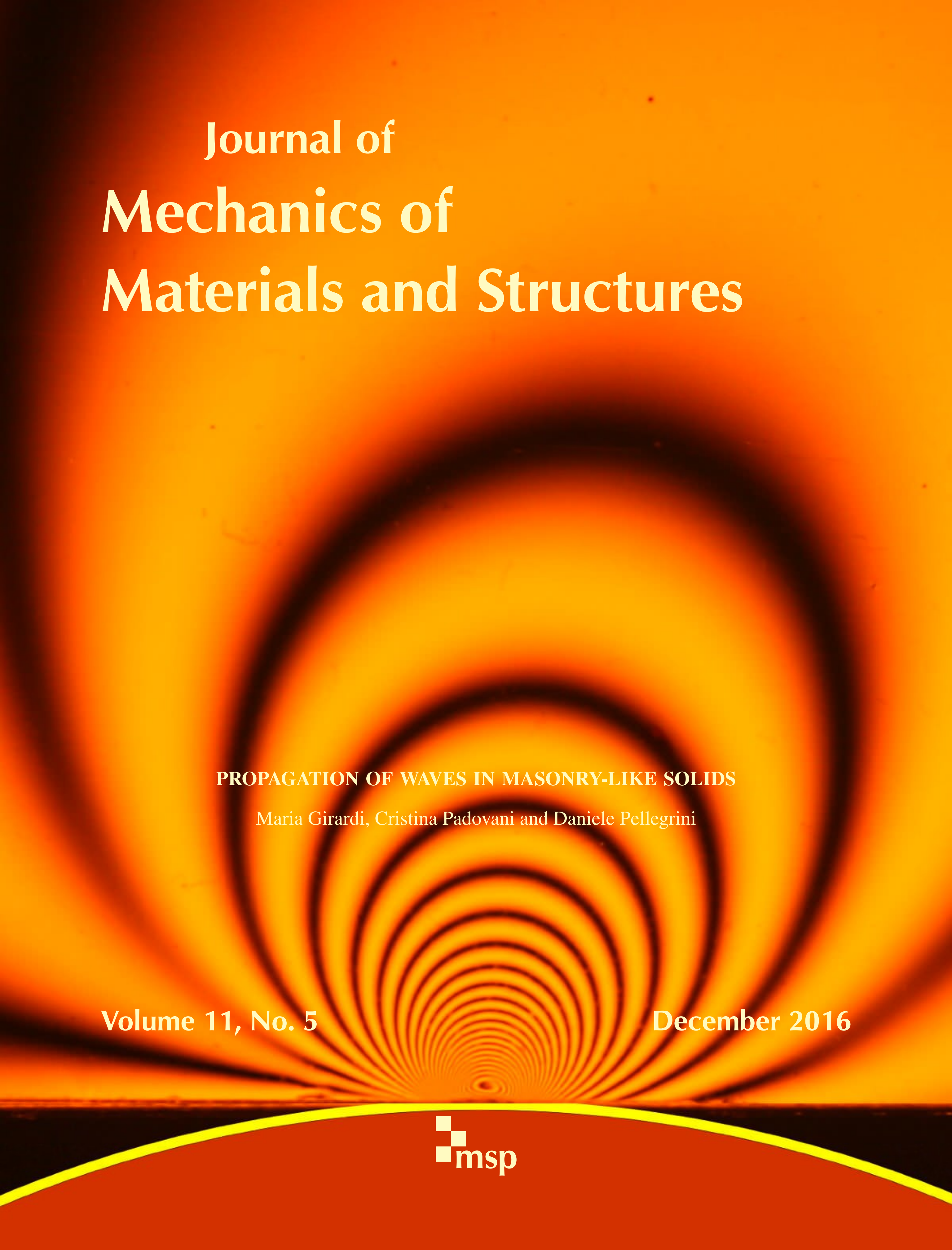

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PROPAGATION OF WAVES IN MASONRY-LIKE SOLIDS

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This paper deals with the propagation of progressive elastic waves in masonry-like solids. The constitutive equation of masonry-like materials models the mechanical behavior of materials (such as masonry, rocks and stones) that do not withstand tensile stresses. The stress function $\mathbb{T}$ delivering the Cauchy stress $\boldsymbol{T}$ corresponding to an infinitesimal strain tensor $\boldsymbol{E}$ is nonlinear and differentiable on an open subset W of the set of all strains. We consider the propagation of small amplitude elastic waves in a masonry-like body subjected to a given homogenous strain field $\boldsymbol{E}$ belonging to W. We obtain the propagation condition, which involves the acoustic tensor $\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$, which depends on both $\boldsymbol{E}$ and the direction of propagation $\boldsymbol{n}$, and prove that, due to the presence of cracks, the wave propagation velocities in masonry are lower than in a linear elastic material.

Introduction

The study of elastic waves finds its main applications in addressing earthquakes and seismological problems [Ewing et al. 1957], in the evaluation of cracks in elastic media [Crampin 1981], as well as in the acoustic determination of third-order elastic constants and residual stresses [Winkler and Liu 1996; Pao et al. 1984; Ogden and Singh 2011]. A further application is in assessing the mechanical behavior of constructions in response to earthquakes by studying the propagation properties of seismic waves. In [Safack 1999] the changes in the propagation characteristics of seismic waves in a building were shown to be more reliable indicators of damage than changes in natural frequencies. In [Ivanovic et al. 2001; Safak et al. 2009] the wave propagation method is used for structural health monitoring purposes.

A detailed treatment of elastic waves is available in [Royer and Dieulesaint 2000], which addresses the different types of waves that propagate in isotropic and anisotropic solids, with particular focus on the propagation and generation of waves in crystals. Progressive waves and the Fresnel–Hadamard condition for their propagation, involving the acoustic tensor, are discussed in [Truesdell and Toupin 1960]. Progressive waves have been studied in [Gurtin 1972; Chadwick 1989] for isotropic and transversely isotropic linear elastic media. Lastly, the acoustic tensor and its eigenvalues and eigenvectors are explicitly calculated in [Chadwick 1989].

This paper deals with the propagation of progressive elastic waves in a masonry-like body subjected to a given homogeneous strain field. Unlike [Gurtin 1972; Chadwick 1989], which deal with linear elasticity, here we consider the constitutive equation of masonry-like materials [Del Piero 1989; Lucchesi et al. 2008], which models the mechanical behavior of materials (such as masonry, rocks and stones) that

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do not withstand tensile stresses. A masonry-like material is a nonlinear hyperelastic material with zero tensile strength and infinite compressive strength. For the Cauchy stress $\boldsymbol{T}$ and the infinitesimal strain $\boldsymbol{E}$ (both belonging to the set Sym of symmetric tensors), the stress-strain relation is determined by the nonlinear relation $\boldsymbol{T} = \mathbb{C}[\mathbb{P}(\boldsymbol{E})]$, where $\mathbb{C}$ is the fourth-order elasticity tensor, and $\mathbb{P}$ is the nonlinear projection of the strain tensor onto the image of the set $\mathbb{C}^{-1}\mathrm{Sym}^{-}$ of negative-semidefinite stresses Sym^{-} under $\mathbb{C}^{-1}$ with respect to the energetic scalar product on Sym. The tensor $\boldsymbol{E}^f = \boldsymbol{E} - \mathbb{P}(\boldsymbol{E})$, which is positive-semidefinite and orthogonal to $\boldsymbol{T}$, is called fracture strain and is different from zero where fractures arise.

The constitutive equation of isotropic masonry-like materials is briefly described in Section 1, which also presents the explicit expression for the stress function $\mathbb{T}(\boldsymbol{E}) = \mathbb{C}[\mathbb{P}(\boldsymbol{E})]$ as $\boldsymbol{E}$ varies in the four regions V_i, $i = 0, 1, 2, 3$. Regions V_i characterize the different types of behavior that a masonry-like material can exhibit. In V_3 the material behaves like a linear elastic material, since the stress is negative-semidefinite. In V_0 the stress tensor is zero and the material can crack in all directions. Regions V_1 and V_2 exhibit mixed behavior: the stress tensor has respectively two and one eigenvectors corresponding to the zero eigenvalue, and the material can fracture orthogonally to these directions. As demonstrated in [Lucchesi et al. 2008; Padovani and Šilhavý 2015], the function $\mathbb{T}$ is differentiable on $W = \bigcup_{i=0}^{3} W_i$, with W_i being the interior of set V_i. The derivative $D_E\mathbb{T}(\boldsymbol{E})$ of $\mathbb{T}$ with respect to $\boldsymbol{E}$ is a symmetric fourth-order tensor from Sym with values in Sym, whose spectral representation has been calculated in [Lucchesi et al. 2008] and is recalled here.

The boundary-initial-value problem of the dynamics of masonry-like solids has been addressed in [Casarosa et al. 1998; Lucchesi et al. 1999; Degl'Innocenti et al. 2006], which deal with the nonlinearity of the equation of motion. The exact solution to the problem of free longitudinal vibrations of both finite and infinite beams has been calculated in [Casarosa et al. 1998; Lucchesi et al. 1999]. The main features of the solution is the development of a shock wave [Šilhavý 1997] at the interface between the cracked and compressed parts of the beam, which determines a loss of mechanical energy and a progressive decay of the solution. As far as the numerical solution of the dynamic problem of masonry structures is concerned, Degl'Innocenti et al. [2006] proposed a method to integrate with respect to time the system of ordinary differential equations obtained by discretizing the structure into finite elements. The method has been implemented in the NOSA–ITACA code [Binante et al. 2014] and was used to study the dynamic behavior of historical masonry buildings [Callieri et al. 2010; De Falco et al. 2014].

The approach followed in this paper is rather different: instead of addressing the boundary-initial-value problem of dynamics, we consider the propagation of progressive elastic waves in a masonry-like body subjected to a given homogeneous strain field $\boldsymbol{E}$ belonging to W. By using the differentiability of the stress function at $\boldsymbol{E}$ and considering elastic displacements superimposed on $\boldsymbol{E}$, we obtain the linearized equation of the motion involving the constant fourth-order tensor $D_E\mathbb{T}(\boldsymbol{E})$. We then consider progressive waves and determine the condition they must satisfy in order to propagate in the masonry body. This condition involves the acoustic tensor $\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$, whose eigenvalues and eigenvectors are calculated in Section 2.

Unlike the linear elastic case, in which the acoustic tensor depends only on the direction of propagation $\boldsymbol{n}$, here it depends on the strain $\boldsymbol{E}$ as well. Moreover, even though symmetric, $\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$ is positive–semidefinite, since its eigenvalues are non-negative.

The acoustic tensor and the properties of progressive waves are analyzed as the given homogeneous strain field $\boldsymbol{E}$ varies in the regions W_i, $i = 0, 1, 2, 3$. For $\boldsymbol{E} \in W_0$, the material is completely cracked and elastic waves cannot propagate. For $\boldsymbol{E} \in W_1, W_2$, due to the presence of cracks, the wave propagation velocities in masonry are lower than in a linear elastic material. Moreover, longitudinal waves propagate only for some values of $\boldsymbol{n}$, depending on the directions of cracking. For the remaining values of $\boldsymbol{n}$ the waves are neither longitudinal nor transverse and propagate with different velocities in different directions. Finally, if $\boldsymbol{E} \in W_3$, masonry behaves like a linear elastic material.

Section 3 provides a detailed description of the two-dimensional case. In order to highlight the difference between a masonry-like and a linear elastic material, pictures illustrating the effects of the presence of cracks on the propagation of elastic waves are presented.

1. The constitutive equation

Let Lin be the set of all second-order tensors with the scalar product

$$\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}^T \boldsymbol{B})$$

for any $\boldsymbol{A}, \boldsymbol{B} \in$ Lin, with $\boldsymbol{A}^T$ the transpose of $\boldsymbol{A}$. For Sym, the subspace of symmetric tensors, Sym^- and Sym^+ are the sets of all negative-semidefinite and positive-semidefinite elements of Sym. Given the symmetric tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, we denote by $\boldsymbol{A} \otimes \boldsymbol{B}$ the fourth-order tensor defined by

$$\boldsymbol{A} \otimes \boldsymbol{B}[\boldsymbol{H}] = (\boldsymbol{B} \cdot \boldsymbol{H})\boldsymbol{A}$$

for $\boldsymbol{H} \in$ Lin, and by $\mathbb{I}_{\mathrm{Sym}}$ the fourth-order identity tensor on Sym. For $\boldsymbol{a}$ and $\boldsymbol{b}$ vectors, the dyad $\boldsymbol{a} \otimes \boldsymbol{b}$ is defined by $\boldsymbol{a} \otimes \boldsymbol{b}\boldsymbol{h} = (\boldsymbol{b} \cdot \boldsymbol{h})\boldsymbol{a}$, for any vector $\boldsymbol{h}$, and $\cdot$ is the scalar product in the space of vectors. We define the subspaces

$$\mathrm{Span}(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{v} = a\boldsymbol{a} + b\boldsymbol{b} : a, b \in \mathbb{R}\},$$
$$\mathrm{Span}(\boldsymbol{a})^\perp = \{\boldsymbol{v} : \boldsymbol{a} \cdot \boldsymbol{v} = 0\}$$

of the three-dimensional vector space.

Now, let $\mathbb{C}$ be the isotropic fourth-order tensor of the elastic constants

$$\mathbb{C} = 2\mu \mathbb{I}_{\mathrm{Sym}} + \lambda \boldsymbol{I} \otimes \boldsymbol{I}, \tag{1-1}$$

where $\boldsymbol{I} \in$ Sym is the identity tensor and μ and λ are the Lamé moduli of the material satisfying the conditions

$$\mu > 0, \quad \lambda \geq 0. \tag{1-2}$$

$\mathbb{C}$ is symmetric,

$$\boldsymbol{A} \cdot \mathbb{C}[\boldsymbol{B}] = \boldsymbol{B} \cdot \mathbb{C}[\boldsymbol{A}] \text{ for all } \boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}, \tag{1-3}$$

and in view of (1-2) is positive-definite on Sym,

$$\boldsymbol{A} \cdot \mathbb{C}[\boldsymbol{A}] > 0 \text{ for all } \boldsymbol{A} \in \mathrm{Sym}, \quad \boldsymbol{A} \neq 0. \tag{1-4}$$

Then $\mathbb{C}$ is invertible, with inverse $\mathbb{C}^{-1}$. We define the energetic scalar product on Sym by setting $(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{A} \cdot \mathbb{C}[\boldsymbol{B}]$ for any $\boldsymbol{A}, \boldsymbol{B} \in$ Sym.

A masonry-like material is a nonlinear elastic material characterized by the fact that, for $\boldsymbol{E} \in \mathrm{Sym}$, there exists a unique triplet $(\boldsymbol{T}, \boldsymbol{E}^e, \boldsymbol{E}^f)$ of elements of Sym such that [Lucchesi et al. 2008]

$$\boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^f, \tag{1-5}$$

$$\boldsymbol{T} = \mathbb{C}[\boldsymbol{E}^e], \tag{1-6}$$

$$\boldsymbol{T} \in \mathrm{Sym}^-, \quad \boldsymbol{E}^f \in \mathrm{Sym}^+, \tag{1-7}$$

$$\boldsymbol{T} \cdot \boldsymbol{E}^f = 0. \tag{1-8}$$

$\boldsymbol{T}$ is the Cauchy stress corresponding to strain $\boldsymbol{E}$; $\boldsymbol{E}^e$ and $\boldsymbol{E}^f$ are respectively the elastic and inelastic parts of $\boldsymbol{E}$; $\boldsymbol{E}^f$ is also called fracture strain. Denoting by $\mathbb{P} : \mathrm{Sym} \to \mathrm{Sym}$ the metric projection onto the closed convex cone $\mathbb{C}^{-1}\mathrm{Sym}^-$ with respect to the energetic scalar product, it is possible to prove that $\boldsymbol{E}^e = \mathbb{P}(\boldsymbol{E})$ and $\boldsymbol{T} = \mathbb{C}[\mathbb{P}(\boldsymbol{E})]$ [Padovani and Šilhavý 2015]. The stress function $\mathbb{T} : \mathrm{Sym} \to \mathrm{Sym}$ is given by

$$\mathbb{T}(\boldsymbol{E}) = \boldsymbol{T} = \mathbb{C}[\mathbb{P}(\boldsymbol{E})] \text{ for any } \boldsymbol{E} \in \mathrm{Sym}. \tag{1-9}$$

The explicit expression for the stress function $\mathbb{T}$, calculated in [Lucchesi et al. 2008], is recalled in the following.

For $\boldsymbol{E} \in \mathrm{Sym}$, let $e_1 \le e_2 \le e_3$ be its ordered eigenvalues and $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ the corresponding eigenvectors. We introduce the orthonormal basis of Sym (with respect to the scalar product $\cdot$)

$$\begin{aligned} &\boldsymbol{O}_{11} = \boldsymbol{q}_1 \otimes \boldsymbol{q}_1, \quad \boldsymbol{O}_{22} = \boldsymbol{q}_2 \otimes \boldsymbol{q}_2, \quad \boldsymbol{O}_{33} = \boldsymbol{q}_3 \otimes \boldsymbol{q}_3, \\ &\boldsymbol{O}_{12} = 1/\sqrt{2}(\boldsymbol{q}_1 \otimes \boldsymbol{q}_2 + \boldsymbol{q}_2 \otimes \boldsymbol{q}_1), \quad \boldsymbol{O}_{13} = 1/\sqrt{2}(\boldsymbol{q}_1 \otimes \boldsymbol{q}_3 + \boldsymbol{q}_3 \otimes \boldsymbol{q}_1), \\ &\boldsymbol{O}_{23} = 1/\sqrt{2}(\boldsymbol{q}_2 \otimes \boldsymbol{q}_3 + \boldsymbol{q}_3 \otimes \boldsymbol{q}_2). \end{aligned} \tag{1-10}$$

Given $\boldsymbol{E}$, the corresponding stress satisfying the constitutive equation of masonry-like materials is given by

$$\text{if } \boldsymbol{E} \in V_0, \text{ then } \boldsymbol{T} = \boldsymbol{0}, \tag{1-11}$$

$$\text{if } \boldsymbol{E} \in V_1, \text{ then } \boldsymbol{T} = E e_1 \boldsymbol{O}_{11}, \tag{1-12}$$

$$\text{if } \boldsymbol{E} \in V_2, \text{ then } \boldsymbol{T} = 2\mu/(2+\alpha)\{[2(1+\alpha)e_1 + \alpha e_2]\boldsymbol{O}_{11} + [\alpha e_1 + 2(1+\alpha)e_2]\boldsymbol{O}_{22}\}, \tag{1-13}$$

$$\text{if } \boldsymbol{E} \in V_3, \text{ then } \boldsymbol{T} = \mathbb{C}[\boldsymbol{E}], \tag{1-14}$$

where the sets V_k are

$$V_0 = \{\boldsymbol{E} \in \mathrm{Sym} : e_1 \ge 0\}, \tag{1-15}$$

$$V_1 = \{\boldsymbol{E} \in \mathrm{Sym} : e_1 \le 0, \quad \alpha e_1 + 2(1+\alpha)e_2 \ge 0\}, \tag{1-16}$$

$$V_2 = \{\boldsymbol{E} \in \mathrm{Sym} : \alpha e_1 + 2(1+\alpha)e_2 \le 0,\ 2e_3 + \alpha \,\mathrm{tr}\, \boldsymbol{E} \ge 0\}, \tag{1-17}$$

$$V_3 = \{\boldsymbol{E} \in \mathrm{Sym} : 2e_3 + \alpha \,\mathrm{tr}\, \boldsymbol{E} \le 0\}, \tag{1-18}$$

with $\alpha = \lambda/\mu$ and the Young's modulus $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$. As for the fracture strain, we have

$$\text{if } \boldsymbol{E} \in V_0, \text{ then } \boldsymbol{E}^f = \boldsymbol{E}, \tag{1-19}$$

$$\text{if } \boldsymbol{E} \in V_1, \text{ then } \boldsymbol{E}^f = \big(e_2 + \alpha/(2(1+\alpha))e_1\big)\boldsymbol{O}_{22} + \big(e_3 + \alpha/(2(1+\alpha))e_1\big)\boldsymbol{O}_{33}, \tag{1-20}$$

$$\text{if } \boldsymbol{E} \in V_2, \text{ then } \boldsymbol{E}^f = [e_3 + \alpha/(2+\alpha)(e_1+e_2)]\boldsymbol{O}_{33}, \tag{1-21}$$

$$\text{if } \boldsymbol{E} \in V_3, \text{ then } \boldsymbol{E}^f = \boldsymbol{0}. \tag{1-22}$$

Thus, as $\boldsymbol{E}$ varies in the four regions V_i, $i = 0, 1, 2, 3$, the corresponding stress tensor $\boldsymbol{T}$ and fracture strain $\boldsymbol{E}^f$ have rank $i = 0, 1, 2, 3$ and $r = 3, 2, 1, 0$, respectively.

For W_k, the interior of V_k, function $\mathbb{T}$ turns out to be differentiable on $W = \bigcup_{i=0}^{3} W_i$ [Lucchesi et al. 2008; Padovani and Šilhavý 2015]. The derivative $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ of $\mathbb{T}(\boldsymbol{E})$ with respect to $\boldsymbol{E}$ in the regions W_i has been calculated in [Lucchesi et al. 2008]. $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ is a symmetric fourth-order tensor from Sym into itself and has the following expressions:

$$\text{if } \boldsymbol{E} \in W_0, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) = \mathbb{O}, \tag{1-23}$$

where $\mathbb{O}$ is the null fourth-order tensor,

$$\text{if } \boldsymbol{E} \in W_1, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) = E\left(\boldsymbol{O}_{11} \otimes \boldsymbol{O}_{11} - \frac{e_1}{e_2 - e_1}\boldsymbol{O}_{12} \otimes \boldsymbol{O}_{12} - \frac{e_1}{e_3 - e_1}\boldsymbol{O}_{13} \otimes \boldsymbol{O}_{13}\right), \tag{1-24}$$

$$\begin{aligned} \text{if } \boldsymbol{E} \in W_2, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) &= 2\mu\boldsymbol{O}_{12} \otimes \boldsymbol{O}_{12} - \frac{2\mu}{2+\alpha}\frac{2(1+\alpha)e_1 + \alpha e_2}{e_3 - e_1}\boldsymbol{O}_{13} \otimes \boldsymbol{O}_{13} \\ &- \frac{2\mu}{2+\alpha}\frac{\alpha e_1 + 2(1+\alpha)e_2}{e_3 - e_2}\boldsymbol{O}_{23} \otimes \boldsymbol{O}_{23} + \frac{2\mu(2+3\alpha)}{2+\alpha}\frac{\boldsymbol{O}_{11} + \boldsymbol{O}_{22}}{\sqrt{2}} \otimes \frac{\boldsymbol{O}_{11} + \boldsymbol{O}_{22}}{\sqrt{2}} \\ &+ 2\mu\frac{\boldsymbol{O}_{11} - \boldsymbol{O}_{22}}{\sqrt{2}} \otimes \frac{\boldsymbol{O}_{11} - \boldsymbol{O}_{22}}{\sqrt{2}}, \end{aligned} \tag{1-25}$$

$$\text{if } \boldsymbol{E} \in W_3, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) = \mathbb{C}. \tag{1-26}$$

From (1-24) and (1-25), bearing in mind that they are the spectral decomposition of $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ for $\boldsymbol{E} \in W_1$ and $\boldsymbol{E} \in W_2$ [Itskov 2015], by taking (1-16) and (1-17) into account, we conclude that $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ has non-negative eigenvalues [Lucchesi et al. 2008] and is positive-semidefinite; hence it satisfies the Legendre–Hadamard condition [Šilhavý 1997]

$$\frac{\boldsymbol{a}\otimes\boldsymbol{b}+\boldsymbol{b}\otimes\boldsymbol{a}}{2} \cdot D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{a}\otimes\boldsymbol{b}+\boldsymbol{b}\otimes\boldsymbol{a}}{2}\right] \geq 0 \quad \text{for each vector } \boldsymbol{a}, \boldsymbol{b}. \tag{1-27}$$

For $\boldsymbol{E} \in W_3$, $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ coincides with the tensor of elastic constants (1-1); it is positive-definite and then strongly elliptic [Gurtin 1972].

2. Progressive waves

We are interested in studying the propagation of small amplitude elastic waves in an infinite masonry-like solid $\mathcal{B}$ with homogeneous mass density ρ, homogeneous material properties μ, and λ satisfying (1-2), subjected to a uniform stress $\boldsymbol{T} = \mathbb{T}(\boldsymbol{E})$, with $\boldsymbol{E}$ being a given uniform strain belonging to $W = \bigcup_{i=0}^{3} W_i$.

From the differentiability of $\mathbb{T}$ at $\boldsymbol{E}$ in W [Padovani and Šilhavý 2015], it follows that

$$\mathbb{T}(\boldsymbol{E} + \boldsymbol{H}) = \boldsymbol{T} + D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})[\boldsymbol{H}] + o(\boldsymbol{H}), \quad \boldsymbol{H} \in \text{Sym}, \quad \boldsymbol{H} \to \boldsymbol{0}. \tag{2-1}$$

We consider small elastic displacements $\boldsymbol{u}$, defined on $\mathcal{B}\times(0,\infty)$, such that their gradient $\nabla\boldsymbol{u}$ is small, and denote by $\partial^2\boldsymbol{u}/\partial t^2$ the acceleration and by div the divergence. For such displacements superimposed on strain $\boldsymbol{E}$, from (2-1), neglecting terms of order $o(\nabla\boldsymbol{u})$, we obtain the linearized equation of motion in the absence of body forces

$$\operatorname{div}\big(D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})[(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)/2]\big)=\rho\partial^2\boldsymbol{u}/\partial t^2 \text{ on } \mathcal{B}. \tag{2-2}$$

A progressive wave has the form

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{m}\,\psi(\boldsymbol{n}\cdot\boldsymbol{x}-vt), \tag{2-3}$$

where the unit vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ are respectively the polarization vector (or direction of motion) and the direction of propagation, v is the wave velocity and ψ an arbitrary function of class C^2 on $(-\infty,\infty)$ such that

$$d^2\psi/ds^2\neq 0. \tag{2-4}$$

The wave $\boldsymbol{u}$ in (2-3) is longitudinal if $\boldsymbol{m}=\pm\boldsymbol{n}$, and transverse if $\boldsymbol{m}\cdot\boldsymbol{n}=0$. Moreover, $\boldsymbol{u}$ is elastic if it satisfies the equation of motion (2-2).

From (2-3) we get [Gurtin 1972]

$$\nabla\boldsymbol{u}=\psi'\boldsymbol{m}\otimes\boldsymbol{n}, \tag{2-5}$$

$$\partial^2\boldsymbol{u}/\partial t^2=\psi'' v^2\boldsymbol{m}, \tag{2-6}$$

with

$$\psi'=d\psi/ds|_{s=\boldsymbol{n}\cdot\boldsymbol{x}-vt}, \tag{2-7}$$

$$\psi''=d^2\psi/ds^2|_{s=\boldsymbol{n}\cdot\boldsymbol{x}-vt}. \tag{2-8}$$

From (2-5) it follows that

$$D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})\left[\frac{\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T}{2}\right]=\psi' D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{m}}{2}\right]. \tag{2-9}$$

Moreover, since $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ is independent of $\boldsymbol{x}$,

$$\operatorname{div}\left(D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})\left[\frac{\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T}{2}\right]\right)=\psi'' D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{m}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{m}}{2}\right]\boldsymbol{n}=\rho\psi''\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})\,\boldsymbol{m}, \tag{2-10}$$

where $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ is the tensor defined by

$$\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{a}=\rho^{-1}D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{a}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{a}}{2}\right]\boldsymbol{n} \text{ for every vector } \boldsymbol{a}. \tag{2-11}$$

We call $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ the acoustic tensor for strain $\boldsymbol{E}$ and direction $\boldsymbol{n}$. From (2-2), by taking (2-6), (2-10) and (2-4) into account, we obtain the condition

$$\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{m}=v^2\boldsymbol{m}, \tag{2-12}$$

which expresses the Fresnel–Hadamard propagation condition [Gurtin 1972]. Thus, for an elastic progressive wave to propagate in a direction $\boldsymbol{n}$, its polarization vector must be an eigenvector of the acoustic

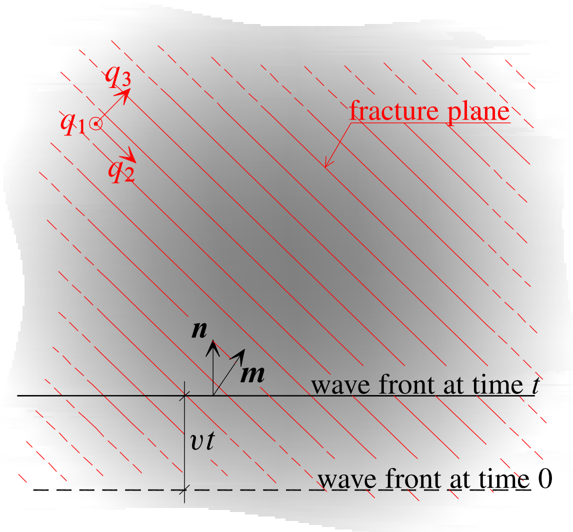

Figure 1. Depiction of a plane wave with direction of propagation $\boldsymbol{n}$ and polarization vector $\boldsymbol{m}$ in a cracked body.

tensor $\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$ and the square of the velocity of propagation must be the associated eigenvalue. Figure 1 depicts a progressive wave with direction of propagation $\boldsymbol{n}$ and polarization $\boldsymbol{m}$ in an infinite body subjected to a homogeneous strain field $\boldsymbol{E} \in W_2$. According to (1-21), the fracture planes are orthogonal to $\boldsymbol{q}_3$. For a given constant ζ, at any time t, the displacement field $\boldsymbol{u}$ in (2-3) is constant on the plane $P_t = \{\boldsymbol{x} : \boldsymbol{x} \cdot \boldsymbol{n} - v\, t = \zeta\}$, called the wave front.

2A. ***The acoustic tensor.*** In this subsection we state some properties of the acoustic tensor $\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$ defined in (2-11) and obtain its expression in the four regions W_i.

Proposition 2.1.

(a) *$\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$ is symmetric.*

(b) *For $\boldsymbol{E}$ in $\bigcup_{i=1}^{2} W_i$, let*

$$D_E \mathbb{T}(\boldsymbol{E}) = \sum_{j=1}^{6} \delta_j(\boldsymbol{E}) \boldsymbol{V}_j(\boldsymbol{E}) \otimes \boldsymbol{V}_j(\boldsymbol{E}), \tag{2-13}$$

be the spectral decomposition of $D_E \mathbb{T}(\boldsymbol{E})$, with $\delta_j(\boldsymbol{E})$ eigenvalues and $\boldsymbol{V}_j(\boldsymbol{E})$ eigentensors of $D_E \mathbb{T}(\boldsymbol{E})$, $j = 1, \ldots, 6$. Thus

$$\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n}) = \rho^{-1} \sum_{j=1}^{6} \delta_j(\boldsymbol{E}) \boldsymbol{V}_j(\boldsymbol{E}) \boldsymbol{n} \otimes \boldsymbol{V}_j(\boldsymbol{E}) \boldsymbol{n}. \tag{2-14}$$

(c) *$\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$ is positive-semidefinite for $\boldsymbol{E} \in \bigcup_{i=0}^{2} W_i$, and positive-definite for $\boldsymbol{E}$ in W_3.*

(d) *$\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{A}(\boldsymbol{E}, -\boldsymbol{n})$ for each unit vector $\boldsymbol{n}$.*

(e) *Let* Orth *be the subset of* Lin *of orthogonal tensors* $\boldsymbol{Q}$, $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$. *The acoustic tensor satisfies the relation*

$$\boldsymbol{Q}\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{Q}^T = \boldsymbol{A}(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{n}) \text{ for each } \boldsymbol{Q} \in \text{Orth}. \tag{2-15}$$

Proof.

(a) By taking the symmetry of $D_E\mathbb{T}(\boldsymbol{E})$ into account, for each vector $\boldsymbol{l}$ and $\boldsymbol{p}$ we have,

$$\begin{aligned}
\boldsymbol{l}\cdot\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{p} &= \boldsymbol{l}\cdot\rho^{-1}D_E\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{p}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{p}}{2}\right]\boldsymbol{n}\\
&= \boldsymbol{l}\otimes\boldsymbol{n}\cdot\rho^{-1}D_E\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{p}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{p}}{2}\right]\\
&= \frac{\boldsymbol{l}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{l}}{2}\cdot\rho^{-1}D_E\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{p}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{p}}{2}\right]\\
&= \rho^{-1}D_E\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{l}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{l}}{2}\right]\cdot\frac{\boldsymbol{p}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{p}}{2}\\
&= \boldsymbol{p}\cdot\rho^{-1}D_E\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{l}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{l}}{2}\right]\boldsymbol{n}\\
&= \boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{l}.
\end{aligned}\tag{2-16}$$

(b) From (2-13) it follows that

$$\begin{aligned}
D_E\mathbb{T}(\boldsymbol{E})\left[\frac{\boldsymbol{p}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{p}}{2}\right]\boldsymbol{n} &= \sum_{j=1}^{6}\delta_j(\boldsymbol{E})\left(\boldsymbol{V}_j(\boldsymbol{E})\cdot\left[\frac{\boldsymbol{p}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{p}}{2}\right]\right)\boldsymbol{V}_j(\boldsymbol{E})\boldsymbol{n}\\
&= \sum_{j=1}^{6}\delta_j(\boldsymbol{E})(\boldsymbol{V}_j(\boldsymbol{E})\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{V}_j(\boldsymbol{E})\boldsymbol{n}\\
&= \sum_{j=1}^{6}\delta_j(\boldsymbol{E})(\boldsymbol{V}_j(\boldsymbol{E})\boldsymbol{n}\otimes\boldsymbol{V}_j(\boldsymbol{E})\boldsymbol{n})\boldsymbol{p},
\end{aligned}\tag{2-17}$$

for all vectors $\boldsymbol{p}$, and (2-14) follows from (2-11).

(c) If $\boldsymbol{E} \in \bigcup_{i=0}^{2} W_i$, from (2-16) for $\boldsymbol{l} = \boldsymbol{p}$, by taking the condition (1-27) into account, we obtain that $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ is positive-semidefinite. If $\boldsymbol{E} \in W_3$, $D_E\mathbb{T}(\boldsymbol{E}) = \mathbb{C}$ is positive-definite and $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ coincides with the acoustic tensor of an isotropic linear elastic material and is positive-definite (see (2-24)).

(d) A trivial consequence of (2-11).

(e) We note that from the isotropy of the stress function $\mathbb{T}$ [Lucchesi et al. 2008],

$$\mathbb{T}(\boldsymbol{E}) = \mathbb{T}(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T) \text{ for each } \boldsymbol{E} \in \text{Sym},\ \boldsymbol{Q} \in \text{Orth}, \tag{2-18}$$

the invariance of its derivative $D_E\mathbb{T}(\boldsymbol{E})$ [Gurtin 1981]

$$\boldsymbol{Q}\,D_E\mathbb{T}(\boldsymbol{E})[\boldsymbol{H}]\boldsymbol{Q}^T = D_E\mathbb{T}(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T)[\boldsymbol{Q}\boldsymbol{H}\boldsymbol{Q}^T], \tag{2-19}$$

follows for each $\boldsymbol{E} \in \bigcup_{i=0}^{3} W_i$, $\boldsymbol{Q} \in \mathrm{Orth}$, $\boldsymbol{H} \in \mathrm{Sym}$.

Thus, for $\boldsymbol{E} \in \bigcup_{i=0}^{3} W_i$ and $\boldsymbol{n}$ unit vector, from (2-11) and (2-19), we obtain

$$\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n}) = \boldsymbol{Q}^T \boldsymbol{A}(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{n})\boldsymbol{Q} \text{ for each } \boldsymbol{Q} \in \mathrm{Orth}. \tag{2-20}$$

Equation (2-20) extends an analogous relation proved in [Gurtin 1972] for isotropic linear elastic materials. From (2-20) it follows that if $\boldsymbol{m}$ is an eigenvector of $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ corresponding to the eigenvalue v^2, then $\boldsymbol{Q}\boldsymbol{m}$ is an eigenvector of $\boldsymbol{A}(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{n})$ corresponding to the same eigenvalue v^2. □

From Proposition 2.1, tensor $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ defined in (2-11) turns out to be symmetric and positive-semidefinite, hence, for each $\boldsymbol{n}$, there exist three orthogonal eigenvectors $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3$, and three associated non-negative eigenvalues v_1^2, v_2^2, v_3^2, whose expressions are calculated in the following.

Due to the different expressions of $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ in the regions W_i, from (2-11) it follows that the acoustic tensor $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$ has different expressions $\boldsymbol{A}_i(\boldsymbol{E},\boldsymbol{n})$ in the four regions W_i. In view of Proposition 2.1(b) we have:

$$\text{if } \boldsymbol{E} \in W_0, \text{ then } \boldsymbol{A}_0(\boldsymbol{E},\boldsymbol{n}) = \boldsymbol{0}, \tag{2-21}$$

$$\text{if } \boldsymbol{E} \in W_1, \text{ then } \boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n}) = E\rho^{-1}\bigg(\boldsymbol{O}_{11}\boldsymbol{n}\otimes\boldsymbol{O}_{11}\boldsymbol{n} - \frac{e_1}{e_2-e_1}\boldsymbol{O}_{12}\boldsymbol{n}\otimes\boldsymbol{O}_{12}\boldsymbol{n} - \frac{e_1}{e_3-e_1}\boldsymbol{O}_{13}\boldsymbol{n}\otimes\boldsymbol{O}_{13}\boldsymbol{n}\bigg), \tag{2-22}$$

$$\begin{aligned}\text{if } \boldsymbol{E} \in W_2, \text{ then } \boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n}) = 2\mu\rho^{-1}\bigg(&\boldsymbol{O}_{12}\boldsymbol{n}\otimes\boldsymbol{O}_{12}\boldsymbol{n}\\ &- \frac{2(1+\alpha)e_1+\alpha e_2}{(2+\alpha)(e_3-e_1)}\boldsymbol{O}_{13}\boldsymbol{n}\otimes\boldsymbol{O}_{13}\boldsymbol{n} - \frac{\alpha e_1+2(1+\alpha)e_2}{(2+\alpha)(e_3-e_2)}\boldsymbol{O}_{23}\boldsymbol{n}\otimes\boldsymbol{O}_{23}\boldsymbol{n}\\ &+ \frac{2+3\alpha}{2+\alpha}\frac{\boldsymbol{O}_{11}+\boldsymbol{O}_{22}}{\sqrt{2}}\boldsymbol{n}\otimes\frac{\boldsymbol{O}_{11}+\boldsymbol{O}_{22}}{\sqrt{2}}\boldsymbol{n} + \frac{\boldsymbol{O}_{11}-\boldsymbol{O}_{22}}{\sqrt{2}}\boldsymbol{n}\otimes\frac{\boldsymbol{O}_{11}-\boldsymbol{O}_{22}}{\sqrt{2}}\boldsymbol{n}\bigg),\end{aligned} \tag{2-23}$$

$$\text{if } \boldsymbol{E} \in W_3 \text{ then } \boldsymbol{A}_3(\boldsymbol{E},\boldsymbol{n}) = (2\mu+\lambda)\rho^{-1}\boldsymbol{n}\otimes\boldsymbol{n} + \mu\rho^{-1}(\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}). \tag{2-24}$$

Our goal is to find the eigenvalues $c_1^{(i)}(\boldsymbol{E},\boldsymbol{n}) \ge c_2^{(i)}(\boldsymbol{E},\boldsymbol{n}) \ge c_3^{(i)}(\boldsymbol{E},\boldsymbol{n})$ and eigenvectors $\boldsymbol{m}_1^{(i)}(\boldsymbol{E},\boldsymbol{n})$, $\boldsymbol{m}_2^{(i)}(\boldsymbol{E},\boldsymbol{n})$, $\boldsymbol{m}_3^{(i)}(\boldsymbol{E},\boldsymbol{n})$ of the acoustic tensor $\boldsymbol{A}_i(\boldsymbol{E},\boldsymbol{n})$, for $i = 1, 2, 3$.

For $\boldsymbol{E} \in W_1$, from (2-22) we get

$$\begin{aligned}\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n}) = E\rho^{-1}\bigg\{&\bigg[(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2 - \frac{e_1}{2(e_2-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_2)^2 - \frac{e_1}{2(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_3)^2\bigg]\boldsymbol{O}_{11}\\ &- \frac{e_1}{2(e_2-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2\boldsymbol{O}_{22} - \frac{e_1}{2(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2\boldsymbol{O}_{33} - \frac{e_1}{\sqrt{2}(e_2-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)(\boldsymbol{n}\cdot\boldsymbol{q}_2)\boldsymbol{O}_{12}\\ &- \frac{e_1}{\sqrt{2}(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)(\boldsymbol{n}\cdot\boldsymbol{q}_3)\boldsymbol{O}_{13}\bigg\}.\end{aligned} \tag{2-25}$$

Thus, if $\boldsymbol{n} = \boldsymbol{q}_1$, then

$$\boldsymbol{A}_1(\boldsymbol{E}, \boldsymbol{q}_1) = E\rho^{-1}\left\{\boldsymbol{O}_{11} - \frac{e_1}{2(e_2 - e_1)}\boldsymbol{O}_{22} - \frac{e_1}{2(e_3 - e_1)}\boldsymbol{O}_{33}\right\}, \tag{2-26}$$

whose eigenvalues and eigenvectors are

$$c_1^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = E\rho^{-1}, \quad c_2^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = -\frac{E\rho^{-1}e_1}{2(e_2 - e_1)}, \tag{2-27}$$

$$c_3^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = -\frac{E\rho^{-1}e_1}{2(e_3 - e_1)}, \tag{2-28}$$

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = \boldsymbol{q}_1, \quad \boldsymbol{m}_2^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = \boldsymbol{q}_2, \quad \boldsymbol{m}_3^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = \boldsymbol{q}_3. \tag{2-29}$$

If $\boldsymbol{n} \cdot \boldsymbol{q}_1 = 0$, then

$$\boldsymbol{A}_1(\boldsymbol{E}, \boldsymbol{n}) = -E\rho^{-1}\left[\frac{e_1}{2(e_2 - e_1)}(\boldsymbol{n} \cdot \boldsymbol{q}_2)^2 + \frac{e_1}{2(e_3 - e_1)}(\boldsymbol{n} \cdot \boldsymbol{q}_3)^2\right]\boldsymbol{O}_{11}, \tag{2-30}$$

whose eigenvalues and eigenvectors are

$$c_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = -\frac{E\rho^{-1}e_1}{2}\left[\frac{1}{e_2 - e_1}(\boldsymbol{n} \cdot \boldsymbol{q}_2)^2 + \frac{1}{e_3 - e_1}(\boldsymbol{n} \cdot \boldsymbol{q}_3)^2\right], \tag{2-31}$$

$$c_2^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = c_3^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = 0, \tag{2-32}$$

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{q}_1, \quad \boldsymbol{m}_2^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{q}_2, \quad \boldsymbol{m}_3^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{q}_3. \tag{2-33}$$

In particular, if $\boldsymbol{n} = \boldsymbol{q}_2$, then

$$c_1^{(1)}(\boldsymbol{E}, \boldsymbol{q}_2) = -\frac{E\rho^{-1}e_1}{2(e_2 - e_1)}, \tag{2-34}$$

and for $\boldsymbol{n} = \boldsymbol{q}_3$, then

$$c_1^{(1)}(\boldsymbol{E}, \boldsymbol{q}_3) = -\frac{E\rho^{-1}e_1}{2(e_3 - e_1)}. \tag{2-35}$$

If $\boldsymbol{n} \cdot \boldsymbol{q}_2 = 0$, then the eigenvalues and eigenvectors of $\boldsymbol{A}_1(\boldsymbol{E}, \boldsymbol{n})$ are

$$c_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \frac{E\rho^{-1}}{2}\left[(\boldsymbol{n} \cdot \boldsymbol{q}_1)^2 - \frac{e_1}{2(e_3 - e_1)} + \sqrt{\left((\boldsymbol{n} \cdot \boldsymbol{q}_1)^2 - \frac{e_1}{2(e_3 - e_1)}\right)^2 + 2\frac{e_1}{e_3 - e_1}(\boldsymbol{n} \cdot \boldsymbol{q}_1)^4}\,\right], \tag{2-36}$$

$$c_2^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = -\frac{E\rho^{-1}e_1}{2(e_2 - e_1)}(\boldsymbol{n} \cdot \boldsymbol{q}_1)^2, \tag{2-37}$$

$$c_3^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \frac{E\rho^{-1}}{2}\left[(\boldsymbol{n} \cdot \boldsymbol{q}_1)^2 - \frac{e_1}{2(e_3 - e_1)} - \sqrt{\left((\boldsymbol{n} \cdot \boldsymbol{q}_1)^2 - \frac{e_1}{2(e_3 - e_1)}\right)^2 + 2\frac{e_1}{e_3 - e_1}(\boldsymbol{n} \cdot \boldsymbol{q}_1)^4}\,\right] \tag{2-38}$$

(where $c_2^{(1)} \leq c_3^{(1)}$ or $c_2^{(1)} \geq c_3^{(1)}$ depending on $\boldsymbol{E}$) and

$$\boldsymbol{m}_2^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{q}_2, \quad \boldsymbol{m}_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}),\ \boldsymbol{m}_3^{(1)}(\boldsymbol{E}, \boldsymbol{n}) \in \mathrm{Span}(\boldsymbol{q}_1, \boldsymbol{q}_3). \tag{2-39}$$

Analogously, if $\boldsymbol{n}\cdot\boldsymbol{q}_3=0$, the eigenvalues and eigenvectors of $\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})$ are

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{n})=\frac{E\rho^{-1}}{2}\left[(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2-\frac{e_1}{2(e_2-e_1)}+\sqrt{\left((\boldsymbol{n}\cdot\boldsymbol{q}_1)^2-\frac{e_1}{2(e_2-e_1)}\right)^2+2\frac{e_1}{e_2-e_1}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^4}\,\right], \tag{2-40}$$

$$c_2^{(1)}(\boldsymbol{E},\boldsymbol{n})=\frac{E\rho^{-1}}{2}\left[(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2-\frac{e_1}{2(e_2-e_1)}-\sqrt{\left((\boldsymbol{n}\cdot\boldsymbol{q}_1)^2-\frac{e_1}{2(e_2-e_1)}\right)^2+2\frac{e_1}{e_2-e_1}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^4}\,\right], \tag{2-41}$$

$$c_3^{(1)}(\boldsymbol{E},\boldsymbol{n})=-\frac{E\rho^{-1}e_1}{2(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2, \tag{2-42}$$

(where $c_2^{(1)}\le c_3^{(1)}$ or $c_2^{(1)}\ge c_3^{(1)}$ depending on $\boldsymbol{E}$) and

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E},\boldsymbol{n}),\quad \boldsymbol{m}_2^{(1)}(\boldsymbol{E},\boldsymbol{n})\in \mathrm{Span}(\boldsymbol{q}_1,\boldsymbol{q}_2),\quad \boldsymbol{m}_3^{(1)}(\boldsymbol{E},\boldsymbol{n})=\boldsymbol{q}_3. \tag{2-43}$$

For $\boldsymbol{n}\cdot\boldsymbol{q}_1\neq 0$, $\boldsymbol{n}\cdot\boldsymbol{q}_2\neq 0$, $\boldsymbol{n}\cdot\boldsymbol{q}_3\neq 0$, the eigenvalues of $\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})$ can be determined by using the formulae in [Kachanov 1974]:

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{n})=\tfrac{2}{\sqrt{3}}\chi^{(1)}\cos(\theta^{(1)}-\tfrac{\pi}{3})+\tfrac{1}{3}I_1^{(1)}, \tag{2-44}$$

$$c_2^{(1)}(\boldsymbol{E},\boldsymbol{n})=\tfrac{2}{\sqrt{3}}\chi^{(1)}\cos(\theta^{(1)}+\tfrac{\pi}{3})+\tfrac{1}{3}I_1^{(1)}, \tag{2-45}$$

$$c_3^{(1)}(\boldsymbol{E},\boldsymbol{n})=-\tfrac{2}{\sqrt{3}}\chi^{(1)}\cos\theta^{(1)}+\tfrac{1}{3}I_1^{(1)}, \tag{2-46}$$

where

$$\chi^{(1)}=\sqrt{\tfrac{1}{3}((I_1^{(1)})^2-3I_2^{(1)})}, \tag{2-47}$$

$$\cos 3\theta^{(1)}=-\frac{3\sqrt{3}\gamma^{(1)}}{2(\chi^{(1)})^3}, \tag{2-48}$$

$$\gamma^{(1)}=I_3^{(1)}-\tfrac{1}{3}I_1^{(1)}I_2^{(1)}+\tfrac{2}{27}(I_1^{(1)})^3, \tag{2-49}$$

with

$$\begin{aligned} I_1^{(1)}&=\mathrm{tr}\,\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})\\ &=E\rho^{-1}\left\{(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2-\frac{e_1}{2(e_2-e_1)}[(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2+(\boldsymbol{n}\cdot\boldsymbol{q}_2)^2]-\frac{e_1}{2(e_3-e_1)}[(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2+(\boldsymbol{n}\cdot\boldsymbol{q}_3)^2]\right\}, \end{aligned} \tag{2-50}$$

$$I_2^{(1)}=\tfrac{1}{2}[(\mathrm{tr}\,\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n}))^2-\mathrm{tr}\,\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})^2], \tag{2-51}$$

$$I_3^{(1)}=\det\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})=\frac{E^3\rho^{-3}e_1^2(\boldsymbol{n}\cdot\boldsymbol{q}_1)^6}{4(e_2-e_1)(e_3-e_1)}, \tag{2-52}$$

the principal invariants of $\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})$. The angle $\theta^{(1)}$ varies between 0 and $\pi/3$.

Since $\boldsymbol{n}\cdot\boldsymbol{q}_1\neq 0$, then $I_3^{(1)}\neq 0$ and $\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})$ has no zero eigenvalues.

For $j=1,2,3$, given $c_j^{(1)}(\boldsymbol{E},\boldsymbol{n})$, the corresponding eigenvector $\boldsymbol{m}_j^{(1)}(\boldsymbol{E},\boldsymbol{n})$ can be calculated by solving the system

$$(\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})-c_j^{(1)}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{I})\boldsymbol{m}_j^{(1)}(\boldsymbol{E},\boldsymbol{n})=\boldsymbol{0}. \tag{2-53}$$

Let us now consider the case in which $\boldsymbol{E} \in W_2$. From (2-23) we obtain

$$\begin{aligned}\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n}) &= \rho^{-1}\left(\varphi(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2+\mu(\boldsymbol{n}\cdot\boldsymbol{q}_2)^2-\mu\frac{2(1+\alpha)e_1+\alpha e_2}{(2+\alpha)(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_3)^2\right)\boldsymbol{O}_{11}\\ &+\rho^{-1}\left(\mu(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2+\varphi(\boldsymbol{n}\cdot\boldsymbol{q}_2)^2-\mu\frac{\alpha e_1+2(1+\alpha)e_2}{(2+\alpha)(e_3-e_2)}(\boldsymbol{n}\cdot\boldsymbol{q}_3)^2\right)\boldsymbol{O}_{22}\\ &-\rho^{-1}\left(\mu\frac{2(1+\alpha)e_1+\alpha e_2}{(2+\alpha)(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2+\mu\frac{\alpha e_1+2(1+\alpha)e_2}{(2+\alpha)(e_3-e_2)}(\boldsymbol{n}\cdot\boldsymbol{q}_2)^2\right)\boldsymbol{O}_{33}\\ &+\mu\rho^{-1}\sqrt{2}\frac{2+3\alpha}{2+\alpha}(\boldsymbol{n}\cdot\boldsymbol{q}_1)(\boldsymbol{n}\cdot\boldsymbol{q}_2)\boldsymbol{O}_{12}\\ &-\mu\rho^{-1}\sqrt{2}\frac{2(1+\alpha)e_1+\alpha e_2}{(2+\alpha)(e_3-e_1)}(\boldsymbol{n}\cdot\boldsymbol{q}_1)(\boldsymbol{n}\cdot\boldsymbol{q}_3)\boldsymbol{O}_{13}\\ &-\mu\rho^{-1}\sqrt{2}\frac{\alpha e_1+2(1+\alpha)e_2}{(2+\alpha)(e_3-e_2)}(\boldsymbol{n}\cdot\boldsymbol{q}_2)(\boldsymbol{n}\cdot\boldsymbol{q}_3)\boldsymbol{O}_{23},\end{aligned} \tag{2-54}$$

with

$$\varphi=\frac{4\mu(1+\alpha)}{2+\alpha}. \tag{2-55}$$

If $\boldsymbol{n}=\boldsymbol{q}_1$, then

$$\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{q}_1)=\varphi\rho^{-1}\boldsymbol{O}_{11}+\mu\rho^{-1}\boldsymbol{O}_{22}-\mu\rho^{-1}\frac{2(1+\alpha)e_1+\alpha e_2}{(2+\alpha)(e_3-e_1)}\boldsymbol{O}_{33}, \tag{2-56}$$

whose eigenvalues and eigenvectors are

$$c_1^{(2)}(\boldsymbol{E},\boldsymbol{q}_1)=\varphi\rho^{-1},\quad c_2^{(2)}(\boldsymbol{E},\boldsymbol{q}_1)=\mu\rho^{-1}, \tag{2-57}$$

$$c_3^{(2)}(\boldsymbol{E},\boldsymbol{q}_1)=-\frac{\mu\rho^{-1}}{2+\alpha}\frac{2(1+\alpha)e_1+\alpha e_2}{e_3-e_1}, \tag{2-58}$$

$$\boldsymbol{m}_1^{(2)}(\boldsymbol{E},\boldsymbol{q}_1)=\boldsymbol{q}_1,\quad \boldsymbol{m}_2^{(2)}(\boldsymbol{E},\boldsymbol{q}_1)=\boldsymbol{q}_2,\quad \boldsymbol{m}_3^{(2)}(\boldsymbol{E},\boldsymbol{q}_1)=\boldsymbol{q}_3. \tag{2-59}$$

If $\boldsymbol{n}=\boldsymbol{q}_2$, then

$$\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{q}_2)=\mu\rho^{-1}\boldsymbol{O}_{11}+\varphi\rho^{-1}\boldsymbol{O}_{22}-\mu\rho^{-1}\frac{\alpha e_1+2(1+\alpha)e_2}{(2+\alpha)(e_3-e_2)}\boldsymbol{O}_{33}, \tag{2-60}$$

whose eigenvalues and eigenvectors are

$$c_1^{(2)}(\boldsymbol{E},\boldsymbol{q}_2)=\varphi\rho^{-1},\quad c_2^{(2)}(\boldsymbol{E},\boldsymbol{q}_2)=\mu\rho^{-1}, \tag{2-61}$$

$$c_3^{(2)}(\boldsymbol{E},\boldsymbol{q}_2)=-\frac{\mu\rho^{-1}}{2+\alpha}\frac{\alpha e_1+2(1+\alpha)e_2}{e_3-e_2}, \tag{2-62}$$

$$\boldsymbol{m}_1^{(2)}(\boldsymbol{E},\boldsymbol{q}_2)=\boldsymbol{q}_2,\quad \boldsymbol{m}_2^{(2)}(\boldsymbol{E},\boldsymbol{q}_2)=\boldsymbol{q}_1,\quad \boldsymbol{m}_3^{(2)}(\boldsymbol{E},\boldsymbol{q}_2)=\boldsymbol{q}_3. \tag{2-63}$$

If $\boldsymbol{n}=\boldsymbol{q}_3$, then

$$\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{q}_3)=-\mu\rho^{-1}\frac{2(1+\alpha)e_1+\alpha e_2}{(2+\alpha)(e_3-e_1)}\boldsymbol{O}_{11}-\mu\rho^{-1}\frac{\alpha e_1+2(1+\alpha)e_2}{(2+\alpha)(e_3-e_2)}\boldsymbol{O}_{22}, \tag{2-64}$$

whose eigenvalues and eigenvectors are

$$c_1^{(2)}(\boldsymbol{E},\boldsymbol{q}_3) = -\frac{\mu\rho^{-1}}{2+\alpha}\frac{2(1+\alpha)e_1+\alpha e_2}{e_3-e_1}, \quad c_2^{(2)}(\boldsymbol{E},\boldsymbol{q}_3) = -\frac{\mu\rho^{-1}}{2+\alpha}\frac{\alpha e_1+2(1+\alpha)e_2}{e_3-e_2}, \tag{2-65}$$

$$c_3^{(2)}(\boldsymbol{E},\boldsymbol{q}_3) = 0, \tag{2-66}$$

$$\boldsymbol{m}_1^{(2)}(\boldsymbol{E},\boldsymbol{q}_3) = \boldsymbol{q}_1, \quad \boldsymbol{m}_2^{(2)}(\boldsymbol{E},\boldsymbol{q}_3) = \boldsymbol{q}_2, \quad \boldsymbol{m}_3^{(2)}(\boldsymbol{E},\boldsymbol{q}_3) = \boldsymbol{q}_3. \tag{2-67}$$

If $\boldsymbol{n} \neq \boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$, then the eigenvalues of $\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n})$ can be determined by using the formulae in [Kachanov 1974]:

$$c_1^{(2)}(\boldsymbol{E},\boldsymbol{n}) = \tfrac{2}{\sqrt{3}}\chi^{(2)}\cos(\theta^{(2)} - \tfrac{\pi}{3}) + \tfrac{1}{3}I_1^{(2)}, \tag{2-68}$$

$$c_2^{(2)}(\boldsymbol{E},\boldsymbol{n}) = \tfrac{2}{\sqrt{3}}\chi^{(2)}\cos(\theta^{(2)} + \tfrac{\pi}{3}) + \tfrac{1}{3}I_1^{(2)}, \tag{2-69}$$

$$c_3^{(2)}(\boldsymbol{E},\boldsymbol{n}) = -\tfrac{2}{\sqrt{3}}\chi^{(2)}\cos\theta^{(2)} + \tfrac{1}{3}I_1^{(2)}, \tag{2-70}$$

where

$$\chi^{(2)} = \sqrt{\tfrac{1}{3}((I_1^{(2)})^2 - 3I_2^{(2)})}, \tag{2-71}$$

$$\cos 3\theta^{(2)} = -\frac{3\sqrt{3}\gamma^{(2)}}{(2\chi^{(2)})^3}, \tag{2-72}$$

$$\gamma^{(2)} = I_3^{(2)} - \tfrac{1}{3}I_1^{(2)}I_2^{(2)} + \tfrac{2}{27}(I_1^{(2)})^3, \tag{2-73}$$

with

$$I_1^{(2)} = \operatorname{tr}\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n}).$$

$$I_2^{(2)} = \tfrac{1}{2}[(\operatorname{tr}\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n}))^2 - \operatorname{tr}\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n})^2], \tag{2-74}$$

$$I_3^{(2)} = \det\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n}). \tag{2-75}$$

the principal invariants of $\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n})$. The angle $\theta^{(2)}$ varies between 0 and $\pi/3$.

As in the case of $\boldsymbol{E} \in W_1$, for $j = 1, 2, 3$, given $c_j^{(2)}(\boldsymbol{E},\boldsymbol{n})$, the corresponding eigenvector $\boldsymbol{m}_j^{(2)}(\boldsymbol{E},\boldsymbol{n})$ can be calculated by solving the system

$$(\boldsymbol{A}_2(\boldsymbol{E},\boldsymbol{n}) - c_j^{(2)}(\boldsymbol{E},\boldsymbol{n})\boldsymbol{I})\boldsymbol{m}_j^{(2)}(\boldsymbol{E},\boldsymbol{n}) = \boldsymbol{0}. \tag{2-76}$$

Note that the acoustic tensor $\boldsymbol{A}_3(\boldsymbol{E},\boldsymbol{n})$ for $\boldsymbol{E} \in W_3$ coincides with the acoustic tensor of a linear elastic material with Lamé moduli μ and λ; its eigenvalues are

$$c_1^{(3)} = \rho^{-1}(2\mu+\lambda), \quad c_2^{(3)} = c_3^{(3)} = \rho^{-1}\mu, \tag{2-77}$$

and the corresponding eigenvectors

$$\boldsymbol{m}_1^{(3)}(\boldsymbol{E},\boldsymbol{n}) = \boldsymbol{n}, \quad \boldsymbol{m}_2^{(3)}(\boldsymbol{E},\boldsymbol{n}) \text{ and } \boldsymbol{m}_3^{(3)}(\boldsymbol{E},\boldsymbol{n}) \text{ belong to } \operatorname{Span}(\boldsymbol{n})^{\perp}. \tag{2-78}$$

2B. ***Behavior of plane waves in the regions W_i.*** Let us now analyze the behavior of progressive waves in a body composed of a masonry-like material with homogeneous stress $\boldsymbol{T}$ associated to a homogeneous strain field $\boldsymbol{E}$. This behavior is different in the four regions W_i introduced in Section 1, since it depends on $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$, which has a different expression $\boldsymbol{A}_i(\boldsymbol{E},\boldsymbol{n})$ in W_i (see (2-21) , (2-22), (2-23) and (2-24)).

If $\boldsymbol{E} \in W_0$, from (2-21) it follows that no waves propagate in the medium.

For $\boldsymbol{E} \in W_i, i = 1, 2, 3$, Tables 1, 2 and 3 report the eigenvalues $c_1^{(i)} \geq c_2^{(i)} \geq c_3^{(i)}$ of $\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})$, for each unit vector $\boldsymbol{n}$. The associated wave velocities are $v_j^{(i)} = \sqrt{c_j^{(i)}}$, for $i, j = 1, 2, 3$.

Let us consider the case of $\boldsymbol{E} \in W_1$.

For the eigenvalues $c_1^{(1)}, c_2^{(1)}, c_3^{(1)}$ in (2-27)–(2-28), it is a simple matter to prove that

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) < c_1^{(3)} = \rho^{-1}(2\mu+\lambda) \tag{2-79}$$

and

$$c_2^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) < c_2^{(3)} = \rho^{-1}\mu, \quad c_3^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) < c_3^{(3)} = \rho^{-1}\mu. \tag{2-80}$$

The first inequality in (2-80) follows from the condition $\alpha e_1 + 2(1+\alpha)e_2 > 0$, which characterizes W_1 (see (1-16)).

As for $c_1^{(1)}$ in (2-31), simple calculations show that

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{n}) < \rho^{-1}\mu. \tag{2-81}$$

For the eigenvalues in (2-36)–(2-38) and (2-40)–(2-42) we have

$$\begin{gathered} c_1^{(1)}(\boldsymbol{E},\boldsymbol{n}) < c_1^{(3)} = \rho^{-1}(2\mu+\lambda), \quad c_2^{(1)}(\boldsymbol{E},\boldsymbol{n}) < c_2^{(3)} = \rho^{-1}\mu, \\ c_3^{(1)}(\boldsymbol{E},\boldsymbol{n}) < c_2^{(3)} = \rho^{-1}\mu. \end{gathered} \tag{2-82}$$

The polarization vector and the squared velocity of waves propagating in the masonry body subjected to a uniform strain field $\boldsymbol{E} \in W_1$ are summarized in Table 1, for varying directions of propagation $\boldsymbol{n}$.

$\boldsymbol{E} \in W_1$	$c_1^{(1)}$	$c_2^{(1)}$	$c_3^{(1)}$
$\boldsymbol{n} = \boldsymbol{q}_1$	$E\rho^{-1}$ longitudinal wave, $\boldsymbol{q}_1$	(2-27) transverse wave, $\boldsymbol{q}_2$	(2-28) transverse wave, $\boldsymbol{q}_3$
$\boldsymbol{n}\cdot\boldsymbol{q}_1 = 0$	(2-31) transverse wave, $\boldsymbol{q}_1$	0 no propagation	0 no propagation
$\boldsymbol{n}\cdot\boldsymbol{q}_2 = 0$	(2-36)	(2-37) transverse wave, $\boldsymbol{q}_2$	(2-38)
$\boldsymbol{n}\cdot\boldsymbol{q}_3 = 0$	(2-40)	(2-41)	(2-42) transverse wave, $\boldsymbol{q}_3$
$\boldsymbol{n}\cdot\boldsymbol{q}_1 \neq 0$ $\boldsymbol{n}\cdot\boldsymbol{q}_2 \neq 0$ $\boldsymbol{n}\cdot\boldsymbol{q}_3 \neq 0$	(2-44)	(2-45)	(2-46)

Table 1. Wave velocities squared for $\boldsymbol{E} \in W_1$.

If $\boldsymbol{n} = \boldsymbol{q}_1$, there are one longitudinal wave and two transverse waves,

$$A_1(\boldsymbol{E}, \boldsymbol{q}_1)\boldsymbol{q}_1 = \rho^{-1} E \boldsymbol{q}_1, \tag{2-83}$$

$$A_1(\boldsymbol{E}, \boldsymbol{q}_1)\boldsymbol{q}_2 = -\frac{E\rho^{-1}e_1}{2(e_2 - e_1)}\boldsymbol{q}_2, \quad A_1(\boldsymbol{E}, \boldsymbol{q}_1)\boldsymbol{q}_3 = -\frac{E\rho^{-1}e_1}{2(e_3 - e_1)}\boldsymbol{q}_3. \tag{2-84}$$

Due to (2-79) and (2-80), the velocities of the longitudinal and transverse waves are lower than those of the corresponding waves in a linear elastic material. In particular, if $e_2 = e_3$, the transverse waves have equal velocities; if not, the lowest velocity is associated with the direction of motion $\boldsymbol{q}_3$, which corresponds to the highest values of the fracture strain (see (1-20)).

If $\boldsymbol{n} \cdot \boldsymbol{q}_1 = 0$, only a transverse wave can propagate with square velocity (2-31), which, because of (2-81), is less than the squared velocity of the transverse linear elastic wave.

In the case of $\boldsymbol{n} \cdot \boldsymbol{q}_2 = 0$ or $\boldsymbol{n} \cdot \boldsymbol{q}_3 = 0$, three waves propagate, a single transverse one with squared velocity less than $\rho^{-1}\mu$, while the other two have squared velocities respectively less than $\rho^{-1}(2\mu + \lambda)$, and less than $\rho^{-1}\mu$ (see (2-82)).

Let us consider $\boldsymbol{E} \in W_2$. For the eigenvalues $c_1^{(2)}, c_2^{(2)}, c_3^{(2)}$, in (2-57)–(2-58), it is an easy matter to prove that

$$c_1^{(2)}(\boldsymbol{E}, \boldsymbol{q}_1) < c_1^{(3)} = \rho^{-1}(2\mu + \lambda), \quad c_3^{(2)}(\boldsymbol{E}, \boldsymbol{q}_1) < c_2^{(3)} = \rho^{-1}\mu \tag{2-85}$$

where the last inequality comes from the condition $2e_3 + \alpha(e_1 + e_2 + e_3) > 0$, which holds in W_2 (see (1-17)).

Analogously, for $c_3^{(2)}$ in (2-62) we have

$$c_3^{(2)}(\boldsymbol{E}, \boldsymbol{n}) < c_3^{(3)} = \rho^{-1}\mu. \tag{2-86}$$

The polarization vector and the squared velocities of waves that propagate in the masonry medium for $\boldsymbol{E} \in W_2$ are summarized in Table 2, for different values of the propagation vector $\boldsymbol{n}$.

If $\boldsymbol{n} = \boldsymbol{q}_1$ or $\boldsymbol{n} = \boldsymbol{q}_2$, there are one longitudinal wave and two transverse waves:

$$A_2(\boldsymbol{E}, \boldsymbol{q}_1)\boldsymbol{q}_1 = \varphi\rho^{-1}\boldsymbol{q}_1, \quad A_2(\boldsymbol{E}, \boldsymbol{q}_1)\boldsymbol{q}_2 = \mu\rho^{-1}\boldsymbol{q}_2, \tag{2-87}$$

$$A_2(\boldsymbol{E}, \boldsymbol{q}_1)\boldsymbol{q}_3 = -\frac{\mu\rho^{-1}}{2+\alpha}\frac{2(1+\alpha)e_1 + \alpha e_2}{e_3 - e_1}\boldsymbol{q}_3; \tag{2-88}$$

$\boldsymbol{E} \in W_2$	$c_1^{(2)}$	$c_2^{(2)}$	$c_3^{(2)}$
$\boldsymbol{n} = \boldsymbol{q}_1$	$\varphi\rho^{-1}$ longitudinal wave, $\boldsymbol{q}_1$	$\mu\rho^{-1}$ transverse wave, $\boldsymbol{q}_2$	(2-58) transverse wave, $\boldsymbol{q}_3$
$\boldsymbol{n} = \boldsymbol{q}_2$	$\varphi\rho^{-1}$ longitudinal wave, $\boldsymbol{q}_2$	$\mu\rho^{-1}$ transverse wave, $\boldsymbol{q}_1$	(2-62) transverse wave, $\boldsymbol{q}_3$
$\boldsymbol{n} = \boldsymbol{q}_3$	(2-65) transverse wave, $\boldsymbol{q}_1$	(2-65) transverse wave, $\boldsymbol{q}_2$	0 no propagation
$\boldsymbol{n} \neq \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$	(2-68)	(2-69)	(2-70)

Table 2. Wave velocities squared for $\boldsymbol{E} \in W_2$.

$$A_2(\boldsymbol{E},\boldsymbol{q}_2)\boldsymbol{q}_2 = \varphi\rho^{-1}\boldsymbol{q}_2, \quad A_2(\boldsymbol{E},\boldsymbol{q}_2)\boldsymbol{q}_1 = \mu\rho^{-1}\boldsymbol{q}_1, \tag{2-89}$$

$$A_2(\boldsymbol{E},\boldsymbol{q}_2)\boldsymbol{q}_3 = -\frac{\mu\rho^{-1}}{2+\alpha}\frac{\alpha e_1 + 2(1+\alpha)e_2}{e_3 - e_2}\boldsymbol{q}_3. \tag{2-90}$$

Due to (2-85) and (2-86), the velocities of longitudinal and transverse waves are less than those of the corresponding waves in a linear elastic material.

If $\boldsymbol{n} = \boldsymbol{q}_3$, no longitudinal waves propagate, and two transverse waves propagate with velocities (2-65) that, in view of (2-85) and (2-86), are less than the velocity of the transverse linear elastic waves.

Note that even though the elasticity tensor is not strongly elliptic in W_1 and W_2, and hence the hypotheses of the Fedorov–Stippes theorem [Gurtin 1972] are not satisfied, longitudinal and transverse progressive waves do exist. In particular, for $\boldsymbol{E} \in W_1$, then a longitudinal wave exists only for $\boldsymbol{n} = \boldsymbol{q}_1$. The other two progressive waves in direction $\boldsymbol{q}_1$, whose directions of motion are equal to $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$, are transverse. For $\boldsymbol{n} \in W_2$, two longitudinal waves exist, one for $\boldsymbol{n} = \boldsymbol{q}_1$, and another for $\boldsymbol{n} = \boldsymbol{q}_2$.

For $\boldsymbol{E} \in W_3$ the material behaves like an isotropic linear elastic material, and there are but two types of progressive waves: longitudinal and transverse, as shown in Table 3.

$\boldsymbol{E} \in W_3$	$c_1^{(3)}$	$c_2^{(3)}$	$c_3^{(3)}$
$\boldsymbol{n}$	$(2\mu+\lambda)\rho^{-1}$ longitudinal wave	$\mu\rho^{-1}$ transverse wave	$\mu\rho^{-1}$ transverse wave

Table 3. Wave velocities squared for $\boldsymbol{E} \in W_3$.

3. The two-dimensional case

Let us consider a plane strain state and, for fixed $\boldsymbol{q}_3$, strain tensors $\boldsymbol{E}$ such that $\boldsymbol{E}\boldsymbol{q}_3 = \boldsymbol{0}$. Let us indicate with the same symbols $\boldsymbol{E}$, $\boldsymbol{T}$ and $\boldsymbol{E}^f$ the restriction of $\boldsymbol{E}$, $\boldsymbol{T}$ and $\boldsymbol{E}^f$ to the two-dimensional subspace of the three-dimensional vector space orthogonal to $\boldsymbol{q}_3$.

Let $e_1 \le e_2$ be the ordered eigenvalues of $\boldsymbol{E}$, and $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ the corresponding eigenvectors and put

$$\boldsymbol{O}_{11} = \boldsymbol{q}_1 \otimes \boldsymbol{q}_1, \quad \boldsymbol{O}_{22} = \boldsymbol{q}_2 \otimes \boldsymbol{q}_2, \quad \boldsymbol{O}_{12} = 1/\sqrt{2}(\boldsymbol{q}_1 \otimes \boldsymbol{q}_2 + \boldsymbol{q}_2 \otimes \boldsymbol{q}_1). \tag{3-1}$$

Define the sets

$$S_0 = \{\boldsymbol{E} : e_1 \ge 0\}, \tag{3-2}$$

$$S_1 = \{\boldsymbol{E} : e_1 \le 0,\ \alpha e_1 + (2+\alpha)e_2 \ge 0\}, \tag{3-3}$$

$$S_2 = \{\boldsymbol{E} : \alpha e_1 + (2+\alpha)e_2 \le 0\}, \tag{3-4}$$

depicted on the next page in Figure 2. Given $\boldsymbol{E}$, the corresponding stress $\boldsymbol{T} = \mathbb{T}(\boldsymbol{E})$ satisfying the constitutive equation of masonry-like materials in the plane strain case is given as follows [Lucchesi et al. 2008]:

$$\text{if } \boldsymbol{E} \in S_0, \text{ then } \boldsymbol{T} = \boldsymbol{0}; \tag{3-5}$$

$$\text{if } \boldsymbol{E} \in S_1, \text{ then } \boldsymbol{T} = \varphi e_1 \boldsymbol{O}_{11}, \quad \text{with } \varphi \text{ as in (2-55)}; \tag{3-6}$$

$$\text{if } \boldsymbol{E} \in S_2, \text{ then } \boldsymbol{T} = \mu[(2+\alpha)e_1 + \alpha e_2]\boldsymbol{O}_{11} + \mu[\alpha e_1 + (2+\alpha)e_2]\boldsymbol{O}_{22}. \tag{3-7}$$

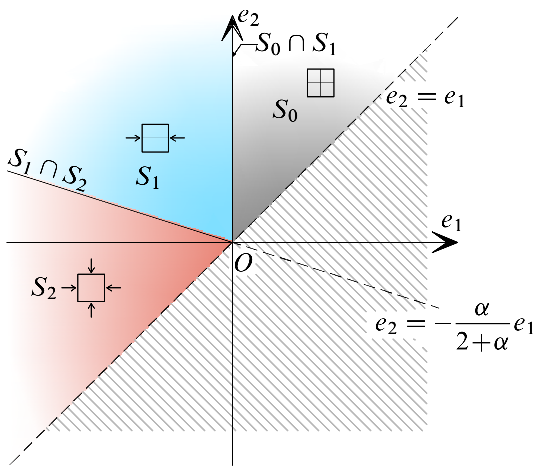

Figure 2. Subdivision of the half-plane $e_1 \leq e_2$ into the regions S_0, S_1 and S_2.

As for the fracture strain, we have

$$\text{if } \boldsymbol{E} \in S_0, \text{ then } \boldsymbol{E}^f = \boldsymbol{E}, \tag{3-8}$$

$$\text{if } \boldsymbol{E} \in S_1, \text{ then } \boldsymbol{E}^f = (e_2 + \frac{\alpha}{2+\alpha} e_1)\boldsymbol{O}_{22}, \tag{3-9}$$

$$\text{if } \boldsymbol{E} \in S_2, \text{ then } \boldsymbol{E}^f = \boldsymbol{0}. \tag{3-10}$$

For Z_i the interior of S_i, the derivative $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ in regions Z_i has been calculated in [Lucchesi et al. 2008] and has the expressions:

$$\text{if } \boldsymbol{E} \in Z_0, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) = \mathbb{O}, \tag{3-11}$$

$$\text{if } \boldsymbol{E} \in Z_1, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) = \varphi \boldsymbol{O}_{11} \otimes \boldsymbol{O}_{11} - \varphi \frac{e_1}{e_2 - e_1} \boldsymbol{O}_{12} \otimes \boldsymbol{O}_{12}, \tag{3-12}$$

$$\text{if } \boldsymbol{E} \in Z_2, \text{ then } D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E}) = \mathbb{C}. \tag{3-13}$$

For $\boldsymbol{E} \in \bigcup_{i=0}^{2} Z_i$ and unit vector $\boldsymbol{n}$, the acoustic tensor $\boldsymbol{A}(\boldsymbol{E}, \boldsymbol{n})$ defined in (2-11) has different expressions $\boldsymbol{A}_i(\boldsymbol{E}, \boldsymbol{n})$ in the three regions Z_i [Degl'Innocenti et al. 2006]:

$$\text{if } \boldsymbol{E} \in Z_0, \text{ then } \boldsymbol{A}_0(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{0}, \tag{3-14}$$

$$\begin{aligned}\text{if } \boldsymbol{E} \in Z_1, \text{ then } \boldsymbol{A}_1(\boldsymbol{E}, \boldsymbol{n}) = \varphi\rho^{-1}\Big[(\boldsymbol{q}_1 \cdot \boldsymbol{n})^2 - \frac{e_1}{2(e_2 - e_1)}(\boldsymbol{q}_2 \cdot \boldsymbol{n})^2\Big]\boldsymbol{O}_{11} \\ - \varphi\rho^{-1}\frac{e_1}{2(e_2 - e_1)}(\boldsymbol{q}_1 \cdot \boldsymbol{n})^2 \boldsymbol{O}_{22} - \varphi\rho^{-1}\frac{e_1}{\sqrt{2}(e_2 - e_1)}(\boldsymbol{q}_1 \cdot \boldsymbol{n})(\boldsymbol{q}_2 \cdot \boldsymbol{n})\boldsymbol{O}_{12},\end{aligned} \tag{3-15}$$

$$\text{if } \boldsymbol{E} \in Z_2, \text{ then } \boldsymbol{A}_2(\boldsymbol{E}, \boldsymbol{n}) = (2\mu + \lambda)\rho^{-1}\boldsymbol{n} \otimes \boldsymbol{n} + \mu\rho^{-1}(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}). \tag{3-16}$$

As in the three-dimensional case, our goal is to determine the eigenvalues $c_1^{(i)}(\boldsymbol{E}, \boldsymbol{n}) \geq c_2^{(i)}(\boldsymbol{E}, \boldsymbol{n})$ and eigenvectors $\boldsymbol{m}_1^{(i)}(\boldsymbol{E}, \boldsymbol{n}), \boldsymbol{m}_2^{(i)}(\boldsymbol{E}, \boldsymbol{n})$ of the acoustic tensor $\boldsymbol{A}_i(\boldsymbol{E}, \boldsymbol{n})$, for $i = 1, 2$.

Let us now consider $\boldsymbol{E} \in Z_1$. The components $A_{ij}(\boldsymbol{E},\boldsymbol{n}) = \boldsymbol{q}_i \cdot \boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})\boldsymbol{q}_j$ of the acoustic tensor $\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})$ in (3-15) with respect to the basis $(\boldsymbol{q}_1, \boldsymbol{q}_2)$ are

$$A_{11}(\boldsymbol{E},\boldsymbol{n}) = \varphi\rho^{-1}(\boldsymbol{q}_1\cdot\boldsymbol{n})^2 - \varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}(\boldsymbol{q}_2\cdot\boldsymbol{n})^2, \tag{3-17}$$

$$A_{22}(\boldsymbol{E},\boldsymbol{n}) = -\varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}(\boldsymbol{q}_1\cdot\boldsymbol{n})^2, \tag{3-18}$$

$$A_{12}(\boldsymbol{E},\boldsymbol{n}) = -\varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}(\boldsymbol{q}_1\cdot\boldsymbol{n})(\boldsymbol{q}_2\cdot\boldsymbol{n}). \tag{3-19}$$

If $\boldsymbol{n} = \boldsymbol{q}_1$, then

$$\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{q}_1) = \varphi\rho^{-1}\boldsymbol{O}_{11} - \varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}\boldsymbol{O}_{22}, \tag{3-20}$$

whose eigenvalues, both greater than zero, and eigenvectors are

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) = \varphi\rho^{-1}, \quad c_2^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) = -\varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}, \tag{3-21}$$

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) = \boldsymbol{q}_1, \qquad \boldsymbol{m}_2^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) = \boldsymbol{q}_2. \tag{3-22}$$

If $\boldsymbol{n} = \boldsymbol{q}_2$, then

$$\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{q}_2) = -\varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}\boldsymbol{O}_{11}, \tag{3-23}$$

whose eigenvalues and eigenvectors are

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_2) = -\varphi\rho^{-1}\frac{e_1}{2(e_2-e_1)}, \quad c_2^{(1)}(\boldsymbol{E},\boldsymbol{q}_2) = 0, \tag{3-24}$$

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_2) = \boldsymbol{q}_1, \qquad \boldsymbol{m}_2^{(1)}(\boldsymbol{E},\boldsymbol{q}_2) = \boldsymbol{q}_2. \tag{3-25}$$

If $\boldsymbol{n} \neq \boldsymbol{q}_1$ and $\boldsymbol{n} \neq \boldsymbol{q}_2$, the eigenvalues and eigenvectors of (3-15) are

$$c_1^{(1)}(\boldsymbol{E},\boldsymbol{n}) = \frac{\varphi\rho^{-1}}{2}\left\{(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2 - \frac{e_1}{2(e_2-e_1)} + \sqrt{\frac{e_2+e_1}{e_2-e_1}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^4 - \frac{e_1}{e_2-e_1}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2 + \frac{e_1^2}{4(e_2-e_1)^2}}\right], \tag{3-26}$$

$$c_2^{(1)}(\boldsymbol{E},\boldsymbol{n}) = \frac{\varphi\rho^{-1}}{2}\left[(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2 - \frac{e_1}{2(e_2-e_1)} - \sqrt{\frac{e_2+e_1}{e_2-e_1}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^4 - \frac{e_1}{e_2-e_1}(\boldsymbol{n}\cdot\boldsymbol{q}_1)^2 + \frac{e_1^2}{4(e_2-e_1)^2}}\right], \tag{3-27}$$

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E},\boldsymbol{n}) = \frac{1}{m_1}\left(\boldsymbol{q}_1 + \frac{c_1^{(1)}(\boldsymbol{E},\boldsymbol{n}) - A_{11}(\boldsymbol{E},\boldsymbol{n})}{A_{12}(\boldsymbol{E},\boldsymbol{n})}\boldsymbol{q}_2\right), \tag{3-28}$$

$$\boldsymbol{m}_2^{(1)}(\boldsymbol{E},\boldsymbol{n}) = \frac{1}{m_2}\left(\frac{c_2^{(1)}(\boldsymbol{E},\boldsymbol{n}) - A_{22}(\boldsymbol{E},\boldsymbol{n})}{A_{12}(\boldsymbol{E},\boldsymbol{n})}\boldsymbol{q}_1 + \boldsymbol{q}_2\right), \tag{3-29}$$

where $A_{11}(\boldsymbol{E},\boldsymbol{n})$, $A_{22}(\boldsymbol{E},\boldsymbol{n})$ and $A_{12}(\boldsymbol{E},\boldsymbol{n})$ (which is different from zero since $\boldsymbol{n} \neq \boldsymbol{q}_1$ and $\boldsymbol{n} \neq \boldsymbol{q}_2$ and

$\boldsymbol{E} \in Z_1$) are given by (3-17), (3-18) and (3-19) and

$$m_1 = \sqrt{1 + \frac{(c_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) - A_{11}(\boldsymbol{E}, \boldsymbol{n}))^2}{A_{12}(\boldsymbol{E}, \boldsymbol{n})^2}}, \tag{3-30}$$

$$m_2 = \sqrt{1 + \frac{(c_2^{(1)}(\boldsymbol{E}, \boldsymbol{n}) - A_{22}(\boldsymbol{E}, \boldsymbol{n}))^2}{A_{12}(\boldsymbol{E}, \boldsymbol{n})^2}}. \tag{3-31}$$

Note that, taking (3-3) into account, it is a simple matter to prove that

$$c_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) < (2\mu + \lambda)\rho^{-1}, \quad c_2^{(1)}(\boldsymbol{E}, \boldsymbol{n}) < \mu\rho^{-1}, \tag{3-32}$$

for $c_1^{(1)}$ and $c_2^{(1)}$ given in (3-21), (3-24), and (3-26)–(3-27).

If $\boldsymbol{E} \in Z_2$, the acoustic tensor $\boldsymbol{A}_2(\boldsymbol{E}, \boldsymbol{n})$ coincides with the acoustic tensor of a linear elastic material subjected to a plane strain state. Its eigenvalues are

$$c_1^{(2)} = (2\mu + \lambda)\rho^{-1}, \quad c_2^{(2)} = c_3^{(3)} = \mu\rho^{-1}, \tag{3-33}$$

and the corresponding eigenvectors are

$$\boldsymbol{m}_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) = \boldsymbol{n}, \quad \boldsymbol{m}_2^{(2)}(\boldsymbol{E}, \boldsymbol{n}) \text{ belonging to } \mathrm{Span}(\boldsymbol{n})^{\perp}. \tag{3-34}$$

Thus, the behavior of progressive waves in a body composed of a masonry-like material with homogeneous stress $\boldsymbol{T}$ associated to a homogeneous plane strain field $\boldsymbol{E} \in \bigcup_{i=0}^{2} Z_i$ can be summarized in Table 4, which reports the polarization vector and the squared velocity of waves propagating in masonry solids for different directions of propagation $\boldsymbol{n}$. If $\boldsymbol{E} \in Z_0$, no propagation occurs. If $\boldsymbol{E} \in Z_1$, for $\boldsymbol{n} = \boldsymbol{q}_1$

$\boldsymbol{E} \in Z_0$	$c_1^{(0)}$	$c_2^{(0)}$
$\boldsymbol{n}$	0 no propagation	0 no propagation

$\boldsymbol{E} \in Z_1$	$c_1^{(1)}$	$c_2^{(1)}$
$\boldsymbol{n} = \boldsymbol{q}_1$	$\varphi\rho^{-1}$ longitudinal wave, $\boldsymbol{q}_1$	$-\varphi\rho^{-1}e_1/(2(e_2 - e_1))$ transverse wave, $\boldsymbol{q}_2$
$\boldsymbol{n} = \boldsymbol{q}_2$	$-\varphi\rho^{-1}e_1/(2(e_2 - e_1))$ transverse wave, $\boldsymbol{q}_1$	0 no propagation
$\boldsymbol{n} \neq \boldsymbol{q}_1, \boldsymbol{q}_2$	(3-26)	(3-27)

$\boldsymbol{E} \in Z_2$	$c_1^{(2)}$	$c_2^{(2)}$
$\boldsymbol{n}$	$(2\mu + \lambda)\rho^{-1}$ longitudinal wave	$\mu\rho^{-1}$ transverse wave

Table 4. Wave velocities squared for $\boldsymbol{E} \in Z_i$, $i = 0, 1, 2$.

there is one longitudinal wave and one transverse wave,

$$\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{q}_1)\boldsymbol{q}_1 = \varphi\rho^{-1}\boldsymbol{q}_1, \tag{3-35}$$

$$\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{q}_1)\boldsymbol{q}_2 = -\frac{\varphi\rho^{-1}e_1}{2(e_2-e_1)}\boldsymbol{q}_2. \tag{3-36}$$

Due to (3-32), the velocities of the longitudinal and transverse waves are less than those of the corresponding waves in a linear elastic material. For $\boldsymbol{n} = \boldsymbol{q}_2$, only one transverse wave propagates, with squared velocity (3-24), which because of (3-32), is less than the squared velocity of the transverse linear elastic wave. For $\boldsymbol{n} \neq \boldsymbol{q}_1$ and $\boldsymbol{n} \neq \boldsymbol{q}_2$, two waves propagate with the squared velocities (3-26) and (3-27), that are less than the squared velocities of the longitudinal and transverse waves in a linear elastic material (see (3-32)).

For $\boldsymbol{E} \in Z_2$, masonry material behaves like a linear elastic material and for each unit vector $\boldsymbol{n}$, there are a longitudinal wave and a transverse wave with the squared velocities in (3-33).

Now, we wish to analyze the behavior of eigenvalues $c_1^{(1)}(\boldsymbol{E},\boldsymbol{n})$ and $c_2^{(1)}(\boldsymbol{E},\boldsymbol{n})$ of $\boldsymbol{A}_1(\boldsymbol{E},\boldsymbol{n})$, as $\boldsymbol{n}$ varies.

Let us put

$$z = (\boldsymbol{n}\cdot\boldsymbol{q}_1)^2, \quad z \in [0,1], \tag{3-37}$$

and

$$k = -e_2/e_1, \tag{3-38}$$

with k satisfying the inequality

$$k > \frac{\alpha}{2+\alpha}, \tag{3-39}$$

because $\boldsymbol{E}$ belongs to Z_1.

By taking (3-37) and (3-38) into account, the eigenvalue $c_1^{(1)}(\boldsymbol{E},\boldsymbol{n})$ in (3-26) can be expressed in terms of z and k via the expression

$$f_1(z;k) = \frac{\varphi\rho^{-1}}{2}\left(z + \frac{1}{2(k+1)} + \sqrt{\frac{k-1}{k+1}z^2 + \frac{1}{k+1}z + \frac{1}{4(k+1)^2}}\right). \tag{3-40}$$

For each k satisfying (3-39), $f_1(z;k)$ is an increasing function of z, with

$$f_1(0;k) = c_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_2) = \frac{\varphi\rho^{-1}}{2(k+1)}, \tag{3-41}$$

$$f_1(1;k) = c_1^{(1)}(\boldsymbol{E},\boldsymbol{q}_1) = \varphi\rho^{-1}, \tag{3-42}$$

thus,

$$f_1(z;k) \le \varphi\rho^{-1} \text{ for each } z \in [0,1], \tag{3-43}$$

and

$$\begin{aligned} &\lim_{k\to\frac{\alpha}{2+\alpha}} f_1(0;k) = \mu\rho^{-1}, \\ &\lim_{k\to\infty} f_1(z;k) = \varphi\rho^{-1}z \quad \text{for each } z \in [0,1]. \end{aligned} \tag{3-44}$$

Analogously, the eigenvalue $c_2^{(1)}(\boldsymbol{E}, \boldsymbol{n})$ in (3-27) can be expressed in terms of z and k via the expression

$$f_2(z;k) = \frac{\varphi\rho^{-1}}{2}\left(z + \frac{1}{2(k+1)} - \sqrt{\frac{k-1}{k+1}z^2 + \frac{1}{k+1}z + \frac{1}{4(k+1)^2}}\right). \tag{3-45}$$

In particular, for each k satisfying (3-39),

$$f_2(0;k) = c_2^{(1)}(\boldsymbol{E}, \boldsymbol{q}_2) = 0, \tag{3-46}$$

$$f_2(1;k) = c_2^{(1)}(\boldsymbol{E}, \boldsymbol{q}_1) = \frac{\varphi\rho^{-1}}{2(k+1)}, \tag{3-47}$$

and

$$f_2(z;k) \le \frac{\varphi\rho^{-1}}{2(k+1)} \text{ for each } z \in [0, 1], \tag{3-48}$$

moreover,

$$\lim_{k\to\infty} f_2(z;k) = 0, \qquad \lim_{k\to\alpha/(2+\alpha)} f_2(1;k) = \mu\rho^{-1} \text{ for each } z \in [0, 1]. \tag{3-49}$$

As for the elastic constants, we have assumed $E/\rho = 500000\,(\text{m/s})^2$ and $\nu = 0.2$ for the Poisson's ratio. Consequently, we have $\mu/\rho = E/(2\rho(1+\nu)) = 208333\,(\text{m/s})^2$, $\lambda/\rho = \nu E/(\rho(1+\nu)(1-2\nu)) = 138889\,(\text{m/s})^2$, $\varphi/\rho = 520833\,(\text{m/s})^2$, $(2\mu+\lambda)/\rho = 555556\,(\text{m/s})^2$ and $\alpha = 0.7$.

Figures 3 and 4 show the behavior of $f_1(z;k)$ and $f_2(z;k)$ versus z for different values of k compared with the eigenvalues $(2\mu+\lambda)\rho^{-1}$ and $\mu\rho^{-1}$ of the acoustic tensor corresponding to a linear elastic material. The dashed line represents the function $\lim_{k\to\infty} f_1(z;k)$ in (3-44). In particular, we have chosen $k = 0.25$ (corresponding to $\alpha/(2+\alpha)$) and $k = 0.3, 0.5, 1, 2, 10$. Both (3-44) and (3-49) are in agreement with the jump conditions (3-65) and (3-67), reported in the Appendix, of the acoustic tensor at the interfaces $S_0 \cap S_1$ and $S_1 \cap S_2$, which are reached when $k \to \infty$ and $k \to \alpha/(2+\alpha)$. Note that waves propagating in a masonry-like material are slower than waves propagating in a linear elastic material and that their velocities decrease as k increases.

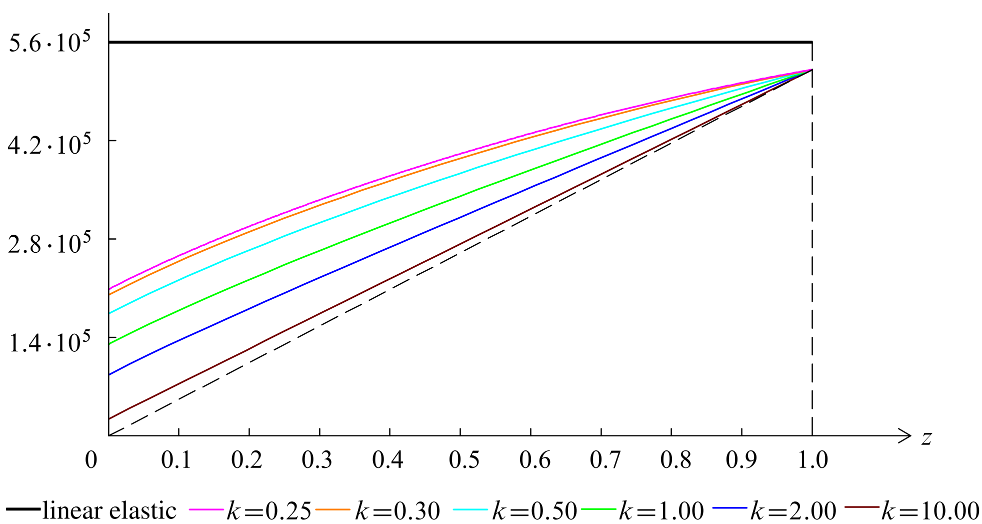

Figure 3. Function $f_1(z;k)$ vs. $z \in [0, 1]$ for different values of k.

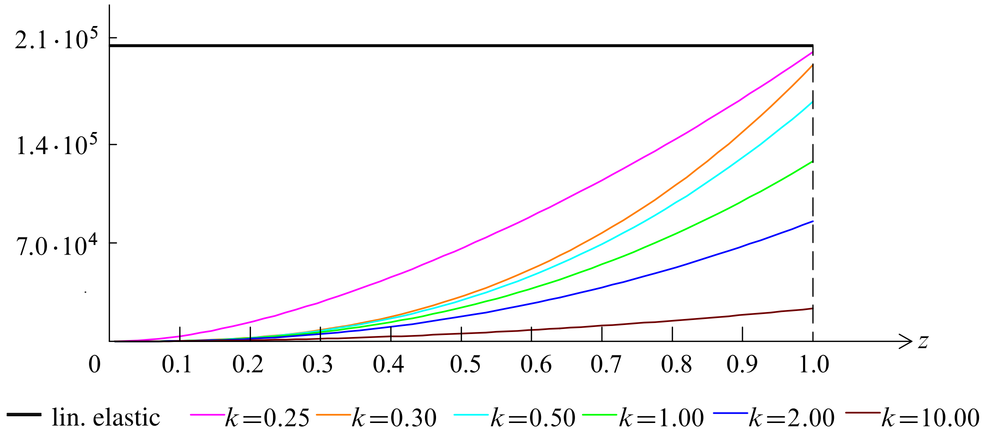

Figure 4. Function $f_2(z;k)$ vs. $z \in [0,1]$ for different values of k.

Figure 5 shows a plot of the curves $\mathscr{C}_1(k)$ composed of the points having coordinates

$$(\tau f_1(\tau^2;k),\ \sqrt{1-\tau^2}\, f_1(\tau^2;k)) \tag{3-50}$$

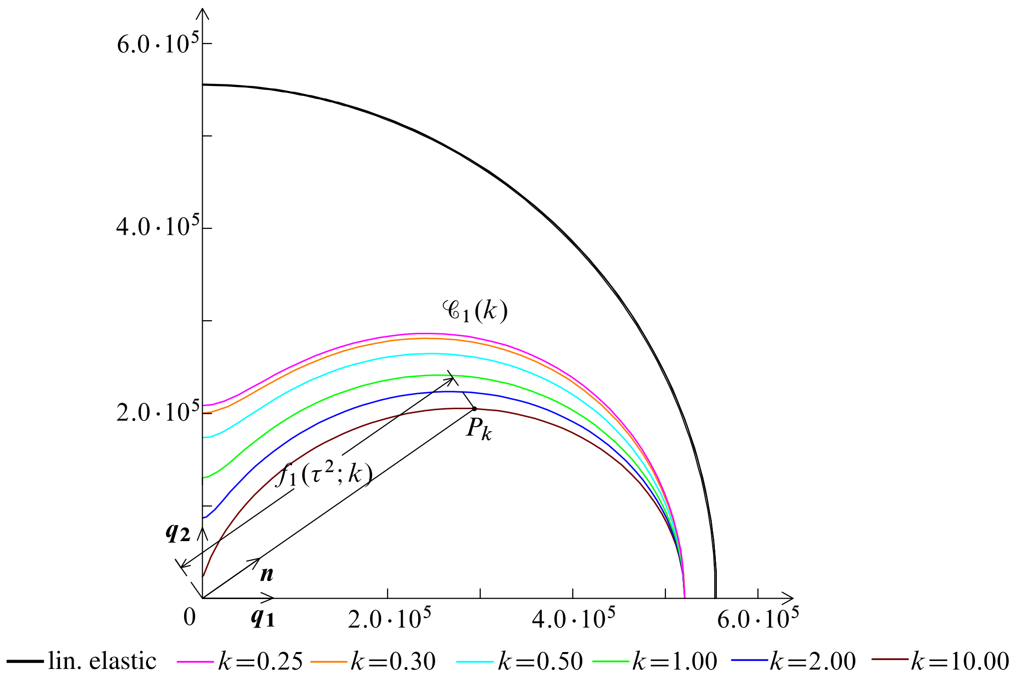

Figure 5. Curve $\mathscr{C}_1(k)$ formed by the points with coordinates in (3-50), for $k = 0.25$, 0.30, 0.50, 1.00, 2.00, and 10.00, from the top down. The thick black curve represents the linear elastic case.

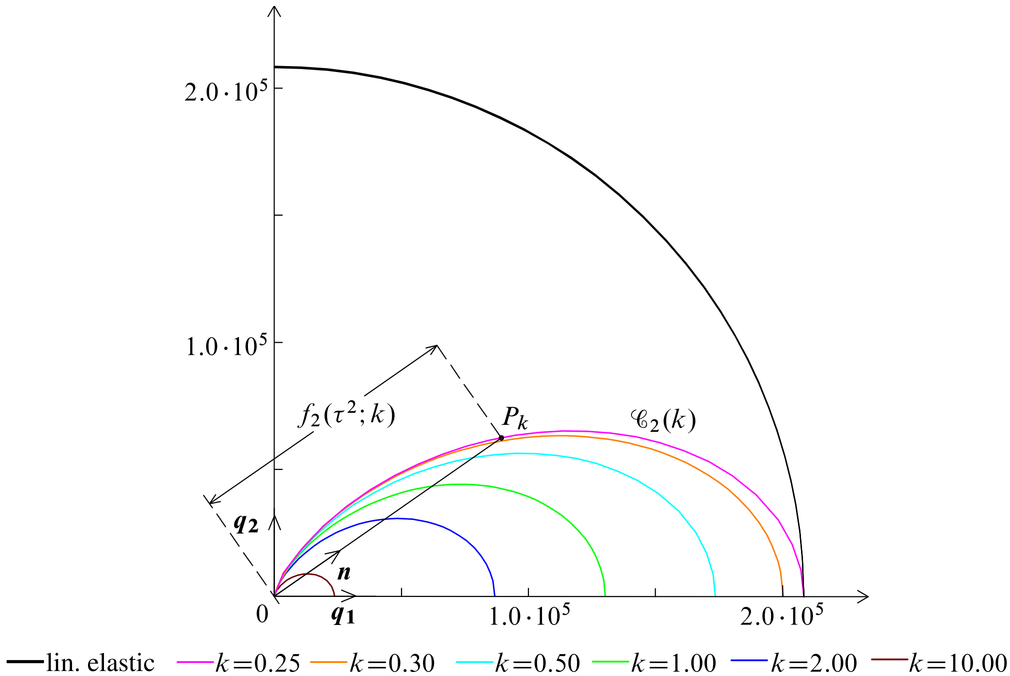

Figure 6. Curve $\mathscr{C}_2(k)$ formed by the points with coordinates in (3-51), for $k = 0.25$, 0.30, 0.50, 1.00, 2.00, and 10.00, from the top down. The thick black curve represents the linear elastic case.

with respect to the Cartesian coordinate frame $\{\boldsymbol{q}_1, \boldsymbol{q}_2, O\}$, with $\tau = \boldsymbol{n} \cdot \boldsymbol{q}_1 \in [0, 1]$ for $k = 0.25, 0.3, 0.5, 1, 2, 10$. For P_k the intersection point of curve $\mathscr{C}_1(k)$ and the line passing from the origin and parallel to the unit vector $\boldsymbol{n}$ of components $(\boldsymbol{n} \cdot \boldsymbol{q}_1, \boldsymbol{n} \cdot \boldsymbol{q}_2)$, the length of the segment OP_k is $f_1((\boldsymbol{n} \cdot \boldsymbol{q}_1)^2; k)$, which coincides with the maximum squared velocity of a wave propagating along $\boldsymbol{n}$.

Analogously, Figure 6 plots the curves $\mathscr{C}_2(k)$, composed of the points having coordinates

$$(\tau f_2(\tau^2; k),\ \sqrt{1-\tau^2}\, f_2(\tau^2; k)) \tag{3-51}$$

with respect to the Cartesian coordinate frame $\{\boldsymbol{q}_1, \boldsymbol{q}_2, O\}$, with $\tau = \boldsymbol{n} \cdot \boldsymbol{q}_1 \in [0, 1]$ for $k = 0.25, 0.3, 0.5, 1, 2, 10$. For P_k the intersection point of curve $\mathscr{C}_2(k)$ and the line passing from the origin and parallel to the unit vector $\boldsymbol{n}$ of components $(\boldsymbol{n} \cdot \boldsymbol{q}_1, \boldsymbol{n} \cdot \boldsymbol{q}_2)$, the length of the segment OP_k is $f_2((\boldsymbol{n} \cdot \boldsymbol{q}_1)^2; k)$, which coincides with the minimum squared velocity of a wave propagating along $\boldsymbol{n}$.

Figure 7 shows the quantities

$$d_1(\tau; k) = \frac{|\sqrt{f_1(\tau^2; k)} - \sqrt{(2\mu+\lambda)\rho^{-1}}|}{\sqrt{(2\mu+\lambda)\rho^{-1}}}, \quad d_2(\tau; k) = \frac{|\sqrt{f_2(\tau^2; k)} - \sqrt{\mu\rho^{-1}}|}{\sqrt{\mu\rho^{-1}}} \tag{3-52}$$

as a function of τ for different values of k. They measure the relative distance between the maximum (top half) and minimum (bottom half) squared velocities of waves propagating in a masonry-like and a linear elastic material. Figure 7 shows that, for a given k, the relative distances $d_1(\tau; k)$ and $d_2(\tau; k)$

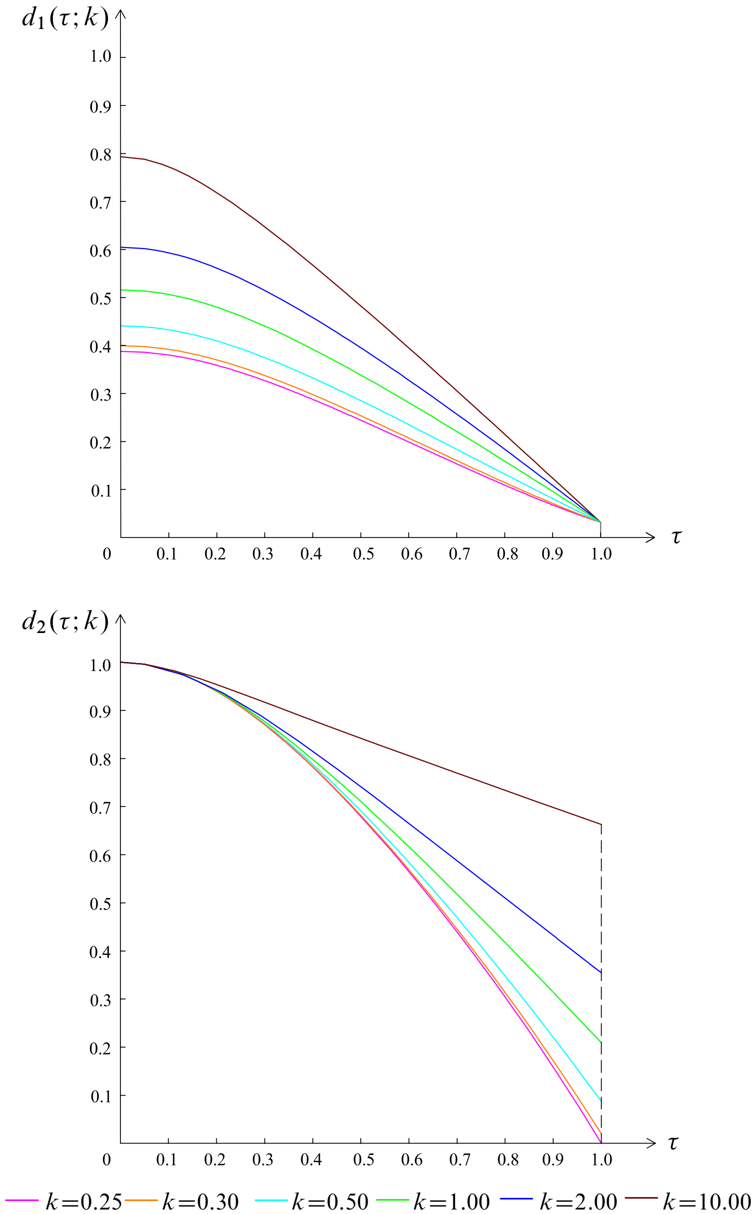

Figure 7. Functions $d_1(\tau; k)$ and $d_2(\tau; k)$ vs. $\tau \in [0, 1]$ for $k = 0.25$, 0.30, 0.50, 1.00, 2.00, and 10.00, from the bottom up.

have a maximum for $\tau = \boldsymbol{n} \cdot \boldsymbol{q}_1 = 0$, namely when the propagation vector is orthogonal to the direction of maximum compression $\boldsymbol{q}_1$, and have a minimum when the propagation vector coincides with $\boldsymbol{q}_1$.

Figure 8 shows the behavior of the scalar product $r(\tau; k) = \boldsymbol{m}_1^{(1)}(\boldsymbol{E}, \boldsymbol{n}) \cdot \boldsymbol{n}$ as a function of $\tau = \boldsymbol{n} \cdot \boldsymbol{q}_1 \in [0, 1]$; it is a measure of the deviation of the eigenvector $\boldsymbol{m}_1^{(1)}(\boldsymbol{E}, \boldsymbol{n})$ from the direction of propagation $\boldsymbol{n}$, along which the longitudinal wave travels in the linear elastic case.

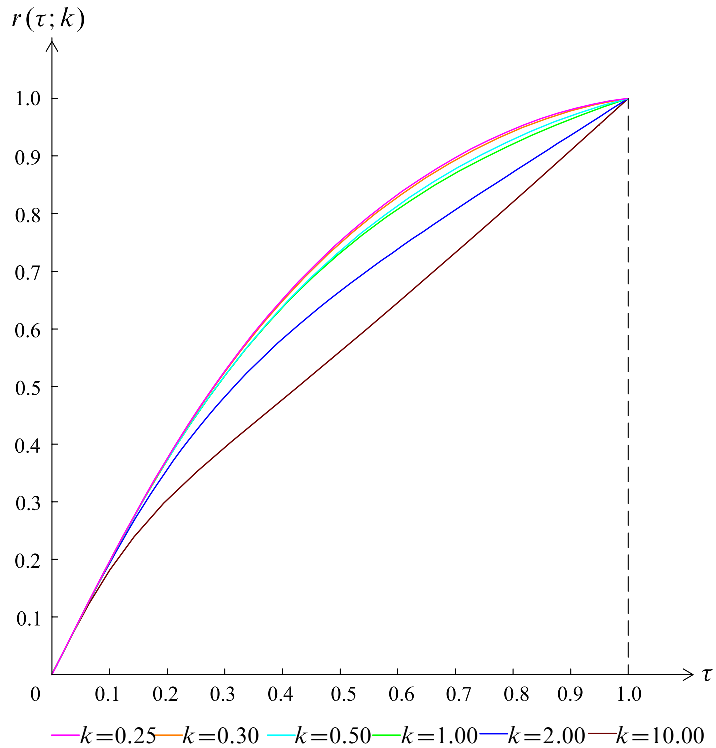

Figure 8. Function $r(\tau; k)$ vs. $\tau \in [0, 1]$ for different values of k.

Let us now consider the homogenous wave

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{m} \sin(\boldsymbol{n} \cdot \boldsymbol{x} - vt). \tag{3-53}$$

We wish to analyze the behavior of $\boldsymbol{u}$ as function of time t in order to highlight the differences between a linear elastic and a masonry-like material. For $\boldsymbol{n} = \boldsymbol{q}_1$, we can distinguish the longitudinal wave

$$\boldsymbol{u}_L(x_1, t) = \boldsymbol{q}_1 \sin(x_1 - \sqrt{\varphi\rho^{-1}}\,t), \tag{3-54}$$

and the transverse wave

$$\boldsymbol{u}_T(x_1, t) = \boldsymbol{q}_2 \sin\left(x_1 - \sqrt{\frac{\varphi\rho^{-1}}{2(k+1)}}\,t\right), \tag{3-55}$$

with $x_1 = \boldsymbol{q}_1 \cdot \boldsymbol{x}$ and $t \geq 0$. The longitudinal and transverse waves in the case of linear elastic materials are respectively

$$\boldsymbol{u}_L^e(x_1, t) = \boldsymbol{q}_1 \sin(x_1 - \sqrt{(2\mu + \lambda)\rho^{-1}}\,t), \tag{3-56}$$

$$\boldsymbol{u}_T^e(x_1, t) = \boldsymbol{q}_2 \sin(x_1 - \sqrt{\mu\rho^{-1}}\,t), \tag{3-57}$$

Figure 9 shows the functions $\boldsymbol{u}_L(x_1, t)$ (red line) and $\boldsymbol{u}_L^e(x_1, t)$ (black line) versus t at $x_1 = 0$. Figure 10

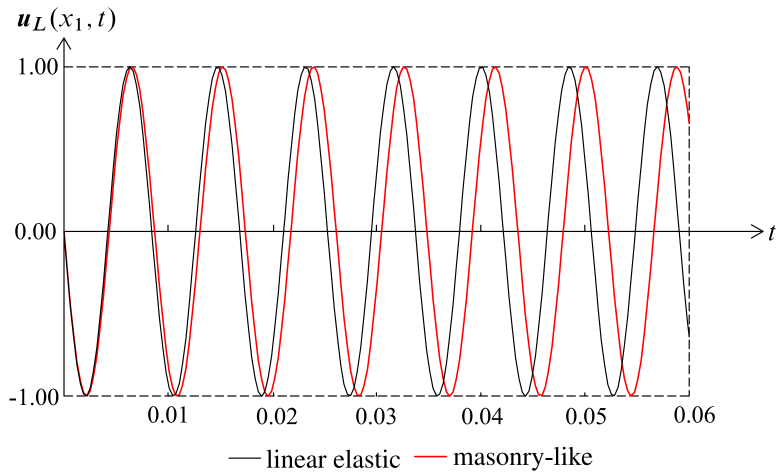

Figure 9. Longitudinal wave propagating along $\boldsymbol{n} = \boldsymbol{q}_1$, $\boldsymbol{u}_L(x_1, t)$ at $x_1 = 0$ vs. t for different values of k.

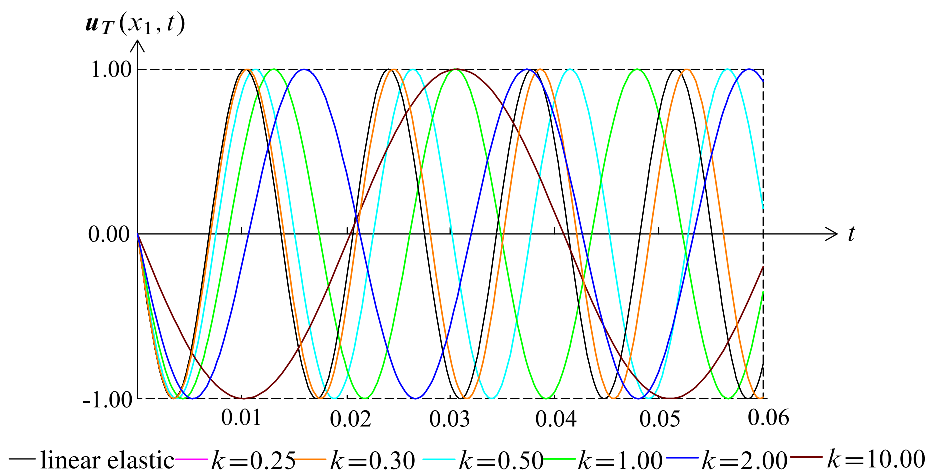

Figure 10. Transverse wave propagating along $\boldsymbol{n} = \boldsymbol{q}_1$, $\boldsymbol{u}_T(x_1, t)$ at $x_1 = 0$ vs. t for different values of k.

shows the functions $\boldsymbol{u}_T(x_1, t)$ and $\boldsymbol{u}^e_T(x_1, t)$ versus t at $x_1 = 0$. Displacement $\boldsymbol{u}_T(x_1, t)$ is plotted for $k = 0.25$, 0.3, 0.5, 1, 2, 10.

For $\boldsymbol{n} = \boldsymbol{q}_2$, there is only one transverse wave,

$$\boldsymbol{u}_T(x_2, t) = \boldsymbol{q}_1 \sin\left(x_2 - \sqrt{\frac{\varphi\rho^{-1}}{2(k+1)}}\,t\right), \tag{3-58}$$

with $x_2 = \boldsymbol{q}_2 \cdot \boldsymbol{x}$, and no longitudinal waves propagate in the material,

$$\boldsymbol{u}_L(x_2, t) = \boldsymbol{0}. \tag{3-59}$$

As for the linear elastic case, we have

$$\boldsymbol{u}_L^e(x_2,t)=\boldsymbol{q}_2\sin(x_2-\sqrt{(2\mu+\lambda)\rho^{-1}}t), \tag{3-60}$$

$$\boldsymbol{u}_T^e(x_2,t)=\boldsymbol{q}_1\sin(x_1-\sqrt{\mu\rho^{-1}}t). \tag{3-61}$$

By replacing φ with E, we obtain the results for the plane stress state. Note that in view of the inequality $\varphi > E$, the velocities of elastic waves, both longitudinal and transverse, in the plane strain state are greater than the velocities in the plane stress state.

Conclusions

The propagation of elastic waves in an infinite masonry-like body subjected to a given homogeneous strain field has been investigated. Masonry-like materials are characterized by the fact that they cannot withstand tensile stresses and, as a consequence, they can crack. The stress function $\mathbb{T}$ defined on Sym with values in the subset of the negative-semidefinite symmetric tensors is nonlinear and differentiable on an open subset W of Sym. Starting with the differentiability of $\mathbb{T}$ with respect to $\boldsymbol{E}$ on W and using the explicit expression for $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$, we obtain the condition that a progressive wave must satisfy in order to propagate in a masonry body subjected to a given homogeneous strain field $\boldsymbol{E}$. The propagation condition involves the acoustic tensor, which is a function of $\boldsymbol{E}$ and the direction of propagation $\boldsymbol{n}$. We show that the behavior of progressive waves propagating in the solid depends on the state of prestrain $\boldsymbol{E}$ and on the corresponding crack distribution. In particular, due to the presence of cracks, the propagation velocity of waves in masonry-like solids is lower than in linear elastic materials. A peculiar aspect of masonry-like solids is that there exist directions $\boldsymbol{n}$ along which waves cannot propagate. The preliminary results obtained in this paper can constitute a basis for the study of the propagation of small elastic waves in masonry constructions. The problem is quite relevant to technical applications: in fact, measurement of the wave propagation velocities in masonry buildings can furnish important information about the mechanical behavior of their undamaged and cracked portions.

Appendix

As pointed out in [Lucchesi et al. 2008], no tangential discontinuity affects $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$ across the interfaces $S_0\cap S_1$ and $S_1\cap S_2$. In fact its jumps are

$$[D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})]=\varphi\boldsymbol{O}_{11}\otimes\boldsymbol{O}_{11}\ \text{for}\ \boldsymbol{E}\in S_0\cap S_1, \tag{3-62}$$

and

$$[D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})]=\mu\Big\{\frac{\alpha^2}{2+\alpha}\boldsymbol{O}_{11}\otimes\boldsymbol{O}_{11}+(2+\alpha)\boldsymbol{O}_{22}\otimes\boldsymbol{O}_{22}+\alpha(\boldsymbol{O}_{11}\otimes\boldsymbol{O}_{22}+\boldsymbol{O}_{22}\otimes\boldsymbol{O}_{11})\Big\}\quad \text{for}\ \boldsymbol{E}\in S_1\cap S_2. \tag{3-63}$$

For the jumps that the acoustic tensor inherits from $D_{\boldsymbol{E}}\mathbb{T}(\boldsymbol{E})$, from (3-14), (3-15) and (3-16) we get

$$[\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})]=\varphi\rho^{-1}(\boldsymbol{q}_1\cdot\boldsymbol{n})^2\boldsymbol{O}_{11}\ \text{for}\ \boldsymbol{E}\in S_0\cap S_1, \tag{3-64}$$

and, in particular,

$$[\boldsymbol{A}(\boldsymbol{E},\boldsymbol{q}_1)]=\varphi\rho^{-1}\boldsymbol{O}_{11},[\boldsymbol{A}(\boldsymbol{E},\boldsymbol{q}_2)]=\boldsymbol{0}. \tag{3-65}$$

Moreover,

$$[\boldsymbol{A}(\boldsymbol{E},\boldsymbol{n})] = \mu\rho^{-1}\left[1+\frac{(1+\alpha)(\alpha-2)}{2+\alpha}(\boldsymbol{q}_1\cdot\boldsymbol{n})^2+\frac{2(1+\alpha)e_1}{(2+\alpha)(e_2-e_1)}(\boldsymbol{q}_2\cdot\boldsymbol{n})^2\right]\boldsymbol{O}_{11} + \mu\rho^{-1}\left[1+\frac{2(1+\alpha)e_1}{(2+\alpha)(e_2-e_1)}(\boldsymbol{q}_1\cdot\boldsymbol{n})^2+(1+\alpha)(\boldsymbol{q}_2\cdot\boldsymbol{n})^2\right]\boldsymbol{O}_{22} - \varphi\rho^{-1}\frac{\alpha e_1}{\sqrt{2}(e_2-e_1)}(\boldsymbol{q}_1\cdot\boldsymbol{n})(\boldsymbol{q}_2\cdot\boldsymbol{n})\boldsymbol{O}_{12} \text{ for } \boldsymbol{E}\in S_1\cap S_2, \quad (3\text{-}66)$$

and

$$[\boldsymbol{A}(\boldsymbol{E},\boldsymbol{q}_1)] = \mu\rho^{-1}\frac{\alpha^2}{2+\alpha}\boldsymbol{O}_{11}, \quad [\boldsymbol{A}(\boldsymbol{E},\boldsymbol{q}_2)] = (2\mu+\lambda)\rho^{-1}\boldsymbol{O}_{22}. \quad (3\text{-}67)$$

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