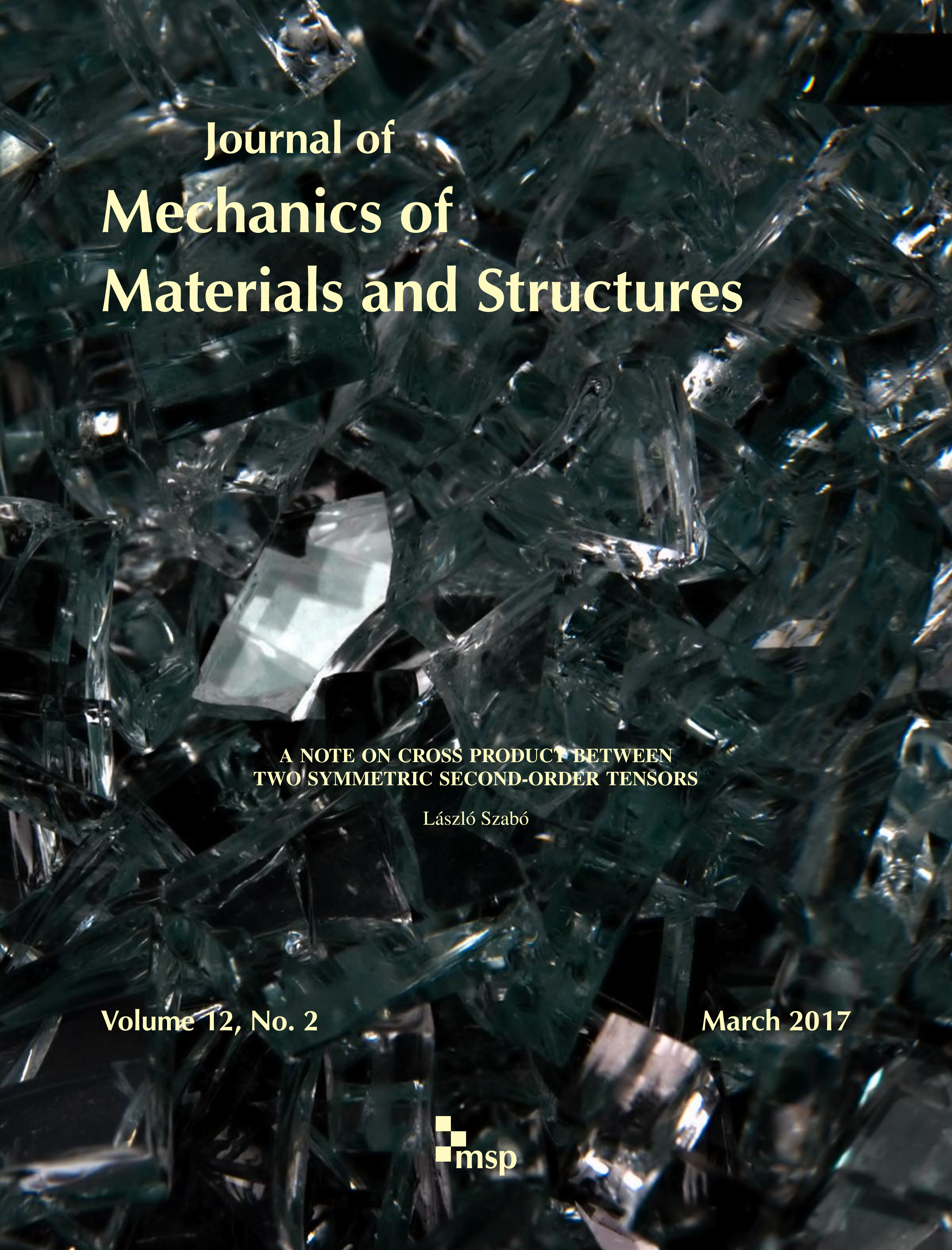

Journal of
Mechanics of
Materials and Structures

A NOTE ON CROSS PRODUCT BETWEEN TWO SYMMETRIC SECOND-ORDER TENSORS

László Szabó

Volume 12, No. 2 March 2017

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 12, No. 2, 2017
dx.doi.org/10.2140/jomms.2017.12.147

A NOTE ON CROSS PRODUCT BETWEEN TWO SYMMETRIC SECOND-ORDER TENSORS

LÁSZLÓ SZABÓ

The present work is concerned with defining the cross product for symmetric, second-order tensors. The operation presented in this paper generalizes the classical vectorial cross product from three-dimensional Euclidean space to symmetric tensor fields on a seven-dimensional vector space. The result of the cross product operation expresses a nonsymmetric tensor as a sum of a symmetric and a skew-symmetric tensor with one parameter, which satisfies the usual properties of the vector cross product except the triple cross product rule. The cross product formulation can be applied to pairs of symmetric or nonsymmetric tensors where the skew-symmetric parts have the same eigenvectors.

1. Introduction

The vectorial cross product in three-dimensional Euclidean space is broadly used in physics, engineering and other fields. It is a bilinear operation on two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, in three-dimensional vector space, that outputs a vector, $\boldsymbol{a} \times \boldsymbol{b}$, that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. The operation is characterized by three basic properties:

(1) skew-symmetry ($\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{a} \times \boldsymbol{b}$),

(2) orthogonality ($\boldsymbol{a} \times \boldsymbol{b} \perp \boldsymbol{a}$ and $\boldsymbol{b}$),

(3) length rule ($|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\alpha$, where α denotes the angle enclosed by $\boldsymbol{a}$ and $\boldsymbol{b}$).

Applying the geometric formula for the Euclidean inner product, $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|||\boldsymbol{b}| \cos\alpha$, the third property can be written in the form of Lagrange's identity: $|\boldsymbol{a}|^2|\boldsymbol{b}|^2 = (\boldsymbol{a} \cdot \boldsymbol{b})^2 + |\boldsymbol{a} \times \boldsymbol{b}|^2$.

An operation similar to the dot product can be defined for two second-order tensors $\boldsymbol{A}$, $\boldsymbol{B}$ defined on the same vector space via the double dot product: $\boldsymbol{A} : \boldsymbol{B} = \|\boldsymbol{A}\| \|\boldsymbol{B}\| \cos\beta$. The analogue of the cross product between $\boldsymbol{A}$ and $\boldsymbol{B}$, however, has not been proposed in literature. Analogous to the vectorial cross product in $\mathbb{R}^3$, a cross product of second-order tensors would plausibly have to satisfy the three properties listed above: skew-symmetry, orthogonality and Lagrange's identity. Defining such a cross product, however, is hindered by the same challenges as extending the cross product to higher-dimensional vector spaces. As a consequence of those challenges, vectorial cross products can only be defined in the three- and seven-dimensional Euclidean spaces (see, e.g., [Eckmann 1943; Brown and Gray 1967; Massey 1983; Shaw 1987]). An exception, however, is the cross product between two general second-order tensors acting on three-dimensional Euclidean space, given that each tensor can be viewed as elements of a nine-dimensional vector space.

In this paper, we introduce a tensorial cross product for symmetric, second-order tensors. When viewed as a subspace of seven-dimensional linear space, the six-dimensional subspace of symmetric

Keywords: vector cross product, second-order tensors, tensorial cross product, seven-dimensional vector space.

tensors inherits a cross product from the ambient seven-dimensional space. The result of this product operation will be a well-defined but nonsymmetric tensor. This construct is analogous to the three-dimensional vectorial cross product restricted to vectors in a two-dimensional plane of $\mathbb{R}^3$, with the resulting cross product normal to the plane in $\mathbb{R}^3$.

In this paper, we first introduce the notation used. In Section 2, the basic properties of cross product in three- and seven-dimensional vector spaces are briefly summarized before some definitions of cross product of second-order tensors are outlined. Sections 3 and 4 present a definition of the cross product between two symmetric second-order tensors. In Section 5, a simple application of the newly developed cross product operation is presented on the field of elastoplasticity. Finally, a conclusion of the results is given in Section 6.

Regarding notation, vectors and tensors are denoted by bold-face characters, the order of which is indicated in text. The cross product is denoted by $\times$, the superscript T denotes the transpose and the trace is indicated by the prefix tr. The tensor product is denoted by $\otimes$ and the following symbolic operations apply:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i \quad \text{and} \quad (\boldsymbol{A}\cdot\boldsymbol{B})_{ij} = \mathrm{A}_{ik}\mathrm{B}_{kj} \quad \text{and} \quad \boldsymbol{A}:\boldsymbol{B} = \mathrm{A}_{ij}\mathrm{B}_{ij},$$

with summation over repeated indices. The symbols

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} \quad \text{and} \quad \|\boldsymbol{A}\| = \sqrt{\boldsymbol{A}:\boldsymbol{A}} \equiv \sqrt{\mathrm{tr}(\boldsymbol{A}\cdot\ \boldsymbol{A}^T)}$$

are used to denote a norm of vector $\boldsymbol{a}$ and second order tensor $\boldsymbol{A}$. The three- and seven-dimensional Euclidean spaces are represented by the symbols $\mathbb{R}^3$ and $\mathbb{R}^7$, and the orthonormal basis vectors for these spaces are denoted by $\boldsymbol{e}_i$ ($i \in [1, 3]$) and $\boldsymbol{E}_\alpha$ ($\alpha \in [1, 7]$), respectively. The notation $[\boldsymbol{a}]$ and $[\boldsymbol{A}]$ represent the column vector and square matrix, respectively, formed by the coordinates of the vector $\boldsymbol{a}$ and the second-order tensor $\boldsymbol{A}$.

2. Theoretical background

2.1. *Cross product of two vectors in a 3D vector space.* In this section, we briefly summarize the basic properties and identities of the cross product defined on the three-dimensional vector space (see, e.g., [Fenn 2001; Itskov 2015; Young 1993]). Let $\boldsymbol{a} = a_i\boldsymbol{e}_i$ and $\boldsymbol{b} = b_i\boldsymbol{e}_i$ be two vectors in $\mathbb{R}^3$. The cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, or $\boldsymbol{a}\times\boldsymbol{b}$, is defined by

$$\boldsymbol{a}\times\boldsymbol{b} = a_i b_j \epsilon_{ijk}\boldsymbol{e}_k, \tag{1}$$

and its coordinates can be expressed as

$$[\boldsymbol{a}\times\boldsymbol{b}]^T = [a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1], \tag{2}$$

where ϵ_{ijk} represents the Levi–Civita permutation symbol, a_i and b_i are the vector coordinates with $i \in [1, 2, 3]$.

The resulting vector has the distinct property of orthogonality between two given vectors, which implies that

$$\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = 0, \quad \boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = 0. \tag{3}$$

From (2), it is readily seen that the cross product is skew-symmetric, that is

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}. \tag{4}$$

The magnitude of vector $\boldsymbol{a} \times \boldsymbol{b}$ is given by

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\alpha, \tag{5}$$

where α is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The angle α can also be defined in the inner or dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\alpha. \tag{6}$$

A well-known consequence of formulas (5) and (6) is that the cross product is related to the inner product via the formula

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2. \tag{7}$$

Finally, we introduce two important identities. Let $\boldsymbol{c}$ be a vector in $\mathbb{R}^3$. The vector triple cross product identity can be written in the form

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}. \tag{8}$$

From (8), it follows that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \mathbb{0}, \tag{9}$$

which is known as the Jacobi identity.

2.2. ***Some further definitions of the cross product in 3D vector space.*** A number of ways to extend the definition of the cross product, in three-dimensional Euclidean space, have been proposed in the literature (see, e.g., [Altenbach 2012; Rubin 2000; Schade and Neemann 2009; Steinmann 2015]). Let's take a brief look at some of these.

2.2.1. *Cross product between a vector and a second-order tensor.* The left cross product of vector $\boldsymbol{a}$ and second-order tensor $\boldsymbol{A}$ is defined by

$$\boldsymbol{a} \times \boldsymbol{A} = (a_i \boldsymbol{e}_i) \times (A_{jl} \boldsymbol{e}_j \otimes \boldsymbol{e}_l) = a_i A_{jl}\, \epsilon_{ijk} \boldsymbol{e}_k \otimes \boldsymbol{e}_l, \tag{10}$$

while the right cross product is given by

$$\boldsymbol{A} \times \boldsymbol{a} = (A_{jl} \boldsymbol{e}_j \otimes \boldsymbol{e}_l) \times (a_i \boldsymbol{e}_i) = a_i A_{jl} \epsilon_{kli} \boldsymbol{e}_k \otimes \boldsymbol{e}_l. \tag{11}$$

2.2.2. *Cross product of second-order tensors.* Define $\boldsymbol{A} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{B} = B_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ in $\mathbb{R}^3$. Then we have the following three definitions of the cross product between two second-order tensors:

$$\boldsymbol{A} \underline{\times} \boldsymbol{B} = A_{ij} B_{kj} \epsilon_{pik} \boldsymbol{e}_p, \tag{12}$$

$$\boldsymbol{A} \overline{\underline{\times}} \boldsymbol{B} = A_{ij} B_{kl}\, \epsilon_{ikm} \epsilon_{jln} \boldsymbol{e}_m \otimes \boldsymbol{e}_n, \tag{13}$$

$$\boldsymbol{A} \overline{\underline{\underline{\times}}} \boldsymbol{B} = A_{ij} B_{kl} \epsilon_{kpj} \boldsymbol{e}_i \otimes \boldsymbol{e}_p \otimes \boldsymbol{e}_l. \tag{14}$$

It is important to note that the operations above do not in general satisfy properties (3)–(9) of the vector product. For more details on these cross products, we refer to [Altenbach 2012; Rubin 2000; Schade and Neemann 2009; Steinmann 2015; de Boer 1982].

Note that the formulation (14) is used in the definition of the homography tensor in the field of computer vision, [Shashua and Wolf 2000] and (13) has been used most recently by Bonet et al. [2015].

2.3. ***Cross product of two vectors in a 7D vector space.*** Three- and seven-dimensional Euclidean spaces are the only Euclidean spaces to have a vector product (see, e.g., [Eckmann 1943; Brown and Gray 1967; Massey 1983; Shaw 1987; Chauhan and Negi 2011; Silagadze 2002; Bisht et al. 2008]). Namely, if $\boldsymbol{u} = u_\alpha \boldsymbol{E}_\alpha$ and $\boldsymbol{v} = v_\alpha \boldsymbol{E}_\alpha$ are two vectors in $\mathbb{R}^7$, then a vector $\boldsymbol{u} \times \boldsymbol{v}$ also in $\mathbb{R}^7$ can be written as

$$\boldsymbol{u} \times \boldsymbol{v} = (u_\alpha \boldsymbol{E}_\alpha) \times (v_\beta \boldsymbol{E}_\beta) = w_\gamma \boldsymbol{E}_\gamma = \boldsymbol{w}, \tag{15}$$

where u_α, v_α, and w_α are the vector coordinates, and $\alpha \in [1, 7]$. In what follows, the subscript index with Greek letters takes the values $1, 2, 3, \dots, 7$.

The coordinates of the resulting vector $\boldsymbol{w}$ can be defined by

$$w_\gamma = u_\alpha v_\beta f_{\alpha\beta\gamma}; \tag{16}$$

here the symbol $f_{\alpha\beta\gamma}$ (see, for example, [Shaw 1987; Chauhan and Negi 2011; Bisht et al. 2008; Baez 2002]) takes the value 1 for the triples

$$f_{\alpha\beta\gamma} = +1 \quad \text{for } (\alpha\beta\gamma) = (123), (471), (257), (165), (624), (543), (736), \tag{17}$$

and is otherwise characterized by being completely antisymmetric in the indices (so for example $f_{123} = +1$ implies $f_{213} = f_{321} = f_{132} = -1$, and $f_{\alpha\beta\gamma} = 0$ unless all three indices are distinct).

For further consideration, the coordinates w_α are listed below:

$$\begin{aligned}
w_1 &= u_2 v_3 - u_3 v_2 + u_6 v_5 - u_5 v_6 + u_4 v_7 - u_7 v_4, \\
w_2 &= u_3 v_1 - u_1 v_3 + u_4 v_6 - u_6 v_4 + u_5 v_7 - u_7 v_5, \\
w_3 &= u_1 v_2 - u_2 v_1 + u_5 v_4 - u_4 v_5 + u_6 v_7 - u_7 v_6, \\
w_4 &= u_3 v_5 - u_5 v_3 + u_6 v_2 - u_2 v_6 + u_7 v_1 - u_1 v_7, \\
w_5 &= u_4 v_3 - u_3 v_4 + u_1 v_6 - u_6 v_1 + u_7 v_2 - u_2 v_7, \\
w_6 &= u_2 v_4 - u_4 v_2 + u_5 v_1 - u_1 v_5 + u_7 v_3 - u_3 v_7, \\
w_7 &= u_1 v_4 - u_4 v_1 + u_2 v_5 - u_5 v_2 + u_3 v_6 - u_6 v_3.
\end{aligned} \tag{18}$$

Note that while the three-dimensional cross product is unique up to a sign, there are many seven-dimensional cross products (see, e.g., [Shaw 1987; Chauhan and Negi 2011; Baez 2002; Dray and Manogue 2015]). These are defined using the symbol $f_{\alpha\beta\gamma}$ and are associated to the octonion multiplication table (see, e.g., [Fenn 2001; Chauhan and Negi 2011; Baez 2002; Dray and Manogue 2015; Ebbinghaus et al. 1991]). In our case, the cross product is consistent with the three-dimensional subspaces, namely if $u_4 = u_5 = u_6 = u_7 = 0$ and $v_4 = v_5 = v_6 = v_7 = 0$, the resulting vector is equivalent to (2). For further details of octonion algebra, the reader is referred to [Baez 2002; Dray and Manogue 2015; Fenn 2001].

Like the cross product in three-dimensions, the seven-dimensional product is anticommutative and $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal both to $\boldsymbol{u}$ and to $\boldsymbol{v}$, namely

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}, \quad \boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0, \quad \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0. \tag{19}$$

Moreover, the Pythagorean formula (6) with (5) and (7) can also be satisfied:

$$|\boldsymbol{u} \times \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2. \tag{20}$$

Finally, it is important to note that the vector product in $\mathbb{R}^7$ does not satisfy the triple cross product (8), nor the Jacobi identity (9). However, the seven-dimensional cross product satisfies the Malcev identity (see, e.g., [Ebbinghaus et al. 1991]):

$$(\boldsymbol{u} \times \boldsymbol{v}) \times (\boldsymbol{u} \times \boldsymbol{t}) + \boldsymbol{u} \times [(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{t}] - \boldsymbol{u} \times [\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{t})] + \boldsymbol{v} \times [\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{t})] = 0, \tag{21}$$

as a generalization of Jacobi identity. Here $\boldsymbol{t} = t_\alpha \boldsymbol{E}_\alpha$ is also a vector in $\mathbb{R}^7$.

3. Cross product between two symmetric second-order tensors

3.1. ***The case of two symmetric second-order tensors.*** In this section we introduce a new definition of cross product on two symmetric second-order tensors. This operation is based on the seven-dimensional Euclidean cross product.

Let $\boldsymbol{A} = A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{B} = B_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ be two symmetric second-order tensors in $\mathbb{R}^3$ ($\boldsymbol{A}^T = \boldsymbol{A}$ and $\boldsymbol{B}^T = \boldsymbol{B}$), with the matrices of these tensors defined by

$$[\boldsymbol{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad [\boldsymbol{B}] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}.$$

Moreover, consider two vectors, $\boldsymbol{u} = u_\alpha \boldsymbol{E}_\alpha$ and $\boldsymbol{v} = v_\alpha \boldsymbol{E}_\alpha$ in $\mathbb{R}^7$, with coordinates defined using Mandel's matrix notation:

$$[\boldsymbol{u}]^T = \begin{bmatrix} A_{11} & A_{22} & A_{33} & \sqrt{2}A_{23} & \sqrt{2}A_{13} & \sqrt{2}A_{12} & 0 \end{bmatrix}, \tag{22}$$

$$[\boldsymbol{v}]^T = \begin{bmatrix} B_{11} & B_{22} & B_{33} & \sqrt{2}B_{23} & \sqrt{2}B_{13} & \sqrt{2}B_{12} & 0 \end{bmatrix}. \tag{23}$$

In this case, the dot products are identical, so that

$$\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{u} \cdot \boldsymbol{v}. \tag{24}$$

Now, the cross product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by an expression similar to (15), but $u_7 = v_7 = 0$:

$$\boldsymbol{u} \times \boldsymbol{v} = (u_\alpha \boldsymbol{E}_\alpha) \times (v_\beta \boldsymbol{E}_\beta) = u_\alpha v_\beta f_{\alpha\beta\gamma} \boldsymbol{E}_\gamma = w_\gamma \boldsymbol{E}_\gamma = \boldsymbol{w}, \tag{25}$$

where the symbol $f_{\alpha\beta\gamma}$ is defined in (17), and the components w_α, where $\alpha \in [1,7]$, using (18) are expressed as

$$\begin{aligned}
w_1 &= A_{22}B_{33} - A_{33}B_{22} + 2(A_{12}B_{13} - A_{13}B_{12}),\\
w_2 &= A_{33}B_{11} - A_{11}B_{33} + 2(A_{23}B_{12} - A_{12}B_{23}),\\
w_3 &= A_{11}B_{22} - A_{22}B_{11} + 2(A_{13}B_{23} - A_{23}B_{13}),\\
w_4 &= \sqrt{2}(A_{12}B_{22} - A_{22}B_{12} + A_{33}B_{13} - A_{13}B_{33}),\\
w_5 &= \sqrt{2}(A_{11}B_{12} - A_{12}B_{11} + A_{23}B_{33} - A_{33}B_{23}),\\
w_6 &= \sqrt{2}(A_{22}B_{23} - A_{23}B_{22} + A_{13}B_{11} - A_{11}B_{13}),\\
w_7 &= \sqrt{2}(A_{11}B_{23} - A_{23}B_{11} + A_{22}B_{13} - A_{13}B_{22} + A_{33}B_{12} - A_{12}B_{33}).
\end{aligned} \tag{26}$$

Finally, we define the cross product as

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C}, \tag{27}$$

where the tensor $\boldsymbol{C}$ can be decomposed into symmetric and a skew-symmetric parts:

$$\boldsymbol{C} = \boldsymbol{C}_S + \boldsymbol{C}_A. \tag{28}$$

The components of these tensors are defined by the components of the resulting vector $\boldsymbol{w}$ (25) as follows:

$$[\boldsymbol{C}_S] = \begin{bmatrix} w_1 & \frac{1}{\sqrt{2}}w_6 & \frac{1}{\sqrt{2}}w_5 \\ \frac{1}{\sqrt{2}}w_6 & w_2 & \frac{1}{\sqrt{2}}w_4 \\ \frac{1}{\sqrt{2}}w_5 & \frac{1}{\sqrt{2}}w_4 & w_3 \end{bmatrix}, \quad [\boldsymbol{C}_A] = \frac{1}{\sqrt{2}}w_7 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

Now, we impose the seven-dimensional vector space structure on 3x3 (real) matrices by looking at symmetric matrices $[\boldsymbol{C}_S]$ with 6 coordinates (w_α, where $\alpha \in [1,6]$) together with one antisymmetric piece (w_7).

As in the preceding Section 2.3, we have the following properties:

(i) skew-symmetry,

$$\boldsymbol{A} \times \boldsymbol{B} = -(\boldsymbol{B} \times \boldsymbol{A}); \tag{30}$$

(ii) orthogonality

$$(\boldsymbol{A} \times \boldsymbol{B}) : \boldsymbol{A} = 0, \quad (\boldsymbol{A} \times \boldsymbol{B}) : \boldsymbol{B} = 0, \tag{31}$$

so $\boldsymbol{A} \times \boldsymbol{B}$ is orthogonal to both $\boldsymbol{A}$ and $\boldsymbol{B}$; and

(iii) the dot product

$$\boldsymbol{A} : \boldsymbol{B} = \|\boldsymbol{A}\| \|\boldsymbol{B}\| \cos\alpha, \tag{32}$$

where norm of the cross product,

$$\|\boldsymbol{A} \times \boldsymbol{B}\| = \|\boldsymbol{A}\| \|\boldsymbol{B}\| \sin\alpha, \tag{33}$$

is associated with the same angle α, from which the Pythagorean formula holds:

$$\|\boldsymbol{A} \times \boldsymbol{B}\|^2 = \|\boldsymbol{A}\|^2 \|\boldsymbol{B}\|^2 - (\boldsymbol{A} : \boldsymbol{B})^2. \tag{34}$$

As a special case, the triple cross product takes the form

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{A} = (\boldsymbol{A} : \boldsymbol{A})\boldsymbol{B} - (\boldsymbol{B} : \boldsymbol{A})\boldsymbol{A}. \tag{35}$$

Note that the triple cross product identity fails for the cross product defined in this section, however, the Malcev identity (21) is satisfied.

3.2. ***The case of a symmetric and a nonsymmetric second-order tensors.*** It is clear that the preceding definition of a cross product generalizes to the case of one symmetric and one nonsymmetric second-order tensor. Namely, let $\boldsymbol{A}^T \neq \boldsymbol{A}$ be a nonsymmetric second-order tensor, and let $\boldsymbol{B}^T = \boldsymbol{B}$ be a symmetric second-order tensor. Then, the tensor $\boldsymbol{A}$ can be decomposed into its symmetric $\boldsymbol{A}_S$ and skew-symmetric $\boldsymbol{A}_A$ parts, i.e.,

$$\boldsymbol{A} = \boldsymbol{A}_S + \boldsymbol{A}_A, \tag{36}$$

such that

$$\boldsymbol{A}_S = \tfrac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T), \quad \boldsymbol{A}_A = \tfrac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T).$$

The tensor $\boldsymbol{A}_A$ has zero diagonal elements and only three independent scalar quantities, given as a, b and c. Hence, the matrix of $\boldsymbol{A}_A$ may be defined in the form

$$[\boldsymbol{A}_A] = \begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}. \tag{37}$$

The eigenvalues and the eigenvectors of $\boldsymbol{A}_A$ are defined as [Itskov 2015]

$$\lambda_1 = 0, \quad \lambda_2 = \omega i, \quad \lambda_3 = -\omega i, \tag{38}$$

and

$$\boldsymbol{A}_A \cdot \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2, \quad \boldsymbol{v}_3 \cdot \boldsymbol{A}_A = -\lambda_3 \boldsymbol{v}_3,$$

where

$$\omega = \sqrt{a^2 + b^2 + c^2}.$$

The spectral representation of $\boldsymbol{A}_A$ takes the form

$$\boldsymbol{A}_A = \omega(\boldsymbol{p} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{p}), \tag{39}$$

where

$$\boldsymbol{p} = \frac{1}{\sqrt{2}}(\boldsymbol{v}_2 + \boldsymbol{v}_3), \quad \boldsymbol{q} = \frac{1}{\sqrt{2}} i(\boldsymbol{v}_3 - \boldsymbol{v}_2).$$

The vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, by a straightforward manipulation, are given by

$$[\boldsymbol{p}]^T = \frac{1}{\omega\sqrt{b^2 + c^2}}[-ac, -ab, b^2 + c^2], \quad [\boldsymbol{q}]^T = \frac{1}{\sqrt{b^2 + c^2}}[-b, c, 0].$$

In addition, we define a third vector

$$\boldsymbol{r} = \boldsymbol{p} \times \boldsymbol{q}. \tag{40}$$

Then, in the coordinate system referred relative to the bases $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$, the matrix of tensor $\boldsymbol{A}_A$ becomes

$$[\widetilde{\boldsymbol{A}}_A] = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

which has the same form as $\boldsymbol{C}_A$ in (29).

Finally, transforming the tensors $\boldsymbol{A}_S$ and $\boldsymbol{B}$ to the same system as in $\widetilde{\boldsymbol{A}}_A$,

$$\widetilde{\boldsymbol{A}}_s = \boldsymbol{Q}^T \boldsymbol{A}_S \boldsymbol{Q}, \quad \widetilde{\boldsymbol{B}} = \boldsymbol{Q}^T \boldsymbol{B} \boldsymbol{Q}, \tag{42}$$

where the first, second and third column of matrix of $\boldsymbol{Q}$ contains the vectors $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$, respectively.

Now, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have the forms

$$\begin{aligned} [\boldsymbol{u}]^T &= [(\widetilde{\boldsymbol{A}}_s)_{11}\ (\widetilde{\boldsymbol{A}}_s)_{22}\ (\widetilde{\boldsymbol{A}}_s)_{33}\ \sqrt{2}(\widetilde{\boldsymbol{A}}_s)_{23}\sqrt{2}(\widetilde{\boldsymbol{A}}_s)_{13}\ \sqrt{2}(\widetilde{\boldsymbol{A}}_s)_{12}\ \sqrt{2}\omega], \\ [\boldsymbol{v}]^T &= [\widetilde{B}_{11}\ \widetilde{B}_{22}\ \widetilde{B}_{33}\ \sqrt{2}\widetilde{B}_{23}\ \sqrt{2}\widetilde{B}_{13}\ \sqrt{2}\widetilde{B}_{12}\ 0], \end{aligned}$$

and the cross product defined previously, using (25) and (18) with $v_7 = 0$ can be determined.

However, the resulting tensor is defined in the rotated system, that is

$$\widetilde{\boldsymbol{A}} \times \widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{C}}, \tag{43}$$

where $\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}_s + \widetilde{\boldsymbol{A}}_A$. In order to complete the procedure, we need to transform the tensor $\widetilde{\boldsymbol{C}}$ back to the original coordinate system. This is accomplished by noting that

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C} = \boldsymbol{Q}\widetilde{\boldsymbol{C}}\boldsymbol{Q}^T. \tag{44}$$

It should be noted that our cross product is also applicable for nonsymmetric tensors as long as their skew-symmetric parts have the same eigenvectors. In this case the matrices of tensor $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$[\boldsymbol{A}] = \begin{bmatrix} A_{11} & A_{12}+\omega_A & A_{13} \\ A_{21}-\omega_A & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad [\boldsymbol{B}] = \begin{bmatrix} B_{11} & B_{12}+\omega_B & B_{13} \\ B_{21}-\omega_B & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}, \tag{45}$$

with $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$ and vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by

$$\begin{aligned} [\boldsymbol{u}]^T &= [A_{11}\ A_{22}\ A_{33}\ \sqrt{2}A_{23}\ \sqrt{2}A_{13}\ \sqrt{2}A_{12}\ \sqrt{2}\omega_A], \\ [\boldsymbol{v}]^T &= [B_{11}\ B_{22}\ B_{33}\ \sqrt{2}B_{23}\ \sqrt{2}B_{13}\ \sqrt{2}B_{12}\ \sqrt{2}\omega_B]. \end{aligned}$$

The cross product, $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C}$, can be calculated by the relations (18), (25) and (27)–(29). The matrix of the resulting tensor is defined by

$$[\boldsymbol{C}] = \begin{bmatrix} C_{11} & C_{12}+\omega_C & C_{13} \\ C_{21}-\omega_C & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}, \tag{46}$$

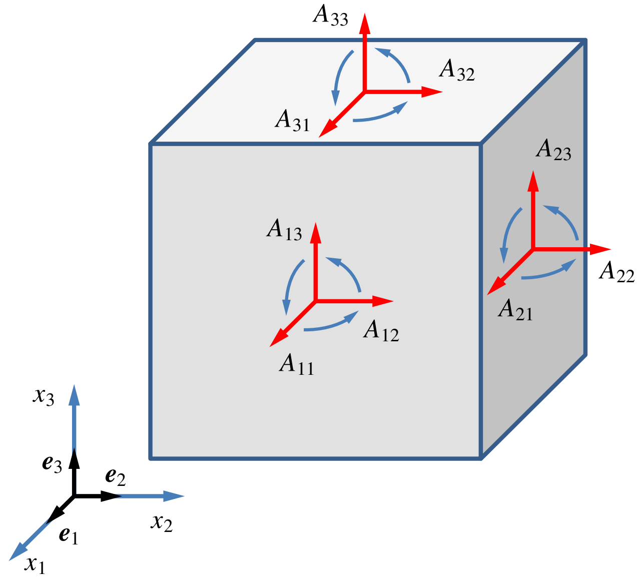

Figure 1. Illustration of the components of a symmetric second-order tensor.

where the components C_{ij}, using (18), are expressed as

$$
\begin{aligned}
C_{11} &= A_{22}B_{33} - A_{33}B_{22} + 2(A_{12}B_{13} - A_{13}B_{12} + A_{23}\omega_B - B_{23}\omega_A),\\
C_{22} &= A_{33}B_{11} - A_{11}B_{33} + 2(A_{23}B_{12} - A_{12}B_{23} + A_{13}\omega_B - B_{13}\omega_A),\\
C_{33} &= A_{11}B_{22} - A_{22}B_{11} + 2(A_{13}B_{23} - A_{23}B_{13} + A_{12}\omega_B - B_{12}\omega_A),\\
C_{23} &= C_{32} = A_{12}B_{22} - A_{22}B_{12} + A_{33}B_{13} - A_{13}B_{33} + A_{11}\omega_B - B_{11}\omega_A,\\
C_{13} &= C_{31} = A_{11}B_{12} - A_{12}B_{11} + A_{23}B_{33} - A_{33}B_{23} + A_{22}\omega_B - B_{22}\omega_A,\\
C_{12} &= C_{21} = A_{22}B_{23} - A_{23}B_{22} + A_{13}B_{11} - A_{11}B_{13} + A_{33}\omega_B - B_{33}\omega_A,\\
\omega_C &= A_{11}B_{23} - A_{23}B_{11} + A_{22}B_{13} - A_{13}B_{22} + A_{33}B_{12} - A_{12}B_{33}.
\end{aligned}
\tag{47}
$$

The cross product formulations (47) can be applied to all the cases introduced in this paper, namely:

(i) when $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric ($\omega_A = \omega_B = 0$),

(ii) when $\boldsymbol{A}$ is nonsymmetric but $\boldsymbol{B}$ is symmetric ($\omega_A \neq 0$, $\omega_B = 0$), and

(iii) $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsymmetric and the skew-symmetric parts of both of them have the same eigenvectors ($\omega_A \neq 0$, $\omega_B \neq 0$).

4. An alternative and equivalent definition

The cross product operation defined in this paper can be introduced in a different way. In many engineering fields, such as solid mechanics, the components of a symmetric second-order tensor (e.g., stress or strain tensors) can be represented on three mutually perpendicular planes, as shown in Figure 1.

Consider the following second-order orthonormal base tensors (see [Itskov 2015; Mehrabadi and Cowin 1990; Moakher and Norris 2006]):

$$\begin{gathered}\boldsymbol{m}_{11}=\boldsymbol{e}_1\otimes\boldsymbol{e}_1,\quad \boldsymbol{m}_{22}=\boldsymbol{e}_2\otimes\boldsymbol{e}_2,\quad \boldsymbol{m}_{33}=\boldsymbol{e}_3\otimes\boldsymbol{e}_3,\\ \boldsymbol{m}_{23}\equiv\boldsymbol{m}_{32}=\frac{1}{\sqrt{2}}(\boldsymbol{e}_2\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_2),\\ \boldsymbol{m}_{13}\equiv\boldsymbol{m}_{31}=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1\otimes\boldsymbol{e}_3+\boldsymbol{e}_3\otimes\boldsymbol{e}_1),\\ \boldsymbol{m}_{12}\equiv\boldsymbol{m}_{21}=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1),\\ \boldsymbol{m}_{\text{skew}}=\frac{1}{\sqrt{2}}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2-\boldsymbol{e}_2\otimes\boldsymbol{e}_1).\end{gathered}\tag{48}$$

These base tensors are associated with the tensor components ($\boldsymbol{m}_{ij}\rightsquigarrow A_{ij}$) and the tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ defined in (45) can be expressed as

$$\boldsymbol{A}=A_{11}\boldsymbol{m}_{11}+A_{22}\boldsymbol{m}_{22}+A_{33}\boldsymbol{m}_{33}+\sqrt{2}(A_{12}\boldsymbol{m}_{12}+A_{13}\boldsymbol{m}_{13}+A_{23}\boldsymbol{m}_{23}+\omega_A\boldsymbol{m}_{\text{skew}})\tag{49}$$

and

$$\boldsymbol{B}=B_{11}\boldsymbol{m}_{11}+B_{22}\boldsymbol{m}_{22}+B_{33}\boldsymbol{m}_{33}+\sqrt{2}(B_{12}\boldsymbol{m}_{12}+B_{13}\boldsymbol{m}_{13}+B_{23}\boldsymbol{m}_{23}+\omega_B\boldsymbol{m}_{\text{skew}}).\tag{50}$$

The following relationships are crucial for our method. We define the cross product of the base tensors with the right-hand rule as follows.

For $\boldsymbol{m}_{11}$, $\boldsymbol{m}_{22}$ and $\boldsymbol{m}_{33}$:

$$\boldsymbol{m}_{11}\times\boldsymbol{m}_{22}=\boldsymbol{m}_{33},\quad \boldsymbol{m}_{22}\times\boldsymbol{m}_{33}=\boldsymbol{m}_{11},\quad \boldsymbol{m}_{33}\times\boldsymbol{m}_{11}=\boldsymbol{m}_{22}.\tag{51}$$

Moreover, for $\boldsymbol{m}_{1i}$, $\boldsymbol{m}_{2i}$ and $\boldsymbol{m}_{3i}$, which are related to three mutually perpendicular planes (Figure 1):

$$\boldsymbol{m}_{1i}\boldsymbol{m}_{1j}=\epsilon_{ijk}\boldsymbol{m}_{1k},\quad \boldsymbol{m}_{2i}\boldsymbol{m}_{2j}=\epsilon_{ijk}\boldsymbol{m}_{2k},\quad \boldsymbol{m}_{3i}\boldsymbol{m}_{3j}=\epsilon_{ijk}\boldsymbol{m}_{3k},\qquad i\in[1,2,3],\tag{52}$$

and

$$\boldsymbol{m}_{11}\times\boldsymbol{m}_{23}=\boldsymbol{m}_{\text{skew}},\quad \boldsymbol{m}_{22}\times\boldsymbol{m}_{31}=\boldsymbol{m}_{\text{skew}},\quad \boldsymbol{m}_{33}\times\boldsymbol{m}_{12}=\boldsymbol{m}_{\text{skew}}.\tag{53}$$

Then the cross product $\boldsymbol{A}\times\boldsymbol{B}=\boldsymbol{C}$, using (51)–(53), can be defined by

$$\boldsymbol{C}=C_{11}\boldsymbol{m}_{11}+C_{22}\boldsymbol{m}_{22}+C_{33}\boldsymbol{m}_{33}+\sqrt{2}(C_{12}\boldsymbol{m}_{12}+C_{13}\boldsymbol{m}_{13}+C_{23}\boldsymbol{m}_{23}+\omega_C\boldsymbol{m}_{\text{skew}}),\tag{54}$$

and the components C_{ij} and ω_C are listed in (47).

Remark. There are many possible cross products in $\mathbb{R}^7$, namely 480 as shown in [Coxeter 1946; Schray and Manogue 1996], and listed by Barber [2015]. In this analysis, we view the symmetric second-order tensor as a column vector of six components. However, the order of these components is subjective. But, when the cross product of the base tensors with the right-hand rules (51)–(53) is taken into account, the resulting tensor of the cross product is uniquely defined by (47) independently of the order of the elements in vector $\boldsymbol{u}$ and $\boldsymbol{v}$. In other words, a given order of the tensor components $(\cdot)_{ij}$ in seven dimensions is always associated with exactly one multiplication set (triples), $f_{\alpha\beta\gamma}$, which satisfies the right-hand rule for the base tensors.

5. A simple application

As an illustration, the applicability of the newly developed cross product operation is demonstrated by reformulating the constitutive relation of classical elastoplasticity. The deviatoric part of the Prandtl–Reuss equation in elastic-perfectly plastic case is defined by (see, e.g., [Simo and Hughes 1998])

$$\dot{\boldsymbol{s}} = 2G[\dot{\boldsymbol{e}} - \boldsymbol{n}(\dot{\boldsymbol{e}} : \boldsymbol{n})], \tag{55}$$

where $\boldsymbol{s}$ is the deviatoric stress tensor, $\dot{\boldsymbol{e}}$ is the rate of the deviatoric strain and

$$\boldsymbol{n} = \frac{1}{R}\boldsymbol{s}, \quad R = \|\boldsymbol{s}\|, \quad \boldsymbol{n} : \boldsymbol{n} = 1. \tag{56}$$

Equation (55), using the identity (35), can be rewritten as

$$\dot{\boldsymbol{s}} = 2G\boldsymbol{n} \times (\dot{\boldsymbol{e}} \times \boldsymbol{n}). \tag{57}$$

6. Conclusion

We have introduced a cross-product operation between two symmetric second-order tensors. This operation generalizes the classical vectorial cross product from a three-dimensional Euclidean space to symmetric tensor fields. The operation developed here utilizes the vectorial cross product defined on a seven-dimensional vector space. The resulting product tensor is the sum of a symmetric and a skew-symmetric tensor with one parameter, satisfying the usual properties of the vector cross product with the exception of the triple cross product rule. Potential application areas for the tensorial cross product developed here could possibly include computer vision, continuum mechanics and electromagnetism.

Acknowledgement

I would like to thank Professor Tevian Dray (Department of Mathematics, Oregon State University, Corvallis, OR 97331) for several helpful discussions regarding this work.

References

[Altenbach 2012] H. Altenbach, *Kontinuumsmechanik: einfüh-rung in die materialunabhängigen und material-abhängigen Gleichungen*, Springer, Berlin, 2012.

[Baez 2002] J. C. Baez, "The octonions", *Bull. Amer. Math. Soc.* **39**:2 (2002), 145–205.

[Barber 2015] D. Barber, "480 varieties of octonion multiplication", 2015, available at http://tamivox.org/eugene/octonion480/index.html.

[Bisht et al. 2008] P. S. Bisht, S. Dangwal, and O. P. S. Negi, "Unified split octonion formulation of dyons", *Int. J. Theoret. Phys.* **47**:9 (2008), 2297–2313.

[Bonet et al. 2015] J. Bonet, A. J. Gil, and R. Ortigosa, "A computational framework for polyconvex large strain elasticity", *Comput. Methods Appl. Mech. Eng.* **283** (2015), 1061–1094.

[Brown and Gray 1967] R. B. Brown and A. Gray, "Vector cross products", *Comment. Math. Helv.* **42** (1967), 222–236.

[Chauhan and Negi 2011] B. Chauhan and O. Negi, "Octonionic formulation of seven dimensional vector space", *Fund. J. Math. Phys.* **1**:1 (2011), 41–52.

[Coxeter 1946] H. S. M. Coxeter, "Integral Cayley numbers", *Duke Math. J.* **13** (1946), 561–578.

[de Boer 1982] R. de Boer, "Vektor- und Tensorrechnung für Ingenieure", 1982.

[Dray and Manogue 2015] T. Dray and C. A. Manogue, *The geometry of the octonions*, World Scientific, Hackensack, NJ, 2015.

[Ebbinghaus et al. 1991] H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R. Remmert, *Numbers*, Springer, New York, 1991.

[Eckmann 1943] B. Eckmann, "Stetige Lösungen linearer Gleichungssysteme", *Comment. Math. Helv.* **15** (1943), 318–339.

[Fenn 2001] R. Fenn, *Geometry*, Springer, London, 2001.

[Itskov 2015] M. Itskov, *Tensor algebra and tensor analysis for engineers: with applications to continuum mechanics*, 4th ed., Springer, Cham, Switzerland, 2015.

[Massey 1983] W. S. Massey, "Cross products of vectors in higher-dimensional Euclidean spaces", *Amer. Math. Monthly* **90**:10 (1983), 697–701.

[Mehrabadi and Cowin 1990] M. M. Mehrabadi and S. C. Cowin, "Eigentensors of linear anisotropic elastic materials", *Quart. J. Mech. Appl. Math.* **43**:1 (1990), 15–41.

[Moakher and Norris 2006] M. Moakher and A. N. Norris, "The closest elastic tensor of arbitrary symmetry to an elasticity tensor of lower symmetry", *J. Elasticity* **85**:3 (2006), 215–263.

[Rubin 2000] M. Rubin, *Cosserat theories: shells, rods and points*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[Schade and Neemann 2009] H. Schade and K. Neemann, *Tensoranalysis*, de Gruyter, Bad Berleburg, Germany, 2009.

[Schray and Manogue 1996] J. Schray and C. A. Manogue, "Octonionic representations of Clifford algebras and triality", *Found. Phys.* **26**:1 (1996), 17–70.

[Shashua and Wolf 2000] A. Shashua and L. Wolf, "Homography tensors: on algebraic entities that represent three views of static or moving planar points", pp. 507–521 in *Computer vision — ECCV 2000: 6th European conference on computer vision proceedings, part I* (Dublin, Ireland), edited by D. Vernon, Springer, Berlin, 2000.

[Shaw 1987] R. Shaw, "Vector cross products in n dimensions", *Int. J. Math. Educ. Sci. Technol.* **18** (1987), 803–816.

[Silagadze 2002] Z. K. Silagadze, "Multi-dimensional vector product", *J. Phys. A* **35**:23 (2002), 4949–4953.

[Simo and Hughes 1998] J. C. Simo and T. J. R. Hughes, *Computational inelasticity*, Interdisciplinary Applied Mathematics **7**, Springer, New York, 1998.

[Steinmann 2015] P. Steinmann, *Geometrical foundations of continuum mechanics: an application to first- and second- order elasticity and elasto-plasticity*, Lecture Notes in Applied Mathematics and Mechanics **2**, Springer, Heidelberg, 2015.

[Young 1993] E. C. Young, *Vector and tensor analysis*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics **172**, Marcel Dekker, New York, 1993.

Received 15 Jan 2016. Revised 11 Oct 2016. Accepted 15 Nov 2016.

LÁSZLÓ SZABÓ: szabo@mm.bme.hu
Department of Applied Mechanics, Budapest University of Technology and Economics, Müegyetem rkp. 5, Budapest, 1111, Hungary

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

Cover photo: Mando Gomez, www.mandolux.com

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2017 is US $615/year for the electronic version, and $775/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 12, No. 2 March 2017