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Carlos Andrade, Chan Soo Ha and Roderic S. Lakes

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EXTREME COSSERAT ELASTIC CUBE STRUCTURE WITH LARGE MAGNITUDE OF NEGATIVE POISSON'S RATIO

CARLOS ANDRADE, CHAN SOO HA AND RODERIC S. LAKES

A structure consisting of pivoting cubes is presented. It has zero Young's modulus and zero bulk modulus. Poisson's ratio has large negative values in all directions; the structure exhibits anisotropy in Poisson's ratio. The structure is compliant in tension but rigid in torsion and bending. The Cosserat characteristic length tends to infinity.

1. Introduction

A 3D negative Poisson's ratio material based on transformed open cell polyurethane foam was reported in [Lakes 1987a]; it had a Poisson's ratio -0.7. It is possible to approach the isotropic lower limit -1 via structures or lattices with hinges. Negative Poisson's ratio was analyzed in a model of rods, hinges and springs [Almgren 1985]; a value of -1 was calculated. A Poisson's ratio of -1 can be achieved in 2D structures containing rotating rigid units such as squares [Grima et al. 2005] connected by ideal hinges. Negative Poisson's ratio was also studied in 2D systems with rotating hexamers [Wojciechowski 1987; 1989] in the context of thermodynamic stability.

More recent designs with bars linked by ideal pivots allow the structure to undergo arbitrarily large volumetric strain with zero bulk modulus [Milton 2013]. Negative Poisson's ratio materials have been called "dilational" [Milton 1992] because they easily undergo volume changes but are difficult to shear.

It is possible to approach the isotropic lower limit -1 at small strain in the analysis of a hierarchical two phase composite [Milton 1992] if there is sufficient contrast between constituent properties. A 2D chiral lattice [Prall and Lakes 1997] exhibits a Poisson's ratio -1 over a range of strain as shown by experiment and analysis.

In the present research, we develop a structure made of cubes connected by pivots at their corners. Poisson's ratio and sensitivity to gradients are studied.

2. Cube structure

A structure is envisaged of cubes of side length a connected by pivots at the corners; see Figure 1.

Views of the $3 \times 3 \times 3$ cube structure along principal directions are shown in Figure 2. The rear layers of the cubes are fully hidden. Deformation results in tilting of the cubes at the pivot points. This tilt causes void space to appear in the structure giving rise to a volume change. Transverse expansion of the structure under tension implies a negative Poisson's ratio.

2.1. *Analysis and interpretation.*

Keywords: auxetic, negative Poisson's ratio, Cosserat, metamaterial.

Figure 1. Cube structure, oblique view.

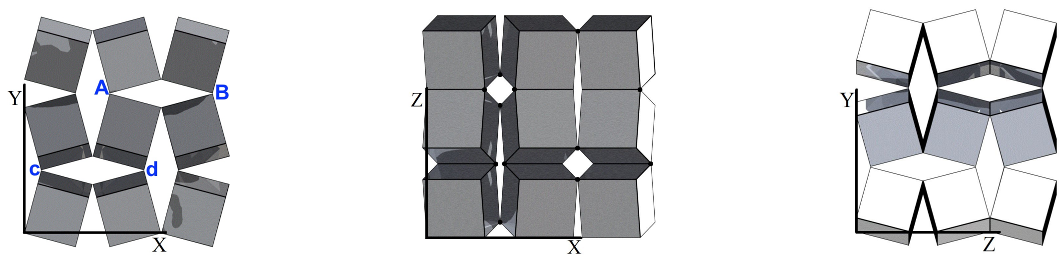

Figure 2. Cube structure, principal direction view.

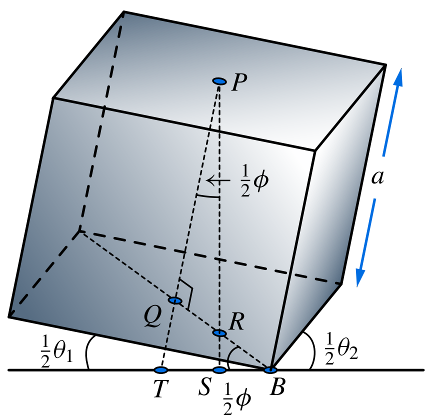

Figure 3. Analysis of cube structure deformation using two angles.

2.1.1. *Elastic moduli and Poisson's ratio: two angles.* We assume that the cubes are rigid and the pivots are ideal and allow frictionless rotation in all directions. Tensile deformation of the structure freely occurs in each axial direction: Young's modulus $E = 0$. Consequently, changes in volume occur with no resistance so the bulk modulus is zero. Shear forces in the X direction (Figure 2, left) on adjacent layers cause no deformation because the cube tilt cannot accommodate such motion. Shear forces in the Y direction (Figure 2, center) cause no deformation because the edges are in contact, forming a hinge.

Consequently, the structure resists shear in all directions but allows tensile deformation, suggesting an extremal negative Poisson's ratio.

Strain ϵ depends on cube tilt angle ϕ, beginning at zero, as follows. Two angles are considered for simplicity and transparency; the third angle shown in Figure 2 right appears to be small and is neglected for the present. In Figure 3 consider the change in the length of a vertical line element during deformation. Points A and B (Figure 2, left) are at pivoted corners; distance a is the cube width. After deformation the vertical line from P, center of top face, intersects the bottom cube face at R; because tilt occurs in two orthogonal directions, the intersection is along a diagonal. Points B, S, and T, not necessarily collinear, are in a horizontal plane. The change in length is $\Delta L = PS - a$. The corresponding strain is $\epsilon_{yy} = (PS - a)/a$.

In triangle PQR, $\cos\frac{1}{2}\phi = PQ/(PR) = a/(PR)$. In triangle BQT, $\tan\frac{1}{2}\phi = QT/(BQ)$ so $QT = a\frac{1}{2}\sqrt{2}\tan\frac{1}{2}\phi$. Also, $PT = a + QT = a(1 + \frac{1}{2}\sqrt{2}\tan\frac{1}{2}\phi)$, but in triangle PST, $\cos\frac{1}{2}\phi = PS/(PT)$ so $PS = PT\cos\frac{1}{2}\phi$ with $PT = a + QT$. $PS = a(1 + \frac{1}{2}\sqrt{2}\tan\frac{1}{2}\phi)\cos\frac{1}{2}\phi$.

So the strain in terms of tilt angle is

$$\epsilon_{yy} = \left(1 + \tfrac{1}{2}\sqrt{2}\tan\tfrac{1}{2}\phi\right)\cos\tfrac{1}{2}\phi - 1. \tag{1}$$

If the angle is sufficiently large, the force has a line of action passing through a pivot. The force generates no moment to cause further rotation. For ϵ_{xx} as seen in the XY plane, the geometry is similar. However, viewed in the z direction, the effect of θ_1 alone gives the following in the linear regime of small angle:

$$\epsilon_{xx}^{\theta_1} = \tfrac{1}{2}\tan\tfrac{1}{2}\phi, \tag{2}$$

but θ_2 rotates the corresponding point on the right face center down, reducing the distance, yielding a strain

$$\epsilon_{xx} = +\frac{1}{2\sqrt{2}}\tan\tfrac{1}{2}\phi. \tag{3}$$

The Poisson's ratio is, for small angle,

$$\nu_{xy} = -\frac{\epsilon_{xx}}{\epsilon_{yy}} = -\frac{\left(\dfrac{1}{2\sqrt{2}}\tan\tfrac{1}{2}\phi\right)}{\left(1 + \dfrac{\sqrt{2}}{2}\tan\tfrac{1}{2}\phi\right)\cos\tfrac{1}{2}\phi - 1}. \tag{4}$$

From the definition, $\nu_{yx} = 1/\nu_{xy}$. For small angle, $\nu_{xy} = -\frac{1}{2}$, $\nu_{yx} = -2$. The structure is therefore anisotropic even though Young's modulus $E = 0$ in all directions.

As for ϵ_{zz} as seen in the YZ plane, $\epsilon_{zz} = \epsilon_{xx}$ by virtue of a similar construction (neglecting tilt in the third orthogonal direction). So $\nu_{zx} = -1$, $\nu_{xz} = -1$.

This analysis makes the simplifying assumption that tilt in the third direction is small compared with tilt in the two directions considered. Tilt in all three directions is considered in the numerical approach below.

2.1.2. *Bending.* The classical bending rigidity of a bar is $M\mathcal{R} = EI$ with $\mathcal{R}$ as the principal bending radius of curvature and I is the area moment of inertia. Moment M is about the y axis; the z axis is along the bar. The rigidity depends only on Young's modulus E not on Poisson's ratio ν. The effect

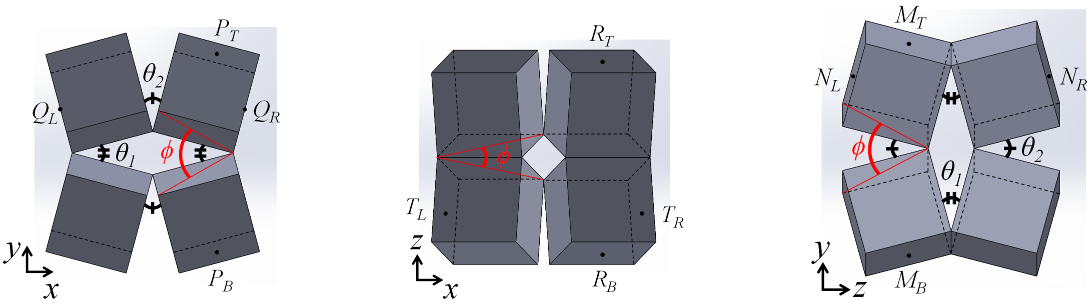

Figure 4. Cube structure viewed along the principal directions. All points are located at the center of cube faces. Here, ϕ is an angle between diagonal lines of adjacent cubes; θ_1 and θ_2 represent angles between cube edges when the structure is viewed in the corresponding principal directions. For the structure shown, $\phi = 42.17°$, $\theta_1 = 30°$ and $\theta_2 = 28.96°$.

of Poisson's ratio is to alter the deformation field. For positive Poisson's ratio the cross sections curve oppositely to the principal bending curve, the familiar anticlastic curvature. A negative Poisson's ratio causes curvature in the same direction as the bending curve, synclastic curvature [Lakes 1987a]. The three-dimensional displacement field for pure bending of a linear elastic homogeneous, isotropic bar of rectangular section is

$$u_x = -\frac{z^2 + \nu(x^2 - y^2)}{2\mathcal{R}}, \quad u_y = -\nu\frac{xy}{\mathcal{R}}, \quad u_z = \frac{xz}{\mathcal{R}}. \tag{5}$$

So if Young's modulus E tends to zero, it should be easy to bend the bar with no effort regardless of Poisson's ratio, provided the bar obeys classical elasticity.

The cube structure, while easy to deform in tension ($E = 0$), is rigid in bending. To visualize this, in the XY plane in the left image in Figure 2, expansion of a line along direction AB due to bending due to a Z moment is accompanied by contraction along line cd. The pivoted cube structure requires either expansion or contraction in all directions, so the structure is rigid to bending. Lines AB and cd are in different planes but that does not affect the argument because the classical motion has the same sign on the front and back.

Bending differs from axial extension in that bending entails gradients in strain and rotation. Classical elasticity is insensitive to gradients but Cosserat elasticity allows such sensitivity. Rigidity of the structure to bending combined with a zero tensile Young's modulus implies a Cosserat characteristic length that tends to infinity. Again, the cubes are assumed to be rigid and the pivots are assumed to be ideal.

2.1.3. *Elastic moduli and Poisson's ratio: numerical model, three angles.* Figure 4 illustrates views of the structure along the principal directions and points on the center of cube faces that were used to compute Poisson's ratio via a numerical model. Moreover, θ_1 and θ_2 represent angles between cube edges when the structure is viewed in the corresponding principal directions.

To determine the effect of motion in all three angles, the cube structure was modeled by SolidWorks commercial CAD software. In this analysis, a cube structure was modeled with a cube side length of 10 mm and with various inclined angles ϕ of 7.07, 14.13, 21.18, 42.17, 62.74, and 82.56 degrees. Poisson's ratio in the principal directions was obtained as follows. The "mate" feature in SolidWorks was used to make hinge constraints on corners. Distance was then measured using the software. The effect is purely geometrical so there was no need to use tools such as ANSYS APDL.

To obtain Poisson's ratio of the cube structure, strain and Poisson's ratio were determined in terms of the distances. First, strains in the principal directions due to the angle ϕ were computed, as given in the following equations:

In xy plane,

$$\epsilon_{xx} = \frac{|\overline{Q_{R,x} Q_{L,x}}| - 2a}{2a}, \tag{6a}$$

$$\epsilon_{yy} = \frac{|\overline{P_{T,y} P_{B,y}}| - 2a}{2a}. \tag{6b}$$

In xz plane,

$$\epsilon_{xx} = \frac{|\overline{T_{R,x} T_{L,x}}| - 2a}{2a}, \tag{6c}$$

$$\epsilon_{zz} = \frac{|\overline{R_{T,z} R_{B,z}}| - 2a}{2a}. \tag{6d}$$

In yz plane,

$$\epsilon_{zz} = \frac{|\overline{N_{R,z} R_{L,z}}| - 2a}{2a}, \tag{6e}$$

$$\epsilon_{yy} = \frac{|\overline{M_{T,y} M_{B,y}}| - 2a}{2a}. \tag{6f}$$

In the above, a is the cube side length and $|\overline{Q_{R,x} Q_{L,x}}|$ denotes a distance between Q_R and Q_L in the x direction in the xy plane after deformation; ϵ_{xx} is then equal to $(|\overline{Q_{R,x} Q_{L,x}}| - 2a)/2a$. With strains found in (6), Poisson's ratio in the principal directions are

$$\nu_{xy} = -\frac{\epsilon_{xx}}{\epsilon_{yy}} = -\frac{|\overline{Q_{R,x} Q_{L,x}}| - 2a}{|\overline{P_{T,y} P_{B,y}}| - 2a}, \tag{7a}$$

$$\nu_{xz} = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = -\frac{|\overline{T_{R,x} T_{L,x}}| - 2a}{|\overline{R_{T,z} R_{B,z}}| - 2a}, \tag{7b}$$

$$\nu_{zy} = -\frac{\epsilon_{zz}}{\epsilon_{yy}} = -\frac{|\overline{N_{R,z} R_{L,z}}| - 2a}{|\overline{M_{T,y} M_{B,y}}| - 2a}. \tag{7c}$$

From numerical results, it was found that $\epsilon_{xx,\text{num}}$ from the xy and the xz planes were identical (i.e., $\epsilon_{xx,\text{num}}^{\text{xy-plane}} = \epsilon_{xx,\text{num}}^{\text{xz-plane}}$). Similarly, $\epsilon_{yy,\text{num}}$ in the xy plane agreed exactly with the one in the yz plane, and $\epsilon_{zz,\text{num}}$ were the same for the xz and the yz planes (i.e., $\epsilon_{yy,\text{num}}^{\text{xy-plane}} = \epsilon_{yy,\text{num}}^{\text{yz-plane}}$ and $\epsilon_{zz,\text{num}}^{\text{xz-plane}} = \epsilon_{zz,\text{num}}^{\text{yz-plane}}$). This confirms that cube structures modeled by the employed CAD software were correctly designed and interpreted since the computed strains were the same regardless of views in different principal directions.

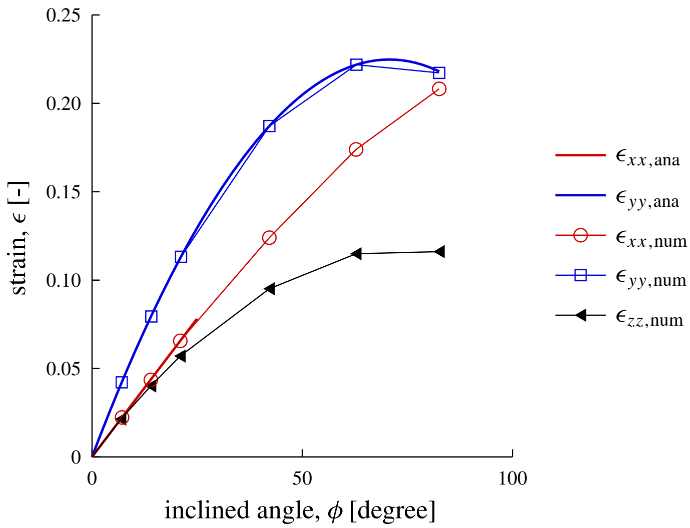

Figure 5. Strains of a cube structure as a function of the angle ϕ.

As a result, the superscript of strains obtained numerically were omitted in this paper as follows, unless stated otherwise.

A comparison of strains between analytical and numerical approaches was made, as shown in Figure 5. The strain ϵ_{yy} between these two approaches agreed well with one another throughout the range of ϕ. The strain ϵ_{xx} also agreed over the narrower range of strain consistent with the simplifying assumptions in that analysis including neglect of the effect of the tilt in the third orthogonal direction and of higher nonlinearity.

As illustrated in Figure 4, left, and Figure 4, right, the effect of the orthogonal tilt is small when the angle ϕ is small. From this, it was expected that $\epsilon_{xx,\text{num}}$ and $\epsilon_{zz,\text{num}}$ be similar when ϕ is small, and this can be observed in Figure 5. In this regime (i.e., for small ϕ), strains are almost linear as a function of angle. In contrast, nonlinearity occurs when ϕ is large. The effect of the tilt in the third orthogonal direction can be quantified by θ_1 and θ_2 that represent angles between adjacent cube edges when the structure is viewed in the corresponding principal directions. For small ϕ, these two angles are similar.

In summary, for small strain, the Poisson's ratios ν_{zx} and ν_{xz} obtained by two-angle analysis and three-angle numerical methods are equal with a value of -1, as illustrated in Figure 6. For other directions, the simple analysis and numerical results agree with the appropriate small angle range.

3. Physical model

Physical models were made to aid visualization and to illustrate the concepts. An initial model was made with cubes cut from polymer foam. A design was assembled digitally using Solidworks 2016. Cubes were prepared using Solidworks in .stl (StereoLithography) format for export to a 3D printer. The method was fused deposition method (FDM). The print resolution (i.e., the minimum size of a stand alone feature) was 0.5 mm. These cubes, 2 cm wide, were manufactured using a Dimension Elite 3D printer, and made of Stratasys ABSplus P430 thermoplastic. Pivots can be made by 3D printing but

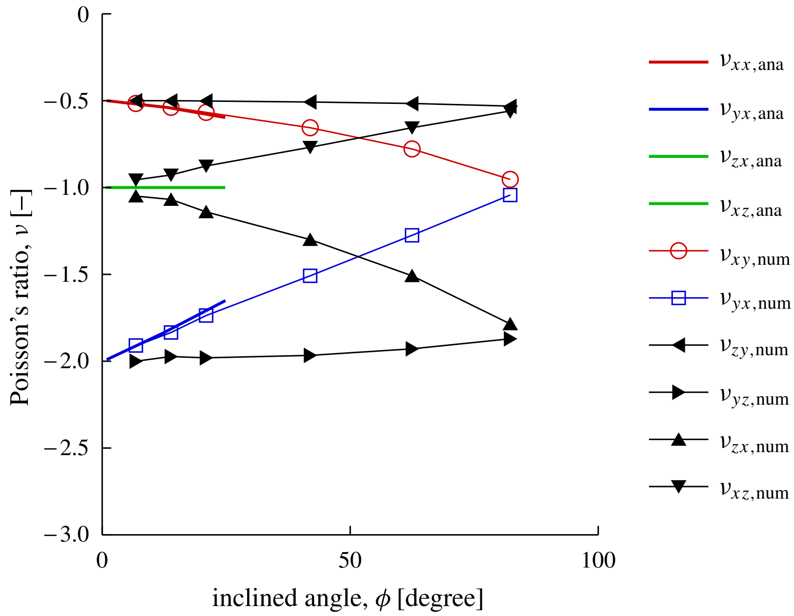

Figure 6. Poisson's ratio in all the principal directions.

they are subject to considerable friction which would interfere with the demonstration of the concept. Therefore a fibrous tape was cut into a dog-bone shape. Segments were taped to adjacent cubes, leaving the slender portion as a pivot. A $3 \times 3 \times 6$ model was made with Z as the long direction.

The physical model was observed to be easy to stretch provided only one or two cubes on each end were held gently and allowed to rotate. The structure expanded fully in tension under its own weight. The model was rigid to torsion as well as to shear in different directions. The model was also rigid to cantilever bending. In both cases, slight movement associated with slack in the pivots was observed.

4. Discussion

Several negative Poisson's ratio structures with rotating hinged elements are known. In addition to rotating squares [Grima et al. 2005], one can have rotating rhombi [Attard and Grima 2008], triangles [Grima and Evans 2006], edge connected cuboids of different sizes [Attard and Grima 2012], and complex hinged structures [Milton 2013]. Such pivoted structures, including the present one, exhibit a hard nonlinearity when the structural elements come into contact and when the lattice is fully extended. Even so, the geometry of hinged structures has been used to help explain [Attard and Grima 2008] the negative Poisson's ratio of materials in which the effects arise on the molecular scale.

For the present ideal structure with rigid cubes and frictionless pivots, Young's modulus is zero in tension provided the end cubes are allowed to rotate as is the case in stress control. Although tensile deformation freely occurs, the structure is rigid in bending and torsion. Classical elasticity cannot account for such behavior but Cosserat [Cosserat and Cosserat 1909] (micropolar [Eringen 1968]) elasticity, which allows sensitivity to gradients, can do so. Cosserat theory provides characteristic length parameters as elastic constants. If the specimen size is not too much greater than the characteristic length, size effects are observed in bending and torsion; the effective modulus in bending exceeds the true Young's modulus in tension. Such effects are known in a variety of foams including negative Poisson's ratio foam [Rueger and Lakes 2016]. However the range of Poisson's ratio is the same in Cosserat solids as in classical ones so coarse cell structure is not needed to control the Poisson's ratio [Lakes 1987b]. The cube structure will be rigid to bending and torsion independent of how small the cubes are in comparison with the specimen size, provided the cubes are rigid and the pivots are ideal. This implies a characteristic length that is infinitely large. Such singular behavior arises from the geometrical constraints in a highly idealized structure. Similar singular behavior likely occurs in other negative Poisson's ratio hinged structures and in structures made using sliding elements [Gourgiotis and Bigoni 2016]. Extremely large Cosserat effects leading to folding and faulting can occur in highly anisotropic materials that admit couple stress [Bigoni and Gourgiotis 2016]. Three-dimensional structures are of particular interest because in 3D, classical bending can occur either via shear at constant volume, as in rubbery materials, or via local volume change with constant shape, as when Poisson's ratio tends to -1. A material or structure that does not allow bending cannot be classically elastic.

5. Conclusions

A structure of pivoting cubes is presented. It has negative Poisson's ratio of large magnitude in each direction and a tensile modulus of zero. It is rigid to bending, therefore it is not classically elastic. The structure behaves as an extreme Cosserat solid.

Acknowledgements

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CARLOS ANDRADE: carlos.andrade.von@gmail.com
University of Wisconsin, Madison, WI, United States

CHAN SOO HA: ha3@wisc.edu
University of Wisconsin, Madison, WI, United States

RODERIC S. LAKES: rlakes@wisc.edu
Department of Engineering Physics, University of Wisconsin-Madison, Madison, WI, United States