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FORMULAS FOR THE H/V RATIO OF RAYLEIGH WAVES IN COMPRESSIBLE PRESTRESSED HYPERELASTIC HALF-SPACES

PHAM CHI VINH, THANH TUAN TRAN, VU THI NGOC ANH AND LE THI HUE

This paper concerns the propagation of Rayleigh waves in compressible prestressed elastic half-spaces. The main aim is to derive formulas for the H/V ratio (ellipticity), which is the ratio of the amplitude of the horizontal displacement to the amplitude of the vertical displacement of Rayleigh waves. First, the equation for the H/V ratio is obtained using the secular equation and the relation between the H/V ratio and the Rayleigh wave velocity. Then, formulas for the H/V ratio are derived for a general strain-energy function. They are then specified for some strain-energy functions. Since the H/V ratio is a good tool for nondestructively evaluating the prestresses of structures before and during loading, the obtained formulas will be very useful in practical applications.

1. Introduction

Nowadays, prestressed materials are widely used in many technical applications. Evaluating nondestructively prestresses of structures before and during loading is necessary and important, and the Rayleigh wave is a convenient tool for this task; see, for example, [Hirao et al. 1981; Delsanto and Clark 1987; Makhort et al. 1990; Duquennoy et al. 1999; 2006] (noting that an error in the second paper was corrected by Song and Fu [2007]). First, a Rayleigh wave is generated (excited) and it propagates in the structure of which the prestresses are needed to evaluate. Then its velocity is measured. Based on the explicit secular equations of the Rayleigh wave and the measured values of the Rayleigh wave velocity, an inverse problem is solved to determine the prestresses. Suppose we have in hand the explicit formulas for the Rayleigh wave velocity, then the inverse problem will be much simpler if it is solved by using them, instead of using the secular equations.

As shown recently by M. Junge and Jacobs [Junge et al. 2006], the H/V ratio is more sensitive to the state of stress than the Rayleigh velocity, and, further, in contrast to the Rayleigh velocity, it is reference-free, so the H/V ratio of Rayleigh waves is a more convenient tool than the Rayleigh wave velocity for characterizing the state of stress. When using the H/V ratio, the explicit H/V ratio equations are employed as a mathematical base for extracting the prestresses from measured values of the H/V ratio. The inverse problem will become much simpler if its formulation is based on the explicit H/V ratio formulas. While the explicit secular equation [Dowaikh and Ogden 1991] and the explicit exact formulas for the velocity [Vinh 2011] of Rayleigh waves propagating in compressible prestressed elastic half-spaces have been found, no explicit equations and explicit exact formulas for the H/V ratio have appeared in the literature so far.

The main aim of this paper is to derive the explicit exact H/V ratio formulas for compressible prestressed elastic half-spaces. First, the H/V ratio equation is established by using the secular equation and

Keywords: Rayleigh waves, prestressed half-spaces, H/V ratio, H/V ratio formulas.

the relation between the H/V ratio and the Rayleigh wave velocity. This relation is obtained by using the surface impedance matrix of Rayleigh waves propagating in compressible prestressed half-spaces. Solving analytically the H/V ratio equation, we arrive at the explicit exact H/V ratio formulas which are valid for a general strain-energy function. They are then written down for some specific strain-energy functions. These obtained formulas express directly the H/V ratio in terms of material parameters and prestresses. Since the obtained H/V ratio formulas are totally explicit, they will be a powerful tool for evaluating the prestresses appearing in structures before and during loading.

It is worth noting that the H/V ratio is an important quantity which reflects fundamental properties of the elastic material [Malischewsky and Scherbaum 2004]. Therefore, it can be used for the nondestructive evaluation of the elastic constants of the material as well.

2. Equations for the H/V ratio

Relation between the H/V ratio and the Rayleigh wave velocity. Consider a Rayleigh wave propagating in an elastic half-space $x_2 \geq 0$ with velocity $c > 0$, wave number $k > 0$ in the x_1-direction, and decaying in the x_2-direction. Then, the displacement vector $\boldsymbol{u}$ and the traction vector $\boldsymbol{t}$ at the planes $x_2 =$ constant of the Rayleigh wave are of the form

$$\boldsymbol{u} = \boldsymbol{U}(y)e^{ik(x_1-ct)}, \quad \boldsymbol{t} = ik\boldsymbol{\Sigma}(y)e^{ik(x_1-ct)}, \quad y = kx_2. \tag{1}$$

Matrix $\boldsymbol{M}$ is called the surface impedance matrix of the Rayleigh wave if it relates the vectors $\boldsymbol{U}(0)$ and $\boldsymbol{\Sigma}(0)$ by the equality [Ingebrigtsen and Tonning 1969; Chadwick and Smith 1977; Barnett and Lothe 1985; Fu and Mielke 2002; Destrade and Fu 2006]

$$\boldsymbol{\Sigma}(0) = i\boldsymbol{M}\boldsymbol{U}(0). \tag{2}$$

It is well-known that matrix $\boldsymbol{M}$ is Hermitian and it is an important tool for studying the existence and uniqueness of Rayleigh waves in generally anisotropic solids [Barnett and Lothe 1985].

Let the half-space $x_2 \geq 0$ be a compressible prestressed isotropic elastic half-space that is assumed to be deformed from the unstressed half-space $X_2 \geq 0$ by application of a pure homogeneous strain of the form

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \quad \lambda_i = \text{constant}, \quad i = 1, 2, 3, \tag{3}$$

where $\lambda_i > 0$ are the principal stretches of the deformation. In the deformed configuration, the elastic half-space is characterized by the fourth-order elasticity tensor A_{ijkl} defined as [Dowaikh and Ogden 1991; Vinh 2011; Ogden 1984; Fu and Ogden 2001]

$$\begin{aligned} JA_{iijj} &= \lambda_i \lambda_j \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}, \\ JA_{ijij} &= \begin{cases} (\lambda_i \, \partial W/\partial \lambda_i - \lambda_j \, \partial W/\partial \lambda_j)\, \lambda_i^2/(\lambda_i^2 - \lambda_j^2) & (i \neq j, \lambda_i \neq \lambda_j), \\ \frac{1}{2}(A_{iiii} - A_{iijj} + \lambda_i \, \partial W/\partial \lambda_i) & (i \neq j, \lambda_i = \lambda_j), \end{cases} \\ JA_{ijji} &= JA_{jiij} = JA_{ijij} - \lambda_i \frac{\partial W}{\partial \lambda_i} \ (i \neq j), \end{aligned} \tag{4}$$

for $i, j \in \{1, 2, 3\}$, $J = \lambda_1\lambda_2\lambda_3$, and where $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function per unit volume in the unstressed state, all other components being zero. For simplicity we introduce the following notations:

$$\alpha_{11} = JA_{1111}, \quad \alpha_{22} = JA_{2222}, \quad \alpha_{12} = JA_{1122},$$
$$\gamma_1 = JA_{1212}, \quad \gamma_2 = JA_{2121}, \quad \gamma_* = JA_{2112}. \tag{5}$$

From the strong-ellipticity condition, α_{ij}, γ_i are required to satisfy the inequalities [Dowaikh and Ogden 1991; Ogden 1984]

$$\alpha_{11} > 0, \quad \alpha_{22} > 0, \quad \gamma_1 > 0, \quad \gamma_2 > 0. \tag{6}$$

According to Dowaikh and Ogden [1991] and Vinh [2011], the Rayleigh wave is a two-component surface wave: $\boldsymbol{U} = [U_1\ U_2]^T$, $\boldsymbol{\Sigma} = [\Sigma_1\ \Sigma_2]^T$. The surface impedance matrix $\boldsymbol{M}$ is therefore a 2×2 matrix

$$\boldsymbol{M} = \begin{bmatrix} M_{11} & \hat{M}_{12} + iM_{12} \\ \hat{M}_{12} - iM_{12} & M_{22} \end{bmatrix}, \tag{7}$$

where $M_{11}, M_{22}, M_{12}, \hat{M}_{12} \in \mathbb{R}$. According to Fu and Mielke [2002, (4.2)–(4.4) and (5.4)],

$$M_{22} = M_{11}\sqrt{\frac{\alpha_{22}}{\gamma_2}}\sqrt{\frac{\gamma_1 - X}{\alpha_{11} - X}}, \quad \hat{M}_{12} = 0, \tag{8}$$

where $X = \rho c^2$ and ρ is the mass density at the unstrained state. It has been shown that [Vinh and Seriani 2009; Vinh 2011], if a Rayleigh wave exists, then

$$0 < X < \min\{\alpha_{11}, \gamma_1\}. \tag{9}$$

Suppose that the half-space is free of traction, i.e., $\boldsymbol{\Sigma}(0) = \boldsymbol{0}$. Then, from (2), (7), and the second part of (8) we have

$$M_{11}\frac{U_1(0)}{U_2(0)} + iM_{12} = 0, \quad -iM_{12}\frac{U_1(0)}{U_2(0)} + M_{22} = 0. \tag{10}$$

These two equations give immediately

$$\left[\frac{U_1(0)}{U_2(0)}\right]^2 = -\frac{M_{22}}{M_{11}}. \tag{11}$$

By κ we denote the H/V ratio: $\kappa = |U_1(0)/U_2(0)|$. From (6), the first part of (8), (9), and (11) it follows that

$$\kappa^2 = \sqrt{\frac{\alpha_{22}}{\gamma_2}}\sqrt{\frac{\gamma_1 - X}{\alpha_{11} - X}}. \tag{12}$$

This is the relation between the H/V ratio and the Rayleigh wave velocity.

Secular equation. According to Dowaikh and Ogden [1991], Fu and Mielke [2002], and Vinh [2011], the secular equation of Rayleigh waves propagating in the x_1-direction and decaying in the x_2-direction is given by

$$\sqrt{\gamma_2}\,[\alpha_{22}(\alpha_{11} - X) - \alpha_{12}^2]\sqrt{\frac{\gamma_1 - X}{\alpha_{11} - X}} + \sqrt{\alpha_{22}}\,[\gamma_2(\gamma_1 - X) - \gamma_*^2] = 0. \tag{13}$$

Equation for the H/V ratio. From (12) it follows that

$$X = \frac{\alpha_{22}\gamma_1 - \alpha_{11}\gamma_2\,\kappa^4}{\alpha_{22} - \gamma_2\kappa^4}. \tag{14}$$

Introducing (12) and (14) into (13) yields the H/V ratio equation, namely,

$$(1-d)w^3 + (1-a\theta)w^2 + b\theta(d-\theta)w + b\theta^2(a-1) = 0, \quad w = \kappa^2 \in (0,+\infty), \tag{15}$$

where the dimensionless parameters a, b, d, and θ are defined as

$$a = 1 - \frac{\gamma_*^2}{\gamma_1\gamma_2}, \quad b = \frac{\alpha_{11}\alpha_{22}}{\gamma_1\gamma_2}, \quad d = 1 - \frac{\alpha_{12}^2}{\alpha_{11}\alpha_{22}}, \quad \theta = \frac{\gamma_1}{\alpha_{11}}. \tag{16}$$

It is clear from (16) and (6) that

$$a \le 1 \ (a = 1 \text{ when } \gamma_* = 0), \quad d \le 1 \ (d = 1 \text{ when } \alpha_{12} = 0), \quad b > 0, \quad \theta > 0. \tag{17}$$

As (15) has no positive roots for the case $d = a = 1$ ($\Leftrightarrow \alpha_{12} = \gamma_* = 0$), we consider three remaining possibilities: $d < 1, a < 1$; $d = 1, a < 1$; $d < 1, a = 1$.

When $d < 1, a < 1$ ($\Leftrightarrow \alpha_{12} \neq 0, \gamma_* \neq 0$), (15) can be rewritten as

$$f(w) := w^3 + a_2 w^2 + a_1 w + a_0 = 0, \tag{18}$$

where the coefficients a_k of (18) are given by

$$a_0 = \frac{b\theta^2(1-a)}{d-1}, \quad a_1 = \frac{b\theta(\theta-d)}{d-1}, \quad a_2 = \frac{1-a\theta}{1-d}. \tag{19}$$

For the cases $d < 1, a = 1$ ($\Leftrightarrow \alpha_{12} \neq 0, \gamma_* = 0$) and $d = 1, a < 1$ ($\Leftrightarrow \alpha_{12} = 0, \gamma_* \neq 0$), (15) is respectively equivalent to

$$(1-d)w^2 + (1-\theta)w + b\theta(d-\theta) = 0, \tag{20}$$

and

$$(1-a\theta)w^2 + b\theta(1-\theta)w + b\theta^2(a-1) = 0. \tag{21}$$

The existence of solution of the H/V ratio equations. Because $w = \sqrt{\alpha_{22}/\gamma_2}$ when $\gamma_1 = \alpha_{11}$ ($\theta = 1$), according to (12), in the rest of the paper we assume that $\gamma_1 \neq \alpha_{11}$ ($\theta \neq 1$). Now we introduce the squared dimensionless Rayleigh wave velocity defined as $x = X/\gamma_1$. From (9) it follows that

$$0 < x < 1 \ \text{ if } \ 0 < \theta < 1, \quad 0 < x < 1/\theta \ \text{ if } \ \theta > 1. \tag{22}$$

In terms of x, the relation (12) is of the form

$$w = \sqrt{b}\,\theta\sqrt{\frac{1-x}{1-\theta x}}\,. \tag{23}$$

It is not difficult to verify that:

(i) The mapping defined by (23) is 1-1.

(ii) If $0 < \theta < 1$, (23) maps $x \in (0, 1)$ to $w \in (0, \sqrt{b}\,\theta)$.

(iii) If $\theta > 1$, (23) maps $x \in (0, 1/\theta)$ to $w \in (\sqrt{b}\,\theta, +\infty)$.

By w_r we denote a solution of H/V equations which belongs to interval $(0, \sqrt{b}\theta)$ if $0 < \theta < 1$ and belongs to interval $(\sqrt{b}\theta, +\infty)$ if $\theta > 1$. It also called a solution corresponding to a Rayleigh wave.

Since the mapping (23) is 1-1 as mentioned above, it follows that (15) has a solution if and only if (13) does as well. From this fact and [Vinh 2011, Propositions 3 and 6] (see also [Dowaikh and Ogden 1991, Equations (5.19), (5.33), and (5.34)]) we have immediately the following results:

Proposition 1. *Suppose that $d < 1$, $a < 1$ and $\theta > 0$, $\theta \neq 1$, $b > 0$. Then, (18) has a unique root w_r if*

$$a + \sqrt{b}d > 0, \tag{24}$$

otherwise, it has no solution corresponding to a Rayleigh wave.

Proposition 2. *Let $d < 1$ and $\theta > 0$, $\theta \neq 1$, $b > 0$:*

(i) *If*

$$\theta - d > 0, \quad 1 + \sqrt{b}d > 0, \tag{25}$$

then (20) has a unique root w_r.

(ii) *If (25) is not valid, then (20) has no root corresponding to a Rayleigh wave.*

Applying Proposition 2 for $d := a$, $\theta := 1/\theta$, $b := 1/b$, and $w := 1/w$ we have:

Proposition 3. *Let $a < 1$ and $\theta > 0$, $\theta \neq 1$, $b > 0$:*

(1) *If*

$$1 - a\theta > 0, \quad a + \sqrt{b} > 0, \tag{26}$$

then (21) has a unique root w_r.

(2) *If (26) is not valid, then (21) has no root corresponding to a Rayleigh wave.*

Proposition 4. *Suppose $d < 1$, $a < 1$, $\theta > 0$, $\theta \neq 1$, $b > 0$, and (24) holds. If (18) has two or three distinct real roots, then its w_r is the largest root.*

Proof. Let $d < 1$, $a < 1$, $\theta > 0$, $\theta \neq 1$, $b > 0$, and (24) holds. If (18) has two or three distinct real roots, then the equation $f'(w) = 0$ has two distinct roots, denoted by $w_{\max}$ and $w_{\min}$, so that either:

(a) $w_{\max} < w_{\min} \leq 0$, or

(b) $w_{\max} < 0 < w_{\min}$ due to $w_{\max} + w_{\min} = -\frac{2}{3}a_2 < 0$ for $0 < \theta < 1$ and $w_{\max}w_{\min} = \frac{1}{3}a_1 < 0$ for $\theta > 1$.

If (a) holds, since $f(w)$ is strictly monotonically increasing in $(0, +\infty)$, $f(0) < 0$ and $f(+\infty) = +\infty$, the equation $f(w) = 0$ has a unique root in $(0, +\infty)$ and it is w_r. If (b) is valid, then $f(w_{\min}) < 0$ because $f(w)$ is strictly monotonically decreasing in $(w_{\max}, w_{\min})$ and $f(0) < 0$. As $f(w)$ is strictly monotonically increasing in $(w_{\min}, +\infty)$, $f(w_{\min}) < 0$, and $f(+\infty) = +\infty$, therefore equation $f(w) = 0$ has a unique root in $(w_{\min}, +\infty)$, so it is a unique root in $(0, +\infty)$ and it is w_r. That means w_r is the largest root. □

3. Formulas for the H/V ratio for a general strain-energy function

Theorem 5. *If there exists a (unique) Rayleigh wave propagating along the x_1-direction, and attenuating in the x_2-direction in a compressible isotropic elastic half-space subject to a homogeneous initial deformation (Equation (3)), then its squared H/V ratio κ^2 is determined as follows:*

(i) *If $d < 1$, $a < 1$ (i.e., $\alpha_{12} \neq 0$, $\gamma_* \neq 0$), then*

$$\kappa^2 = -\tfrac{1}{3}a_2 + \sqrt[3]{R+\sqrt{D}} + \frac{(a_2^2 - 3a_1)}{9\sqrt[3]{R+\sqrt{D}}}, \tag{27}$$

where each radical is understood as the complex root taking its principal value; R and D are given by

$$R = \tfrac{1}{54}(9a_1a_2 - 27a_0 - 2a_2^3), \quad D = \tfrac{1}{108}(4a_0a_2^3 - a_1^2a_2^2 - 18a_0a_1a_2 + 27a_0^2 + 4a_1^3), \tag{28}$$

where a_0, a_1, and a_2 are determined by (19).

(ii) *If $d < 1$, $a = 1$ (i.e., $\alpha_{12} \neq 0$, $\gamma_* = 0$), then*

$$\kappa^2 = \frac{\theta - 1 + \sqrt{(1-\theta)^2 + 4b\theta(\theta - d)(1-d)}}{2(1-d)}. \tag{29}$$

(iii) *If $d = 1$, $a < 1$ (i.e., $\alpha_{12} = 0$, $\gamma_* \neq 0$), then*

$$\kappa^2 = \frac{b\theta(\theta-1) + \sqrt{b}\theta\sqrt{b(1-\theta)^2 + 4(1-a)(1-a\theta)}}{2(1-a\theta)}. \tag{30}$$

Proof.

(i): Suppose $d < 1$, $a < 1$, ($\theta > 0$, $\theta \neq 1$, $b > 0$) and (24) holds. According to Proposition 1, a unique Rayleigh wave can propagate in the half-space, and its H/V ratio is a root of (18). Let $z = w + \frac{1}{3}a_2$, then in terms of z (18) takes the form

$$z^3 - 3q^2z + r = 0, \tag{31}$$

where

$$r = -2R, \quad q^2 = \tfrac{1}{9}(a_2^2 - 3a_1). \tag{32}$$

According to the theory of cubic equations, three roots z_k ($k = 1, 2, 3$) of (31) are calculated by [Cowles and Thompson 1947]

$$z_1 = S + T, \quad z_2 = -\tfrac{1}{2}(S+T) + \tfrac{1}{2}i\sqrt{3}(S-T), \quad z_3 = -\tfrac{1}{2}(S+T) - \tfrac{1}{2}i\sqrt{3}(S-T), \tag{33}$$

where

$$S = \sqrt[3]{R+\sqrt{D}}, \quad T = \sqrt[3]{R-\sqrt{D}}, \quad D = R^2 + Q^3, \quad Q = -q^2. \tag{34}$$

In relation to (34), we emphasize two points:

- The cube root of a negative real number is taken as the real negative root.
- If, in the expression S, $R + \sqrt{D}$ is complex, the phase angle in T is taken as the negative of the phase angle in S so that $T = S^*$, where S^* is the complex conjugate of S.

Remark 6. • If $D > 0$, then (31) has one real root and two complex conjugate roots.

- If $D = 0$, this equation has three real roots, at least two of which are equal.
- If $D < 0$, it has three real distinct roots.

Let $z_r = \frac{1}{3}a_2 + w_r$, then z_r is a real root of (31) and if (31) has two or three real roots, z_r is the largest real root, according to Proposition 4. We will show that z_r is given by

$$z_r = \sqrt[3]{R+\sqrt{D}} + \frac{q^2}{\sqrt[3]{R+\sqrt{D}}}, \tag{35}$$

where each radical is understood as a complex root taking its principal value, R and D are calculated by (28), and q^2 is given by $(32)_2$. Then (27) is derived directly from (35) taking into account $w_r = -\frac{1}{3}a_2 + z_r$. In order to prove (35) we consider the distinct cases dependent on the values of D:

- If $D > 0$, according to Remark 6, (31) has a unique real root, which is z_r, calculated by $(33)_1$; in particular,

$$z_r = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}}, \tag{36}$$

  in which the radicals are understood as real. To prove (35) we have to show that the right-hand side of (36), in which the radicals are understood as real, coincides with the right-hand side of (35), where each radical is understood as complex taking its principal value. Since

$$\sqrt[3]{R-\sqrt{D}} = \frac{q^2}{\sqrt[3]{R+\sqrt{D}}}, \tag{37}$$

  this will be proved if we show that $R+\sqrt{D} > 0$. Note that, because (31) has a unique real root, (18) must have a unique real root. To prove $R+\sqrt{D} > 0$, we examine the distinct cases dependent on the values of $\Delta' = a_2^2 - 3a_1$, the discriminant of $f'(w) = 0$:
  - If $\Delta' \leq 0$, then $f(w)$ is strictly monotonically increasing in $(-\infty, +\infty)$ and $0 < \theta < 1$ because if $\theta > 1$ $(> d) \Longrightarrow a_1 < 0 \Longrightarrow \Delta' > 0$. By w_N we denote the abscissa of the point of inflexion N of the cubic curve $y = f(w)$, then $w_N = -\frac{2}{3}a_2 < 0$ because $a_2 > 0$ (noting that $0 < \theta < 1$). This fact along with $f(0) = a_0 < 0$ and the strictly increasing monotonousness of $f(w)$ lead to $f(w_N) < 0$. Since $r = f(w_N)$ it follows $r < 0$, or equivalently $R > 0$. This gives $R+\sqrt{D} > 0$.
  - If $\Delta' > 0$, $f'(w) = 0$ has two distinct roots, $w_{\max}$ and $w_{\min}$, and either $w_{\max} < w_{\min} \leq 0$ or $w_{\max} < 0 < w_{\min}$ (see the proof of Proposition 4). In both cases we always have that $f(w_{\min}) < 0$, as shown in the proof of Proposition 4. As (18) has a unique real root as addressed above, $f(w_{\max})f(w_{\min}) > 0$, otherwise it has two or three real roots; consequently, $f(w_{\max}) < 0$. This and $f(w_{\min}) < 0$ provides $r = f(w_N) < 0 \Longrightarrow R = -r/2 > 0$; therefore we have $R+\sqrt{D} > 0$.
- If $D = 0$, analogously as above, one can see that $r < 0$, and consequently $R > 0$. When $D = 0$ we have $R^2 = -Q^3 = q^6$ $(q > 0) \Longrightarrow R = q^3 \Longrightarrow r = -2R = -2q^3$, so (31) becomes $z^3 - 3q^2z - 2q^3 = 0$ whose roots are $z_1 = 2q$, $z_2 = -q$ (double root). This says $z_r = 2q$, since it is the largest root. With the help of $q > 0$ and $D = 0$ it is easy to see that z_r calculated by (35) is $2q$.
- If $D < 0$, according to Remark 6, (31) has three distinct real roots, and z_r is the largest one according to Proposition 4. Using the arguments presented in [Vinh and Ogden 2004, p. 255], one can verify

that in this case the largest real root of (31) is

$$z_r = \sqrt[3]{R+\sqrt{D}} + \sqrt[3]{R-\sqrt{D}}, \tag{38}$$

in which each radical is understood as a complex root taking its principal value. By 3θ we denote the phase angle of $R + i\sqrt{-D}$. Then, it is not difficult to prove that

$$\sqrt[3]{R+\sqrt{D}} = qe^{i\theta}, \quad \sqrt[3]{R-\sqrt{D}} = qe^{-i\theta}, \tag{39}$$

where radicals are understood as complex roots taking their principal value. From (39) we have immediately (37) and then (35) by taking into account (38).

(ii), (iii): Let $d < 1$, $a = 1$ (and $\theta > 0$, $\theta \neq 1$). According to Proposition 2, a Rayleigh wave can propagate in the half-space if and only if (25) holds, and the H/V ratio is computed by (20). It is easy to see that with (25), Equation (20) has two distinct real roots w_1 and w_2 so that $w_1 < 0 < w_2$, $w_2 = w_r$, and w_r is calculated by (29). Equation (30) is proved analogously. □

4. Formulas for the H/V ratio for specific strain-energy functions

4.1. ***Neo-Hookean material.*** For the neo-Hookean material, the strain-energy function is of the form [Roxburgh and Ogden 1994]

$$W = \tfrac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\ln(\lambda_1\lambda_2\lambda_3)), \tag{40}$$

where μ is a Lame coefficient and λ_i $(i = \overline{1,3})$ are the principal stretches of the deformation. From (5), (4), and (40) we have

$$\alpha_{11} = \mu(\lambda_1^2 + 1), \quad \alpha_{22} = \mu(\lambda_2^2 + 1), \quad \gamma_1 = \mu\lambda_1^2, \quad \gamma_2 = \mu\lambda_2^2, \quad \gamma_* = \mu, \quad \alpha_{12} = 0. \tag{41}$$

In this case, $\alpha_{12} = 0$, $\gamma_* = \mu \neq 0$; therefore, the H/V ratio equation is of the form (21) with

$$a = 1 - \frac{1}{\lambda_1^2\lambda_2^2}, \quad b = \frac{(1+\lambda_1^2)(1+\lambda_2^2)}{\lambda_1^2\lambda_2^2}, \quad \theta = \frac{\lambda_1^2}{1+\lambda_1^2}, \tag{42}$$

and according to Theorem 5 (iii), the squared H/V ratio is calculated by (30). Introducing (42) into (30) yields

$$\kappa^2 = \tfrac{1}{2}(\sqrt{1 + 4/\lambda_2^2} - 1). \tag{43}$$

The necessary and sufficient condition for a (unique) Rayleigh wave to exist in this case, according to Proposition 3, (26), is

$$1 - \frac{1}{\lambda_1^2\lambda_2^2} + \frac{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}{\lambda_1\lambda_2} > 0. \tag{44}$$

Figure 1, left, shows the existence domain of Rayleigh waves (shaded) in the space of λ_1 and λ_2 and some contour lines of the squared H/V ratio in this domain. They are horizontal lines parallel to λ_1 axis since the H/V ratio depends only on λ_2 in this case. The right figure shows the graph of the squared H/V ratio as a function of λ_2. The graph is decreasing and asymptotes to both axes. That means there is no horizontal vibration on the surface, when the material is stretched remarkably along the x_2-direction. And there is no vertical vibration when the material is significantly squeezed along the x_2-direction.

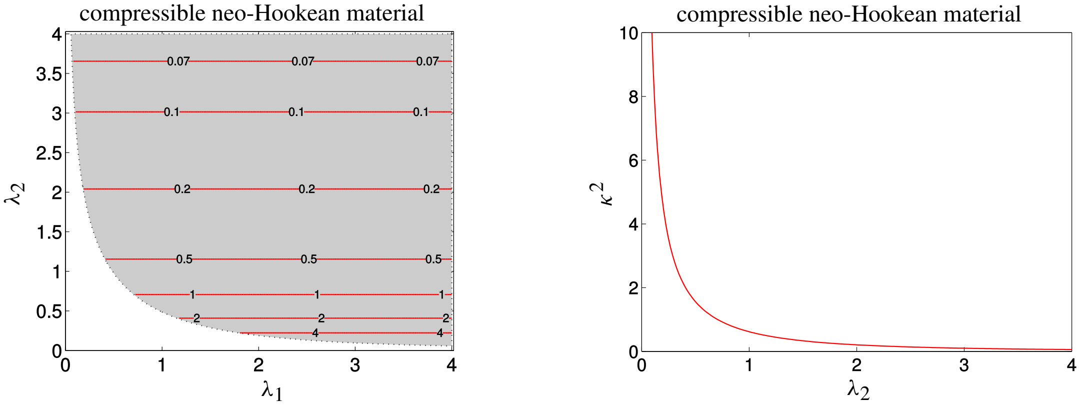

Figure 1. Left: some contours of the squared H/V ratio in the space of λ_1 and λ_2 for the neo-Hookean material. The shaded area is the domain where a Rayleigh wave is possible. Right: the dependence of the squared H/V ratio on λ_2.

4.2. ***Varga material.*** The strain-energy function for this material is [Roxburgh and Ogden 1994]

$$W = \mu(\lambda_1 + \lambda_2 + \lambda_3 - 3 - \ln(\lambda_1\lambda_2\lambda_3)). \tag{45}$$

From (5), (4), and (45) we have

$$\alpha_{11} = \mu, \quad \alpha_{22} = \mu, \quad \gamma_1 = \frac{\mu\lambda_1^2}{\lambda_1+\lambda_2}, \quad \gamma_2 = \frac{\mu\lambda_2^2}{\lambda_1+\lambda_2}, \quad \gamma_* = \mu\left(1 - \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right), \quad \alpha_{12} = 0. \tag{46}$$

Substituting (46) into (16) gives

$$a = 1 - \frac{(\lambda_1+\lambda_2-\lambda_1\lambda_2)^2}{\lambda_1^2\lambda_2^2}, \quad b = \frac{(\lambda_1+\lambda_2)^2}{\lambda_1^2\lambda_2^2}, \quad \theta = \frac{\lambda_1^2}{\lambda_1+\lambda_2}. \tag{47}$$

Similar to the neo-Hookean material, $\alpha_{12} = 0$ and $\gamma_* \neq 0$ except at points where $1 - \lambda_1\lambda_2/(\lambda_1+\lambda_2) = 0$. As noted in Section 2, at these points a Rayleigh wave is impossible due to $\alpha_{12} = \gamma_* = 0$. According to Theorem 5 (iii), the squared H/V ratio is determined by (30). Introducing (47) into (30) yields

$$\kappa^2 = \frac{1}{2}\frac{\lambda_1+\lambda_2-\lambda_1^2}{\lambda_1+\lambda_2+\lambda_2^2-2\lambda_1\lambda_2} \times \left(\sqrt{1 + \frac{4(\lambda_1+\lambda_2-\lambda_1\lambda_2)^2(\lambda_1-\lambda_2)^2}{\lambda_2^2(\lambda_1+\lambda_2-\lambda_1^2)^2} + \frac{4(\lambda_1+\lambda_2-\lambda_1\lambda_2)^2}{\lambda_2^2(\lambda_1+\lambda_2-\lambda_1^2)}} - 1\right). \tag{48}$$

The existence condition of Rayleigh waves for this material, according to (26), is

$$3\lambda_1\lambda_2 - (\lambda_1+\lambda_2) > 0, \quad 2\lambda_1\lambda_2 - (\lambda_1+\lambda_2+\lambda_2^2) < 0. \tag{49}$$

The H/V ratio for this case is a function of two variables: λ_1 and λ_2. Figure 2 shows some contour lines with different values of the squared H/V ratio computed by (48) in the (shaded) domain of variables λ_1 and λ_2, where a Rayleigh wave is possible. The left-lower dotted line represents the condition $(49)_1$, and the right-upper one shows the condition $(49)_2$. The H/V ratio increases and approaches infinity at the upper boundary line since the denominator of (48) goes to zero. The thick curve is the collection of points $1 - \lambda_1\lambda_2/(\lambda_1+\lambda_2) = 0$, where a Rayleigh wave is impossible, as mentioned above. The H/V ratio reduces to zero at points approaching to this curve. This could be seen from (48) by letting $\lambda_1 + \lambda_2$

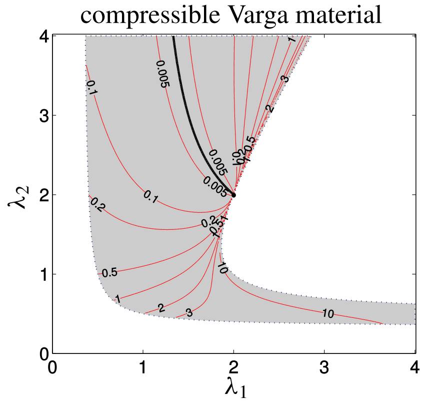

Figure 2. Some contour lines with different values of the squared H/V ratio in the space of λ_1 and λ_2 for the Varga material.

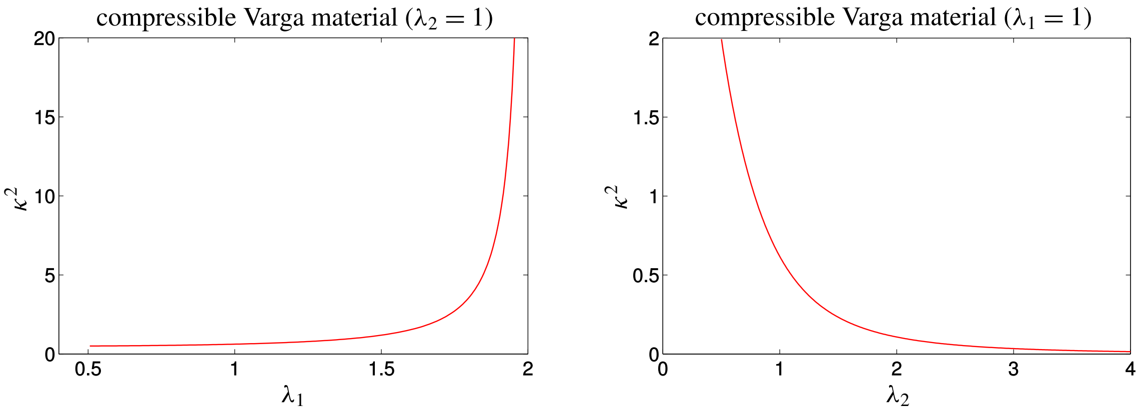

Figure 3. The squared H/V ratio as a function of λ_1 with $\lambda_2 = 1$ (left) and of λ_2 with $\lambda_1 = 1$ (right) for the Varga material.

go to $\lambda_1\lambda_2$. Therefore, the H/V ratio could vary from 0^+ to infinity for λ_1 and λ_2, varying between the thick curve and the right-upper boundary curve.

Figure 3 shows the dependence of the squared H/V ratio on λ_1 (left) and λ_2 (right) when the other λ is fixed and equals 1, i.e., there is no stretch in the other direction. When $\lambda_2 = 1$, λ_1 varies from 0.5 to 2, and the squared H/V ratio increases from 0.5 to infinity. This means there are no vertical vibrations of particles on the surface when the material is stretched along the x_1-direction by a factor of nearly 2. When $\lambda_1 = 1$, λ_2 varies from 0.5 to infinity, and the squared H/V ratio decreases from 2 to 0.

4.3. ***Blatz–Ko material.*** The strain-energy function of this material is [Roxburgh and Ogden 1994]

$$W = \tfrac{1}{2}\mu(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5). \tag{50}$$

From (5), (4), and (50) we have

$$\alpha_{11} = \frac{3\mu}{\lambda_1^2}, \quad \alpha_{22} = \frac{3\mu}{\lambda_2^2}, \quad \gamma_1 = \frac{\mu}{\lambda_2^2}, \quad \gamma_2 = \frac{\mu}{\lambda_1^2},$$

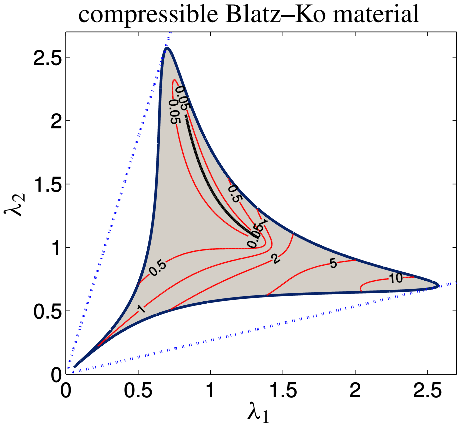

Figure 4. Several contour lines of the squared H/V ratio in the space of λ_1 and λ_2 with $\lambda_3 = 1$ for the Blatz–Ko material.

$$\gamma_* = \mu\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \lambda_1\lambda_2\lambda_3\right), \quad \alpha_{12} = \mu\lambda_1\lambda_2\lambda_3. \tag{51}$$

Consider $\lambda_3 = 1$. It is clear from (51) that $\alpha_{12} \neq 0$ ($\Leftrightarrow d < 1$) for this case. Hence, the H/V ratio equation is the cubic equation (15) with

$$a = 1 - \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} - \lambda_1^2\lambda_2^2\right)^2, \quad b = 9, \quad d = 1 - \tfrac{1}{9}\lambda_1^4\lambda_2^4, \quad \theta = \frac{\lambda_1^2}{3\lambda_2^2}. \tag{52}$$

If $\gamma_* \neq 0$ ($\Leftrightarrow a < 1$), i.e.,

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \lambda_1\lambda_2 \neq 0, \tag{53}$$

then by Theorem 5 (i), the squared H/V ratio κ^2 is given by (27) and (28) with

$$\begin{aligned} a_2 &= 9\left\{\frac{1}{\lambda_1^4\lambda_2^4} - \frac{1}{3\lambda_1^2\lambda_2^6}\left[1 - \lambda_1^2\lambda_2^2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \lambda_1\lambda_2\right)^2\right]\right\}, \\ a_1 &= \frac{3}{\lambda_2^8}\left(-3 + \frac{9\lambda_2^2}{\lambda_1^2} - \lambda_1^2\lambda_2^6\right), \quad a_0 = -\frac{9\lambda_1^2}{\lambda_2^6}\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \lambda_1\lambda_2\right)^2, \end{aligned} \tag{54}$$

and the existence condition of Rayleigh waves, according to Proposition 1, (24), is

$$4 - \tfrac{1}{3}\lambda_1^4\lambda_2^4 - \lambda_1^2\lambda_2^2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \lambda_1\lambda_2\right)^2 > 0. \tag{55}$$

It could be easily checked from the equation of the H/V ratio (18) that the squared H/V ratio takes value $\sqrt{b}\theta$ when either $\theta = 1$ or $a + \sqrt{b}d = 0$. Therefore, the squared H/V ratio goes to $\sqrt{b}\theta$ at the boundary of the existence domain of Rayleigh waves given in (55).

Figure 4 shows some contour lines of the squared H/V ratio on the (shaded) domain of λ_1 and λ_2, defined by the condition (55), where a Rayleigh wave can exist. The two dotted lines are expressed by the equations $\lambda_1/\lambda_2 = 2 + \sqrt{3}$ and $\lambda_1/\lambda_2 = 2 - \sqrt{3}$, and in the domain located between them the strong

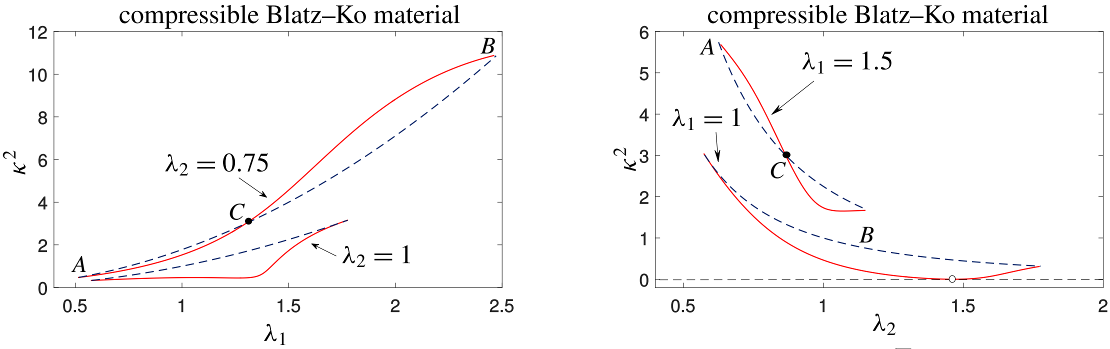

Figure 5. The squared H/V ratio (continuous line) and the quantity $\sqrt{b}\theta$ (dotted line) as a function of λ_1 (left) and λ_2 (right) with $\lambda_3 = 1$ for the Blatz–Ko material.

ellipticity condition stated by Roxburgh and Ogden [1994] is satisfied. The existence domain of Rayleigh waves is a subset of this domain.

When $\gamma_* = 0$, i.e., the left-hand side of (53) equals zero, for a Rayleigh wave to exist, λ_1 and λ_2 must satisfy the two conditions (25) given in Proposition 2. The thick continuous curve in the shaded area in Figure 4 shows the set of points which do not satisfy the first condition. The two end-points of this thick curve are $(\lambda_1, \lambda_2) = (0.8223, 2.0770)$ and $(1.3450, 1.0657)$. A Rayleigh wave is impossible on this curve.

The left figure in Figure 5 shows the dependence of κ^2 on λ_1 for two fixed values of $\lambda_2 = 0.75$ and $\lambda_2 = 1$ plotted by continuous curves. The dotted curves show the value of $\sqrt{b}\theta$, which is λ_1^2/λ_2^2 in this case. This value is mentioned in Section 2 as the separation of two ranges of κ^2 depending on if θ is less than or greater than one. When $\lambda_2 = 0.75$, for a Rayleigh wave to exist, it is deduced from (55) that $0.51 < \lambda_1 < 2.48$. The two end-points are denoted by A and B. The continuous curve and the dotted curve meet each other at these end-points because the squared H/V ratio goes to $\sqrt{b}\theta$ at the boundary of the existence domain of Rayleigh waves, as mentioned above. These two curves cross at point C at which $\theta = 1$. One can see that the phase velocity $x < 1$ and the squared H/V ratio $\kappa^2 < \sqrt{b}\theta$ in the segment AC, and $x < 1/\theta$, $\kappa^2 > \sqrt{b}\theta$ in the segment CB. When $\lambda_2 = 1$, the possible range of λ_1 is $(0.57, 1.78)$ and $\theta < 1$ in this range, therefore $\kappa^2 < \sqrt{b}\theta$. The right figure in Figure 5 shows the dependence of κ^2 on λ_2 for two fixed values of $\lambda_1 = 1$ and $\lambda_1 = 1.5$ with the same description as the left figure. When $\lambda_1 = 1$, there is an undefined point of the H/V ratio at $\lambda_2 = 1.465$. This is a point of the thick curve shown in Figure 4 where the H/V ratio does not exist.

4.4. ***Foam rubber strain-energy function.*** According to Murphy and Destrade [2009], in the plane-strain $\lambda_3 = 1$, the foam rubbers are well-characterized by the strain-energy function

$$W = \tfrac{1}{2}\mu\Big(I - 2 + \frac{\epsilon}{1-\epsilon}\big(J^{2(\epsilon-1)/\epsilon} - 1\big)\Big), \tag{56}$$

where

$$I = \lambda_1^2 + \lambda_2^2, \quad J = \lambda_1\lambda_2, \quad \lambda_2 = \lambda_1^{\epsilon-1}, \quad 0 < \epsilon < 1. \tag{57}$$

From (5), (4) and (56), (57) it is not difficult to verify that

$$\alpha_{11} = \mu\lambda_1^2[1 + (2/\epsilon - 1)\lambda_1^{2(\epsilon-2)}], \quad \alpha_{22} = (2\mu/\epsilon)\lambda_1^{2(\epsilon-1)},$$
$$\alpha_{12} = (1-\epsilon)\alpha_{22}, \quad \gamma_1 = \mu\lambda_1^2, \quad \gamma_2 = \mu\lambda_1^{2(\epsilon-1)}, \quad \gamma_* = \gamma_2. \tag{58}$$

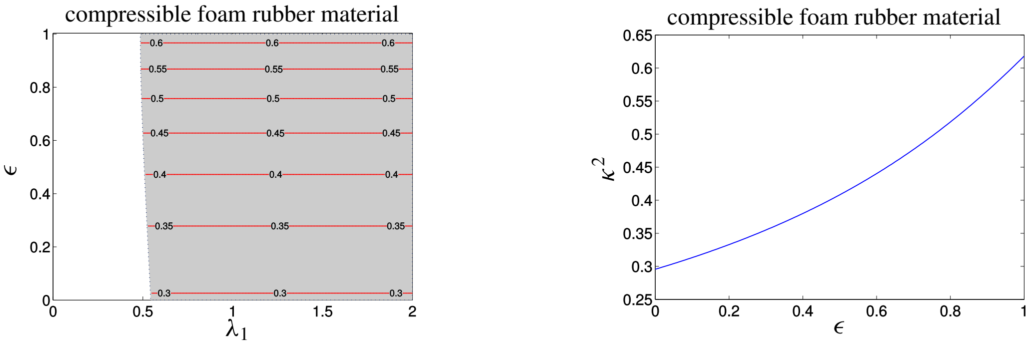

Figure 6. Some contour lines of κ^2 in the possible domain of λ_1 and ϵ (left). The dependence of the squared H/V ratio κ^2 on $\epsilon \in (0, 1)$ for the foam rubber material (right).

By using (16), (19) along with (58), the H/V ratio equation for this material is (18) with

$$a_0 = -\frac{1}{(1-\epsilon)^2}, \quad a_1 = \frac{3-2\epsilon}{(1-\epsilon)^2}, \quad a_2 = \frac{1}{(1-\epsilon)^2}. \tag{59}$$

As $0 < \epsilon < 1$ and $\lambda_1 > 0$ it is clear from (58) that $\alpha_{12} > 0$ ($\Rightarrow d < 1$) and $\gamma_* > 0$ ($\Rightarrow a < 1$). According to Theorem 5 (i), κ^2 is calculated by (27) in which a_k is given by (59), and R, D are given by (28); in particular,

$$\begin{aligned} R &= \frac{(3\epsilon-2)(9\epsilon^3-36\epsilon^2+51\epsilon-26)}{54(1-\epsilon)^6}, \\ D &= -\frac{(2-\epsilon)^2(32\epsilon^3-107\epsilon^2+124\epsilon-44)}{108(1-\epsilon)^8}. \end{aligned} \tag{60}$$

It is interesting that H/V ratio depends only on ϵ, it does not depend on λ_1. The condition for a Rayleigh wave to exist according to Proposition 1 is

$$1 - \lambda_1^{2(\epsilon-2)} + \sqrt{2}\frac{1+\lambda_1^{2(\epsilon-2)}(3-2\epsilon)}{\sqrt{\epsilon+\lambda_1^{2(\epsilon-2)}(2-\epsilon)}} > 0. \tag{61}$$

The left figure in Figure 6 shows the existence domain of Rayleigh waves in the space of ϵ and λ_1 (shaded) and some contour lines of κ^2 in this domain. Since κ^2 is a function of ϵ only, these contour lines are horizontal. The right figure in Figure 6 shows the value of κ^2 depending on the whole range of ϵ from 0 to 1.

5. Conclusions

In this paper, the relation between the H/V ratio and the velocity of Rayleigh waves propagating in a compressible prestressed elastic half-space is established. Based on it and the secular equation, the equation determining the H/V ratio is obtained. Then, the exact H/V ratio formulas have been derived by solving analytically the H/V ratio equation. These formulas are valid for a general strain-energy function. Some particular strain-energy functions are employed to specify these formulas. Since the

H/V ratio is a convenient tool for nondestructively evaluating the prestresses of structures before and during loading, the obtained formulas will be very useful in practical applications.

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