in viscoplasticity modeling of three-dimensional continua. One novel feature of this new formulation is that it is Eulerian and does not depend on convected Lagrangian coordinates from a fixed reference configuration for both elastic and inelastic responses.

The membrane is modeled as a composite of an elastic component and an elastic-inelastic dissipative component. Specifically, the elastic response of each component is taken to be isotropic. Evolution equations are proposed for elastic deformations of both the elastic and dissipative components. In particular, the relaxation effect of inelasticity in the evolution equation for the elastic deformation of the dissipative component is introduced through a rate of relaxation term. Using the recent developments in [Hollenstein et al. 2013] the rate of relaxation term exhibits a smooth elastic-inelastic transition and models both rate-independent and rate-dependent inelasticity. Simple functional forms are proposed for this inelastic model with only five material constants. Two constants control rate-independent inelastic response, two constants control rate-dependent inelastic response and an additional constant controls the yield strain. Moreover, these simple functional forms yield robust, strongly objective [Papes 2012; Rubin and Papes 2011] closed form (i.e., noniterative) numerical integration algorithms for the evolution equations. Also, an evolution equation for an isotropic hardening variable could be proposed which includes both hardening and softening without difficulty [Hollenstein et al. 2013], but is not included here.

One challenge in developing an Eulerian formulation of deformation of a membrane is the removal of dependence on a reference configuration in the formulation of the balance laws and constitutive equations for the membrane. Membrane theory is typically formulated using the same convected (i.e., Lagrangian) coordinates θ^α ($\alpha = 1, 2$) for all time. This means that the director vectors $\boldsymbol{a}_\alpha$, determined by differentiating the position vector $\boldsymbol{x}$ of material points on the surface of the membrane with respect to θ^α, identify the same material line elements for all time. In order to develop the Eulerian formulation presented here, use is made of the fact that the convected coordinates are arbitrary and thus can be changed with time. In this formulation, at each instant of time, the director vectors can be chosen to be any two linearly independent vectors in the tangent plane of the membrane's surface. These director vectors characterize different material line elements at each instant of time and can be used to define coordinates that are instantaneously convected.

The example of an initially flat circular membrane subjected to a follower pressure normal on its surface is considered to examine: rate-independent elastic and elastic-plastic responses, as well as rate-dependent inelastic relaxation effects. This problem has important applications to the bursting disks of safety valves and the "bulge test" used to obtain the mechanical properties of thin films and ductile materials. The response to small deformations was analyzed in [Hill and Storåkers 1980]. Large deformations of a thin shell with explicit modeling of changes in thickness of the shell have been considered in [Storåkers 1966; Chater and Neale 1983a; 1983b; Ilahi and Paul 1985]. Cristescu [1967] discusses wave propagation problems for thin elastic-plastic plates that are deformed by dynamic loads.

More recently, Atai and Steigmann [2014] have developed a model for finite deformations of an elastic-viscoplastic thin sheet directly from the three-dimensional theory. This model uses a Lagrangian formulation with multiplicative relations between total, elastic and plastic tensorial deformation measures connecting the reference, intermediate and present configurations. Also, the model characterizes a generalized membrane since it introduces a director vector through the thickness of the sheet that is determined by conditions based on the assumption of generalized plane stress in the sheet. In contrast, here the membrane is modeled as a two-dimensional surface with no thickness and the formulation is Eulerian. The deformation tensor of the membrane's surface has only two invariants: one characterizing area change and the other characterizing surface distortion. Although the thickness of the membrane is not modeled explicitly, the constitutive equations are appropriate for generalized plane-stress response in the surface of the membrane.

An outline of the paper is as follows. Section 2 discusses mathematical aspects of a two-dimensional surface embedded in three-dimensional space with associated tensor analysis. The balance laws of a membrane are presented in Section 3, and Section 4 develops constitutive equations for an elastic-viscoplastic membrane. Invariance under superposed rigid body motions (SRBM) is discussed in Section 5 and numerical integration algorithms are detailed in Section 6. Section 7 discusses an example of axisymmetric deformation and conclusions are presented in Section 8. Also, details of invariance under SRBM are presented in the Appendix.

2. Tensor preliminaries

This paper is concerned with a membrane that is a surface $\mathcal{P}$ with a closed boundary $\partial\mathcal{P}$, which at time t is embedded in three-dimensional Euclidean space. The usual summation convention is used for repeated indices with Greek letters having the range ($\alpha = 1, 2$) and Latin indices having the range ($i = 1, 2, 3$). Moreover, use is made of a triad $\boldsymbol{a}_i$ of linearly independent vectors, with $\boldsymbol{a}_\alpha$ being tangent to the membrane's surface and $\boldsymbol{a}_3$ being a unit vector normal to $\mathcal{P}$, such that

$$a^{1/2} = \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 > 0, \quad \boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}, \tag{2-1}$$

where it is noted that $\boldsymbol{a}_3$ is determined by the tangent vectors $\boldsymbol{a}_\alpha$. The associated reciprocal vectors $\boldsymbol{a}^i$ are defined by

$$\boldsymbol{a}^1 = a^{-1/2}\boldsymbol{a}_2 \times \boldsymbol{a}_3, \quad \boldsymbol{a}^2 = a^{-1/2}\boldsymbol{a}_3 \times \boldsymbol{a}_1, \quad \boldsymbol{a}^3 = \boldsymbol{a}_3. \tag{2-2}$$

Let $\boldsymbol{a} \otimes \boldsymbol{b}$ denote the tensor product between two vectors $\{\boldsymbol{a}, \boldsymbol{b}\}$ and $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^T)$ denote the inner product between two second order tensors $\{\boldsymbol{A}, \boldsymbol{B}\}$. The second order three-dimensional unit tensor $\boldsymbol{I}^*$ has the properties that for an arbitrary vector $\boldsymbol{c}$

$$\boldsymbol{c}\boldsymbol{I}^* = \boldsymbol{I}^*\boldsymbol{c} = \boldsymbol{c}, \quad \boldsymbol{I}^* \cdot \boldsymbol{I}^* = 3. \tag{2-3}$$

The second order tensor $\boldsymbol{A}$ is denoted as a surface tensor if it has the properties

$$\boldsymbol{a}_3\boldsymbol{A} = \boldsymbol{A}\boldsymbol{a}_3 = 0. \tag{2-4}$$

Then, the associated second order surface unit tensor $\boldsymbol{I}$ is defined by

$$\boldsymbol{I} = \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\alpha = \boldsymbol{a}^\alpha \otimes \boldsymbol{a}_\alpha = \boldsymbol{I}^* - \boldsymbol{a}_3 \otimes \boldsymbol{a}_3, \tag{2-5}$$

and has the properties

$$\boldsymbol{c}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{c} = (\boldsymbol{c}\cdot\boldsymbol{a}_\alpha)\boldsymbol{a}^\alpha = (\boldsymbol{c}\cdot\boldsymbol{a}^\alpha)\boldsymbol{a}_\alpha, \quad \boldsymbol{a}_3\boldsymbol{I} = \boldsymbol{I}\boldsymbol{a}_3 = 0, \quad \boldsymbol{I}\cdot\boldsymbol{I} = 2. \tag{2-6}$$

In this paper, tensors will be considered to be three-dimensional unless specifically stated otherwise.

The surface deviatoric operator generates the deviatoric part $\mathrm{dev}(\boldsymbol{A})$ of a second order surface tensor $\boldsymbol{A}$ and is defined by

$$\mathrm{dev}(\boldsymbol{A}) = \boldsymbol{A} - \tfrac{1}{2}(\boldsymbol{A}\cdot\boldsymbol{I})\boldsymbol{I}, \quad \mathrm{dev}(\boldsymbol{A})\cdot\boldsymbol{I} = 0. \tag{2-7}$$

Also, the surface determinant $\det(\boldsymbol{A})$ of the surface tensor $\boldsymbol{A}$ is defined by

$$\det(\boldsymbol{A}) = \frac{\boldsymbol{A}\boldsymbol{a}_1\times\boldsymbol{A}\boldsymbol{a}_2\cdot\boldsymbol{a}_3}{\boldsymbol{a}_1\times\boldsymbol{a}_2\cdot\boldsymbol{a}_3}. \tag{2-8}$$

If the surface determinant of the surface tensor $\boldsymbol{A}$ is nonzero, then $\boldsymbol{A}$ has a surface inverse $\mathrm{inv}(\boldsymbol{A})$ defined by

$$\boldsymbol{A}\,\mathrm{inv}(\boldsymbol{A}) = \mathrm{inv}(\boldsymbol{A})\boldsymbol{A} = \boldsymbol{I}, \quad \boldsymbol{A}\cdot\mathrm{inv}(\boldsymbol{A}) = 2. \tag{2-9}$$

In addition, the three-dimensional determinant of a general three-dimensional second order tensor $\boldsymbol{B}$ is defined by

$$\det{}^*(\boldsymbol{B}) = \frac{\boldsymbol{B}\boldsymbol{a}\times\boldsymbol{B}\boldsymbol{b}\cdot\boldsymbol{B}\boldsymbol{c}}{\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}} \quad \text{for } \boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}\neq 0, \tag{2-10}$$

where $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ are arbitrary linearly independent vectors.

3. Basic equations

Let $\boldsymbol{x}$ locate an arbitrary material point on the surface $\mathcal{P}$ of a membrane. Also, let θ^α be arbitrary convected coordinates at time t that map θ^α to the material point $\boldsymbol{x}$ on the surface $\mathcal{P}$

$$\boldsymbol{x} = \boldsymbol{x}(\theta^\alpha, t). \tag{3-1}$$

The velocity $\boldsymbol{v}$ of this material point is given by

$$\boldsymbol{v} = \dot{\boldsymbol{x}}, \tag{3-2}$$

where $(\dot{\ })$ denotes material time differentiation following the material point, which corresponds to partial differentiation of $\boldsymbol{x}$ in (3-1) with respect to time holding θ^α fixed. The director vectors $\boldsymbol{a}_\alpha$, which are tangent to the surface $\mathcal{P}$, are defined by

$$\boldsymbol{a}_\alpha = \boldsymbol{x}_{,\alpha}, \tag{3-3}$$

where a comma denotes partial differentiation with respect to θ^α. Then, the director $\boldsymbol{a}_3$, which is the unit normal to the surface $\mathcal{P}$, and the reciprocal vectors $\boldsymbol{a}^i$ are defined by (2-1) and (2-2), respectively.

Next, the rate tensor $\boldsymbol{L}$, its symmetric part $\boldsymbol{D}$ and its skew-symmetric part $\boldsymbol{W}$ are defined by

$$\boldsymbol{L} = \boldsymbol{v}_{,\alpha}\otimes\boldsymbol{a}^\alpha = \boldsymbol{D}+\boldsymbol{W}, \quad \boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L}+\boldsymbol{L}^T), \quad \boldsymbol{W} = \tfrac{1}{2}(\boldsymbol{L}-\boldsymbol{L}^T). \tag{3-4}$$

It follows that the material derivative of an arbitrary material line element $d\boldsymbol{x}$ in the membrane's surface $\mathcal{P}$ can be determined by the expression

$$\dot{\overline{d\boldsymbol{x}}} = \boldsymbol{L}\,d\boldsymbol{x}. \tag{3-5}$$

The formulation is Lagrangian when the tensors are expressed in terms of the same convected coordinates for all time. For an Eulerian formulation it is necessary to be able to change the choice of the convected coordinates with time. In particular, let $\boldsymbol{b}_\beta$ be two linearly independent vectors defined in the surface $\mathcal{P}$ satisfying the conditions

$$\boldsymbol{b}_\beta \cdot \boldsymbol{a}_3 = 0, \quad \boldsymbol{b}_1 \times \boldsymbol{b}_2 \cdot \boldsymbol{a}_3 > 0. \tag{3-6}$$

Then, another set of convected coordinate y^α can be defined so that

$$\begin{gathered}\theta^\alpha = \theta^\alpha(y^\beta), \quad y^\beta = y^\beta(\theta^\alpha), \quad \boldsymbol{x} = \boldsymbol{x}(\theta^\alpha, t) = \tilde{\boldsymbol{x}}(y^\beta, t), \quad \boldsymbol{v} = \boldsymbol{v}(\theta^\alpha, t) = \tilde{\boldsymbol{v}}(y^\beta, t), \\ \frac{\partial \tilde{\boldsymbol{x}}}{\partial y^\beta} = (\partial\theta^\alpha/\partial y^\beta)\boldsymbol{a}_\alpha = \boldsymbol{b}_\beta.\end{gathered} \tag{3-7}$$

Next, taking the material derivative of this expression yields

$$\dot{\boldsymbol{b}}_\beta = \frac{\partial \tilde{\boldsymbol{v}}}{\partial y^\beta} = (\partial\theta^\alpha/\partial y^\beta)\boldsymbol{v}_{,\alpha} = \boldsymbol{L}\boldsymbol{b}_\beta. \tag{3-8}$$

Thus, $\boldsymbol{b}_\beta$ can be identified with material line elements in $\mathcal{P}$ at each instant of time. In this sense, y^β can be thought of as coordinates that are instantaneously convected.

For later reference, it is noted that the current element of area $d\sigma$ of the surface $\mathcal{P}$, the current element of arc length ds of the boundary $\partial\mathcal{P}$ and the unit outward normal vector $\boldsymbol{n}$ to $\partial\mathcal{P}$ tangent to the surface $\mathcal{P}$ are defined by

$$d\sigma = a^{1/2} d\theta^1 d\theta^2, \quad \boldsymbol{n}\, ds = \boldsymbol{a}_\alpha d\theta^\alpha \times \boldsymbol{a}_3, \quad \boldsymbol{n}\cdot\boldsymbol{n} = 1, \tag{3-9}$$

so that $a^{1/2}$ is the element of area per unit $d\theta^1 d\theta^2$.

From [Rubin 2000], the conservation of mass for the membrane can be expressed in the form

$$m = \rho a^{1/2}, \quad \dot{m} = 0, \tag{3-10}$$

where ρ is the mass per unit present element of area $d\sigma$. Using the fact that $\boldsymbol{a}_3$ is a unit vector, the director velocity $\boldsymbol{w}_3$ satisfies the conditions

$$\boldsymbol{w}_3 = \dot{\boldsymbol{a}}_3, \quad \boldsymbol{w}_3 \cdot \boldsymbol{a}_3 = 0, \quad \boldsymbol{w}_3 = -(\boldsymbol{w}_\alpha \cdot \boldsymbol{a}_3)\boldsymbol{a}^\alpha = -\boldsymbol{L}^T\boldsymbol{a}_3 = -\boldsymbol{a}_3\boldsymbol{L}. \tag{3-11}$$

Then, taking the material derivative of the expression (2-1) for $a^{1/2}$ yields

$$\frac{d}{dt}(a^{1/2}) = a^{1/2}\boldsymbol{D}\cdot\boldsymbol{I}, \tag{3-12}$$

which allows the conservation of mass equation (3-10) to be rewritten in the form

$$\dot{\rho} + \rho\boldsymbol{D}\cdot\boldsymbol{I} = 0. \tag{3-13}$$

To develop constitutive equations for the response to area changes it is convenient to define the area dilatation J by

$$\rho J = \rho_0, \tag{3-14}$$

where ρ_0 is the constant density of the membrane in its zero stress state ($J = 1$). Then, using (3-13), J satisfies the evolution equation

$$\dot{J} = J\boldsymbol{D}\cdot\boldsymbol{I}. \tag{3-15}$$

Next, using the work in [Rubin 2000], the balance of linear momentum for a membrane can be expressed in the form

$$\rho \dot{\boldsymbol{v}} = \rho \boldsymbol{b} + \operatorname{div}(\boldsymbol{T}), \tag{3-16}$$

where $\boldsymbol{b}$ is the external assigned force per unit mass due to body force and surface tractions on the membrane's surface $\mathcal{P}$ and $\boldsymbol{T}$ is a second order tensor, which has the units of force per unit current length in $\mathcal{P}$, to be determined by constitutive equations. Moreover, the surface divergence $\operatorname{div}(\boldsymbol{T})$ of the $\boldsymbol{T}$ can be defined by [Rubin 2000, Section 4.4]

$$a^{1/2} \operatorname{div}(\boldsymbol{T}) = \boldsymbol{t}^{\alpha}_{,\alpha}, \quad \boldsymbol{t}^{\alpha} = a^{1/2} \boldsymbol{T} \boldsymbol{a}^{\alpha}, \quad \boldsymbol{T} = a^{-1/2} \boldsymbol{t}^{\alpha} \otimes \boldsymbol{a}_{\alpha}, \tag{3-17}$$

so that (3-16) can be multiplied by $a^{1/2}$ to obtain the alternative form

$$m \dot{\boldsymbol{v}} = m \boldsymbol{b} + \boldsymbol{t}^{\alpha}_{,\alpha}. \tag{3-18}$$

Also, the force $\boldsymbol{t}$ and rate of work $\mathcal{W}_s$ done on the membrane, per unit current arc length ds, applied to the boundary $\partial\mathcal{P}$ can be expressed in the forms

$$\boldsymbol{t} = \boldsymbol{T}\boldsymbol{n}, \quad \mathcal{W}_s = \boldsymbol{t} \cdot \boldsymbol{v} \quad \text{on } \partial\mathcal{P}. \tag{3-19}$$

In addition, the balance of angular momentum requires the second order tensor $\boldsymbol{T}$ to be symmetric

$$\boldsymbol{T} = \boldsymbol{T}^T. \tag{3-20}$$

Then, the rate of material dissipation $\mathfrak{D}$ is given by

$$a^{1/2}\mathfrak{D} = a^{1/2} \boldsymbol{T} \cdot \boldsymbol{D} - m\dot{\Sigma} \geq 0, \tag{3-21}$$

where Σ is the strain energy function per unit mass.

4. Constitutive equations

Consider a membrane which is a composite of an elastic component and a dissipative component. Specifically, the elastic component resists both total area changes and total distortional deformations. The total area changes are characterized by the area dilatation J, which satisfies the evolution equation (3-15). Motivated by the work of Flory [1961], the total distortional deformations of $\mathcal{P}$ are characterized by the symmetric, unimodular, positive definite surface tensor $\boldsymbol{B}'$. This tensor satisfies the restrictions

$$\boldsymbol{a}_3 \boldsymbol{B}' = \boldsymbol{B}' \boldsymbol{a}_3 = 0, \quad \boldsymbol{B}' \cdot \boldsymbol{I} > 0, \quad \det(\boldsymbol{B}') = 1, \tag{4-1}$$

where $\boldsymbol{B}'$ and its invariant β, satisfy the equations

$$\dot{\boldsymbol{B}}' = \boldsymbol{L}\boldsymbol{B}' + \boldsymbol{B}'\boldsymbol{L}^T - (\boldsymbol{D} \cdot \boldsymbol{I})\boldsymbol{B}', \quad \beta = \boldsymbol{B}' \cdot \boldsymbol{I}, \quad \dot{\beta} = 2 \operatorname{dev}(\boldsymbol{B}') \cdot \boldsymbol{D}. \tag{4-2}$$

In addition, the elastic area changes and the elastic distortional deformations of $\mathcal{P}$ of the dissipative component are characterized by the elastic area dilatation J_d and the symmetric, unimodular, positive definite surface tensor $\boldsymbol{B}'_d$, which satisfies the restrictions

$$\boldsymbol{a}_3 \boldsymbol{B}'_d = \boldsymbol{B}'_d \boldsymbol{a}_3 = 0, \quad \boldsymbol{B}'_d \cdot \boldsymbol{I} > 0, \quad \det(\boldsymbol{B}'_d) = 1, \tag{4-3}$$

where J_d, $\boldsymbol{B}_d'$ and its invariant α satisfy the evolution equations

$$\dot{J}_d = J_d[\boldsymbol{D}\cdot\boldsymbol{I} - \Gamma_d \ln(J_d)], \quad \dot{\boldsymbol{B}}_d' = \boldsymbol{L}\boldsymbol{B}_d' + \boldsymbol{B}_d'\boldsymbol{L}^T - (\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}_d' - \Gamma\boldsymbol{A}_d,$$
$$\boldsymbol{A}_d = \boldsymbol{B}_d' - \left[\frac{2}{\mathrm{inv}(\boldsymbol{B}_d')\cdot\boldsymbol{I}}\right]\boldsymbol{I}, \tag{4-4}$$
$$\alpha = \boldsymbol{B}_d'\cdot\boldsymbol{I}, \quad \dot{\alpha} = 2\,\mathrm{dev}(\boldsymbol{B}_d')\cdot\boldsymbol{D} - \Gamma\boldsymbol{A}_d\cdot\boldsymbol{I}.$$

In these equations, Γ_d determines the magnitude of the rate of inelastic area changes and $\boldsymbol{A}_d$ determines the direction and Γ determines the magnitude of the rate of inelastic distortional deformations. When Γ_d vanishes, the evolution equation (4-4) for J_d reduces to the same form as (3-15) for the total area dilation J. Also, when Γ vanishes, the evolution equation (4-4) for $\boldsymbol{B}_d'$ reduces to the same form as (4-2) for $\boldsymbol{B}'$. Thus, when $\{\Gamma_d, \Gamma\}$ both vanish, the instantaneous response of the dissipative component becomes elastic with zero rate of dissipation. Moreover, the scalars $\{\Gamma_d, \Gamma\}$ need to be specified by constitutive equations which will be discussed later in this section. In addition, the term $\ln(J_d)$ used for the inelastic area change in (4-4) is similar to the term used in [Rubin 2015] for the inelastic contribution of the active stretch in cardiac muscle and is introduced for simplification of the numerical integration algorithm discussed in Section 6. The inelastic response due to area dilatation can be used to model inelastic distortional deformations due to area and thickness changes of a thin three-dimensional structure within the context of a pure two-dimensional membrane model.

To analyze the rate of material dissipation of the dissipative component use is made of (3-11), (4-2) and (4-4) to deduce the results

$$\frac{d}{dt}(\boldsymbol{B}'\boldsymbol{a}_3) = 0, \quad \frac{d}{dt}(\boldsymbol{a}_3\boldsymbol{B}') = 0, \quad \frac{d}{dt}(\boldsymbol{B}_d'\boldsymbol{a}_3) = 0, \quad \frac{d}{dt}(\boldsymbol{a}_3\boldsymbol{B}_d') = 0, \tag{4-5}$$

which show that the evolution equations (4-2) and (4-4) are consistent with $\{\boldsymbol{B}', \boldsymbol{B}_d'\}$ remaining surface tensors that satisfy (4-1) and (4-3). These evolution equations for $\{\boldsymbol{B}', \boldsymbol{B}_d'\}$ also satisfy the conditions

$$\dot{\boldsymbol{B}}'\cdot\mathrm{inv}(\boldsymbol{B}') = 0, \quad \dot{\boldsymbol{B}}_d'\cdot\mathrm{inv}(\boldsymbol{B}_d') = 0, \tag{4-6}$$

which ensure that $\{\boldsymbol{B}', \boldsymbol{B}_d'\}$ remain unimodular $\{\det(\boldsymbol{B}') = \det(\boldsymbol{B}_d') = 1\}$. In deriving the evolution equations for $\{\beta, \alpha\}$ use has been made of (2-5), (3-11), the restrictions (4-1), (4-3) and the results that

$$\dot{\boldsymbol{I}} = -(\boldsymbol{w}_3\otimes\boldsymbol{a}_3 + \boldsymbol{a}_3\otimes\boldsymbol{w}_3), \quad \boldsymbol{B}'\cdot\dot{\boldsymbol{I}} = 0, \quad \boldsymbol{B}_d'\cdot\dot{\boldsymbol{I}} = 0. \tag{4-7}$$

For the class of membranes under consideration here, the strain energy Σ is additively separated into an elastic part Σ_e and a dissipative part Σ_d, with

$$\Sigma = \Sigma_e(J, \beta) + \Sigma_d(J_d, \alpha). \tag{4-8}$$

This considers the membrane to be modeled like an elastic component in parallel with a dissipative component that is composed of an elastic element in series with a dissipative element (see Figure 1). Here, the dissipative component is introduced to model dissipation, which includes rate-independent or rate-dependent hysteresis in cyclic loadings. Also, the kinetic quantity $\boldsymbol{T}$ separates additively into its elastic part $\boldsymbol{T}_e$ and its dissipative part $\boldsymbol{T}_d$

$$\boldsymbol{T} = \boldsymbol{T}_e + \boldsymbol{T}_d. \tag{4-9}$$

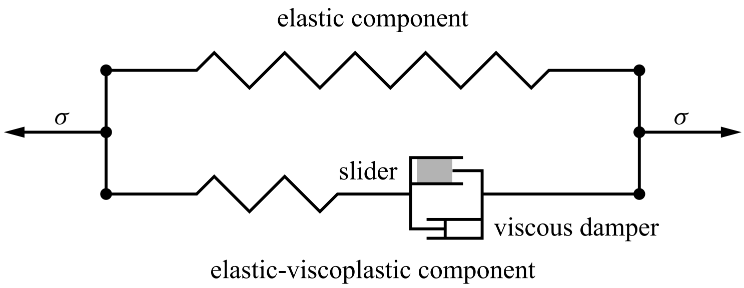

Figure 1. Sketch of an elastic-viscoplastic model with an elastic component and an elastic-viscoplastic dissipative component in parallel.

Now, taking the constitutive equations for $\{\boldsymbol{T}_e, \boldsymbol{T}_d\}$ in the forms

$$\begin{aligned}
&\boldsymbol{T}_e = T_e \boldsymbol{I} + \mathrm{dev}(\boldsymbol{T}_e), \quad T_e = \rho_0 \frac{\partial \Sigma_e}{\partial J}, \qquad \mathrm{dev}(\boldsymbol{T}_e) = 2J^{-1}\rho_0 \frac{\partial \Sigma_e}{\partial \beta}\,\mathrm{dev}(\boldsymbol{B}'),\\
&\boldsymbol{T}_d = T_d \boldsymbol{I} + \mathrm{dev}(\boldsymbol{T}_d), \quad T_d = \left(\frac{J_d}{J}\right)\rho_0 \frac{\partial \Sigma_d}{\partial J_d}, \quad \mathrm{dev}(\boldsymbol{T}_d) = 2J^{-1}\rho_0 \frac{\partial \Sigma_d}{\partial \alpha}\,\mathrm{dev}(\boldsymbol{B}'_d),
\end{aligned} \tag{4-10}$$

and using (3-15), (4-2) and (4-4), it can be shown that the rate of material dissipation (3-21) requires

$$\mathfrak{D} = \Gamma_d \left(\frac{J_d}{J}\right)\rho_0 \frac{\partial \Sigma_d}{\partial J_d}\ln(J_d) + \Gamma J^{-1}\rho_0 \frac{\partial \Sigma_d}{\partial \alpha}\boldsymbol{A}_d \cdot \boldsymbol{I} \geq 0. \tag{4-11}$$

Furthermore, with the help of (2-6) and (4-4) it follows that

$$\boldsymbol{A}_d \cdot \boldsymbol{I} = \boldsymbol{B}'_d \cdot \boldsymbol{I} - \frac{4}{\mathrm{inv}(\boldsymbol{B}'_d)\cdot \boldsymbol{I}}. \tag{4-12}$$

However, by expressing $\boldsymbol{B}'_d$ in its spectral form and using (4-3), the fact that $\boldsymbol{B}'_d$ is a positive definite, symmetric, unimodular surface tensor with eigenvalues $\{\lambda^2, 1/\lambda^2\}$ it follows that

$$\boldsymbol{B}'_d \cdot \boldsymbol{I} = \mathrm{inv}(\boldsymbol{B}'_d)\cdot \boldsymbol{I} = \lambda^2 + \frac{1}{\lambda^2}, \tag{4-13}$$

which can be used to deduce the result

$$\boldsymbol{A}_d \cdot \boldsymbol{I} \geq 0. \tag{4-14}$$

Then, using (3-14) and the expression for $\boldsymbol{T}_d$ in (4-10), it is assumed that the strain energy of the dissipative component satisfies the restrictions

$$\frac{\partial \Sigma_d}{\partial J_d} = 0 \text{ for } J_d = 1, \quad \rho_0 \frac{\partial \Sigma_d}{\partial J_d}\ln(J_d) > 0 \text{ for } J_d \neq 1, \quad \rho_0 \frac{\partial \Sigma_d}{\partial \alpha} > 0. \tag{4-15}$$

It then follows that sufficient, but not necessary, conditions for inelasticity to be dissipative and the restriction (4-11) to be satisfied are

$$\Gamma_d \geq 0, \quad \Gamma \geq 0. \tag{4-16}$$

Moreover, in the absence of deformation rate $\boldsymbol{L}$, the evolution equations (4-4) cause J_d to relax towards 1 and $\boldsymbol{B}'_d$ to relax towards $\boldsymbol{I}$, which cause $\boldsymbol{T}_d$ to relax towards zero.

For the examples considered later in the text, the strain energy functions are specified by

$$\begin{aligned}\rho_0\Sigma_e(J,\beta) &= \tfrac{1}{4}K_e[J^2-1-2\ln(J)]+\tfrac{1}{2}\mu_e(\beta-2),\\ \rho_0\Sigma_d(\alpha) &= \tfrac{1}{4}K_d[J_d^2-1-2\ln(J_d)]+\tfrac{1}{2}\mu_d(\alpha-2),\end{aligned}\tag{4-17}$$

where the constants $\{K_e, \mu_e\}$ are the zero stress bulk and shear moduli of the elastic component and the constants $\{K_d, \mu_d\}$ are the zero stress bulk and shear moduli of the dissipative component. These constants $\{K_e, \mu_e, K_d, \mu_d\}$ have the units of force per unit length [N/m]. Then, with the help of (4-10) the kinetic quantities are given by

$$\begin{aligned}\boldsymbol{T}_e &= T_e\boldsymbol{I}+\mathrm{dev}(\boldsymbol{T}_e), & T_e &= \frac{1}{2}K_e\left(J-\frac{1}{J}\right), & \mathrm{dev}(\boldsymbol{T}_e) &= J^{-1}\mu_e\,\mathrm{dev}(\boldsymbol{B}'),\\ \boldsymbol{T}_d &= T_d\boldsymbol{I}+\mathrm{dev}(\boldsymbol{T}_d), & T_d &= \frac{1}{2}K_d\left(\frac{J_d}{J}\right)\left(J_d-\frac{1}{J_d}\right), & \mathrm{dev}(\boldsymbol{T}_d) &= J^{-1}\mu_d\,\mathrm{dev}(\boldsymbol{B}_d').\end{aligned}\tag{4-18}$$

These constitutive equations are similar to those of a compressible neo-Hookean material except that $\{\boldsymbol{T}_e, \boldsymbol{T}_d\}$ are surface tensors instead of three-dimensional tensors. Also, the constants $\{K_e, \mu_e, K_d, \mu_d\}$ are taken to be nonnegative so the restrictions (4-15) are satisfied.

Moreover, following [Hollenstein et al. 2013], the constitutive equation for the function Γ in (4-4), which controls the rate of inelastic response, is proposed in the form

$$\Gamma=\Gamma_0+\Gamma_1\langle g\rangle,\quad \Gamma_i=a_i+b_i\dot{\epsilon}\ (i=0,1),\quad g=1-\frac{\kappa}{\gamma_d},\quad a_i\ge 0,\quad b_i\ge 0,\tag{4-19}$$

where $\{a_i, b_i\}$ are nonnegative material constants, g is the yield function with κ being a constant yield strain and the equivalent elastic strain γ_d of the dissipative component being defined by

$$\gamma_d=\sqrt{\tfrac{3}{2}\boldsymbol{g}_d\cdot\boldsymbol{g}_d},\quad \boldsymbol{g}_d=\tfrac{1}{2}\,\mathrm{dev}(\boldsymbol{B}_d').\tag{4-20}$$

Also, the equivalent rate $\dot{\epsilon}$ of total distortional strain is defined by

$$\dot{\epsilon}=\sqrt{\tfrac{2}{3}\,\mathrm{dev}(\boldsymbol{D})\cdot\mathrm{dev}(\boldsymbol{D})},\tag{4-21}$$

and the Macaulay brackets $\langle g\rangle$ are defined by

$$\langle g\rangle=\max(g,0).\tag{4-22}$$

This constitutive equation produces a smooth elastic-inelastic transition, which is rate-independent when a_i vanish. The constants a_i control relaxation effects for vanishing $\dot{\epsilon}$ and the constants b_i control the magnitude of overstrain when $\dot{\epsilon}$ is nonzero and g is positive. In particular, large values of b_i cause g to be limited by a small positive value since the relaxation effects of inelasticity tend to cause g to decrease. Similar constitutive equations for Γ_d and a yield function g_d for inelasticity of the elastic area dilatation J_d could be proposed, but are not considered here since the example problems considered in Section 7 do not depend on J_d.

For the analysis of plasticity it is common to analyze a measure of accumulated plastic strain. Since the rate of inelastic area dilatation and the rate of inelastic distortional deformation in the evolution equations (4-4) model different physical mechanisms it is natural to introduce two measures of accumulated plastic strain. The formulation of inelastic response in this paper is based on a rate of inelastic deformation and not on an inelastic deformation tensor. In this regard, it is noted from (4-4) and (4-20) that for small elastic strains

$$\boldsymbol{A}_d \approx 2\boldsymbol{g}_d. \tag{4-23}$$

Therefore, the accumulated distortional plastic strain ϵ_p, defined in [Rubin and Attia 1996], is determined by integrating the evolution equation

$$\dot{\epsilon}_p = \Gamma\sqrt{\tfrac{2}{3}\boldsymbol{g}_d\cdot\boldsymbol{g}_d} = \tfrac{2}{3}\Gamma\gamma_d, \tag{4-24}$$

subject to the initial condition that ϵ_p vanishes. Since $\dot{\epsilon}_p$ is nonnegative, a positive value of ϵ_p denotes regions which have experienced some inelastic deformations even though they may be responding elastically in the present state. An additional measure of inelastic area dilatation ϵ_d can be defined by integrating the evolution equation

$$\dot{\epsilon}_d = \Gamma_d|\ln(J_d)|. \tag{4-25}$$

It can be seen from (4-4) that this corresponds to a logarithmic rate of inelastic area dilatation.

5. Superposed rigid body motions (SRBM)

Under SRBM the position vector $\boldsymbol{x}$, time t and the directors $\boldsymbol{a}_i$, transform to their superposed values $\{\boldsymbol{x}^+, t^+, \boldsymbol{a}_i^+\}$, such that

$$\boldsymbol{x}^+ = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{x}, \quad t^+ = t + c, \quad \boldsymbol{a}_i^+ = \boldsymbol{Q}\boldsymbol{a}_i, \tag{5-1}$$

where $\boldsymbol{c}(t)$ is an arbitrary translation vector, c is an arbitrary constant time shift and $\boldsymbol{Q}(t)$ is an arbitrary proper orthogonal tensor

$$\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}^*, \quad \det{}^*(\boldsymbol{Q}) = +1, \quad \boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T = -\boldsymbol{\Omega}^T, \tag{5-2}$$

with $\boldsymbol{\Omega}$ being a skew symmetric tensor. It can be shown that the velocity $\boldsymbol{v}$, the reciprocal vectors $\boldsymbol{a}^\alpha$ and the director velocities $\boldsymbol{w}_\alpha$ transform under SRBM to $\{\boldsymbol{v}^+, \boldsymbol{a}^{\alpha+}, \boldsymbol{w}_\alpha^+\}$, such that

$$\boldsymbol{v}^+ = \dot{\boldsymbol{c}} + \boldsymbol{Q}\boldsymbol{v} + \boldsymbol{\Omega}(\boldsymbol{x}^+ - \boldsymbol{c}), \quad \boldsymbol{a}^{\alpha+} = \boldsymbol{Q}\boldsymbol{a}^\alpha, \quad \boldsymbol{w}_\alpha^+ = \boldsymbol{Q}\boldsymbol{w}_\alpha + \boldsymbol{\Omega}\boldsymbol{a}_\alpha^+. \tag{5-3}$$

Here, and throughout the text, a superposed $(^+)$ is added to a symbol to denote the value of a quantity in the superposed configuration. Moreover, in the Appendix it is shown that

$$\boldsymbol{L}^+ = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T + \boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T, \quad \boldsymbol{D}^+ = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T + \tfrac{1}{2}(\boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T\boldsymbol{\Omega}). \tag{5-4}$$

Additional transformations under SRBM are given by

$$\begin{gathered} a^{1/2+} = a^{1/2}, \quad \rho^+ = \rho, \quad J^+ = J, \quad \boldsymbol{B}'^+ = \boldsymbol{Q}\boldsymbol{B}'\boldsymbol{Q}^T, \quad J_d^+ = J_d, \quad \boldsymbol{B}_d'^+ = \boldsymbol{Q}\boldsymbol{B}_d'\boldsymbol{Q}^T, \\ \boldsymbol{b}^+ = \dot{\boldsymbol{v}}^+ + \boldsymbol{Q}(\boldsymbol{b} - \dot{\boldsymbol{v}}), \quad \boldsymbol{T}^+ = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \quad \boldsymbol{n}^+ = \boldsymbol{Q}\boldsymbol{n}, \quad \boldsymbol{t}^+ = \boldsymbol{Q}\boldsymbol{t}, \\ \Gamma_d^+ = \Gamma_d, \quad \Gamma^+ = \Gamma, \quad \epsilon_p^+ = \epsilon_p, \quad \epsilon_d^+ = \epsilon_d. \end{gathered} \tag{5-5}$$

Then, using these expressions and other results recorded in the Appendix, it can be shown that the entire theory discussed in the previous sections is properly invariant under SRBM.

6. Robust, strongly objective numerical integration algorithms

The objective of this section is to develop robust, strongly objective numerical algorithms for integrating the evolution equations (3-15) for J, (4-2) for $\boldsymbol{B}'$ and (4-4) for J_d and $\boldsymbol{B}'_d$ over a time step which begins at $t = t_1$, ends at $t = t_2$, with time increment $\Delta t = t_2 - t_1$. Specifically, given the values

$$\{J(t_1),\ \boldsymbol{B}'(t_1),\ J_d(t_1),\ \boldsymbol{B}'_d(t_1)\} \tag{6-1}$$

at the beginning of the time step, the numerical algorithm predicts the values

$$\{J(t_2),\ \boldsymbol{B}'(t_2),\ J_d(t_2),\ \boldsymbol{B}'_d(t_2)\} \tag{6-2}$$

at the end of the time step.

Motivated by the work in [Simo 1992; Simo and Hughes 1998, p. 315; Rubin and Papes 2011; Papes 2012; Hollenstein et al. 2013], robust, strongly object numerical algorithms are based on the relative deformation gradient $\boldsymbol{F}_r$, which satisfies the evolution equation and initial conditions

$$\dot{\boldsymbol{F}}_r = \boldsymbol{L}\boldsymbol{F}_r, \quad \boldsymbol{F}_r(t_1) = \boldsymbol{I}. \tag{6-3}$$

In particular, the solution of this equation can be expressed in the form

$$\boldsymbol{F}_r(t) = \boldsymbol{a}_\alpha(t) \otimes \boldsymbol{a}^\alpha(t_1), \tag{6-4}$$

and the relative dilatation J_r satisfies the equations

$$J_r(t) = \det(\boldsymbol{F}_r), \quad \dot{J}_r = J_r \boldsymbol{D}\cdot\boldsymbol{I}, \quad J_r(t_1) = 1, \tag{6-5}$$

Moreover, the unimodular part $\boldsymbol{F}'_r$ of $\boldsymbol{F}_r$ satisfies the equations

$$\boldsymbol{F}'_r = J_r^{-1/2}\boldsymbol{F}_r, \quad \dot{\boldsymbol{F}}'_r = [\boldsymbol{L} - \tfrac{1}{2}(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{I}]\boldsymbol{F}'_r, \quad \boldsymbol{F}'_r(t_1) = \boldsymbol{I}. \tag{6-6}$$

Using (6-5) and (6-6), the exact solutions of (3-15) and (4-2), which satisfy the initial conditions (6-1), are given by

$$J(t_2) = J_r(t_2)J(t_1), \quad \boldsymbol{B}'(t_2) = \boldsymbol{F}'_r(t_2)\boldsymbol{B}'(t_1)\boldsymbol{F}'^T_r(t_2). \tag{6-7}$$

Also, the elastic trial values J_d^* of J_d and $\boldsymbol{B}'^*_d(t)$ of $\boldsymbol{B}'_d$, defined by

$$J_d^*(t) = J_r(t)J_d(t_1), \quad \boldsymbol{B}'^*_d(t) = \boldsymbol{F}'_r(t)\boldsymbol{B}'_d(t_1)\boldsymbol{F}'^T_r(t), \tag{6-8}$$

satisfy the evolution equations and initial conditions

$$\begin{aligned} \dot{J}_d^* &= J_d^*(\boldsymbol{D}\cdot\boldsymbol{I}), & J_d^*(t_1) &= J_d(t_1), \\ \dot{\boldsymbol{B}}'^*_d &= \boldsymbol{L}\boldsymbol{B}'^*_d + \boldsymbol{B}'^*_d\boldsymbol{L}^T - (\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}'^*_d, & \boldsymbol{B}'^*_d(t_1) &= \boldsymbol{B}'_d(t_1), \end{aligned} \tag{6-9}$$

where $\boldsymbol{B}'^*_d$ in these equations is a surface tensor. Next, it is convenient to use an implicit backward Euler approximation of the evolution equation (4-4) for J_d of the form

$$\ln\{J_d(t_2)\} = \ln\{J_d^*(t_2)\} - \Delta t\,\Gamma_d(t_2)\ln[J_d(t_2)], \tag{6-10}$$

where $\Gamma_d(t_2)$ denotes the value of Γ_d at the end of the time step. Then, the value $J_d(t_2)$ at the end of the time step is given by

$$J_d(t_2) = [J_d^*(t_2)]^{1/\{1+\Delta t \Gamma_d(t_2)\}}. \tag{6-11}$$

This value of $J_d(t_2)$ represents the exact solution of (4-4) when Γ_d vanishes and it represents an approximate solution of (4-4) if the rate of inelastic area dilatation is nonzero. A discussion similar to the one below for determining $\Gamma_d(t_2)$ could be presented but is not pursued further here since the constitutive response for the example considered in the Section 7 does not depend on J_d.

Similarly, it is convenient to use an implicit backward Euler approximation of the evolution equation (4-4) for $\boldsymbol{B}'_d$ by introducing the auxiliary tensor $\bar{\boldsymbol{B}}'_d$ defined by

$$\bar{\boldsymbol{B}}'_d(t_2) = \boldsymbol{B}'^*_d(t_2) - \Delta t \Gamma(t_2) \left[\bar{\boldsymbol{B}}'_d(t_2) - \left(\frac{2}{\mathrm{inv}[\bar{\boldsymbol{B}}'_d(t_2)] \cdot \boldsymbol{I}} \right) \boldsymbol{I} \right], \tag{6-12}$$

where $\Gamma(t_2)$ denotes the value of Γ at the end of the time step. The tensor $\bar{\boldsymbol{B}}'_d(t_2)$ represents the exact solution of (4-4) if Γ vanishes and it represents an approximate solution of (4-4) if the rate of inelastic distortional deformation is nonzero. Moreover, $\bar{\boldsymbol{B}}'_d(t_2)$ is introduced as an auxiliary tensor since it is not necessarily unimodular and therefore is only an approximation of $\boldsymbol{B}'_d(t_2)$. Now, using the surface deviatoric operator (2-7), the deviatoric part of $\boldsymbol{B}'_d(t_2)$ is set equal to the deviatoric part of $\bar{\boldsymbol{B}}'_d(t_2)$ to obtain

$$\mathrm{dev}\{\boldsymbol{B}'_d(t_2)\} = \mathrm{dev}\{\bar{\boldsymbol{B}}'_d(t_2)\} = \mathrm{dev}\{\boldsymbol{B}'^*_d(t_2)\} - \Delta t \Gamma(t_2)\, \mathrm{dev}\{\boldsymbol{B}'_d(t_2)\}. \tag{6-13}$$

This equation can be solved to obtain

$$\begin{aligned} \mathrm{dev}\{\boldsymbol{B}'_d(t_2)\} &= \frac{\mathrm{dev}\{\boldsymbol{B}'^*_d(t_2)\}}{1+\Delta t \Gamma(t_2)}, & \gamma_d(t_2) &= \frac{\gamma_d^*(t_2)}{1+\Delta t \Gamma(t_2)}, \\ \gamma_d^*(t_2) &= \sqrt{\tfrac{3}{2} \boldsymbol{g}_d^* \cdot \boldsymbol{g}_d^*}, & \boldsymbol{g}_d^* &= \tfrac{1}{2}\, \mathrm{dev}\{\boldsymbol{B}'^*_d(t_2)\}, \\ \gamma_d(t_2) &= \sqrt{\tfrac{3}{2} \boldsymbol{g}_d \cdot \boldsymbol{g}_d}, & \boldsymbol{g}_d &= \tfrac{1}{2}\, \mathrm{dev}\{\boldsymbol{B}'_d(t_2)\}, \end{aligned} \tag{6-14}$$

where use has been made of (4-20). Once the value $\Gamma(t_2)$ is known, the value $\boldsymbol{B}'_d(t_2)$ can be determined using $\mathrm{dev}\{\boldsymbol{B}'_d(t_2)\}$ and the expression

$$\boldsymbol{B}'_d(t_2) = \tfrac{1}{2}\alpha(t_2)\boldsymbol{I} + \mathrm{dev}\{\bar{\boldsymbol{B}}'_d(t_2)\}, \tag{6-15}$$

together with the condition that $\boldsymbol{B}'_d(t_2)$ is unimodular, which requires

$$\alpha(t_2) = 2\sqrt{1 - \det[\mathrm{dev}\{\boldsymbol{B}'_d(t_2)\}]}, \tag{6-16}$$

where it is noted that $\det[\mathrm{dev}\{\boldsymbol{B}'_d(t_2)\}]$ is nonpositive so that $\alpha(t_2)$ is real and positive.

More specifically, introducing the relative total distortional deformation measures $\{\boldsymbol{C}'_r, \boldsymbol{B}'_r\}$

$$\boldsymbol{C}'_r = \boldsymbol{F}'^T_r \boldsymbol{F}'_r, \quad \boldsymbol{B}'_r = \boldsymbol{F}'_r \boldsymbol{F}'^T_r, \tag{6-17}$$

and using (2-7), (3-4) and (6-6) it can be shown that

$$\dot{\boldsymbol{C}}'_r = 2\boldsymbol{F}'^T_r\, \mathrm{dev}(\boldsymbol{D}) \boldsymbol{F}'_r, \quad \mathrm{dev}(\boldsymbol{D}) = \tfrac{1}{2} \boldsymbol{F}'^{-T}_r \dot{\boldsymbol{C}}'_r \boldsymbol{F}'^{-1}_r. \tag{6-18}$$

Then, the value of $\operatorname{dev}(\boldsymbol{D})$ can be approximated by

$$\operatorname{dev}(\boldsymbol{D}) \approx \operatorname{dev}(\bar{\boldsymbol{D}}) = \frac{1}{2\Delta t}[\boldsymbol{F}_r'^{-T}(t_2)\{\boldsymbol{C}_r'(t_2) - \boldsymbol{I}\}\boldsymbol{F}_r'^{-1}(t_2)] = \frac{1}{2\Delta t}[\boldsymbol{I} - \boldsymbol{B}_r'^{-1}(t_2)], \tag{6-19}$$

so the equivalent rate $\dot{\epsilon}$ in (4-21) can be approximated by

$$\dot{\epsilon}(t_2) \approx \dot{\bar{\epsilon}} = \sqrt{\tfrac{2}{3}\operatorname{dev}(\bar{\boldsymbol{D}}) \cdot \operatorname{dev}(\bar{\boldsymbol{D}})}, \qquad \Delta\bar{\epsilon} = \Delta t \dot{\bar{\epsilon}}. \tag{6-20}$$

Also, with the help of (4-19) and (6-14), the value of the yield function g at the end of the time step is given by

$$g(t_2) = 1 - \frac{\kappa[1 + \Delta t \Gamma(t_2)]}{\gamma_d^*(t_2)}. \tag{6-21}$$

Moreover, using (4-19) and (6-20), the value of Γ at the end of the time step is given by

$$\Delta t \Gamma(t_2) = \Delta\Gamma_0 + \Delta\Gamma_1 \langle g(t_2) \rangle, \qquad \Delta\Gamma_i = \Delta t a_i + b_i \Delta\bar{\epsilon} \qquad (i = 0, 1). \tag{6-22}$$

In order to solve these equations for $\Delta t \Gamma(t_2)$ it is convenient to introduce the auxiliary variable $\bar{g}$ defined by

$$\bar{g} = 1 - \frac{\kappa(1 + \Delta\Gamma_0)}{\gamma_d^*(t_2)}, \tag{6-23}$$

to obtain

$$g(t_2) = \frac{\langle \bar{g} \rangle}{1 + \dfrac{\kappa \Delta\Gamma_1}{\gamma_d^*(t_2)}}, \qquad \Delta t \Gamma(t_2) = \Delta\Gamma_0 + \frac{\Delta\Gamma_1 \langle \bar{g} \rangle}{1 + \dfrac{\kappa \Delta\Gamma_1}{\gamma_d^*(t_2)}}. \tag{6-24}$$

Furthermore, the evolution equations (4-24) and (4-25) are integrated by taking

$$\epsilon_p(t_2) = \epsilon_p(t_1) + \tfrac{2}{3}\Delta t \Gamma(t_2)\gamma_d(t_2), \qquad \epsilon_d(t_2) = \epsilon_d(t_1) + \Delta t \Gamma_d(t_2)|\ln(J_d(t_2)|, \tag{6-25}$$

with $\gamma_d(t_2)$ given by (6-14) and $\Delta t \Gamma(t_2)$ given by (6-22). Next, introducing the relative director displacements $\boldsymbol{\delta}_\alpha$ and the relative displacement gradient $\boldsymbol{H}_r$ by the expressions

$$\boldsymbol{a}_\alpha(t_2) = \boldsymbol{a}_\alpha(t_1) + \boldsymbol{\delta}_\alpha, \qquad \boldsymbol{H}_r = \boldsymbol{\delta}_\alpha \otimes \boldsymbol{a}^\alpha(t_1), \tag{6-26}$$

it follows that the relative deformation gradient at the end of the time step is given by

$$\boldsymbol{F}_r(t_2) = \boldsymbol{I} + \boldsymbol{H}_r. \tag{6-27}$$

Thus, within the context of standard finite element procedures, the value of $\boldsymbol{F}_r(t_2)$ can be expressed in terms of nodal displacements which are determined at each iteration step.

In summary, given the displacements $\boldsymbol{\delta}_\alpha$: the relative deformation gradient $\boldsymbol{F}_r$ during the time step is determined by (6-26) and (6-27); the exact values $\{J(t_2), \boldsymbol{B}'(t_2)\}$ are determined by (6-7); the elastic trial values $\{J_d^*(t_2), \boldsymbol{B}_d'^*(t_2), \gamma_d'^*(t_2)\}$ are determined by (6-8) and (6-14); the final value $J_d(t_2)$ is determined by (6-11); $\{\Delta\Gamma_i, \Delta t \Gamma(t_2)\}$ are determined by (6-20)–(6-24); and the final value $\boldsymbol{B}_d'(t_2)$ is determined by (6-14)–(6-16). Finally, using these values, the kinetic quantities are determined by the constitutive equations (4-18), which are then used to check that the balance laws are satisfied to the desired accuracy and to update the estimates of the displacements if additional corrections are needed.

The expressions in this section are strongly objective with the numerical estimates of tensor quantities having the same invariance properties under SRBM as the exact values of the tensors (see [Rubin and Papes 2011]).

7. An example of axisymmetric deformation

As an example, the finite deformation of an initially flat circular membrane subjected to pressure normal to its surface, which was analyzed in [Chater and Neale 1983a; 1983b], is used here to illustrate the inelastic response of the proposed model. Figure 2 shows a sketch of a deformed membrane which in its initial unstressed reference configuration at $t = 0$ is a flat circular disk of radius B. Its edges are simply supported by a rigid ring of radius B. The position vector $\boldsymbol{x}$ for this axisymmetric problem is expressed in terms of the cylindrical polar base vectors $\{\boldsymbol{e}_r(\theta), \boldsymbol{e}_\theta(\theta), \boldsymbol{e}_3\}$ in the form

$$\boldsymbol{x} = r(R,t)\boldsymbol{e}_r + z(R,t)\boldsymbol{e}_3, \quad 0 \le R \le \frac{B}{1+b}, \qquad b = 0.01, \tag{7-1}$$

where R is the convected (i.e., Lagrangian) radial coordinate, θ is the circumferential angle, $r(R,t)$ is the current radial position and $z(R,t)$ is the current axial position. In order to avoid numerical problems associated with zero stiffness to normal displacements of a flat membrane, the membrane is slightly stretched in the radial direction in its initial configuration at $t = 0$ with

$$r(R,0) = (1+b)R, \quad z(R,0) = 0. \tag{7-2}$$

A uniform follower force pressure p is applied normal to the surface of the membrane which inflates it to its deformed configuration. For this example, the external assigned force is given by

$$\rho\boldsymbol{b} = p\boldsymbol{a}_3. \tag{7-3}$$

Also, the constitutive equations are specified by (4-17)–(4-19) with K_d set equal to zero.

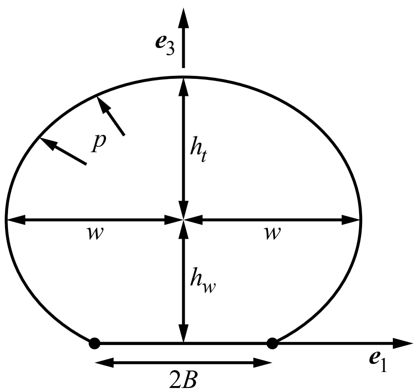

Figure 2. Sketch of inflation of an initially flat circular membrane of radius B which is loaded by an internal pressure p. The maximum width is denoted by $2w$, the height of this maximum width is denoted by h_w and the height of the top of the membrane is denoted by $h_w + h_t$.

In order to emphasize the influence of inelasticity, the elastic component is taken to have no resistance to distortional deformations and the rate of inelasticity is simplified by taking

$$\mu_e = 0, \quad b_0 = 0, \quad a_1 = 0. \tag{7-4}$$

The remaining constants are specified by

$$\mu_d = 100 K_e, \quad b_1 = 0 \text{ or } 10^6, \quad \kappa = 0.001. \tag{7-5}$$

Also, the positive value of B is used to normalize the length measures $\{r, z\}$. A positive value of a_0 causes rate-dependent inelasticity which tends to cause the stress $\boldsymbol{T}_d$ in the dissipative component to relax towards zero. When a_0 vanishes and b_1 is positive, then the dissipative component has a finite elastic range and rate-independent plasticity only occurs when the yield function g becomes positive. Moreover, when b_1 attains the large value in (7-5) it causes the yield function g in (4-19) to remain very close to zero during inelastic loading. Also, the applied pressure p is increased to the maximum value $p_{\max}$ given by

$$p_{\max} = 100 \frac{K_e}{B}, \tag{7-6}$$

and then is held constant. Here, the rate of loading need not be specified since attention is limited to elastic response and elastic-plastic response, which are both rate-independent, and to fully relaxed viscoplastic response with zero elastic distortional deformation.

The equations of equilibrium [(3-18) with $\dot{\boldsymbol{v}} = 0$] were formulated in terms of $\{R, t\}$ and were solved numerically using finite differences for the spatial dependence at each time. The equilibrium equations for the axisymmetric problem considered here simplify into a system of two second order nonlinear ordinary differential equations in the spatial domain, which can be expressed as a system of four nonlinear first order ordinary differential equations. The system of nonlinear first order ordinary differential equations can be efficiently solved numerically using the shooting method for the spatial dependence at each time. The solution procedure is iterative with the elastic distortional deformation (6-12) of the dissipative component being based on the same initial value $\boldsymbol{B}'_d(t_1)$ in (6-8) until an equilibrium configuration has been obtained and the next time step is analyzed.

7.1. ***Elastic response (E).*** When, b_1 vanishes and the loading is rapid (i.e., $\dot{\epsilon} \gg a_0$), the influence of viscoplasticity is negligible and the solution is purely elastic. This solution is denoted by (E).

7.2. ***Rate-independent elastic-plastic response (P).*** When, b_1 is positive and the loading is rapid (i.e., $b_1 \dot{\epsilon} \gg a_0$), the influence of viscoplasticity is negligible and the solution exhibits rate-independent elastic-plastic response. This solution is denoted by (P).

7.3. ***Rate-dependent viscoplastic relaxation (R).*** When the pressure is held constant, the viscoplastic response, controlled by a_0, causes the elastic distortional deformation γ_d to asymptotically approach zero with time. This causes $\boldsymbol{T}_d$ to vanish so the final state is controlled only by the elastic area changes associated with T_e in (4-18). This asymptotic relaxed solution is denoted by (R). Since only the fully relaxed asymptotic solution for viscoplasticity is presented, it is sufficient to solve the elastic problem with vanishing μ_d.

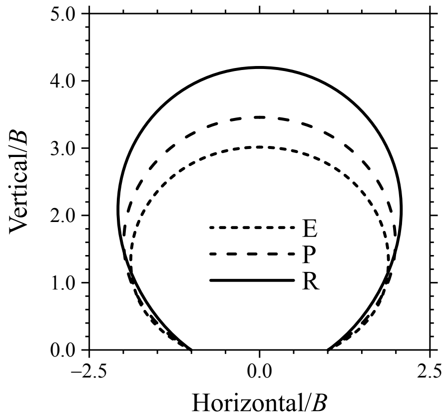

Figure 3. Deformed shapes of the membrane loaded by the constant uniform internal pressure $p = 10K_e/B$ for the elastic solution (E), the plastic solution (P) and the fully relaxed viscoplastic solution (R).

Figure 3 plots the shapes of the membrane normalized by the radius B of the ring for the elastic solution (E), the plastic solution (P) and the viscoplastic relaxed solution (R) with $p = p_{\max}$. From this figure it can be seen that the deformations of the solution (E) are smaller than those of (P) and (R) because the resistance to distortional deformations in the solution (P) is reduced relative to that in the solution (E) due to plasticity. Also, the solution (R) is fully relaxed with zero resistance to distortional deformations. Figure 4 plots the geometric parameters $\{h_t/w, h_w/w\}$ (see Figure 2) as functions of the pressure for the three solutions (E, P, R). From Figure 4 (left) it can be seen that the normalized heights h_t/w of each of the solutions (E, P) are very close to each other and the top part of the fully relaxed membrane attains a near circular shape (i.e., $h_t/w = 1$). Figure 4 (right) shows that the ring controls the maximum width w of the membrane until the pressure attains a critical value when the normalized height h_w/w becomes nonzero. This figure also shows that the effect of distortional strength of the membrane near the constraining ring significantly changes the shape of the lower part of the membrane, with only the solution (R) approaching that of a spherical membrane (i.e., $h_w/w = 1$).

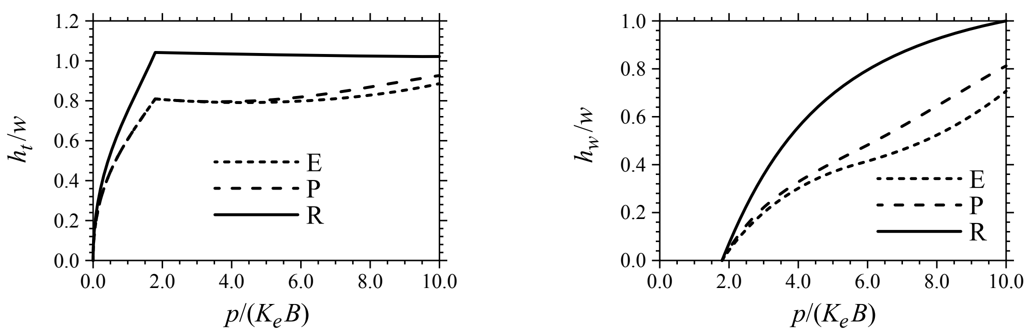

Figure 4. Geometric properties of the membrane as functions of the internal pressure p for the elastic solution (E), the plastic solution (P) and the fully relaxed viscoplastic solution (R).

8. Conclusions

The balance laws and constitutive equations for large deformations of an elastically isotropic elastic-viscoplastic membrane have been developed based on an Eulerian formulation. Specifically, the membrane is considered to be a composite of an elastic component and a dissipative elastic-viscoplastic component. The constitutive equations (4-10) are hyperelastic in the sense that the kinetic quantities $\{\boldsymbol{T}_e, \boldsymbol{T}_d\}$ are determined by derivatives of the strain energy functions $\{\Sigma_e, \Sigma_d\}$, respectively. The response of the elastic component depends on the area dilatation J and on the total elastic distortional deformation $\boldsymbol{B}'_e$, which are determined by the evolution equations (3-15) and (4-2). In addition, the response of the dissipative component depends on the elastic area dilatation J_d and the elastic distortional deformation $\boldsymbol{B}'_d$ of the dissipative component, which are determined by the evolution equations (4-4).

The rate of inelastic deformation in (4-4) is based on the work in [Hollenstein et al. 2013] which models a smooth elastic-inelastic transition, with rate-independent plasticity and rate-dependent viscoplasticity included as special cases. Specifically, this rate of inelasticity depends on five material constants: two $\{a_0, a_1\}$ which control rate-dependent response; two $\{b_0, b_1\}$ which control rate-independent response; and one $\{\kappa\}$ which controls the yield strain.

Numerical algorithms have been developed which are robust and strongly objective. Also, these algorithms produce exact solutions for purely elastic response. Finally, the example of large axisymmetric deformations of an initially flat circular membrane subjected to a follower pressure normal to its surface is considered to examine: the elastic and elastic-plastic responses and the fully relaxed viscoplastic response of the membrane.

This model can be implemented into a general purpose computer program. Then, the influence of dissipation and hysteresis during cyclic loading due to the dissipative component can be examined. Moreover, since the model uses an Eulerian formulation, it would be interesting to use it to model some features of lipid membranes with dissipation to the distortional motion of lipid reorganization in the surface of the membrane.

Appendix: Details of invariance properties under SRBM

With the help of (2-5), (5-1) and (5-3) it follows that

$$\boldsymbol{I}^+ = \boldsymbol{a}^+_\alpha \otimes \boldsymbol{a}^{\alpha+} = \boldsymbol{a}^{\alpha+} \otimes \boldsymbol{a}^+_\alpha = \boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T. \tag{A.1}$$

Then, from (3-4) and (5-3) it can be shown that

$$\boldsymbol{L}^+ = \boldsymbol{v}^+_{,\alpha} \otimes \boldsymbol{a}^{\alpha+} = (\boldsymbol{Q}\boldsymbol{w}_\alpha + \boldsymbol{\Omega}\boldsymbol{a}^+_\alpha) \otimes \boldsymbol{a}^{\alpha+} = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T + \boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T. \tag{A.2}$$

Also, using (5-2) it follows that

$$(\boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T\boldsymbol{\Omega})^T = (\boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T\boldsymbol{\Omega}), \tag{A.3}$$

which verifies (5-4).

Next, using (5-4) and (A.1) it can be shown that

$$\begin{aligned} \boldsymbol{D}^+\cdot\boldsymbol{I}^+ &= [\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T+\tfrac{1}{2}(\boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T-\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T\boldsymbol{\Omega})]\cdot\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T,\\ \boldsymbol{D}^+\cdot\boldsymbol{I}^+ &= \boldsymbol{D}\cdot\boldsymbol{I}+\tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T(\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T)^T-\tfrac{1}{2}\boldsymbol{\Omega}\cdot(\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T)^T\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T,\\ \boldsymbol{D}^+\cdot\boldsymbol{I}^+ &= \boldsymbol{D}\cdot\boldsymbol{I}+\tfrac{1}{2}(\boldsymbol{\Omega}-\boldsymbol{\Omega})\cdot\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T=\boldsymbol{D}\cdot\boldsymbol{I}. \end{aligned} \tag{A.4}$$

It then follows that the invariance properties (5-5) for $\{\rho, J\}$ are consistent with proper invariance under SRBM of the evolution equation (3-15) and the conservation of mass equation (3-16). Moreover, using (5-1) and (5-5) it can be shown that

$$\begin{aligned} \operatorname{inv}(\boldsymbol{B}_d^{\prime+}) &= \boldsymbol{Q}\operatorname{inv}(\boldsymbol{B}_d^{\prime})\boldsymbol{Q}^T,\\ \operatorname{inv}(\boldsymbol{B}_d^{\prime+})\cdot\boldsymbol{I}^+ &= \boldsymbol{Q}\operatorname{inv}(\boldsymbol{B}_d^{\prime})\boldsymbol{Q}^T\cdot(\boldsymbol{Q}\boldsymbol{I}\boldsymbol{Q}^T)=\operatorname{inv}(\boldsymbol{B}_d^{\prime})\cdot\boldsymbol{I}, \end{aligned} \tag{A.5}$$

so with the help of (4-4) it follows that

$$\boldsymbol{A}_d^+ = \boldsymbol{B}_d^{\prime+}-\left[\frac{2}{\operatorname{inv}(\boldsymbol{B}_d^{\prime+})\cdot\boldsymbol{I}^+}\right]\boldsymbol{I}^+=\boldsymbol{Q}\boldsymbol{A}_d\boldsymbol{Q}^T. \tag{A.6}$$

Next, use is made of (5-4) and (5-5) to deduce that

$$\begin{aligned} \boldsymbol{L}^+\boldsymbol{B}^{\prime+}+\boldsymbol{B}^{\prime+}\boldsymbol{L}^{+T}-(\boldsymbol{D}^+\cdot\boldsymbol{I}^+)\boldsymbol{B}^{\prime+} &= \boldsymbol{Q}[\boldsymbol{L}\boldsymbol{B}^{\prime}+\boldsymbol{B}^{\prime}\boldsymbol{L}^T-(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}^{\prime}]\boldsymbol{Q}^T+\boldsymbol{\Omega}\boldsymbol{B}^{\prime+}-\boldsymbol{B}^{\prime+}\boldsymbol{\Omega},\\ \boldsymbol{L}^+\boldsymbol{B}_d^{\prime+}+\boldsymbol{B}_d^{\prime+}\boldsymbol{L}^{+T}-(\boldsymbol{D}^+\cdot\boldsymbol{I}^+)\boldsymbol{B}_d^{\prime+} &= \boldsymbol{Q}[\boldsymbol{L}\boldsymbol{B}_d^{\prime}+\boldsymbol{B}_d^{\prime}\boldsymbol{L}^T-(\boldsymbol{D}\cdot\boldsymbol{I})\boldsymbol{B}_d^{\prime}]\boldsymbol{Q}^T+\boldsymbol{\Omega}\boldsymbol{B}_d^{\prime+}-\boldsymbol{B}_d^{\prime+}\boldsymbol{\Omega}. \end{aligned} \tag{A.7}$$

Then, with the help of (5-2) and (5-5) it can be shown that

$$\frac{d}{dt}(\boldsymbol{B}^{\prime+})=\boldsymbol{Q}\dot{\boldsymbol{B}}^{\prime}\boldsymbol{Q}^T+\boldsymbol{\Omega}\boldsymbol{B}^{\prime+}-\boldsymbol{B}^{\prime+}\boldsymbol{\Omega},\qquad \frac{d}{dt}(\boldsymbol{B}_d^{\prime+})=\boldsymbol{Q}\dot{\boldsymbol{B}}_d^{\prime}\boldsymbol{Q}^T+\boldsymbol{\Omega}\boldsymbol{B}_d^{\prime+}-\boldsymbol{B}_d^{\prime+}\boldsymbol{\Omega}, \tag{A.8}$$

which can be used together with (A.4) to deduce that the evolution equations (4-2) for $\boldsymbol{B}^{\prime}$ and (4-5) for $\boldsymbol{B}_d^{\prime}$ are properly invariant under SRBM.

The invariance properties (5-5) for $\boldsymbol{T}$ follow directly from the constitutive equations (4-10). Next, using (3-17), (5-3) and (5-5) it follows that

$$\operatorname{div}^+(\boldsymbol{T}^+)=a^{-1/2}(a^{1/2}\boldsymbol{T}^+\boldsymbol{a}^{\alpha+})_{,\alpha}=\boldsymbol{Q}\operatorname{div}(\boldsymbol{T}), \tag{A.9}$$

which with the help of (5-5) can be used to show that the balance of linear momentum (3-16) is properly invariant under SRBM.

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