Journal of Mechanics of Materials and Structures

UNIFORM STRESS RESULTANTS INSIDE TWO NONELLIPTICAL INHOMOGENEITIES IN ISOTROPIC LAMINATED PLATES

Xu Wang, Liang Chen and Peter Schiavone

Volume 13, No. 4 July 2018

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 13, No. 4, 2018
dx.doi.org/10.2140/jomms.2018.13.531

UNIFORM STRESS RESULTANTS INSIDE TWO NONELLIPTICAL INHOMOGENEITIES IN ISOTROPIC LAMINATED PLATES

XU WANG, LIANG CHEN AND PETER SCHIAVONE

We use complex variable methods to establish two sets of specific conditions which ensure the existence of uniform and hydrostatic internal membrane stress resultants and bending moments inside two through-thickness nonelliptical elastic inhomogeneities embedded in an infinite isotropic laminated Kirchhoff plate subjected to uniform remote membrane stress resultants and bending moments. These conditions can be interpreted as restrictions on the remote membrane stress resultants and bending moments for the given material and geometric parameters. We show that when these conditions are met, explicit expressions are available for the uniform stress resultants inside the two inhomogeneities and the constant hoop stress resultants on the matrix side along the two interfaces.

1. Introduction

Establishing uniformity of stresses inside multiple elastic inhomogeneities is both a fascinating and challenging area of study which continues to attract considerable attention in the literature (see, for example, Kang et al. 2008; Liu 2008; Wang 2012; Wang and Schiavone 2016; Dai et al. 2015; 2016). The majority of these investigations although confined to planar elasticity, antiplane elasticity, or conductivity have generated ideas and procedures which lend themselves well to other areas which play an equally important role in the engineering sciences. One such area concerns the analysis of laminated plate structures which are used extensively in mechanical, civil, aviation, and aerospace applications. A simple and elegant complex variable formulation, originally presented by Beom and Earmme [1998], was recently developed by Wang and Zhou [2014] to tackle the coupled stretching and bending deformations of isotropic laminated plates within the context of the celebrated Kirchhoff plate theory [Timoshenko and Woinowsky-Krieger 1959; Reddy 1997].

In this paper, we adopt the complex variable formulation of Wang and Zhou [2014] and the conformal mapping in Wang [2012] to study the coupled stretching and bending deformations of an infinite isotropic laminated plate in which there are embedded two through-thickness nonelliptical elastic inhomogeneities when the surrounding (plate) matrix is subjected to uniform remote membrane stress resultants and bending moments. The internal stress resultants (here, internal membrane stress resultants and bending moments for the plate) inside the two inhomogeneities are uniform and hydrostatic when either a set of three or two conditions on the remote loading is satisfied for the given material and geometric parameters. In addition, the hoop membrane stress resultant and hoop bending moment on the matrix side are uniformly distributed along the two inhomogeneity-matrix interfaces and the two inhomogeneities also satisfy the "harmonic field condition" of Bjorkman and Richards [1976]. In contrast to previous results

Keywords: isotropic laminated plate, uniform stress resultants, complex variable method, conformal mapping.

in [Wang 2012; Wang and Schiavone 2016], the coefficient Λ appearing in the mapping function should be determined through the solution of a generalized eigenvalue problem for two 2×2 real symmetric matrices. Both cases of when the two real symmetric matrices are either proportional or nonproportional to each other have been discussed in detail.

2. Complex variable formulation for isotropic laminated plates

In this section, we review the complex variable formulation for an isotropic laminated plate. Consider an undeformed plate of uniform thickness h in a Cartesian coordinate system $\{x_i\}$ $(i=1,2,3)$ with its reference plane (not the midplane) at $x_3=0$. The plate is composed of an isotropic, linearly elastic material which can be inhomogeneous in the thickness direction. In what follows, Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively and we sum over repeated indices.

The displacement field in the Kirchhoff plate theory is assumed to take the form

$$\tilde{u}_\alpha(x_i)=u_\alpha+x_3\vartheta_\alpha,\qquad \tilde{u}_3(x_i)=w, \tag{1}$$

where the two in-plane displacements u_α, the deflection w, and the slopes $\vartheta_\alpha=-w_{,\alpha}$ on the reference plane are all independent of x_3.

The coordinate system is chosen judiciously so that the two in-plane displacements and the deflection on the reference plane are *decoupled* in the equilibrium equations [Beom and Earmme 1998]. We introduce the integral operator $Q(\cdots)=\int_{-h_0}^{h-h_0}(\cdots)\,\mathrm{d}x_3$ in which h_0 is the distance between the reference plane and the lower surface of the plate. Accordingly, the membrane stress resultants and bending moments defined by $N_{\alpha\beta}=Q\sigma_{\alpha\beta}$, $M_{\alpha\beta}=Qx_3\sigma_{\alpha\beta}$ (with $\sigma_{\alpha\beta}$ being the in-plane stress components), the transverse shearing forces $\Re_\beta=M_{\alpha\beta,\alpha}$, in-plane displacements, deflection, and slopes on the reference plane of the plate as well as the four stress functions φ_α and η_α can be expressed concisely in terms of four analytic functions $\phi(z)$, $\psi(z)$, $\Phi(z)$, and $\Psi(z)$ of the complex variable $z=x_1+ix_2$ as [Beom and Earmme 1998; Wang and Zhou 2014]

$$\begin{aligned} N_{11}+N_{22}&=4\,\mathrm{Re}\{\phi'(z)+B\Phi'(z)\},\\ N_{22}-N_{11}+2iN_{12}&=2[\bar{z}\phi''(z)+\psi'(z)+B\bar{z}\Phi''(z)+B\Psi'(z)],\end{aligned} \tag{2}$$

$$\begin{aligned} M_{11}+M_{22}&=4D(1+\nu^D)\,\mathrm{Re}\{\Phi'(z)\}+\frac{B(\kappa^A-1)}{\mu}\,\mathrm{Re}\{\phi'(z)\},\\ M_{22}-M_{11}+2iM_{12}&=-2D(1-\nu^D)[\bar{z}\Phi''(z)+\Psi'(z)]-\frac{B}{\mu}[\bar{z}\phi''(z)+\psi'(z)],\\ \Re_1-i\Re_2&=4D\Phi''(z)+\frac{B(\kappa^A+1)}{2\mu}\phi''(z),\end{aligned} \tag{3}$$

$$\begin{aligned} 2\mu(u_1+iu_2)&=\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)},\\ \vartheta_1+i\vartheta_2&=\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)},\qquad w=-\,\mathrm{Re}\{\bar{z}\Phi(z)+\gamma(z)\},\\ \varphi_1+i\varphi_2&=i[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}]+iB[\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}],\\ \eta_1+i\eta_2&=iD(1-\nu^D)[\kappa^D\Phi(z)-z\overline{\Phi'(z)}-\overline{\Psi(z)}]+i\frac{B}{2\mu}[\kappa^A\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}],\end{aligned} \tag{4}$$

in which $\Psi(z) = \gamma'(z)$, and

$$\mu = \tfrac{1}{2}(A_{11} - A_{12}), \quad B = B_{12}, \quad D = D_{11}, \quad \nu^A = \frac{A_{12}}{A_{11}}, \quad \nu^D = \frac{D_{12}}{D_{11}},$$
$$\kappa^A = \frac{3A_{11} - A_{12}}{A_{11} + A_{12}} = \frac{3 - \nu^A}{1 + \nu^A}, \quad \kappa^D = \frac{3D_{11} + D_{12}}{D_{11} - D_{12}} = \frac{3 + \nu^D}{1 - \nu^D}, \tag{5}$$

with $A_{ij} = QC_{ij}$, $B_{ij} = Qx_3C_{ij}$, and $D_{ij} = Qx_3^2C_{ij}$ $(ij = 11, 12)$. The parameters C_{11} and C_{12} can be expressed in terms of the Young's modulus $E = E(x_3)$ and Poisson's ratio $\nu = \nu(x_3)$ of the plate as $C_{11} = E/(1-\nu^2)$ and $C_{12} = \nu E/(1-\nu^2)$. The distance h_0 is determined as $h_0 = \int_0^h X_3C_{11}\mathrm{d}X_3 / \int_0^h C_{11}\mathrm{d}X_3$ with $X_3 = x_3 + h_0$ being the vertical coordinate of the given point from the lower surface of the plate.

In addition, the membrane stress resultants, bending moments, transverse shearing forces, and modified Kirchhoff transverse shearing forces $V_1 = \mathfrak{R}_1 + M_{12,2}$ and $V_2 = \mathfrak{R}_2 + M_{21,1}$ (which apply exclusively to free edges), can be expressed in terms of the four stress functions φ_α and η_α [Cheng and Reddy 2002] as

$$N_{\alpha\beta} = -\in_{\beta\omega}\varphi_{\alpha,\omega}, \quad M_{\alpha\beta} = -\in_{\beta\omega}\eta_{\alpha,\omega} - \tfrac{1}{2}\in_{\alpha\beta}\eta_{\omega,\omega}, \quad \mathfrak{R}_\alpha = -\tfrac{1}{2}\in_{\alpha\beta}\eta_{\omega,\omega\beta}, \quad V_\alpha = -\in_{\alpha\omega}\eta_{\omega,\omega\omega}, \tag{6}$$

with $\in_{\alpha\beta}$ denoting the components of the two-dimensional permutation tensor.

In a new coordinate system $\{\hat{x}_i\}$ $(i = 1, 2, 3)$ in which $\hat{x}_3 = 0$ lies on an arbitrary plane parallel to the reference plane and $\hat{x}_\alpha = x_\alpha$, the in-plane displacements $\hat{u}_\alpha$ and slopes $\hat{\vartheta}_\alpha$ on $\hat{x}_3 = 0$ and the stress functions $\hat{\varphi}_\alpha$ and $\hat{\eta}_\alpha$ in the new coordinate system can be given quite simply as

$$\hat{\vartheta}_1 + i\hat{\vartheta}_2 = \vartheta_1 + i\vartheta_2, \quad \hat{u}_1 + i\hat{u}_2 = u_1 + iu_2 - \hat{h}(\vartheta_1 + i\vartheta_2),$$
$$\hat{\varphi}_1 + i\hat{\varphi}_2 = \varphi_1 + i\varphi_2, \quad \hat{\eta}_1 + i\hat{\eta}_2 = \eta_1 + i\eta_2 + \hat{h}(\varphi_1 + i\varphi_2). \tag{7}$$

Here,

$$\hat{h} = h_0 - h_1, \tag{8}$$

and h_1 is the distance between $\hat{x}_3 = 0$ and the lower surface of the plate (we note that h_1 is positive or negative, respectively, if $\hat{x}_3 = 0$ is above or below the lower surface of the plate). In the new coordinate system, the stress resultants $\hat{N}_{\alpha\beta} = \hat{Q}\sigma_{\alpha\beta}$ and $\hat{M}_{\alpha\beta} = \hat{Q}\hat{x}_3\sigma_{\alpha\beta}$ with $\hat{Q}(\cdots) = \int_{-h_1}^{h-h_1}(\cdots)\,\mathrm{d}\hat{x}_3$, the transverse shearing forces $\hat{\mathfrak{R}}_\beta = \hat{M}_{\alpha\beta,\alpha}$, and the modified Kirchhoff transverse shearing forces $\hat{V}_1 = \hat{\mathfrak{R}}_1 + \hat{M}_{12,2}$ and $\hat{V}_2 = \hat{\mathfrak{R}}_2 + \hat{M}_{21,1}$ can also be expressed in terms of the newly introduced stress functions $\hat{\varphi}_\alpha$ and $\hat{\eta}_\alpha$ as

$$\hat{N}_{\alpha\beta} = -\in_{\beta\omega}\hat{\varphi}_{\alpha,\omega}, \quad \hat{M}_{\alpha\beta} = -\in_{\beta\omega}\hat{\eta}_{\alpha,\omega} - \tfrac{1}{2}\in_{\alpha\beta}\hat{\eta}_{\omega,\omega}, \quad \hat{\mathfrak{R}}_\alpha = -\tfrac{1}{2}\in_{\alpha\beta}\hat{\eta}_{\omega,\omega\beta}, \quad \hat{V}_\alpha = -\in_{\alpha\omega}\hat{\eta}_{\omega,\omega\omega}. \tag{9}$$

3. Uniform stress resultants inside two nonelliptical inhomogeneities

Consider an infinite isotropic laminated plate containing two through-thickness nonelliptical elastic inhomogeneities. Let S_1, S_2, and S_3 denote the left inhomogeneity, the surrounding matrix, and the right inhomogeneity, respectively, all of which are perfectly bonded through the respective left and right interfaces L_1 and L_2. The matrix is subjected to uniform remote membrane stress resultants $(N_{11}^\infty, N_{22}^\infty, N_{12}^\infty)$ and bending moments $(M_{11}^\infty, M_{22}^\infty, M_{12}^\infty)$. Throughout the paper, the subscripts 1, 2, and 3 are used to identify the quantities in S_1, S_2, and S_3, respectively. In what follows, the new coordinate system $\{\hat{x}_i\}$ $(i = 1, 2, 3)$ is common to all three phases and is chosen such that $\hat{x}_3 = 0$ is on the reference plane of the matrix.

Applying the aforementioned conditions, the corresponding boundary value problem reduces to the following system of equations in the analytic functions $\phi(z)$, $\psi(z)$, $\Phi(z)$, and $\Psi(z)$ defined in each of S_1, S_2, and S_3:

$$\begin{aligned}
&\frac{1}{2\mu_2}[\kappa_2^A\phi_2(z)-z\overline{\phi_2'(z)}-\overline{\psi_2(z)}]\\
&\qquad=\frac{1}{2\mu_1}[\kappa_1^A\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)}]-\hat{h}_1[\Phi_1(z)+z\overline{\Phi_1'(z)}+\overline{\Psi_1(z)}],\\
&\Phi_2(z)+z\overline{\Phi_2'(z)}+\overline{\Psi_2(z)}=\Phi_1(z)+z\overline{\Phi_1'(z)}+\overline{\Psi_1(z)},\\
&\phi_2(z)+z\overline{\phi_2'(z)}+\overline{\psi_2(z)}+B_2[\Phi_2(z)+z\overline{\Phi_2'(z)}+\overline{\Psi_2(z)}]\\
&\qquad=\phi_1(z)+z\overline{\phi_1'(z)}+\overline{\psi_1(z)}+B_1[\Phi_1(z)+z\overline{\Phi_1'(z)}+\overline{\Psi_1(z)}],\\
&\frac{B_2}{2\mu_2}[\kappa_2^A\phi_2(z)-z\overline{\phi_2'(z)}-\overline{\psi_2(z)}]+D_2(1-\nu_2^D)[\kappa_2^D\Phi_2(z)-z\overline{\Phi_2'(z)}-\overline{\Psi_2(z)}]\\
&\quad=\frac{B_1}{2\mu_1}[\kappa_1^A\phi_1(z)-z\overline{\phi_1'(z)}-\overline{\psi_1(z)}]+D_1(1-\nu_1^D)[\kappa_1^D\Phi_1(z)-z\overline{\Phi_1'(z)}-\overline{\Psi_1(z)}]\\
&\qquad+\hat{h}_1[\phi_1(z)+z\overline{\phi_1'(z)}+\overline{\psi_1(z)}]+\hat{h}_1B_1[\Phi_1(z)+z\overline{\Phi_1'(z)}+\overline{\Psi_1(z)}],\quad z\in L_1,
\end{aligned}\tag{10a}$$

$$\begin{aligned}
&\frac{1}{2\mu_2}[\kappa_2^A\phi_2(z)-z\overline{\phi_2'(z)}-\overline{\psi_2(z)}]\\
&\qquad=\frac{1}{2\mu_3}[\kappa_3^A\phi_3(z)-z\overline{\phi_3'(z)}-\overline{\psi_3(z)}]-\hat{h}_3[\Phi_3(z)+z\overline{\Phi_3'(z)}+\overline{\Psi_3(z)}],\\
&\Phi_2(z)+z\overline{\Phi_2'(z)}+\overline{\Psi_2(z)}=\Phi_3(z)+z\overline{\Phi_3'(z)}+\overline{\Psi_3(z)},\\
&\phi_2(z)+z\overline{\phi_2'(z)}+\overline{\psi_2(z)}+B_2[\Phi_2(z)+z\overline{\Phi_2'(z)}+\overline{\Psi_2(z)}]\\
&\qquad=\phi_3(z)+z\overline{\phi_3'(z)}+\overline{\psi_3(z)}+B_3[\Phi_3(z)+z\overline{\Phi_3'(z)}+\overline{\Psi_3(z)}],\\
&\frac{B_2}{2\mu_2}[\kappa_2^A\phi_2(z)-z\overline{\phi_2'(z)}-\overline{\psi_2(z)}]+D_2(1-\nu_2^D)[\kappa_2^D\Phi_2(z)-z\overline{\Phi_2'(z)}-\overline{\Psi_2(z)}]\\
&\quad=\frac{B_3}{2\mu_3}[\kappa_3^A\phi_3(z)-z\overline{\phi_3'(z)}-\overline{\psi_3(z)}]+D_3(1-\nu_3^D)[\kappa_3^D\Phi_3(z)-z\overline{\Phi_3'(z)}-\overline{\Psi_3(z)}]\\
&\qquad+\hat{h}_3[\phi_3(z)+z\overline{\phi_3'(z)}+\overline{\psi_3(z)}]+\hat{h}_3B_3[\Phi_3(z)+z\overline{\Phi_3'(z)}+\overline{\Psi_3(z)}],\quad z\in L_2,
\end{aligned}\tag{10b}$$

$$\begin{aligned}
&\phi_2(z)\cong\delta_1z+O(1),\quad \psi_2(z)\cong\delta_2z+O(1),\\
&\Phi_2(z)\cong\gamma_1z+O(1),\quad \Psi_2(z)\cong\gamma_2z+O(1),\quad |z|\to\infty,
\end{aligned}\tag{10c}$$

where

$$\begin{aligned}
\delta_1&=\frac{\mu_2D_2(1+\nu_2^D)(N_{11}^\infty+N_{22}^\infty)-B_2\mu_2(M_{11}^\infty+M_{22}^\infty)}{4\mu_2D_2(1+\nu_2^D)-B_2^2(\kappa_2^A-1)},\\
\gamma_1&=\frac{4\mu_2(M_{11}^\infty+M_{22}^\infty)-B_2(\kappa_2^A-1)(N_{11}^\infty+N_{22}^\infty)}{16\mu_2D_2(1+\nu_2^D)-4B_2^2(\kappa_2^A-1)},\\
\delta_2&=\frac{\mu_2D_2(1-\nu_2^D)(N_{22}^\infty-N_{11}^\infty+2iN_{12}^\infty)+B_2\mu_2(M_{22}^\infty-M_{11}^\infty+2iM_{12}^\infty)}{2\mu_2D_2(1-\nu_2^D)-B_2^2},\\
\gamma_2&=\frac{-2\mu_2(M_{22}^\infty-M_{11}^\infty+2iM_{12}^\infty)-B_2(N_{22}^\infty-N_{11}^\infty+2iN_{12}^\infty)}{4\mu_2D_2(1-\nu_2^D)-2B_2^2}.
\end{aligned}\tag{11}$$

We assume the matrix S_2 in the z-plane is mapped onto an annulus $1 \le |\xi| \le \rho^{-1/2}$ in the ξ-plane by the following conformal mapping function [Wang 2012]

$$z = \omega(\xi) = R\left[\frac{1}{\xi-\lambda} + \frac{p}{\xi-\lambda^{-1}} + \frac{\Lambda^{-1}p}{\rho\xi-\lambda^{-1}} + \sum_{n=1}^{+\infty}(a_n\xi^n + a_{-n}\xi^{-n})\right], \qquad (12)$$
$$\xi(z) = \omega^{-1}(z), \quad 1 \le |\xi| \le \rho^{-1/2},$$

where R is a real scaling constant; λ $(1 < \lambda < \rho^{-1/2})$ is a real constant; p is a complex constant; Λ, a_n, and a_{-n} are unknown complex coefficients to be determined. Using the mapping function in (12), the two interfaces L_1 and L_2 in the z-plane are mapped onto two coaxial circles with radii 1 and $\rho^{-1/2}$ in the ξ-plane, respectively. In addition, $z = \infty$ is mapped to the point $\xi = \lambda$.

In order to ensure that the stress resultants inside the two inhomogeneities are uniform, the analytic functions defined in the two elastic inhomogeneities should take the following form:

$$\phi_1(z) = \frac{4\mu_1(\kappa_2^A+1)[D_1(1+\nu_1^D)+D_2(1-\nu_2^D)+\hat{h}_1(B_1+B_2)]X - 16\mu_1 D_2(B_1-B_2-2\mu_2\hat{h}_1)Y}{\Delta_1}z,$$

$$\Phi_1(z) = \frac{-(\kappa_2^A+1)[(B_1-B_2)(\kappa_1^A-1)+4\mu_1\hat{h}_1]X + 8D_2[\mu_2(\kappa_1^A-1)+2\mu_1]Y}{\Delta_1}z,$$

$$\psi_1(z) = \Psi_1(z) = 0, \quad z \in S_1; \qquad (13)$$

$$\phi_3(z) = \frac{4\mu_3(\kappa_2^A+1)[D_3(1+\nu_3^D)+D_2(1-\nu_2^D)+\hat{h}_3(B_3+B_2)]X - 16\mu_3 D_2(B_3-B_2-2\mu_2\hat{h}_3)Y}{\Delta_3}z,$$

$$\Phi_3(z) = \frac{-(\kappa_2^A+1)[(B_3-B_2)(\kappa_3^A-1)+4\mu_3\hat{h}_3]X + 8D_2[\mu_2(\kappa_3^A-1)+2\mu_3]Y}{\Delta_3}z,$$

$$\psi_3(z) = \Psi_3(z) = 0, \quad z \in S_3; \qquad (14)$$

where X and Y are two real coefficients, and

$$\Delta_1 = 4[\mu_2(\kappa_1^A-1)+2\mu_1][D_1(1+\nu_1^D)+D_2(1-\nu_2^D)+\hat{h}_1(B_1+B_2)] - 2(B_1-B_2-2\mu_2\hat{h}_1)[(B_1-B_2)(\kappa_1^A-1)+4\mu_1\hat{h}_1],$$
$$\Delta_3 = 4[\mu_2(\kappa_3^A-1)+2\mu_3][D_3(1+\nu_3^D)+D_2(1-\nu_2^D)+\hat{h}_3(B_3+B_2)] - 2(B_3-B_2-2\mu_2\hat{h}_3)[(B_3-B_2)(\kappa_3^A-1)+4\mu_3\hat{h}_3]. \qquad (15)$$

By enforcing continuity of displacements and stress resultants across the left interface L_1 in (10a), we arrive at

$$\phi_2(\xi) = \phi_2(\omega(\xi)) = X\omega(\xi),$$
$$\Phi_2(\xi) = \Phi_2(\omega(\xi)) = Y\omega(\xi),$$

$$
\begin{aligned}
\psi_2(\xi) &= \psi(\omega(\xi)) \\
&= \Big\{ \frac{X}{\Delta_1}\Big(8[\mu_1(\kappa_2^A-1)-\mu_2(\kappa_1^A-1)][D_1(1+\nu_1^D)+D_2(1-\nu_2^D)+\hat{h}_1(B_1+B_2)] \\
&\qquad -2[(\kappa_1^A-1)(B_1-B_2)+4\mu_1\hat{h}_1][(\kappa_2^A-1)(B_1-B_2)+4\mu_2\hat{h}_1]\Big) \\
&\qquad + \frac{Y}{\Delta_1}\big(16\mu_2 D_2[(\kappa_1^A-1)(B_1-B_2)+4\mu_1\hat{h}_1]\big)\Big\}\bar{\omega}\Big(\frac{1}{\xi}\Big), \\
\Psi_2(\xi) &= \Psi_2(\omega(\xi)) \\
&= \Big\{ -\frac{X}{\Delta_1}\big(2(\kappa_2^A+1)[(\kappa_1^A-1)(B_1-B_2)+4\mu_1\hat{h}_1]\big) \\
&\qquad - \frac{Y}{\Delta_1}\Big(8[\mu_2(\kappa_1^A-1)+2\mu_1][D_1(1+\nu_1^D)-D_2(1+\nu_2^D)+\hat{h}_1(B_1+B_2)] \\
&\qquad -4(B_1-B_2-2\mu_2\hat{h}_1)[(\kappa_1^A-1)(B_1-B_2)+4\mu_1\hat{h}_1]\Big)\Big\}\bar{\omega}\Big(\frac{1}{\xi}\Big), \\
&\qquad 1 \le |\xi| \le \rho^{-1/2}.
\end{aligned} \tag{16}
$$

Similarly, by enforcing continuity of displacements and stress resultants across the right interface L_2 in (10b), we arrive at

$$
\begin{aligned}
\phi_2(\xi) &= X\omega(\xi), \\
\Phi_2(\xi) &= Y\omega(\xi), \\
\psi_2(\xi) &= \Big\{ \frac{X}{\Delta_3}\Big(8[\mu_3(\kappa_2^A-1)-\mu_2(\kappa_3^A-1)][D_3(1+\nu_3^D)+D_2(1-\nu_2^D)+\hat{h}_3(B_3+B_2)] \\
&\qquad -2[(\kappa_3^A-1)(B_3-B_2)+4\mu_3\hat{h}_3][(\kappa_2^A-1)(B_3-B_2)+4\mu_2\hat{h}_3]\Big) \\
&\qquad + \frac{Y}{\Delta_3}(16\mu_2 D_2[(\kappa_3^2-1)(B_3-B_2)+4\mu_3\hat{h}_3])\Big\}\bar{\omega}\Big(\frac{1}{\rho\xi}\Big), \\
\Psi_2(\xi) &= \Big\{ -\frac{X}{\Delta_3}2(\kappa_2^A+1)[(\kappa_3^A-1)(B_3-B_2)+4\mu_3\hat{h}_3] \\
&\qquad - \frac{Y}{\Delta_3}\Big(8[\mu_2(\kappa_3^A-1)+2\mu_3][D_3(1+\nu_3^D)-D_2(1+\nu_2^D)+\hat{h}_3(B_3+B_2)] \\
&\qquad -4(B_3-B_2-2\mu_2\hat{h}_3)[(\kappa_3^A-1)(B_3-B_2)+4\mu_3\hat{h}_3]\Big)\Big\}\bar{\omega}\Big(\frac{1}{\rho\xi}\Big), \\
&\qquad 1 \le |\xi| \le \rho^{-1/2}.
\end{aligned} \tag{17}
$$

In order to ensure that the elastic field in the matrix is unique, the two sets of functions $\phi_2(\xi)$, $\Phi_2(\xi)$, $\psi_2(\xi)$, and $\Psi_2(\xi)$ obtained in (16) and (17) should coincide. Consequently, we find that

$$
\Lambda = \frac{c_{11}X + c_{12}Y}{d_{11}X + d_{12}Y} = \frac{c_{12}X + c_{22}Y}{d_{12}X + d_{22}Y}, \tag{18}
$$

and

$$
a_n = \frac{\lambda^{-n-1} + p\Lambda^{-1}\rho^n\lambda^{n+1}}{1-\Lambda\rho^{-n}}, \quad a_{-n} = \frac{\lambda^{n-1} + p\lambda^{1-n}}{\Lambda^{-1}\rho^{-n}-1}, \quad n = 1, 2, \ldots, +\infty, \tag{19}
$$

where

$$c_{11} = \frac{1}{\Delta_3}\Big(4[\mu_3(\kappa_2^A - 1) - \mu_2(\kappa_3^A - 1)][D_3(1+\nu_3^D) + D_2(1-\nu_2^D) + \hat{h}_3(B_3+B_2)] \\ - [(\kappa_3^A - 1)(B_3 - B_2) + 4\mu_3\hat{h}_3][(\kappa_2^A - 1)(B_3 - B_2) + 4\mu_2\hat{h}_3]\Big),$$

$$c_{12} = \frac{8\mu_2 D_2}{\Delta_3}[(\kappa_3^A - 1)(B_3 - B_2) + 4\mu_3\hat{h}_3],$$

$$c_{22} = \frac{16\mu_2 D_2}{\Delta_3(\kappa_2^A + 1)}\Big(2[\mu_2(\kappa_3^A - 1) + 2\mu_3][D_3(1+\nu_3^D) - D_2(1+\nu_2^D) + \hat{h}_3(B_3+B_2)] \\ - (B_3 - B_2 - 2\mu_2\hat{h}_3)[(\kappa_3^A - 1)(B_3 - B_2) + 4\mu_3\hat{h}_3]\Big), \tag{20a}$$

$$d_{11} = \frac{1}{\Delta_1}\Big(4[\mu_1(\kappa_2^A - 1) - \mu_2(\kappa_1^A - 1)][D_1(1+\nu_1^D) + D_2(1-\nu_2^D) + \hat{h}_1(B_1+B_2)] \\ - [(\kappa_1^A - 1)(B_1 - B_2) + 4\mu_1\hat{h}_1][(\kappa_2^A - 1)(B_1 - B_2) + 4\mu_2\hat{h}_1]\Big),$$

$$d_{12} = \frac{8\mu_2 D_2}{\Delta_1}[(\kappa_1^A - 1)(B_1 - B_2) + 4\mu_1\hat{h}_1],$$

$$d_{22} = \frac{16\mu_2 D_2}{\Delta_1(\kappa_2^A + 1)}\Big(2[\mu_2(\kappa_1^A - 1) + 2\mu_1][D_1(1+\nu_1^D) - D_2(1+\nu_2^D) + \hat{h}_1(B_1+B_2)] \\ - (B_1 - B_2 - 2\mu_2\hat{h}_1)[(\kappa_1^A - 1)(B_1 - B_2) + 4\mu_1\hat{h}_1]\Big). \tag{20b}$$

Equation (18) can be rewritten in the form

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \Lambda \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}, \tag{21}$$

which is a generalized eigenvalue problem for the two 2×2 real symmetric matrices

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}, \tag{22}$$

with Λ the eigenvalue and $\boldsymbol{v} = [X\ Y]^T$ the associated eigenvector. In what follows, we first address the case in which the above two matrices are not proportional to each other, i.e.,

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \neq k \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix}, \tag{23}$$

where k is an arbitrary real constant.

In this case, the two eigenvalues of (21) can be determined explicitly as

$$\Lambda_{1,2} = \frac{c_{11}d_{22} + c_{22}d_{11} - 2c_{12}d_{12} \pm \sqrt{(c_{11}d_{22} - c_{22}d_{11})^2 + 4(c_{11}d_{12} - c_{12}d_{11})(c_{22}d_{12} - c_{12}d_{22})}}{2(d_{11}d_{22} - d_{12}^2)}, \tag{24}$$

and the two eigenvectors associated with the two eigenvalues are then

$$\boldsymbol{v}_1 = \begin{bmatrix} c_{12} - \Lambda_1 d_{12} \\ \Lambda_1 d_{11} - c_{11} \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} c_{12} - \Lambda_2 d_{12} \\ \Lambda_2 d_{11} - c_{11} \end{bmatrix}, \tag{25}$$

which implies that the two coefficients X and Y are not independent. It is necessary that the two eigenvalues Λ_1 and Λ_2 should be real. Consequently, it is seen from (24) that the following inequality must be satisfied:

$$(c_{11}d_{22}-c_{22}\,d_{11})^2+4(c_{11}d_{12}-c_{12}\,d_{11})(c_{22}\,d_{12}-c_{12}\,d_{22})\geq 0.$$

A comparison of (16) with the asymptotic behaviors at infinity specified in (10c) leads to

$$\delta_1=X,\quad \gamma_1=Y,\quad \delta_2=-2\bar{p}\lambda^2(d_{11}X+d_{12}Y),\quad \gamma_2=\bar{p}\lambda^2\,\frac{\kappa_2^A+1}{4\mu_2 D_2}(d_{12}X+d_{22}Y). \tag{26}$$

The necessary and sufficient condition for the existence of the real coefficient X (or Y) simultaneously satisfying the four conditions in (26) is found to be

$$\begin{aligned}
&\frac{N_{22}^{\infty}-N_{11}^{\infty}+2iN_{12}^{\infty}}{N_{11}^{\infty}+N_{22}^{\infty}}\\
&\quad=\frac{\bar{p}\lambda^2}{8\mu_2 D_2[c_{12}-\Lambda_j d_{12}+B_2(\Lambda_j d_{11}-c_{11})]}\\
&\qquad\qquad\times\left\{8\mu_2 D_2(c_{11}d_{12}-c_{12}d_{11})+B_2(\kappa_2^A+1)[c_{12}d_{12}-c_{11}d_{22}+\Lambda_j(d_{11}d_{22}-d_{12}^2)]\right\},\\
&\frac{M_{11}^{\infty}+M_{22}^{\infty}}{N_{11}^{\infty}+N_{22}^{\infty}}=\frac{4\mu_2 D_2(1+\nu_2^D)(\Lambda_j d_{11}-c_{11})+B_2(\kappa_2^A-1)(c_{12}-\Lambda_j d_{12})}{4\mu_2[c_{12}-\Lambda_j d_{12}+B_2(\Lambda_j d_{11}-c_{11})]},\\
&\frac{M_{22}^{\infty}-M_{11}^{\infty}+2iM_{12}^{\infty}}{N_{11}^{\infty}+N_{22}^{\infty}}\\
&\quad=-\frac{\bar{p}\lambda^2}{8\mu_2[c_{12}-\Lambda_j d_{12}+B_2(\Lambda_j d_{11}-c_{11})]}\\
&\qquad\qquad\times\left\{(1-\nu_2^D)(\kappa_2^A+1)[c_{12}d_{12}-c_{11}d_{22}+\Lambda_j(d_{11}d_{22}-d_{12}^2)]+4B_2(c_{11}d_{12}-c_{12}d_{11})\right\},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,2.
\end{aligned} \tag{27}$$

For the given material and geometric parameters, the three conditions in (27) can be considered as restrictions on the remote loading.

The internal uniform hydrostatic stress resultants inside the two nonelliptical inhomogeneities can be expressed in terms of the two loading parameters δ_1 and γ_1 defined in (11) as

$$\begin{aligned}
&N_{11}=N_{22}=f_{11}\delta_1+f_{12}\gamma_1,\quad N_{12}=0,\\
&M_{11}=M_{22}=g_{11}\delta_1+g_{12}\gamma_1,\quad M_{12}=0,\quad z\in S_1,
\end{aligned} \tag{28}$$

$$\begin{aligned}
&N_{11}=N_{22}=f_{31}\delta_1+f_{32}\gamma_1,\quad N_{12}=0,\\
&M_{11}=M_{22}=g_{31}\delta_1+g_{32}\gamma_1,\quad M_{12}=0,\quad z\in S_3,
\end{aligned} \tag{29}$$

where the eight coefficients f_{j1}, f_{j2}, g_{j1}, g_{j2}, $j=1,3$ are defined by

$$\frac{f_{j1}}{\kappa_2^A+1}=\frac{1}{\Delta_j}\big\{8\mu_j[D_j(1+\nu_j^D)+D_2(1-\nu_2^D)+\hat{h}_j(B_j+B_2)]-2B_j[(B_j-B_2)(\kappa_j^A-1)+4\mu_j\hat{h}_j]\big\},$$
$$\frac{f_{j2}}{16D_2}=\frac{1}{\Delta_j}[B_j\mu_2(\kappa_2^A-1)+2B_2\mu_j+4\mu_j\mu_2\hat{h}_j],$$
$$\frac{g_{j1}}{\kappa_2^A+1}=\frac{1}{\Delta_j}\big\{2B_j(\kappa_j^A-1)[D_j(1+\nu_j^D)+D_2(1-\nu_2^D)+\hat{h}_j(B_j+B_2)]$$
$$-2D_j(1+\nu_j^D)[(B_j-B_2)(\kappa_j^A-1)+4\mu_j\hat{h}_j]\big\},$$
$$\frac{g_{j2}}{8D_2}=\frac{1}{\Delta_j}\big\{2D_j(1+\nu_j^D)[\mu_2(\kappa_j^A-1)+2\mu_j]-B_j(\kappa_j^A-1)[(B_j-B_2)-2\mu_2\hat{h}_j]\big\},\quad j=1,3. \tag{30}$$

It should be emphasized that, in view of (27)$_2$, δ_1 and γ_1 can ultimately be expressed in terms of only $N_{11}^\infty+N_{22}^\infty$. In addition, $N_{11}+N_{22}$ and $M_{11}+M_{22}$ are uniformly distributed in the matrix as

$$N_{11}+N_{22}=4(\delta_1+B_2\gamma_1),\quad M_{11}+M_{22}=\frac{B_2(\kappa_2^A-1)}{\mu_2}\delta_1+4D_2(1+\nu_2^D)\gamma_1,\quad z\in S_2. \tag{31}$$

The above result implies that the two elastic inhomogeneities will not disturb the quantities $N_{11}+N_{22}$ and $M_{11}+M_{22}$ when inserted into the surrounding (uncut) matrix (i.e., $N_{11}+N_{22}=N_{11}^\infty+N_{22}^\infty$ and $M_{11}+M_{22}=M_{11}^\infty+M_{22}^\infty$ for $z\in S_2$). Thus the two elastic inhomogeneities are "harmonic" [Bjorkman and Richards 1976; 1979; Richards and Bjorkman 1980; Wang and Schiavone 2015]. In addition, it follows from (28), (29), and (31) that the hoop membrane stress resultant and hoop bending moment are constant along the two interfaces L_1 and L_2 on the matrix side and are given by

$$\begin{aligned}N_{tt}&=(4-f_{11})\delta_1+(4B_2-f_{12})\gamma_1,\\ M_{tt}&=\left[\frac{B_2(\kappa_2^A-1)}{\mu_2}-g_{11}-\hat{h}_1f_{11}\right]\delta_1+[4D_2(1+\nu_2^D)-g_{12}-\hat{h}_1f_{12}]\gamma_1,\quad z\in L_1,\end{aligned} \tag{32}$$

$$\begin{aligned}N_{tt}&=(4-f_{31})\delta_1+(4B_2-f_{32})\gamma_1,\\ M_{tt}&=\left[\frac{B_2(\kappa_2^A-1)}{\mu_2}-g_{31}-\hat{h}_3f_{31}\right]\delta_1+[4D_2(1+\nu_2^D)-g_{32}-\hat{h}_3f_{32}]\gamma_1,\quad z\in L_2.\end{aligned} \tag{33}$$

Next, we address the case in which the two matrices in (22) are proportional to each other, i.e.,

$$\begin{bmatrix}c_{11}&c_{12}\\c_{12}&c_{22}\end{bmatrix}=k\begin{bmatrix}d_{11}&d_{12}\\d_{12}&d_{22}\end{bmatrix}. \tag{34}$$

In this case, it is seen from (18) that $\Lambda=k$, and that the two coefficients X and Y are now independent of each other. The necessary and sufficient condition for the existence of the two real coefficients X and Y simultaneously satisfying the four conditions in (26) is quite simply derived as

$$\delta_2=-2\bar{p}\lambda^2(d_{11}\delta_1+d_{12}\gamma_1),\quad \gamma_2=\bar{p}\lambda^2\frac{\kappa_2^A+1}{4\mu_2D_2}(d_{12}\delta_1+d_{22}\gamma_1), \tag{35}$$

or more explicitly

$$\begin{aligned}N_{11}^\infty-N_{22}^\infty+2iN_{12}^\infty&=p\lambda^2[\chi_{11}(N_{11}^\infty+N_{22}^\infty)+\chi_{12}(M_{11}^\infty+M_{22}^\infty)],\\ M_{11}^\infty-M_{22}^\infty+2iM_{12}^\infty&=p\lambda^2[\chi_{21}(N_{11}^\infty+N_{22}^\infty)+\chi_{22}(M_{11}^\infty+M_{22}^\infty)],\end{aligned} \tag{36}$$

with the four coefficients χ_{11}, χ_{12}, χ_{21}, and χ_{22} being defined by

$$\begin{aligned}\chi_{11} &= \frac{1}{4\mu_2 D_2[8\mu_2 D_2(1+\nu_2^D) - 2B_2^2(\kappa_2^A - 1)]} \\ &\qquad \times \big\{8\mu_2 D_2[4d_{11}\mu_2 D_2(1+\nu_2^D) - d_{12}B_2(\kappa_2^A - 1)] \\ &\qquad\qquad - B_2(\kappa_2^A + 1)[4d_{12}\mu_2 D_2(1+\nu_2^D) - d_{22}B_2(\kappa_2^A - 1)]\big\}, \\ \chi_{12} &= \frac{8\mu_2 D_2(d_{12} - d_{11}B_2) - B_2(\kappa_2^A + 1)(d_{22} - d_{12}B_2)}{D_2[8\mu_2 D_2(1+\nu_2^D) - 2B_2^2(\kappa_2^A - 1)]}, \\ \chi_{21} &= \frac{1}{4\mu_2[8\mu_2 D_2(1+\nu_2^D) - 2B_2^2(\kappa_2^2 - 1)]} \\ &\qquad \times \big\{(1-\nu_2^D)(\kappa_2^A + 1)[4d_{12}\mu_2 D_2(1+\nu_2^D) - d_{22}B_2(\kappa_2^A - 1)] \\ &\qquad\qquad - 4B_2[4d_{11}\mu_2 D_2(1+\nu_2^D) - d_{12}B_2(\kappa_2^A - 1)]\big\}, \\ \chi_{22} &= \frac{(1-\nu_2^D)(\kappa_2^A + 1)(d_{22} - d_{12}B_2) - 4B_2(d_{12} - d_{11}B_2)}{8\mu_2 D_2(1+\nu_2^D) - 2B_2^2(\kappa_2^A - 1)}. \end{aligned} \tag{37}$$

An example of when the two 2×2 real symmetric matrices are proportional to each other arises when the two inhomogeneities have identical elastic properties (i.e., $\mu_1 = \mu_3$, $D_1 = D_3$, $B_1 = B_3$, $\nu_1^A = \nu_3^A$, $\nu_1^D = \nu_3^D$) and $\hat{h}_1 = \hat{h}_3$. In this example, we will have $c_{11} = d_{11}$, $c_{12} = d_{12}$, $c_{22} = d_{22}$. Consequently, $\Lambda = k = 1$. Various shapes of the two nonelliptical inhomogeneities in the case $\Lambda = 1$ (in (12)) have been illustrated numerically in [Wang 2012].

For given material and geometric parameters, the two conditions in (36) can be considered as constraints on the remote loading. Once the two conditions in (36) are satisfied, (28) and (29) for the internal uniform hydrostatic stress resultants and (32) and (33) for the constant hoop stress resultants on the matrix side along the two interfaces remain valid for this case. Recall that now δ_1 and γ_1 can be expressed in terms of $N_{11}^\infty + N_{22}^\infty$ and $M_{11}^\infty + M_{22}^\infty$ in view of the fact that X and Y are independent.

4. Conclusions

We have identified the shapes of the two nonelliptical elastic inhomogeneities and the conditions leading to uniform interior stress resultants inside the two inhomogeneities. When the inequality in (23) is satisfied, two values of the real coefficient Λ are determined from (24) for the given material parameters of the composite plate. Three conditions on remote loading for the given material and geometric parameters are derived in (27). Once these conditions are satisfied, elementary expressions of the internal uniform hydrostatic stress resultants and constant hoop stress resultants on the matrix side along the two interfaces in terms of only $N_{11}^\infty + N_{22}^\infty$ are presented in (28), (29), (32), and (33). When (34) is valid, a single value of $\Lambda = k$ is found. Two conditions on remote loading for the given material and geometric parameters are derived in (36). In this case, the internal uniform hydrostatic stress resultants and constant hoop stress resultants on the matrix side along the two interfaces can be expressed in terms of both $N_{11}^\infty + N_{22}^\infty$ and $M_{11}^\infty + M_{22}^\infty$.

The complex coefficients a_n and a_{-n} can be uniquely determined from (19) for given values of λ, ρ, p, and Λ. Consequently, the shapes of the two inhomogeneities are known. In addition, the two inhomogeneities are "harmonic" in the sense that they satisfy the harmonic field condition of Bjorkman and Richards [1976] in that their introduction will not cause any disturbance of the fields $N_{11}+N_{22}$ and $M_{11}+M_{22}$ in the surrounding matrix.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 11272121) and through a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (Grant No: RGPIN — 2017-03716115112).

References

[Beom and Earmme 1998] H. G. Beom and Y. Y. Earmme, "Complex variable method for problems of a laminate composed of multiple isotropic layers", *Int. J. Fract.* **92**:4 (1998), 305–324.

[Bjorkman and Richards 1976] G. S. Bjorkman and R. Richards, "Harmonic holes — an inverse problem in elasticity", *J. Appl. Mech. (ASME)* **43**:3 (1976), 414–418.

[Bjorkman and Richards 1979] G. S. Bjorkman and R. Richards, "Harmonic holes for nonconstant fields", *J. Appl. Mech. (ASME)* **46**:3 (1979), 573–576.

[Cheng and Reddy 2002] Z.-Q. Cheng and J. N. Reddy, "Octet formalism for Kirchhoff anisotropic plates", *Proc. R. Soc. Lond. A* **458**:2022 (2002), 1499–1517.

[Dai et al. 2015] M. Dai, C.-F. Gao, and C. Q. Ru, "Uniform stress fields inside multiple inclusions in an elastic infinite plane under plane deformation", *Proc. R. Soc. Lond. A* **471**:2177 (2015), art. id. 20140933, 23 pp.

[Dai et al. 2016] M. Dai, P. Schiavone, and C.-F. Gao, "Uniform strain fields inside periodic inclusions incorporating interface effects in anti-plane shear", *Acta Mech.* **227**:10 (2016), 2795–2803.

[Kang et al. 2008] H. Kang, E. Kim, and G. W. Milton, "Inclusion pairs satisfying Eshelby's uniformity property", *SIAM J. Appl. Math.* **69**:2 (2008), 577–595.

[Liu 2008] L. P. Liu, "Solutions to the Eshelby conjectures", *Proc. R. Soc. Lond. A* **464**:2091 (2008), 573–594.

[Reddy 1997] J. N. Reddy, *Mechanics of laminated composite plates: theory and analysis*, CRC Press, Boca Raton, FL, 1997.

[Richards and Bjorkman 1980] R. Richards and G. S. Bjorkman, "Harmonic shapes and optimum design", *J. Eng. Mech. Div. (ASCE)* **106**:6 (1980), 1125–1134.

[Timoshenko and Woinowsky-Krieger 1959] S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of plates and shells*, 2nd ed., McGraw-Hill, New York, 1959.

[Wang 2012] X. Wang, "Uniform fields inside two non-elliptical inclusions", *Math. Mech. Solids* **17**:7 (2012), 736–761.

[Wang and Schiavone 2015] X. Wang and P. Schiavone, "Harmonic shapes in isotropic laminated plates", *J. Mech. Mater. Struct.* **10**:3 (2015), 433–445.

[Wang and Schiavone 2016] X. Wang and P. Schiavone, "Two inhomogeneities of irregular shape with internal uniform stress fields interacting with a screw dislocation", *Compt. Rend. Mécaniq.* **344**:7 (2016), 532–538.

[Wang and Zhou 2014] X. Wang and K. Zhou, "An inclusion of arbitrary shape in an infinite or semi-infinite isotropic multi-layered plate", *Int. J. Appl. Mech.* **6**:1 (2014), 1450001.

Received 18 Apr 2018. Revised 13 Aug 2018. Accepted 18 Aug 2018.

XU WANG: xuwang@ecust.edu.cn
School of Mechanical and Power Engineering, East China University of Science and Technology, Shanghai, China

LIANG CHEN: liangchen5962@mail.ecust.edu.cn
School of Mechanical and Power Engineering, East China University of Science and Technology, Shanghai, China

PETER SCHIAVONE: p.schiavone@ualberta.ca
Department of Mechanical Engineering, University of Alberta, Edmonton, AB, Canada

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2018 is US $615/year for the electronic version, and $775/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 13, No. 4 July 2018