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ANALYTICAL AND NUMERICAL SOLUTION OF THE FRACTIONAL EULER–BERNOULLI BEAM EQUATION

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In this paper a new formulation of the Euler–Bernoulli beam equation is proposed, which is based on fractional calculus. The fractional Euler–Bernoulli beam equation is derived by using a variational approach. Such formulation leads to an equation containing left and right fractional Caputo derivatives simultaneously. The obtained equation is transformed into an integral equation and then is solved analytically and numerically. Finally, examples of computations and error analysis are shown.

1. Introduction

Fractional calculus has recently played a very important role in various fields of science [Baleanu et al. 2015; Błasik and Klimek 2015; Leszczyński 2011; Klimek 2001; Kukla and Siedlecka 2015; Podlubny 1999; Torres 2015; Zhang et al. 2016; Zingales and Failla 2015]. It is caused largely by the fact that the fractional derivatives are nonlocal operators and depend on the past values of a function (the left derivative) or the future values of a function (the right derivative) [Baleanu et al. 2012; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993]. On the other hand, nonlocal formulations play an essential role in the description of the material deformation. Considering the description of the material deformation, including length scale, we are able to describe the phenomena (such as scale effects) where the classical approach is no longer valid [Sumelka 2014a; 2014b; Sumelka and Błaszczyk 2014].

The first approaches to link fractional calculus and nonlocal continuum mechanics come from Vazquez [2004], Lazopoulos [2006] and Di Paola and Zingales [2008]. Later, Carpinteri et al. [2011; 2014] used spatial fractional calculus to examine a material whose nonlocal stress is defined as the fractional integral of the strain field, highlighting its connection with Eringen [2002] nonlocal elasticity. A similar problem was analyzed by Atanackovic & Stankovic [2009] starting from fractional nonlocal strain measure. Other fractional approaches to nonlocal elasticity are presented in [Alotta et al. 2015; Drapaca and Sivaloganathan 2012; Tarasov 2006; Zingales and Failla 2015].

In addition, recent research [Paola et al. 2013; Pirrotta et al. 2015] on the response evaluation of a viscoelastic Euler–Bernoulli or Timoshenko beam under quasistatic and dynamic loads have shown that for a better understanding of the viscoelastic behavior, a fractional constitutive law should be considered. Vibration analysis of a simply supported beam with a fractional order viscoelastic material model is presented in [Freundlich 2013].

As a concluding remark, it should be emphasised that there are many concepts dealing with nonlocal formulations. However, they often require a large number of parameters. As a response to this inconvenience, new models based on fractional calculus have been proposed.

Keywords: Euler–Bernoulli beam equation, Fractional Euler–Lagrange equation, Analytical and numerical solution, Caputo derivatives.

A good example where fractional models have been successfully used is in a group of viscoelastic materials. These materials are very attractive for civil engineering applications, and thus need special attention. One of the advantages of the fractional models is that they require a smaller number of parameters than classical models containing operators of integer order.

Due to these facts, fractional calculus looks to be a promising tool for modeling scale-dependent material behavior.

In our previous work [Sumelka et al. 2015] the classical Euler–Bernoulli beam theory was reformulated utilizing fractional Riesz–Caputo derivatives. In this paper we propose a new formulation of the Euler–Bernoulli beam equation based on fractional variational calculus. The fractional derivative included in a functional (see Section 3) is applied as a spatial derivative, and influences spatial response. In this sense the fractional order of derivation α is a new material parameter. The parameter α controls the way in which information is governed from the region of influence [Sumelka 2014a; Sumelka and Błaszczyk 2014].

We transform the obtained Euler–Lagrange equation into an integral equation and we find the exact solution. Next, we present two numerical schemes. The first one is based on the discretization of the analytical solution and the second one is based on the discretization of the Euler–Lagrange equation. We calculated the errors generated by both schemes and estimated the rates of convergences of the presented methods for a particular case. Finally, we show a few computational examples of static deflections for various types of loads.

2. Fractional operators

In this section, we only recall necessary definitions of fractional operators and their properties [Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993]. The left and right Caputo derivatives of order $1 < \alpha \le 2$ are respectively defined as

$$ {}^{C}D^{\alpha}_{0^+}u(x) := \begin{cases} I^{2-\alpha}_{0^+}D^2u(x) & \text{for } 1 < \alpha < 2, \\ D^2u(x) & \text{for } \alpha = 2, \end{cases} \tag{2-1} $$

$$ {}^{C}D^{\alpha}_{L^-}u(x) := \begin{cases} I^{2-\alpha}_{L^-}D^2u(x) & \text{for } 1 < \alpha < 2, \\ D^2u(x) & \text{for } \alpha = 2, \end{cases} \tag{2-2} $$

where D^2 is the operator of the second order derivative and operators $I^{\alpha}_{0^+}$ and $I^{\alpha}_{b^-}$ are respectively the left and right fractional Riemann–Liouville integrals of order $\alpha > 0$ defined by

$$ I^{\alpha}_{0^+}u(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{u(\tau)}{(x-\tau)^{1-\alpha}}\, \mathrm{d}\tau \quad (x > 0), \tag{2-3} $$

$$ I^{\alpha}_{L^-}u(x) := \frac{1}{\Gamma(\alpha)} \int_x^L \frac{u(\tau)}{(\tau-x)^{1-\alpha}}\, \mathrm{d}\tau \quad (x < L), \tag{2-4} $$

where Γ is the Euler Gamma function.

The composition rules of the fractional operators (for $\alpha \in (1,2]$) are

$$I_{0^+}^{\alpha}\,{}^{C}D_{0^+}^{\alpha}\,u(x) = u(x) - xu'(a) - u(a), \tag{2-5}$$

$$I_{L^-}^{\alpha}\,{}^{C}D_{L^-}^{\alpha}\,u(x) = u(x) + (L-x)u'(L) - u(L). \tag{2-6}$$

The left and right fractional integrals of the constant C have the following form:

$$I_{0^+}^{\alpha}\,C = Cx^{\alpha}/(\Gamma(1+\alpha)), \tag{2-7}$$

$$I_{L^-}^{\alpha}C = C(L-x)^{\alpha}/(\Gamma(\alpha+1)), \tag{2-8}$$

and for power functions we have

$$I_{0^+}^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}, \tag{2-9}$$

$$I_{0^+}^{\alpha}x^{\beta}(L-x)^{\gamma} = \frac{\Gamma(\beta+1)x^{\alpha+\beta}L^{\gamma}}{\Gamma(\alpha+\beta+1)}\,{}_2F_1(\beta+1, -\gamma; \alpha+\beta+1; x/L), \tag{2-10}$$

where ${}_2F_1$ is a hypergeometric function [Kilbas et al. 2006; Samko et al. 1993].

3. Mathematical model

In classical mechanics, the minimization of the potential energy for a fixed supported beam of length L with a downward transverse load per unit length $f(x)$ requires that the functional

$$V = \int_0^L F(x, u, u'')\,\mathrm{d}x = \int_0^L \left[\frac{1}{2}EI(u''(x))^2 - f(x)\,u(x)\right]\mathrm{d}x \tag{3-1}$$

be minimized, where EI is the bending stiffness (which is constant) and $u(x)$ is the static deflection of the beam.

The individual terms $1/2EI(u''(x))^2$ and $f(x), u(x)$ represent potential (strain) energy due to bending and potential energy due to the lateral deflection, respectively. The boundary conditions for the fixed supported beam are

$$u(0) = u'(0) = u(L) = u'(L) = 0. \tag{3-2}$$

The corresponding Euler–Lagrange equation for the considered problem has the form

$$\frac{\partial F(x,u,u'')}{\partial u} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F(x,u,u'')}{\partial u''}\right) = 0, \tag{3-3}$$

which leads to the Euler–Bernoulli beam equation

$$EI\frac{\mathrm{d}^4}{\mathrm{d}x^4}u(x) - f(x) = 0. \tag{3-4}$$

In this manuscript, we propose a procedure for constructing the fractional Euler–Bernoulli beam equation utilizing the fractional variational calculus. We start from the functional (3-1) and we replace the second order derivative by the left Caputo derivative (2-1) in the following way:

$$\begin{aligned} V_{\mathrm{frac}} &= \int_0^L F_{\mathrm{frac}}(x, u, {}^C D_{0+}^{\alpha} u)\, \mathrm{d}x \\ &= \int_0^L \left[\frac{1}{2} E I (\ell^{\alpha-2} {}^C D_{0+}^{\alpha} u(x))^2 - f(x)\, u(x) \right] \mathrm{d}x. \end{aligned} \tag{3-5}$$

Next, by using results presented in [Lazo and Torres 2013; Malinowska et al. 2015] we get the corresponding form of the fractional Euler–Lagrange equation for the problem (3-5), which has the form

$$\frac{\partial F_{\mathrm{frac}}(x, u, {}^C D_{0+}^{\alpha} u)}{\partial u} + {}^C D_{L-}^{\alpha} \left(\frac{\partial F_{\mathrm{frac}}(x, u, {}^C D_{0+}^{\alpha} u)}{\partial {}^C D_{0+}^{\alpha} u} \right) = 0, \tag{3-6}$$

and leads to the fractional Euler–Bernoulli beam equation

$$\ell^{2(\alpha-2)} E I\, {}^C D_{L-}^{\alpha}\, {}^C D_{0+}^{\alpha} u(x) - f(x) = 0, \tag{3-7}$$

where ℓ is a length scale [Sumelka 2014a; Sumelka and Błaszczyk 2014; Sumelka et al. 2015].

It should be highlighted that if we put $\alpha = 2$ into (3-6) and (3-7) we obtain equations (3-3) and (3-4), respectively. Therefore, we do not need to look at the proposed model as competitive to the classical Euler–Bernoulli model. Because of this, we should treat this model as a complement to the classical theory.

4. Analytical solution

Let us start with the denotation

$$f^*(x) = \frac{f(x)}{\ell^{2(\alpha-2)} E I}. \tag{4-1}$$

By using the denotation above we can rewrite the fractional Euler–Bernoulli Equation (3-7) as

$${}^C D_{L-}^{\alpha}\, {}^C D_{0+}^{\alpha} u(x) = f^*(x). \tag{4-2}$$

We start with the transformation of (4-2) into an integral equation [Błaszczyk and Ciesielski 2015; 2016; Ciesielski and Błaszczyk 2015]. We integrate (4-2) two times by using fractional integral operators (2-3) and (2-4), giving

$$I_{0+}^{\alpha} I_{L-}^{\alpha}\, {}^C D_{L-}^{\alpha}\, {}^C D_{0+}^{\alpha} u(x) = I_{0+}^{\alpha} I_{L-}^{\alpha} f^*(x). \tag{4-3}$$

After the first integration in regards to the property (2-6), we get

$$I_{0+}^{\alpha} \left({}^C D_{0+}^{\alpha} u(x) - {}^C D_{0+}^{\alpha} u(x)|_{x=L} + (L-x)\, {}^C D_{0+}^{\alpha+1} u(x)|_{x=L} \right) = I_{0+}^{\alpha} I_{L-}^{\alpha} f^*(x). \tag{4-4}$$

Next, we have to calculate the expression

$$I_{0+}^{\alpha}(L-x) = I_{0+}^{\alpha} L - I_{0+}^{\alpha} x = L \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{\alpha+1}}{\Gamma(2+\alpha)}, \tag{4-5}$$

and on the basis of the composition rule (2-5) we obtain the following form of the considered equation:

$$u(x)-u(0)-u'(0)x - \left(\frac{{}^{C}D^{\alpha}_{0^+}u(x)|_{x=L}}{\Gamma(\alpha+1)}x^{\alpha}+\frac{{}^{C}D^{\alpha+1}_{0^+}u(x)|_{x=L}}{\Gamma(\alpha+2)}\big(-(\alpha+1)Lx^{\alpha}+x^{\alpha+1}\big)\right)=I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x). \quad (4\text{-}6)$$

Now, we can write the above equation in a simpler form:

$$u(x)-(C_1x^{\alpha}+C_2x^{\alpha}(x-(\alpha+1)L))=I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x), \quad (4\text{-}7)$$

where $C_1={}^{C}D^{\alpha}_{0^+}u(x)|_{x=L}/\,\Gamma(\alpha+1)$ and $C_2={}^{C}D^{\alpha+1}_{0^+}u(x)|_{x=L}/\,\Gamma(\alpha+2)$.

In order to determine the constants C_1 and C_2, we differentiate (4-7):

$$u'(x)-C_1\alpha x^{\alpha-1}+C_2(\alpha+1)x^{\alpha-1}(\alpha L-x)=I^{\alpha-1}_{0^+}I^{\alpha}_{L^-}f^*(x), \quad (4\text{-}8)$$

and in accordance with the boundary conditions (3-2) we write the adequate system of equations as

$$\begin{cases} C_1L^{\alpha}-C_2\alpha L^{\alpha+1}=u(L)-I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}, \\ C_1\alpha L^{\alpha-1}-C_2(1-\alpha^2)L^{\alpha}=u'(L)-I^{\alpha-1}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}. \end{cases} \quad (4\text{-}9)$$

We determine the values C_1 and C_2 as

$$C_1=\frac{\big(u(L)-I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}\big)(1-\alpha^2)-\big(u'(L)-I^{\alpha-1}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}\big)\alpha L}{L^{\alpha}}, \quad (4\text{-}10)$$

$$C_2=\frac{\big(u'(L)-I^{\alpha-1}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}\big)-\big(u(L)-I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}\big)\alpha/L}{L^{\alpha}}. \quad (4\text{-}11)$$

Substituting the right-hand side of the expressions (4-10) and (4-11) into (4-7) we get

$$u(x)-(x/L)^{\alpha}\big[(\alpha x/L-\alpha-1)I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}+(L-x)I^{\alpha-1}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}\big] = -(x/L)^{\alpha}\big[(\alpha+1-\alpha x/L)u(L)+(L-x)u'(L)\big]+I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x). \quad (4\text{-}12)$$

Taking into account the boundary conditions (3-2) we obtain the final integral form of the fractional Euler–Bernoulli Equation (4-2):

$$u(x)-(x/L)^{\alpha}\big[(\alpha x/L-\alpha-1)I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}+(L-x)I^{\alpha-1}_{0^+}I^{\alpha}_{L^-}f^*(x)\big|_{x=L}\big]=I^{\alpha}_{0^+}I^{\alpha}_{L^-}f^*(x). \quad (4\text{-}13)$$

The analytical solution of the considered problem (4-2) and (3-2) is

$$u(x)=(x/L)^{\alpha}\big[(\alpha x/L-\alpha-1)g_{\alpha,\alpha}(L)+(L-x)g_{\alpha-1,\alpha}(L)\big]+g_{\alpha,\alpha}(x), \quad (4\text{-}14)$$

where $g_{\eta,\mu}(x)=I^{\eta}_{0^+}I^{\mu}_{L^-}f^*(x)$.

5. Numerical solution

In this section we present two numerical schemes for (3-7). The first one is dedicated to the integral equation and the second one is dedicated to the differential equation (the Euler–Lagrange equation).

The ability to calculate the composition of left and right fractional operators is very important to get a graphical interpretation of solutions for fractional variational differential equations. The analytical evaluations for any function of such composition are difficult to achieve. In some cases, we can express them through special functions, but this causes difficulties in calculating the function values. Therefore, numerical methods are a useful tool to obtain an approximation of integral or differential operators with different types of kernel [Błaszczyk and Ciesielski 2015; 2016; Ciesielski and Błaszczyk 2015; Durajski 2014; 2015; Siedlecki et al. 2015].

Let us start with introducing the following grid of $n+1$ nodes with the constant step $\Delta x = L/n$: $0 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n = L$, and $x_i = i\,\Delta x$, $i = 0, 1, \ldots, n$.

5.1. ***Method I: discretization of the integral equation.*** In our previous works [Błaszczyk and Ciesielski 2015; 2016] we determined the discrete form of the composition of the left and right fractional integrals of order α. On the basis of these results, we present the composition of fractional integrals (2-3) and (2-4):

$$I^{\alpha}_{0^+} I^{\alpha}_{L^-} f^*(x)\big|_{x=x_i} \approx \sum_{j=0}^{i} w^{(\alpha)}_{i,j} \sum_{k=j}^{n} v^{(\alpha)}_{j,k}\, f^*_k, \tag{5-1}$$

where coefficients $w^{(\alpha)}_{i,j}$ and $v^{(\alpha)}_{i,j}$ have the form

$$w^{(\alpha)}_{i,j} = \frac{(\Delta x)^{\alpha}}{\Gamma(\alpha+2)} \begin{cases} 0 & \text{for } i=0 \text{ and } j=0 \\ (i-1)^{\alpha+1} - i^{\alpha+1} + i^{\alpha}(\alpha+1) & \text{for } i>0 \text{ and } j=0, \\ (i-j+1)^{\alpha+1} - 2(i-j)^{\alpha+1} + (i-j-1)^{\alpha+1} & \text{for } i>0 \text{ and } 0<j<i, \\ 1 & \text{for } i>0 \text{ and } j=i \end{cases} \tag{5-2}$$

$$v^{(\alpha)}_{i,j} = \frac{(\Delta x)^{\alpha}}{\Gamma(\alpha+2)} \begin{cases} 0 & \text{for } i=n \text{ and } j=n, \\ (n-i-1)^{\alpha+1} - (n-i)^{\alpha+1} + (n-i)^{\alpha}(\alpha+1) & \text{for } i<n \text{ and } j=n, \\ (j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1} & \text{for } i<n \text{ and } i<j<n, \\ 1 & \text{for } i<n \text{ and } j=i. \end{cases} \tag{5-3}$$

Now we present the discrete form of the integral equation (4-13). For every grid node x_i, $i = 0, 1, \ldots, n$, we write the equation as

$$u_i = -\left(\frac{i}{n}\right)^{\alpha}\left[\left(\frac{\alpha i}{n} - \alpha - 1\right)\sum_{j=0}^{n} w^{(\alpha)}_{n,j} \sum_{k=j}^{n} v^{(\alpha)}_{j,k} f^*_k + (n-i)\Delta x \sum_{j=0}^{n} w^{(\alpha-1)}_{n,j} \sum_{k=j}^{n} v^{(\alpha)}_{j,k} f^*_k\right] + \sum_{j=0}^{i} w^{(\alpha)}_{i,j} \sum_{k=j}^{n} v^{(\alpha)}_{j,k} f^*_k. \tag{5-4}$$

One can observe that for the calculation of values $u_0, u_1, \ldots, u_n$ we do not need to solve the system of $n+1$ linear equations. This is an important advantage of the proposed scheme from the computational point of view.

5.2. ***Method II: discretization of the differential equation.*** In this case, we based the calculation on the discrete form of the composition of the left and right fractional derivatives of order α presented in

[Błaszczyk et al. 2011]:

$$ {}^{C}D^{\alpha}_{L^-}\,{}^{C}D^{\alpha}_{0^+}u(x)\big|_{x=x_i} \approx \sum_{j=i-1}^{n} q^{(\alpha)}_{i,j}\sum_{k=0}^{j+1} r^{(\alpha)}_{j,k}\,u_k, \tag{5-5}$$

where coefficients $r^{(\alpha)}_{i,j}$ and $q^{(\alpha)}_{i,j}$ have the following forms:

$$ r^{(\alpha)}_{i,j} = \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)}\begin{cases} 0 & \text{for } i=0 \text{ and } j=0,\\ i^{2-\alpha}-(i-1)^{2-\alpha} & \text{for } i>0 \text{ and } j=0,\\ 3(i-1)^{2-\alpha}-2i^{2-\alpha}-(i-2)^{2-\alpha} & \text{for } i>0 \text{ and } j=1,\\ (i-j+2)^{2-\alpha}-3(i-j+1)^{2-\alpha} & \\ \qquad +3(i-j)^{2-\alpha}-(i-j)^{2-\alpha} & \text{for } i>0 \text{ and } 1<j<i,\\ 2^{2-\alpha}-3 & \text{for } i<n \text{ and } j=i,\\ 1 & \text{for } i<n \text{ and } j=i+1 \end{cases} \tag{5-6}$$

$$ q^{(\alpha)}_{i,j} = \frac{(\Delta x)^{-\alpha}}{\Gamma(3-\alpha)}\begin{cases} 0 & \text{for } i=n \text{ and } j=n,\\ (n-i)^{2-\alpha}-(n-i-1)^{2-\alpha} & \text{for } i<n \text{ and } j=n,\\ 3(n-i-1)^{2-\alpha}-2(n-i)^{2-\alpha}-(n-i-2)^{2-\alpha} & \text{for } i<n \text{ and } j=n-1,\\ (j-i+2)^{2-\alpha}-3(j-i+1)^{2-\alpha} & \\ \qquad +3(j-i)^{2-\alpha}-(j-i-1)^{2-\alpha} & \text{for } i<n \text{ and } i<j<n-1,\\ 2^{2-\alpha}-3 & \text{for } i<n \text{ and } j=i,\\ 1 & \text{for } i<n \text{ and } j=i-1 \end{cases} \tag{5-7}$$

Now we present the discrete form of the Euler–Lagrange Equation (3-7). For every grid node x_i, $i = 2, 3, \dots, n-2$, we write the equation as

$$ \sum_{j=i-1}^{n} q^{(\alpha)}_{i,j}\sum_{k=0}^{j+1} r^{(\alpha)}_{j,k}\,u_k = f_i^{*}. \tag{5-8}$$

In this approach we have to solve the system of $n+1$ algebraic equations.

6. Error analysis

Let us consider the following example:

$$ {}^{C}D^{\alpha}_{1^-}\,{}^{C}D^{\alpha}_{0^+}u(x) = C, \tag{6-1}$$

where C is a constant. The exact solution of the considered problem (6-1) with boundary conditions (3-2) is

$$ u(x) = x^{\alpha}\big[(\alpha x-\alpha-1)g_{\alpha,\alpha}(1)+(1-x)g_{\alpha-1,\alpha}(1)\big]+g_{\alpha,\alpha}(x), \tag{6-2}$$

where

$$ g_{\eta,\mu}(x) = I^{\eta}_{0^+}I^{\mu}_{1^-}C = I^{\eta}_{0^+}\frac{C(1-x)^{\mu}}{\Gamma(\mu+1)} = \frac{Cx^{\eta}}{\Gamma(\eta+1)\Gamma(\mu+1)}\,{}_2F_1(1,-\mu;\eta+1;x). \tag{6-3}$$

Method I										
n	$\alpha = 1.5$	p	$\alpha = 1.6$	p	$\alpha = 1.7$	p	$\alpha = 1.8$	p	$\alpha = 1.9$	p
80	$2.27 \cdot 10^{-6}$	–	$1.08 \cdot 10^{-6}$	–	$4.72 \cdot 10^{-7}$	-	$1.84 \cdot 10^{-7}$	–	$5.42 \cdot 10^{-8}$	–
160	$7.01 \cdot 10^{-7}$	1.70	$3.10 \cdot 10^{-7}$	1.80	$1.29 \cdot 10^{-7}$	1.87	$4.84 \cdot 10^{-8}$	1.93	$1.39 \cdot 10^{-8}$	1.96
320	$2.08 \cdot 10^{-7}$	1.75	$8.61 \cdot 10^{-8}$	1.84	$3.42 \cdot 10^{-8}$	1.91	$1.25 \cdot 10^{-8}$	1.95	$3.53 \cdot 10^{-9}$	1.98
640	$6.04 \cdot 10^{-8}$	1.79	$2.34 \cdot 10^{-8}$	1.88	$8.93 \cdot 10^{-9}$	1.94	$3.18 \cdot 10^{-9}$	1.97	$8.91 \cdot 10^{-10}$	1.99
1280	$1.72 \cdot 10^{-8}$	1.81	$6.27 \cdot 10^{-9}$	1.90	$2.30 \cdot 10^{-9}$	1.95	$8.07 \cdot 10^{-10}$	1.98	$2.24 \cdot 10^{-10}$	1.99

Method II										
n	$\alpha = 1.5$	p	$\alpha = 1.6$	p	$\alpha = 1.7$	p	$\alpha = 1.8$	p	$\alpha = 1.9$	p
80	$8.74 \cdot 10^{-3}$	–	$7.97 \cdot 10^{-3}$	–	$7.13 \cdot 10^{-4}$	-	$5.61 \cdot 10^{-4}$	–	$3.05 \cdot 10^{-4}$	–
160	$5.15 \cdot 10^{-4}$	0.75	$5.07 \cdot 10^{-4}$	0.65	$5.11 \cdot 10^{-4}$	0.48	$4.65 \cdot 10^{-4}$	0.27	$2.91 \cdot 10^{-4}$	0.07
320	$3.09 \cdot 10^{-4}$	0.74	$3.20 \cdot 10^{-4}$	0.66	$3.59 \cdot 10^{-4}$	0.51	$3.75 \cdot 10^{-4}$	0.31	$2.72 \cdot 10^{-4}$	0.09
640	$1.89 \cdot 10^{-4}$	0.70	$2.01 \cdot 10^{-4}$	0.67	$2.47 \cdot 10^{-4}$	0.54	$2.96 \cdot 10^{-4}$	0.34	$2.49 \cdot 10^{-4}$	0.13
1280	$1.20 \cdot 10^{-4}$	0.65	$1.27 \cdot 10^{-4}$	0.67	$1.68 \cdot 10^{-4}$	0.55	$2.31 \cdot 10^{-4}$	0.35	$2.25 \cdot 10^{-4}$	0.15

Table 1. Maximum absolute errors and experimental estimation of rates of convergence p for $\alpha \in \{1.5; 1.6; 1.7; 1.8; 1.9\}$ and $C = -1/\ell^{2(\alpha-2)}$.

In this case (when the exact solution is available) we can compute errors generated by the presented methods (5-4) and (5-8). Maximum errors were calculated based on the standard expression

$$\operatorname{err}(\Delta x) = \max_{i=0,\ldots,n} (|u(x_i) - u_i|), \tag{6-4}$$

and the experimental rate of convergence was estimated by using the formula

$$p = \log_2\left(\frac{\operatorname{err}(\Delta x)}{\operatorname{err}(\Delta x/2)}\right). \tag{6-5}$$

The maximum absolute errors and the experimental rate of convergence p for various values of α and n are included in Table 1. One can note that errors decrease by increasing the number n. We also conclude that the results obtained by using Method I are much better than those from Method II. From these numerical tests, one may see that the rate of convergence p for scheme (5-4) is significantly higher than for scheme (5-8).

7. Example of computations

On the basis of the numerical scheme (5-4) presented in the paper, we implemented an algorithm in Maple and carried out computational simulations for various values of parameters α, ℓ, and a different type of function f. In all presented examples we consider a nondimensional case and assumed $L = 1$, $E = 1$, $I = 1$, $\varepsilon = 0.01$ and $\Delta x = 0.001$. Two cases are considered:

(i) The beam with central load

$$f(x) = \begin{cases} -1/\varepsilon, & \text{for } L/2 - \varepsilon/2 \le x \le L/2 + \varepsilon/2 \\ 0, & \text{otherwise} \end{cases}. \tag{7-1}$$

(ii) The beam with asymmetric load

$$f(x) = \begin{cases} -1/\varepsilon, & \text{for } 3L/4 - \varepsilon/2 \le x \le 3L/4 + \varepsilon/2 \\ 0, & \text{otherwise} \end{cases}. \tag{7-2}$$

The numerical results are presented in Figures 1 and 2.

Analyzing the results presented in Figures 1 and 2, we observe that when length scale ℓ increases in comparison to beam length L, the nonlocal effects are more noticeable. If ℓ is close to 0 or if the order of Caputo derivative α tends to 2, the fractional Euler–Bernoulli beam equation reduces to the classical model. It is also clearly seen that when values of α decrease the difference between the classical and fractional result increases.

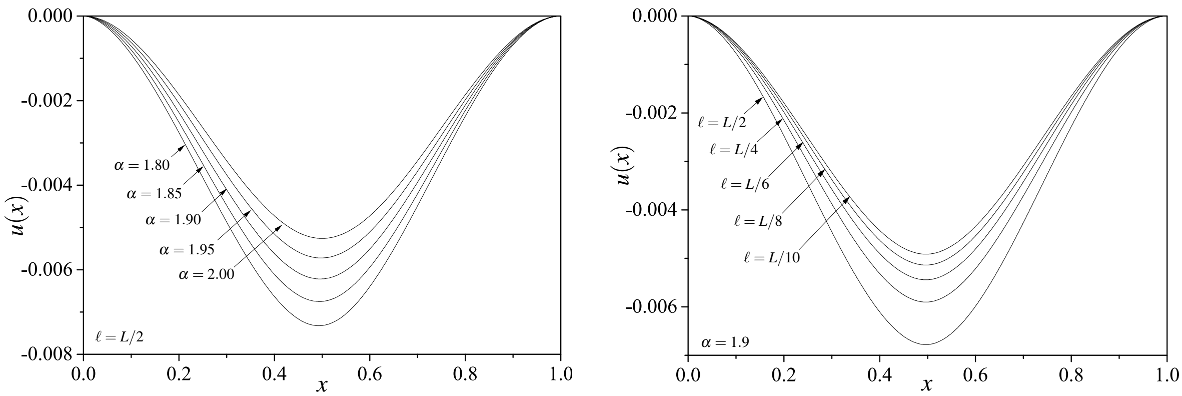

Figure 1. Comparison of the static deflections for a beam with central load (case (i)) for a various order of fractional derivatives α and the length scale ℓ.

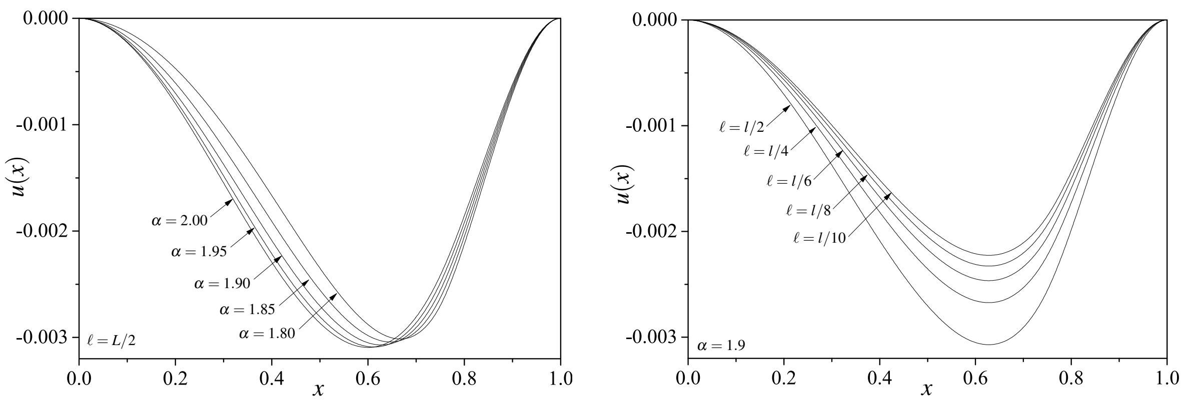

Figure 2. Comparison of the static deflections for a beam with asymmetric load (case (ii)) for a various order of fractional derivatives α and the length scale ℓ.

8. Conclusions

In this paper a new formulation of the Euler–Bernoulli beam equation utilizing fractional calculus was presented. It should be stressed that this is the first time the fractional variational approach was used to obtain the Euler–Bernoulli beam equation. Therefore, the spatial fractional operator appearing in this differential equation is the composition of left and right Caputo derivatives and this operator is defined on the finite interval. The obtained equation was transformed into its equivalent integral form and then was solved. We received both the exact and numerical solution of the integral equation. The errors generated by the presented numerical scheme (Method I) and the experimental rate of convergence were calculated and compared with numerical results obtained by discretizing the Euler–Lagrange equation (Method II) for a particular case. We also carried out simulations to demonstrate how parameters ℓ and α affect the static deflections of the beam.

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