Journal of
Mechanics of
Materials and Structures

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Volume 12, No. 5 December 2017

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 12, No. 5, 2017
dx.doi.org/10.2140/jomms.2017.12.603

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STATIC ANALYSIS OF NANOBEAMS USING RAYLEIGH–RITZ METHOD

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Boundary characteristic orthogonal polynomials have been used as shape functions in the Rayleigh–Ritz method for static analysis of nanobeams. The formulation is based on Euler–Bernoulli and Timoshenko beam theories in conjunction with nonlocal elasticity theory of Eringen. Application of Rayleigh–Ritz method converts the problem into a system of linear equations. Some of the parametric studies have been carried out. The novelty of the method is that it can handle any set of classical boundary conditions (viz., clamped, simply supported and free) with ease. Although the assumed shape functions need to satisfy the geometric boundary condition only, the final solution is for the targeted boundary condition of the problem or domain. Deflection and rotation shapes for some of the boundary conditions have also been illustrated.

1. Introduction

Nanosized structures such as nanobeams, nanoplates and nanoshells are commonly used as components in nanoelectromechanical systems (NEMS) devices. The most distinct characteristic of nanostructures is that their mechanical properties are size dependent [Ansari et al. 2013; Miller and Shenoy 2000; Xu et al. 2010]. Fundamental knowledge of their mechanical behavior is needed for proper design and application of nanostructured materials; however, conducting experiments at nanoscale size is quite difficult. In this regard, size dependent continuum theories came into existence. Among these theories, nonlocal elasticity theory, pioneered by Eringen [1972], has received much attention in modeling small sized structures. According to this theory, the stress at a specific point depends on the strain tensors of the entire body. As such, the nonlocal stress tensor σ at a point x is expressed as [Reddy 2007]

$$\sigma = \int_V K(|x' - x|, \tau) t(x') \, \mathrm{d}x',$$

where V is the volume occupied by the elastic body, τ the material constant which depends on both internal length (lattice spacing) and external characteristic length (wavelength) and $K(|x' - x|, \tau)$ denotes the nonlocal modulus. Also, $|x' - x|$ is the Euclidean distance and $t(x)$ is the classical macroscopic stress tensor at a point x and is related to strain $\varepsilon(x)$ by Hooke's law:

$$t(x) = C(x) : \varepsilon(x),$$

where C is the fourth-order elasticity tensor.

Since it is difficult to solve the integral constitutive relation, an equivalent differential form was proposed [Reddy 2007],

$$(1 - \tau^2 L^2 \nabla^2)\sigma = t, \quad \tau = e_0 a/L,$$

Keywords: Rayleigh–Ritz method, boundary characteristic orthogonal polynomial, nonlocal elasticity theory.

where e_0 is material constant, a the internal characteristic length and L the external characteristic length.

Nonlocal effects considered in the nonlocal elasticity theory play an important role in the analysis and is determined by the magnitude of nonlocal parameter $e_0 a$. The parameter $e_0 a$ is the scale coefficient that incorporates the small scale [Wang et al. 2006]. When the nonlocal parameter is zero, we obtain the constitutive relations of the local theories. Since classical continuum theories do not consider size effects arising from the small scale, so application of classical continuum theory is not appropriate for the nanostructures. In this regard, nonlocal elasticity theory has been widely used in the analysis of nanostructures.

Researchers have applied nonlocal elasticity theory in buckling [Wang et al. 2006; Mohammadi and Ghannadpour 2010] and vibration [Peddieson et al. 2003; Xu 2006] analyses of beams. Few authors have also applied nonlocal elasticity theory in bending analysis of beams. Some of them have been cited below.

Reddy and Pang [2008] presented analytical solutions for bending analysis of beams subjected to four sets of boundary conditions. Aydogdu [2009] developed a general nonlocal beam theory to derive governing equations from which all the well-known beam theories may be obtained. A nonlocal shear deformation beam theory has been proposed by Thai [2012]. Analytical solutions have also been presented for nonlocal sinusoidal shear deformation beam theory [Thai and Vo 2012]. Şimşek and Yurtcu [2013] examined bending and buckling of functionally graded (FG) nanobeams. Bending solutions have been presented analytically by Wang et al. [2008] for nanobeams. Some of the numerical methods such as the Ritz [Ghannadpour et al. 2013], the differential quadrature [Civalek and Demir 2011] and the finite element method [Alshorbagy et al. 2013; Eltaher et al. 2013] have also been developed for the bending analysis of nanobeams. Civalek and Akgöz [Civalek et al. 2009] presented deflection shapes and bending moments for nonlocal Euler–Bernoulli beams subjected to different boundary conditions.

The literature reveals that few works have been done on bending analysis of nanobeams based on Euler–Bernoulli and Timoshenko beam theories. It is also revealed that few numerical methods have also been developed for the above mentioned problem. In this article, authors have implemented Rayleigh–Ritz method with orthogonal polynomials as basis functions. The novelty of the method is that it may handle any set of boundary conditions with ease. Though this method has been used in classical beams and plates [Civalek et al. 2009; Behera and Chakraverty 2014; Bhat 1985; 1991; Chakraverty et al. 1999; 2007; Chakraverty and Petyt 1997; Singh and Chakraverty 1994; Hu et al. 2004], no works have been done in bending analysis of nanobeams. Boundary characteristic orthogonal polynomials have been applied in the Rayleigh–Ritz method to analyze effects of nonlocal, boundary condition and slenderness ratio on the deflection. Nondimensional deflection and rotation shapes have also been shown for three sets of boundary conditions.

2. Problem formulation

The study is carried out on the basis of Euler–Bernoulli and Timoshenko beam theories in conjunction with nonlocal elasticity theory of Eringen.

A straight uniform beam with the length L and a rectangular cross-section of thickness h is considered, as shown in Figure 1. A Cartesian coordinate system (x, y, z) is fixed on the central axis of the beam, where x, y and z coordinates are taken along the length, width and thickness of the beams [Ansari et al. 2013]. The Rayleigh–Ritz method has been employed for bending analysis. To apply the present method,

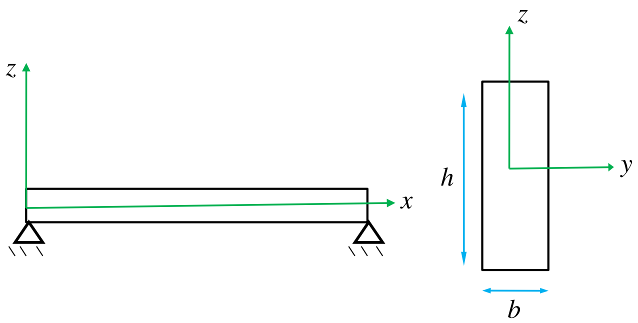

Figure 1. Uniform beam with rectangular cross section and its coordinate system [Ansari et al. 2013].

we have given a summary of the energies of the structures based on Euler–Bernoulli and Timoshenko beam theories.

2.1. ***Euler–Bernoulli beam theory (EBT).*** The strain energy u_s may be written as [Wang et al. 2000]

$$u_s = \frac{1}{2}\int_0^L \int_A \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}A\,\mathrm{d}x, \tag{1}$$

where L is the length of nanobeam, A is the cross sectional area, σ_{xx} is the axial stress and ε_{xx} is the normal strain.

Normal strain ε_{xx} is given by the relation

$$\varepsilon_{xx} = -z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{2}$$

where w is the transverse deflection at the point $(x, 0)$ on the midplane of the beam.

Substituting (2) into (1), we get

$$u_s = -\frac{1}{2}\int_0^L M\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x, \tag{3}$$

where M is the bending moment and is defined as

$$M = \int_A z\sigma_{xx}\,\mathrm{d}A. \tag{4}$$

Assuming that the beam is subjected to a transverse load $q(x)$, the potential energy u_p may be given as [Wang et al. 2000]

$$u_p = -\frac{1}{2}\int_0^L qw\,\mathrm{d}x. \tag{5}$$

Applying the principle of virtual displacement, we may obtain the following governing equation:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q = 0. \tag{6}$$

According to Eringen's nonlocal elasticity theory, the moment-curvature relation has the following form:

$$M-(e_0a)^2\frac{\mathrm{d}^2M}{\mathrm{d}x^2}=-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}, \tag{7}$$

where a is the internal characteristic length (e.g., lattice parameter, C-C bond length and granular distance) and e_0 is a constant appropriate to each material. The magnitude of e_0 is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. Here e_0a is the scale coefficient that incorporates the small scale effect [Wang et al. 2007]. Also E is the Young's modulus and I the second moment of area.

Using (6) and (7), M may be obtained as

$$M=-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-\mu q, \tag{8}$$

where $\mu=(e_0a)^2$ is the nonlocal parameter.

Combining (3) and (5), the total potential energy of the system may be written as

$$U=\frac{1}{2}\int_0^L\left(EI\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right)^2+\mu q\frac{\mathrm{d}^2w}{\mathrm{d}x^2}-qw\right)\mathrm{d}x. \tag{9}$$

2.2. *Timoshenko beam theory.* Based on Timoshenko beam theory, the strain energy u_s may be given as [Ansari et al. 2013]

$$u_s=\frac{1}{2}\int_0^L\int_A(\sigma_{xx}\varepsilon_{xx}+\sigma_{xz}\gamma_{xz})\,\mathrm{d}A\,\mathrm{d}x, \tag{10}$$

where σ_{xx} is the normal stress, σ_{xz} is the transverse shear stress, L is the length of the beam and A is the cross sectional area of the beam.

In (10), ε_{xx} and γ_{xz} are the normal and transverse shear strains respectively and are given by

$$\varepsilon_{xx}=z\frac{\mathrm{d}\phi}{\mathrm{d}x}, \tag{11}$$

$$\gamma_{xz}=\phi+\frac{\mathrm{d}w}{\mathrm{d}x}, \tag{12}$$

where ϕ is the rotation due to bending and w the transverse displacement.

Substituting (11) and (12) into (10), one may obtain

$$u_s=\frac{1}{2}\int_0^L\left(M\frac{\mathrm{d}\phi}{\mathrm{d}x}+Q\left(\phi+\frac{\mathrm{d}w}{\mathrm{d}x}\right)\right)\mathrm{d}x, \tag{13}$$

where M and Q are the bending moment and shear force respectively and are defined as

$$M=\int_A\sigma_{xx}z\,\mathrm{d}A,\qquad Q=\int_A\sigma_{xz}\,\mathrm{d}A.$$

The potential energy of the transverse load u_p may be described as [Ansari et al. 2013]

$$u_p=-\frac{1}{2}\int_0^L qw\,\mathrm{d}x. \tag{14}$$

Based on nonlocal elasticity theory, nonlocal constitutive equations are as follows:

$$M-(e_0a)^2\frac{\mathrm{d}^2M}{\mathrm{d}x^2}=EI\frac{\mathrm{d}\phi}{\mathrm{d}x} \tag{15}$$

$$Q=k_sGA\Big(\phi+\frac{\mathrm{d}w}{\mathrm{d}x}\Big), \tag{16}$$

where I is the second moment of area, E is the Young's modulus, G is the shear modulus and k_s is the shear correction factor in the Timoshenko beam theory to compensate the for error in assuming a constant shear strain (stress) through the thickness of the beam.

Applying the principle of virtual displacement, one may obtain the following governing equations for bending analysis:

$$\frac{\mathrm{d}M}{\mathrm{d}x}=Q, \tag{17}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}x}=-q. \tag{18}$$

Using (15)–(18), bending moment M may be obtained as

$$M=EI\frac{\mathrm{d}\phi}{\mathrm{d}x}-(e_0a)^2q. \tag{19}$$

Combining (13) and (14), the total potential energy of the system may be written as

$$U=\frac{1}{2}\int_0^L\Big(EI\Big(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big)^2-\mu q\frac{\mathrm{d}\phi}{\mathrm{d}x}+k_sGA\Big(\phi+\frac{\mathrm{d}w}{\mathrm{d}x}\Big)^2-qw\Big)\,\mathrm{d}x. \tag{20}$$

3. Solution methodology

Since conducting experiments at nanoscale size is quite difficult, the development of mathematical models has become quite important. In this paper, we have studied bending of beams based on Euler–Bernoulli and Timoshenko beam theories in conjunction with nonlocal elasticity theory. For doing so, we have applied the Rayleigh–Ritz method with boundary characteristic orthogonal polynomials as shape functions. Thus, displacement and rotation functions are represented by a series of admissible functions. Substituting the unknown functions and minimizing the potential energy of the system as a function of constants, one may find the system of linear equations. The above system of linear equations has been solved by using MATLAB and the solutions give the deflection parameter.

We define the nondimensional variable X as

$$X=x/L.$$

Each of the unknown functions w and ϕ may be expressed as the sum of series of polynomials, viz.,

$$w(X)=\sum_{k=1}^{n}c_k\hat{\varphi}_k(X), \tag{21}$$

$$\phi(X)=\sum_{k=1}^{n}d_k\hat{\psi}_k(X), \tag{22}$$

where n is the number of terms taken for computation, c_k, d_k are unknowns and $\hat{\varphi}_k$, $\hat{\psi}_k$ are orthonormal polynomials. First, orthogonal polynomials φ_k have been obtained from a linearly independent set of functions $\theta_k = F_u l_k$, $k = 1, 2, 3 \dots, n$ with $l_k = X^{k-1}$ using the Gram–Schmidt process as follows [Chakraverty and Petyt 1997]:

$$\varphi_1 = \theta_1, \quad \varphi_k = \theta_k - \sum_{j=1}^{k-1} \beta_{kj} \varphi_j, \tag{23}$$

where

$$\beta_{kj} = \frac{\langle \theta_k, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle}, \quad k = 2, 3, \dots, n, \quad j = 1, 2, \dots, k-1.$$

Here, $\langle \, , \rangle$ denotes the inner product of two functions and we define inner product of two functions, say φ_i and φ_k, as

$$\langle \varphi_i, \varphi_k \rangle := \int_0^1 \varphi_i(X) \varphi_k(X) \, \mathrm{d}X. \tag{24}$$

Similarly, the norm of the function φ_k is defined as

$$\|\varphi_k\| = \sqrt{\int_0^1 \varphi_k^2(X) \, \mathrm{d}X}.$$

Then normalized functions $\hat{\varphi}_k$ may be obtained by using the following relation:

$$\hat{\varphi}_k = \frac{\varphi_k}{\|\varphi_k\|}$$

One may note that same procedure may be followed to obtain $\hat{\psi}_k$. F_u and F_v are the boundary functions corresponding to unknown functions w and ϕ, respectively. It may be noted that the boundary polynomial specifies support conditions, particularly essential boundary conditions. Since $\hat{\varphi}_k$ and $\hat{\psi}_k$ are sets of orthogonal polynomials in the interval [0, 1], more rapid convergence and better stability in the numerical computation may be accomplished.

In Euler–Bernoulli beam theory, $F_u = X^r (1 - X)^s$, where r will take values of 0, 1, 2 accordingly as the edge $X = 0$ is free, simply supported or clamped, respectively. The same justification can be given to s for the edge $X = 1$. For Timoshenko beam theory, the following conditions should be satisfied by the boundary conditions; as such, the boundary functions used for the above said boundary conditions are given in Table 1:

- $W = M = 0$ at $X = 0$ and 1 for simply supported-simply supported (SS),
- $W = \phi = 0$ at $X = 0$ and 1 for clamped-clamped (CC), and
- $W = \phi = 0$ at $X = 0$ and $W = M = 0$ at $X = 1$ for clamped-simply supported (CS).

Substituting (21) into (9) and minimizing the potential energy of the system as a function of constants (i.e., $\partial U / \partial c_j = 0$), one may obtain following system of linear equations for EBT:

$$\sum_{j=1}^{n} a_{ij} c_j = P b_i, \tag{25}$$

boundary condition	F_u	F_v
S-S	$X(1-X)$	1
C-S	$X(1-X)$	X
C-C	$X(1-X)$	$X(1-X)$

Table 1. Boundary functions used for different edge conditions (TBT).

where $a_{ij} = \int_0^1 \hat{\varphi}_i{}'' \hat{\varphi}_j{}'' \,\mathrm{d}X$, $b_i = \int_0^1 \hat{\varphi}_i - \mu/L^2 \hat{\varphi}_i{}'' \,\mathrm{d}X$, $i = 1, 2, \ldots, n$ and $P = qL^4/(EI)$

Similarly, substituting (21) and (22) into (20) and minimizing the potential energy of the system as a function of constants (i.e., $\partial U/\partial c_j = 0$ and $\partial U/\partial d_j = 0$; $j = 1, 2, \ldots, n$), one may find the following system of linear equations for Timoshenko beam theory:

$$[K]\{Y\} = P\{B\}, \quad \text{where} \quad K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}. \tag{26}$$

Here, k_1, k_2, k_3 and k_4 are submatrices and are given by

$$k_1(i, j) = \int_0^1 2k_s GA \hat{\varphi}_i{}' \hat{\varphi}_j{}' \,\mathrm{d}X, \qquad k_2(i, j) = \int_0^1 2k_s GAL \hat{\varphi}_i{}' \hat{\psi}_j \,\mathrm{d}X,$$

$$k_3(i, j) = \int_0^1 2k_s GAL \hat{\psi}_i \hat{\varphi}_j{}' \,\mathrm{d}X, \quad k_4(i, j) = \int_0^1 (2k_s GAL^2 \hat{\psi}_i \hat{\psi}_j + 2EI \hat{\psi}_i{}' \hat{\psi}_j{}') \,\mathrm{d}X.$$

In (26), $Y = \{c_1\, c_2\, \ldots\, c_n\, d_1\, d_2\, \ldots\, d_n\}^T$ and $B = \{b_1\, b_2\}^T$, where

$$b_1(i) = \int_0^1 \varphi_i \,\mathrm{d}X, \quad b_2(i) = \int_0^1 \mu q L \psi_i{}' \,\mathrm{d}X.$$

4. Results and discussions

A numerical code has been developed in MATLAB to compute numerical results. Material and geometric properties of the carbon nanotubes are taken from [Alshorbagy et al. 2013], and are given in Table 2. A uniformly distributed load ($q = 1$) has been taken into consideration for three different boundary conditions. The letters C, S and F refer to clamped, simply supported and free edge conditions, respectively. It is a well-known fact that nondimensional maximum deflection is evaluated at the center of the beam, which is given by $W_{\max} = -w \times 10^2 (EI/(qL^4))$. Before presenting and discussing the new results, it is necessary to perform a convergence study and also to validate the present method with other methods presented in the literature.

Therefore, a convergence study has been carried out for the nondimensional maximum deflection $W_{\max}$ of the EBT nanobeam with C-S support. As such, Figure 2 shows convergence of the nanobeam with $L/h = 10$ and $\mu = 1.5\,\mathrm{nm}^2$. As can be seen from the figure, $n = 4$ is sufficient for converged results. It may be noted that previous published results [Ghannadpour et al. 2013] also show the same number of terms required for computation.

In order to validate the results obtained by the present method, the nondimensional maximum deflection is compared in Table 3 with those reported in [Alshorbagy et al. 2013]. In this table, the results are presented for nonlocal beams with boundary conditions at two ends which are of a variety of combinations

properties	value
E	$30 \cdot 10^6$
h	1
ks	$5/6$
ν	0.19

Table 2. Material properties of the carbon nanotubes.

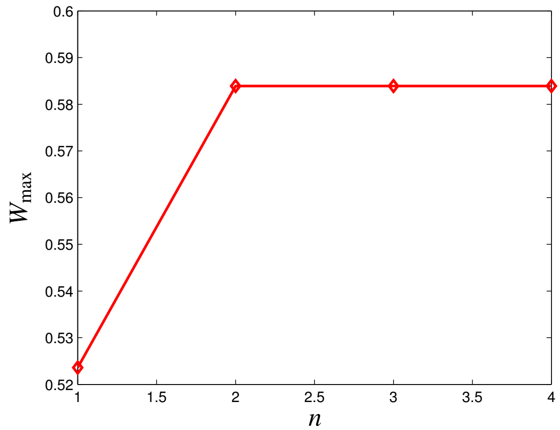

Figure 2. Convergence of nondimensional maximum center deflection for EBT.

	C-S		C-C	
μ	present	[Alshorbagy et al. 2013]	present	[Alshorbagy et al. 2013]
0	0.50	0.54	0.24	0.26
1	0.52	0.58	0.24	0.26
2	0.59	0.61	0.24	0.26
3	0.60	0.65	0.24	0.26

Table 3. Comparison of nondimensional maximum center deflection (W_{max}) for C-S and C-C boundary conditions.

such as C-S and C-C. Results have been shown for different values of a nonlocal parameter. It is noted that the results reported by Alshorbagy et al. [2013] are obtained by the finite element model. It can be seen that there is an excellent agreement between the obtained results in this paper and those reported in the previous work.

Next, we have carried out some of the parametric studies which are discussed below. One may note unless mentioned that deflection and rotation would refer to nondimensional maximum center deflection and nondimensional maximum center rotation respectively.

4.1. ***Effect of slenderness ratio.*** Figure 3 illustrates the effect of the slenderness ratio (L/h) on the deflection of nanobeams. In this figure, we have shown the variation of deflection with slenderness ratio

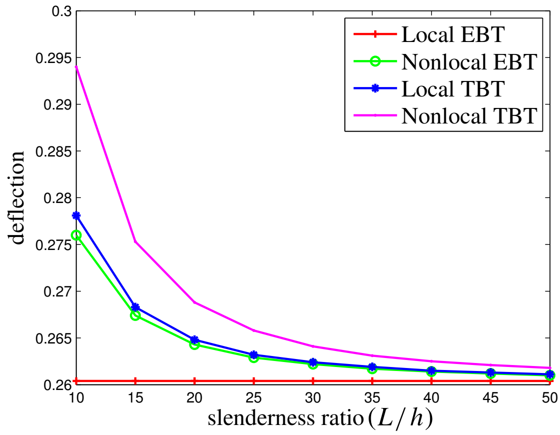

Figure 3. Effect of the slenderness ratio on the dimensionless deflection.

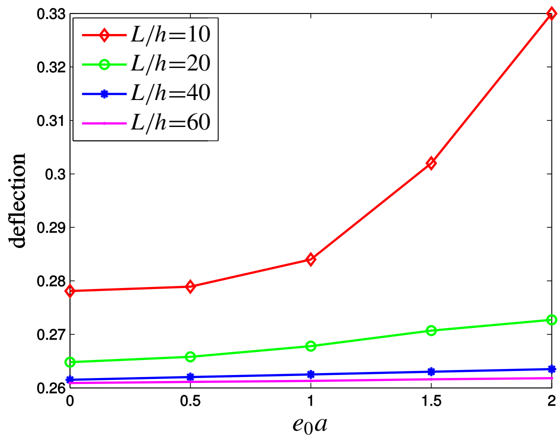

Figure 4. Effect of the nonlocal parameter on the dimensionless deflection.

for both local and nonlocal theories. Here, the slenderness ratio varies from 10 to 50 and the boundary condition is considered as C-S. Local results may be computed by taking the nonlocal parameter (μ) as zero. One may note that nonlocal results have been computed for $\mu = 1\,\text{nm}^2$. We have presented the graphical results for nanobeams based on both EBT and TBT beam theories. One may observe that in the case of nanobeams based on local EBT, the slenderness ratio has no effect on the beam deflection whereas in nonlocal EBT, deflection is dependent on the slenderness ratio. It may also be noticed that in case of nanobeams based on both local and nonlocal TBT, deflection is dependent on the slenderness ratio. The dependency of the responses on the slenderness ratio for local TBT is uniquely due to the effect of shear deformation and this dependency becomes strong with the effect of small scale. As slenderness ratio decreases, the difference between the solutions of EBT and TBT becomes highly important.

4.2. ***Nonlocal parameter effect.*** In order to investigate the effect of the nonlocal parameter on the deflection, variation of deflection with the scale coefficient has been demonstrated in Figure 4 for different values of the slenderness ratio (L/h). In this figure, we have considered TBT nanobeams with the C-S edge condition. Graphical results have been shown for different values of slenderness ratio. It is seen

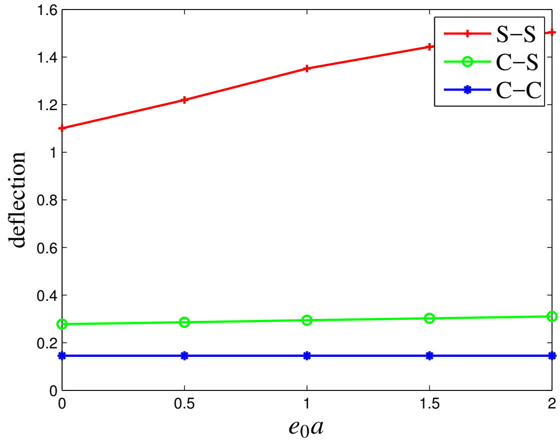

Figure 5. Effect of the nonlocal parameter on the dimensionless deflection for different boundary conditions.

from the figure that deflection varies nonlinearly with the scale coefficient. One may also observe that all responses of nanobeams with lower aspect ratios are strongly affected by the nonlocal parameter than those of nanobeams with relatively higher aspect ratios. From these computations, it may be explained that modeling based on the local beam models may not be suitable, whereas the nonlocal beam models show an adequate approximation for the nanosized structures [Şimşek and Yurtcu 2013]. It is also noticed that deflection increases with the scale coefficient, while it is not true in case of buckling and vibration [Şimşek and Yurtcu 2013]. One may conclude here that the nonlocal beam model produces a larger deflection than the classical (local) beam model. Therefore, the small scale effects (or nonlocal effects) should be considered in the analysis of the mechanical behavior of nanostructures.

4.3. ***Boundary condition effect.*** Deflections of nanobeams under uniform load have been computed for different boundary conditions and are shown in Figure 5. In this figure, the effect of deflection on the scale coefficient has been shown for three sets of boundary conditions, viz., S-S, C-S and C-C. In doing so, we have taken the slenderness ratio as 10. We observe that C-C has the smallest deflection for a particular value of the nonlocal parameter. One may note that in the case of C-C edge condition, there is no effect of the nonlocal parameter on the deflection, whereas in the case of S-S and C-S supports, deflection increases with an increase in the nonlocal parameter. Hence, the effect of the nonlocal parameter on the deflection is inconsistent for different boundary conditions. We state some other observations in Section 4.4.

4.4. ***Deflection and rotation shapes.*** In this subsection, we examine the behavior of deflection and rotation shapes of nanobeams along its length for different boundary conditions. Figures 6–8 show variation of deflection with length for S-S, C-S and C-C edge conditions, respectively. It is observed from the figures that deflection of S-S and C-S nanobeams increases with increases in the nonlocal parameter. It is due to the fact that increasing nonlocal parameter causes an increase in the bonding force of atoms and this force is constrained by its boundaries, which increases deflection. Another observation is seen in that the nonlocal parameter has no effect on the deflection of C-C nanobeams because of its constrained nature [Alshorbagy et al. 2013]. Next, we have shown variation of rotation with length for S-S, C-S and C-C edge conditions, respectively in Figures 9–11. It may be noticed that the rotation behaves differently

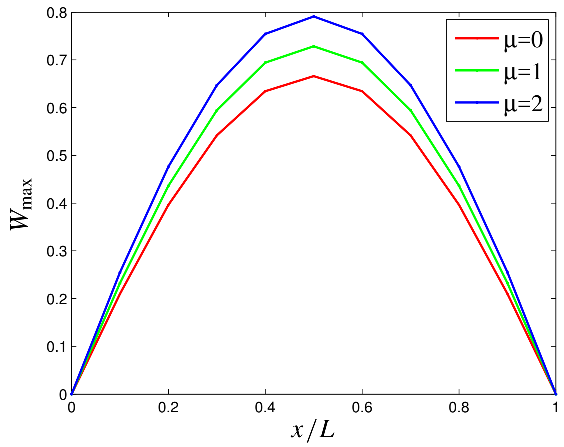

Figure 6. Static deflection of S-S nanobeams for different nonlocal parameters.

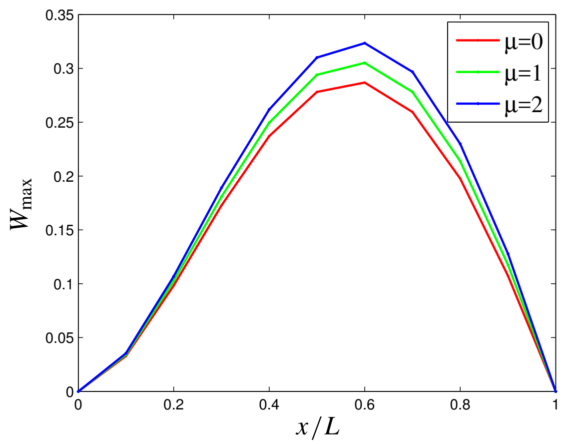

Figure 7. Static deflection of C-S nanobeams for different nonlocal parameters.

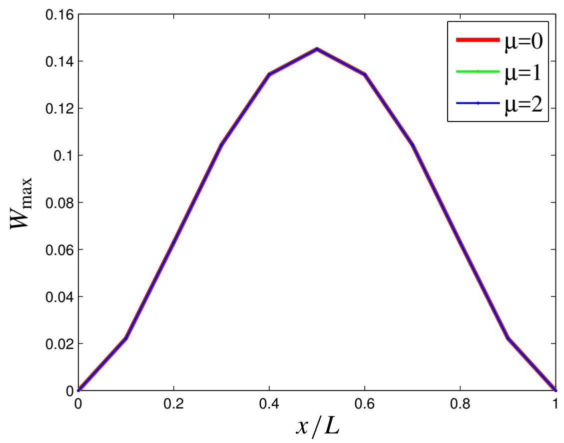

Figure 8. Static deflection of C-C nanobeams for different nonlocal parameters.

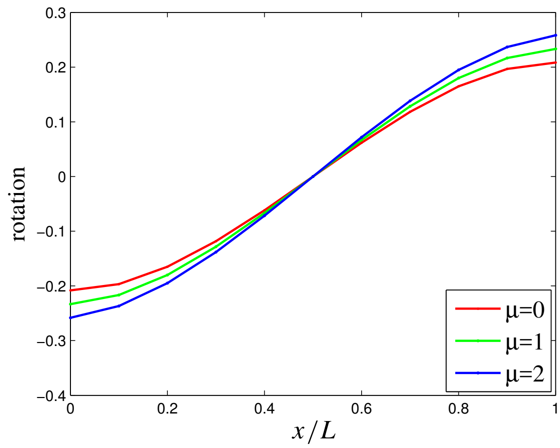

Figure 9. Static rotation of S-S nanobeams for different nonlocal parameters.

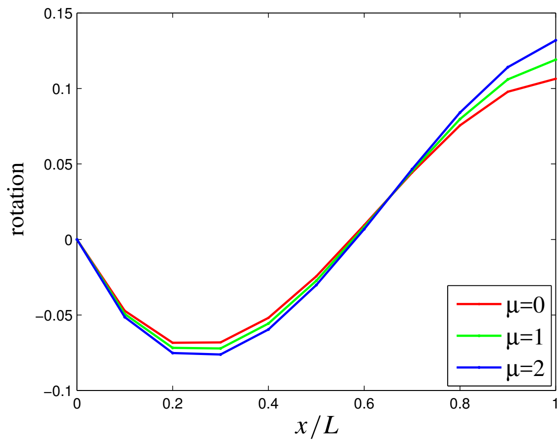

Figure 10. Static rotation of C-S nanobeams for different nonlocal parameters.

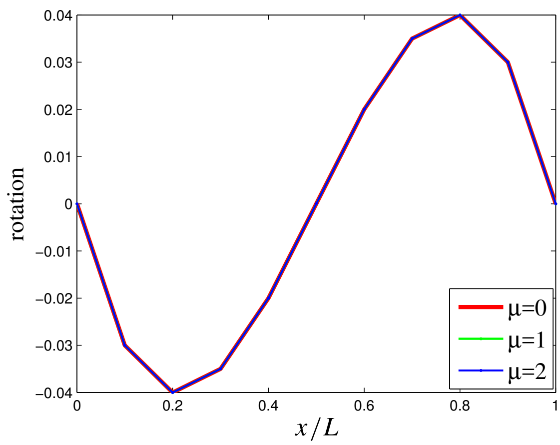

Figure 11. Static rotation of C-C nanobeams for different nonlocal parameters.

than that of deflection. Increasing the nonlocal parameter decreases rotation of S-S and C-C nanobeams up to midlength and afterwards increases in the nonlocal parameter increases rotation. One may also notice that the nonlocal parameter has no effect on the rotation of C-C nanobeams.

5. Concluding remarks

Boundary characteristic orthogonal polynomials as shape functions have been implemented in the Rayleigh–Ritz method for static analysis of nanobeams. The formulation is based on both Euler–Bernoulli and Timoshenko beam theories in conjunction with nonlocal elasticity of Eringen. A system of linear equations is formed by the applying the present method. The following conclusions may be derived from the present analysis:

- Slenderness ratio has no effect on the beam deflection in the case of local EBT, whereas in the case of nonlocal EBT, deflection is dependent on the slenderness ratio.
- It is seen that bending responses vary nonlinearly with the nonlocal parameter. One may also observe that bending responses of nanobeams with lower aspect ratios are strongly affected by the nonlocal parameter than those of the nanobeams with relatively higher aspect ratios.
- The nonlocal parameter has no effect on the deflection of C-C nanobeams, whereas in case of S-S and C-S supports, deflection increases with increases in the nonlocal parameter.

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Received 11 Nov 2016. Revised 2 Jun 2017. Accepted 13 Jun 2017.

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Journal of Mechanics of Materials and Structures

Volume 12, No. 5 December 2017