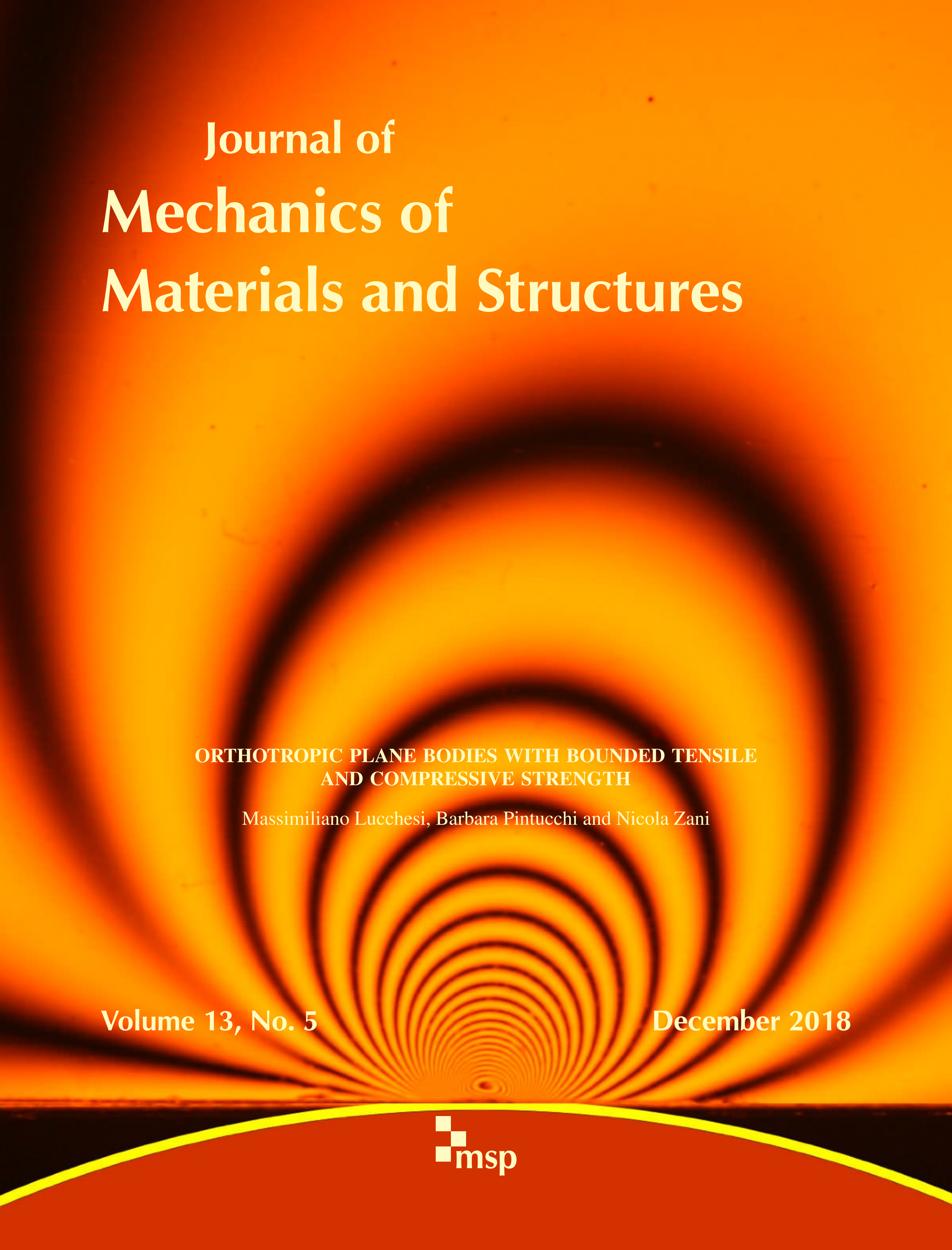

Journal of
Mechanics of
Materials and Structures

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Volume 13, No. 5 December 2018

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 13, No. 5, 2018
dx.doi.org/10.2140/jomms.2018.13.691

ORTHOTROPIC PLANE BODIES WITH BOUNDED TENSILE AND COMPRESSIVE STRENGTH

MASSIMILIANO LUCCHESI, BARBARA PINTUCCHI AND NICOLA ZANI

The constitutive equation of the nonlinear elastic material with limited tensile and compressive strength has been generalized to account for an orthotropic elasticity tensor. The main difference between this case and the simpler isotropic one is the loss of the coaxiality between the strain and the stress tensor, which leads the principal directions of the stress to become an unknown of the problem. The proposed constitutive equation has been implemented in the finite element code Mady and applied to the study of a masonry panel.

1. Introduction

The mechanical behavior of many elastic materials which are unable to withstand certain types of stress is generally described by the constitutive equation of the so-called normal elastic material [Del Piero 1989]. For these materials the stress $\boldsymbol{T}$ must belong to the stress range, a closed and convex subset $\mathcal{K}$ of all symmetric second-order tensors. Consequently, $\boldsymbol{T}$ does not coincide with the image of the strain $\boldsymbol{E}$ via the elasticity tensor $\mathbb{C}$, but with the projection on $\mathcal{K}$ of $\mathbb{C}\boldsymbol{E}$ with respect to a suitable scalar product. It is useful to represent the deformation as decomposed additively in an "elastic part" of which the stress is the image through $\mathbb{C}$ and in an "inelastic part" that belongs to the normal cone to $\mathcal{K}$ at $\boldsymbol{T}$ [Lucchesi et al. 2008; Šilhavý 2014; Angelillo 2014]. If the elasticity tensor is (symmetric and) positive definite, for each assigned strain, the solution of the constitutive equation exists and is unique in virtue of the minimum norm theorem. Moreover, this solution can be obtained quite easily when $\mathbb{C}$ is isotropic, as in this case the coaxiality of the stress and strain tensors allows to solve the problem in their common characteristic space.

To date, materials with limited resistance to tensile, compressive, and shear stress have been successfully modeled via this approach [Lucchesi et al. 2018a; 2018b] and many different types of structures such as arches, towers, and churches studied. Indeed, this approach may still be effective for studying certain monumental constructions, where the texture and the properties of masonry are not easily evaluable. Nevertheless, for several applications, a model that accounts for different properties of the material in various directions is undoubtedly more realistic [Lourenço et al. 1998; Berto et al. 2002; Pelà et al. 2011; Lishak et al. 2012].

Since the constraints on the stress are expressed in terms of its invariants, the tensors belonging to the boundary of $\mathcal{K}$ and the elements of the corresponding normal cone are coaxial, even if $\mathbb{C}$ is not isotropic. Of course, in this case there is no coaxiality between the stress and the strain tensor, and the characteristic space of the stress is part of the unknowns of the problem.

As in the isotropic case, there exists a natural partition of Sym, the space of all symmetric second-order tensors, to which corresponds a partition of $\mathcal{K}$. The belonging of $\mathbb{C}\boldsymbol{E}$ to one of the regions of the partition

Keywords: orthotropic materials, masonry panels.

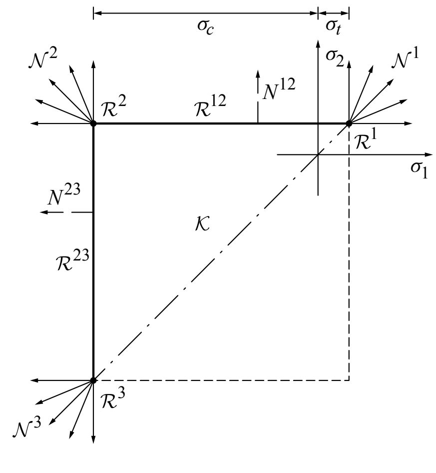

Figure 1. The elastic range.

of Sym implies that its projection (i.e., the associated stress) belongs to the corresponding region of the partition of $\mathcal{K}$. When the material is orthotropic, however, the assignment of $\mathbb{C}\boldsymbol{E}$ to one of these regions is not immediate and requires a more complex procedure with respect to the isotropic case.

The proposed constitutive model, implemented in the finite element Mady code [Lucchesi et al. 2017], has been applied to the study of a panel subjected firstly to its own weight and to a uniformly distributed vertical load, and then to a progressively increased horizontal displacement imposed at its top.

2. Materials with bounded tensile and compressive strength

Let σ_t and σ_c be two nonnegative constants and $\boldsymbol{I}$ the identity tensor. Consider a plane stress state for a material whose stress tensor $\boldsymbol{T}$ must belong to the closed and convex set

$$\mathcal{K} = \{\boldsymbol{T} \in \mathrm{Sym} : \boldsymbol{T} - \sigma_t \boldsymbol{I} \in \mathrm{Sym}^-,\ \boldsymbol{T} + \sigma_c \boldsymbol{I} \in \mathrm{Sym}^+\},$$

where Sym^- and Sym^+ are the cones of the seminegative and semipositive definite elements of Sym, respectively. The following regions make up a partition of the boundary $\partial\mathcal{K}$ of $\mathcal{K}$ (Figure 1):

$$\mathcal{R}^1 = \{\sigma_t \boldsymbol{I}\}, \quad \mathcal{R}^2 = \{\boldsymbol{T} \in \mathrm{Sym} : \mathrm{tr}\,\boldsymbol{T} = -\sigma_c + \sigma_t,\ \det \boldsymbol{T} = -\sigma_c\sigma_t\}, \quad \mathcal{R}^3 = \{-\sigma_c \boldsymbol{I}\},$$
$$\mathcal{R}^{12} = \{\boldsymbol{T} \in \mathrm{Sym} : -\sigma_c + \sigma_t < \mathrm{tr}\,\boldsymbol{T} < 2\sigma_t,\ \gamma^{12}(\boldsymbol{T}) = -\det(\boldsymbol{T} - \boldsymbol{\sigma}_t \boldsymbol{I}) = 0\},$$

and

$$\mathcal{R}^{23} = \{\boldsymbol{T} \in \mathrm{Sym} : -2\sigma_c < \mathrm{tr}\,\boldsymbol{T} < \sigma_t - \sigma_c,\ \gamma^{23}(\boldsymbol{T}) = -\det(\boldsymbol{T} + \boldsymbol{\sigma}_c \boldsymbol{I}) = 0\}.$$

Then, if $\boldsymbol{T} \in \mathcal{R}^{12}$, σ_t is principal stress, while if $\boldsymbol{T} \in \mathcal{R}^{23}$, $-\sigma_c$ is principal stress. Moreover, if σ denotes the other principal value of $\boldsymbol{T}$, in both cases it holds that

$$-\sigma_c < \sigma < \sigma_t.$$

By noting that, in view of the Hamilton–Cayley theorem, this results in

$$\gamma^{12}(\boldsymbol{T}) = -\sigma_t^2 + \sigma_t\,\mathrm{tr}\,\boldsymbol{T} - \tfrac{1}{2}((\mathrm{tr}\,\boldsymbol{T})^2 - \|\boldsymbol{T}\|^2)$$

and

$$\gamma^{23}(\boldsymbol{T}) = -\sigma_c^2 - \sigma_c \operatorname{tr} \boldsymbol{T} - \tfrac{1}{2}((\operatorname{tr} \boldsymbol{T})^2 - \|\boldsymbol{T}\|^2)$$

and, moreover,

$$D\gamma^{12}(\boldsymbol{T}) = \boldsymbol{T} + (\sigma_t - \operatorname{tr} \boldsymbol{T})\boldsymbol{I}, \quad D\gamma^{23}(\boldsymbol{T}) = \boldsymbol{T} - (\sigma_c + \operatorname{tr} \boldsymbol{T})\boldsymbol{I}$$

and

$$\|\boldsymbol{T} + (\sigma_t - \operatorname{tr} \boldsymbol{T})\boldsymbol{I}\| = 2\sigma_t - \operatorname{tr} \boldsymbol{T}, \quad \|\boldsymbol{T} - (\sigma_c + \operatorname{tr} \boldsymbol{T})\boldsymbol{I}\| = 2\sigma_c + \operatorname{tr} \boldsymbol{T},$$

the outward unit normal vectors to $\mathcal{K}$ at $\mathcal{R}^{12}$ and $\mathcal{R}^{23}$ are

$$\boldsymbol{N}^{12} = \frac{\boldsymbol{T} + (\sigma_t - \operatorname{tr} \boldsymbol{T})\boldsymbol{I}}{2\sigma_t - \operatorname{tr} \boldsymbol{T}} \quad \text{and} \quad \boldsymbol{N}^{23} = \frac{\boldsymbol{T} - (\sigma_c + \operatorname{tr} \boldsymbol{T})\boldsymbol{I}}{2\sigma_c + \operatorname{tr} \boldsymbol{T}}$$

respectively. It is worth observing that $\boldsymbol{N}^{12}$ and $\boldsymbol{N}^{23}$ commute with the corresponding stress, so that they have the same characteristic space.

Moreover, it is easy to verify that the normal cones to $\mathcal{K}$ at $\mathcal{R}^1$, $\mathcal{R}^2$, and $\mathcal{R}^3$ are respectively

$$\mathcal{N}^1 = \mathrm{Sym}^+, \quad \mathcal{N}^2 = \{\boldsymbol{N} \in \mathrm{Sym} : N = \alpha N^{12} - \omega N^{23}, \alpha \geq 0,\ \omega \geq 0\}, \quad \mathcal{N}^3 = \mathrm{Sym}^- .$$

Let $\mathbb{C}$ be the elasticity tensor, which is hypothesized to be symmetric and positive definite and $\boldsymbol{E}$ the assigned strain tensor. Moreover, let $\|\bullet\|_E$ be the energy norm, defined in Sym by $\|\boldsymbol{S}\|_E^2 = \boldsymbol{S} \cdot \mathbb{C}^{-1}\boldsymbol{S}$. In order to determine $\boldsymbol{T} \in \mathcal{K}$ having the minimum distance from $\mathbb{C}\boldsymbol{E}$ with respect to $\|\bullet\|_E$, let us denote $\boldsymbol{E}^e = \mathbb{C}^{-1}\boldsymbol{T}$ and $\boldsymbol{E}^a = \boldsymbol{E} - \boldsymbol{E}^e$ so that

$$\boldsymbol{T} = \mathbb{C}[\boldsymbol{E} - \boldsymbol{E}^a].$$

If $\mathbb{C}\boldsymbol{E} \in \mathcal{K}$ then the response of the material is linear elastic, i.e.,

$$\boldsymbol{E}^a = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{T} = \mathbb{C}\boldsymbol{E}.$$

Otherwise, it is necessary to consider the partition of $\mathrm{Sym}\backslash\mathcal{K}$ in the five regions specified below, as the stress depends on which region contains $\mathbb{C}\boldsymbol{E}$.

(i) $\mathbb{C}\boldsymbol{E} \in \mathcal{T}^1$ if $\boldsymbol{E} - \sigma_t\mathbb{C}^{-1}\boldsymbol{I} \in \mathcal{N}^1$, i.e., $\det(\boldsymbol{E} - \sigma_t\mathbb{C}^{-1}\boldsymbol{I}) \geq 0$ and $\operatorname{tr}(\boldsymbol{E} - \sigma_t\mathbb{C}^{-1}\boldsymbol{I}) \geq 0$, then

$$\boldsymbol{E}^a = \boldsymbol{E} - \sigma_t\mathbb{C}^{-1}\boldsymbol{I} \quad \text{and} \quad \boldsymbol{T} = \sigma_t\boldsymbol{I}; \tag{2-1}$$

(ii) $\mathbb{C}\boldsymbol{E} \in \mathcal{T}^{12}$ if there exists $\alpha > 0$ such that $\mathbb{C}[\boldsymbol{E} - \alpha\boldsymbol{N}^{12}] \in \mathcal{R}^{12}$, then

$$\boldsymbol{E}^a = \alpha\boldsymbol{N}^{12} \quad \text{and} \quad \boldsymbol{T} = \mathbb{C}[\boldsymbol{E} - \alpha\boldsymbol{N}^{12}]; \tag{2-2}$$

(iii) $\mathbb{C}\boldsymbol{E} \in \mathcal{T}^2$ if there exist $\alpha > 0$ and $\omega > 0$ such that $\operatorname{tr}\mathbb{C}[\boldsymbol{E} - \alpha\boldsymbol{N}^{12} - \omega\boldsymbol{N}^{23}] = \sigma_t - \sigma_c$ and $\det\mathbb{C}[\boldsymbol{E} - \alpha\boldsymbol{N}^{12} - \omega\boldsymbol{N}^{23}] = -\sigma_t\sigma_c$, then

$$\boldsymbol{E}^a = \alpha\boldsymbol{N}^{12} - \omega\boldsymbol{N}^{23} \quad \text{and} \quad \boldsymbol{T} = \mathbb{C}[\boldsymbol{E} - \alpha\boldsymbol{N}^{12} + \omega\boldsymbol{N}^{23}]; \tag{2-3}$$

(iv) $\mathbb{C}\boldsymbol{E} \in \mathcal{T}^{23}$ if there exists $\omega > 0$ such that $\mathbb{C}[\boldsymbol{E} - \omega\boldsymbol{N}^{23}] \in \mathcal{R}^{23}$, then

$$\boldsymbol{E}^a = \omega\boldsymbol{N}^{23} \quad \text{and} \quad \boldsymbol{T} = \mathbb{C}[\boldsymbol{E} - \omega\boldsymbol{N}^{23}]; \tag{2-4}$$

(v) $\mathbb{C}\boldsymbol{E} \in \mathcal{T}^3$ if $\boldsymbol{E} + \sigma_c\mathbb{C}^{-1}\boldsymbol{I} \in \mathcal{N}^3$, i.e., $\det(\boldsymbol{E} + \sigma_c\mathbb{C}^{-1}\boldsymbol{I}) \geq 0$ and $\operatorname{tr}(\boldsymbol{E} + \sigma_c\mathbb{C}^{-1}\boldsymbol{I}) \leq 0$, then

$$\boldsymbol{E}^a = \boldsymbol{E} + \sigma_c\mathbb{C}^{-1}\boldsymbol{I} \quad \text{and} \quad \boldsymbol{T} = -\sigma_c\boldsymbol{I}. \tag{2-5}$$

3. Orthotropic materials

Let the orthonormal vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ define the symmetry directions of a plane orthotropic body [Mallick 1988], with E_{11}, E_{22} and ν_{12}, ν_{21} the corresponding Young and Poisson moduli, respectively. Then the elasticity tensor is

$$\mathbb{C} = \begin{pmatrix} C_{1111} & C_{1122} & 0 \\ C_{1112} & C_{2222} & 0 \\ 0 & 0 & C_{2323} \end{pmatrix}$$

with

$$C_{1111} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \quad C_{1122} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1-\nu_{12}\nu_{21}}, \quad C_{2222} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}, \quad C_{2323} = 2G.$$

By denoting

$$E_{22} = \beta E_{11}, \quad \nu_{21} = \beta\nu_{12}, \quad 2G = \frac{\phi E_{11}}{1-\beta\nu_{12}^2},$$

and writing E for E_{11} and ν for ν_{12}, we obtain

$$C_{1111} = \frac{E}{1-\beta\nu^2}, \quad C_{1122} = \frac{\beta\nu E}{1-\beta\nu^2}, \quad C_{2222} = \frac{\beta E}{1-\beta\nu^2}, \quad C_{2323} = \frac{\phi E}{1-\beta\nu^2},$$

that is,

$$\mathbb{C} = \frac{E}{1-\beta\nu^2}\begin{pmatrix} 1 & \beta\nu & 0 \\ \beta\nu & \beta & 0 \\ 0 & 0 & \phi \end{pmatrix} \tag{3-1}$$

and

$$\mathbb{C}^{-1} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1/\beta & 0 \\ 0 & 0 & (1-\beta\nu^2)/\phi \end{pmatrix}. \tag{3-2}$$

In the particular case when $\beta = 1$ and $\phi = 1-\nu$, the material is isotropic. In order to guarantee that $\mathbb{C}$ is positive definite, it is assumed that

$$E > 0,\ \phi > 0,\ \nu \in \left(0, \tfrac{1}{2}\right) \text{ and } \beta \in (0, 1].$$

Let

$$\boldsymbol{E} = \epsilon_{11}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \epsilon_{22}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \epsilon_{12}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$$

be the assigned strain tensor. Then

$$\mathbb{C}\boldsymbol{E} = \frac{E}{1-\beta\nu^2}\big((\epsilon_{11}+\beta\nu\epsilon_{22})\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \beta(\epsilon_{22}+\nu\epsilon_{11})\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \phi\epsilon_{12}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\big), \tag{3-3}$$

so that

$$\operatorname{tr}\mathbb{C}\boldsymbol{E} = \left(\frac{E}{1-\beta\nu^2}\right)((\epsilon_{11}+\beta\nu\epsilon_{22}) + \beta(\epsilon_{22}+\nu\epsilon_{11})),$$

$$\det\mathbb{C}\boldsymbol{E} = \left(\frac{E}{1-\beta\nu^2}\right)^2(\beta(\epsilon_{11}+\beta\nu\epsilon_{22})(\epsilon_{22}+\nu\epsilon_{11}) - \phi^2\epsilon_{12}^2).$$

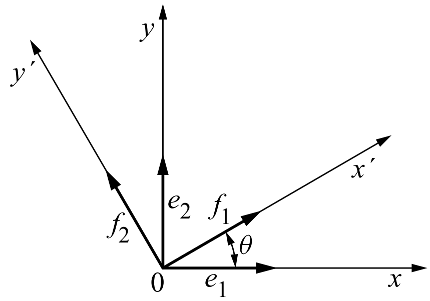

Figure 2. Reference systems.

Moreover, as

$$\mathbb{C}^{-1}\boldsymbol{I}=\frac{1}{E}\Big((1-\nu)\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\frac{1-\beta\nu}{\beta}\boldsymbol{e}_2\otimes\boldsymbol{e}_2\Big),$$

it holds

$$\operatorname{tr}(\sigma_t\mathbb{C}^{-1}\boldsymbol{I})=\frac{\sigma_t(1-2\beta\nu+\beta)}{\beta E},\quad \det(\sigma_t\mathbb{C}^{-1}\boldsymbol{I})=\frac{\sigma_t^2}{\beta E^2}(1-\nu)(1-\beta\nu).$$

and similar expressions are obtained for $-\sigma_c\mathbb{C}^{-1}\boldsymbol{I}$.

4. Determination of the stress

At any point of $\partial\mathcal{K}$, the stress $\boldsymbol{T}$ and the elements of the corresponding normal cone can be expressed with respect to an orthonormal basis of their characteristic space, that will be denoted by $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ (Figure 2).

Then, $\boldsymbol{T}\in\mathcal{R}^{12}$ implies

$$\boldsymbol{T}=\sigma\boldsymbol{f}_1\otimes\boldsymbol{f}_1+\sigma_t\boldsymbol{f}_2\otimes\boldsymbol{f}_2,\quad \boldsymbol{N}^{12}=\boldsymbol{f}_2\otimes\boldsymbol{f}_2 \tag{4-1}$$

and $\boldsymbol{T}\in\mathcal{R}^{23}$ implies

$$\boldsymbol{T}=-\sigma_c\boldsymbol{f}_1\otimes\boldsymbol{f}_1+\sigma\boldsymbol{f}_2\otimes\boldsymbol{f}_2,\quad \boldsymbol{N}^{23}=-\boldsymbol{f}_1\otimes\boldsymbol{f}_1 \tag{4-2}$$

Let $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ be the angle between the vectors $\boldsymbol{e}_1$ and $\boldsymbol{f}_1$, with $\boldsymbol{e}_1\wedge\boldsymbol{e}_2=\boldsymbol{e}_1\wedge\boldsymbol{f}_1/\,|\boldsymbol{e}_1\wedge\boldsymbol{f}_1|$, so that

$$\boldsymbol{f}_1\cdot\boldsymbol{e}_1=\boldsymbol{f}_2\cdot\boldsymbol{e}_2=\cos\theta,\quad \boldsymbol{f}_1\cdot\boldsymbol{e}_2=-\boldsymbol{f}_2\cdot\boldsymbol{e}_1=\sin\theta$$

from which

$$\boldsymbol{f}_1\otimes\boldsymbol{f}_1=\frac{1}{1+t^2}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1+t(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)+t^2\boldsymbol{e}_2\otimes\boldsymbol{e}_2) \tag{4-3}$$

and

$$\boldsymbol{f}_2\otimes\boldsymbol{f}_2=\frac{1}{1+t^2}(t^2\boldsymbol{e}_1\otimes\boldsymbol{e}_1-t(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)+\boldsymbol{e}_2\otimes\boldsymbol{e}_2), \tag{4-4}$$

with $t=\tan\theta$. The expressions of $\boldsymbol{T}$, $\boldsymbol{N}^{12}$, $\boldsymbol{N}^{23}$, $\mathbb{C}\boldsymbol{N}^{12}$, and $\mathbb{C}\boldsymbol{N}^{23}$ with respect to the basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ can now be deduced by (4-1)–(4-4), and (3-1). If $\boldsymbol{T}\in\mathcal{R}^{12}$, then

$$\begin{aligned}\boldsymbol{T}&=\frac{1}{1+t^2}\big((\sigma+\sigma_t t^2)\boldsymbol{e}_1\otimes\boldsymbol{e}_1+t(\sigma-\sigma_t)(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)+(\sigma t^2+\sigma_t)\boldsymbol{e}_2\otimes\boldsymbol{e}_2\big),\\ \boldsymbol{N}^{12}&=\frac{1}{1+t^2}(t^2\,\boldsymbol{e}_1\otimes\boldsymbol{e}_1-t(\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\boldsymbol{e}_2\otimes\boldsymbol{e}_1)+\boldsymbol{e}_2\otimes\boldsymbol{e}_2),\end{aligned} \tag{4-5}$$

and, with the help of (3-3), the components of the stress can be deduced from the equation $\boldsymbol{T} = \mathbb{C}\boldsymbol{E} - \alpha\mathbb{C}\boldsymbol{N}^{12}$, as functions of α, σ and t:

$$T_{11} = \frac{\sigma + \sigma_t t^2}{1+t^2} = \frac{E}{1-\beta\nu^2}\left(\epsilon_{11} + \beta\nu\epsilon_{22} - \alpha\frac{t^2+\beta\nu}{1+t^2}\right), \tag{4-6}$$

$$T_{22} = \frac{\sigma t^2 + \sigma_t}{1+t^2} = \frac{E}{1-\beta\nu^2}\left(\beta(\epsilon_{22} + \nu\epsilon_{11}) - \alpha\frac{\beta(1+\nu t^2)}{1+t^2}\right), \tag{4-7}$$

$$T_{12} = \frac{(\sigma - \sigma_t)t}{1+t^2} = \frac{\phi E}{1-\beta\nu^2}\left(\epsilon_{12} + \frac{\alpha t}{1+t^2}\right). \tag{4-8}$$

If $\boldsymbol{T} \in \mathcal{R}^{23}$, then

$$\boldsymbol{T} = \frac{1}{1+t^2}\big((\sigma t^2 - \sigma_c)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - t(\sigma_c + \sigma)(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + (\sigma - \sigma_c t^2)\boldsymbol{e}_2 \otimes \boldsymbol{e}_2\big),$$

$$\boldsymbol{N}^{23} = -\frac{1}{1+t^2}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + t(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + t^2\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \tag{4-9}$$

and the components of the stress, deduced from $\boldsymbol{T} = \mathbb{C}\boldsymbol{E} - \omega\mathbb{C}\boldsymbol{N}^{23}$, are

$$T_{11} = \frac{\sigma t^2 - \sigma_c}{1+t^2} = \frac{E}{1-\beta\nu^2}\left(\epsilon_{11} + \beta\nu\epsilon_{22} + \omega\frac{1+\beta\nu t^2}{1+t^2}\right), \tag{4-10}$$

$$T_{22} = \frac{\sigma - \sigma_c t^2}{1+t^2} = \frac{E}{1-\beta\nu^2}\left(\beta(\epsilon_{22} + \nu\epsilon_{11}) + \omega\frac{\beta(t^2+\nu)}{1+t^2}\right), \tag{4-11}$$

$$T_{12} = -\frac{(\sigma + \sigma_c)t}{1+t^2} = \frac{\phi E}{1-\beta\nu^2}\left(\epsilon_{12} + \frac{\omega t}{1+t^2}\right). \tag{4-12}$$

It is worth noting that, by taking into account that $\alpha > 0$, $\omega > 0$ and $\sigma - \sigma_t < 0$, $-(\sigma + \sigma_c) < 0$, from (4-8) and (4-12), it follows that $\epsilon_{12} = 0$ if and only if $t/(1+t^2) = 0$, i.e., $\theta = 0$ or $\theta = \frac{\pi}{2}$. Otherwise, t and ϵ_{12} must have opposite sign.

From (4-6) and (4-7), α and σ can be determined easily, as functions of t:

$$\alpha = \frac{(1+t^2)[t^2(\beta\nu\epsilon_{22} + \epsilon_{11} - \beta(\nu\epsilon_{11} + \epsilon_{22})) + \bar{\sigma}_t(1-t^2)(1-\beta\nu^2)]}{t^4 - \beta}, \tag{4-13}$$

$$\sigma = E\frac{\beta(1+t^2)(t^2\epsilon_{22} - \epsilon_{11}) + \bar{\sigma}_t[\beta\nu t^4 - t^2(1-\beta) - \beta\nu]}{t^4 - \beta}, \tag{4-14}$$

where $\bar{\sigma}_t = \sigma_t/E$. Substituting these expressions in (4-8) gives the algebraic equation

$$t^4 + \frac{\beta^2\nu^2(\epsilon_{22} + \nu\sigma\bar{\sigma}_t) + \beta[-\epsilon_{22}(1-\nu\phi) - \nu\sigma\bar{\sigma}_t(1+\nu(1-\phi))] + \phi\epsilon_{11} + \sigma\bar{\sigma}_t(1-\phi)}{\phi\epsilon_{12}}t^3$$
$$- \frac{\beta\nu^2(\epsilon_{11} - \sigma\bar{\sigma}_t(1-\nu)) + \beta[-\epsilon_{11}(1-\nu\phi) + \phi\epsilon_{22} + \sigma\bar{\sigma}_t(1-\nu+\phi\nu^2)] - \sigma\bar{\sigma}_t\phi}{\phi\epsilon_{12}}t - \beta = 0. \tag{4-15}$$

Similarly, (4-10) and (4-11) give

$$\omega = \frac{(1+t^2)[t^2\beta(\nu\epsilon_{11} + \epsilon_{22}) - \beta\nu\epsilon_{22} - \epsilon_{11} - \sigma\bar{\sigma}_c(1-t^2)(1-\beta\nu^2)]}{1-\beta t^4}, \tag{4-16}$$

and

$$\sigma = E\frac{\beta(1+t^2)(t^2\epsilon_{11}-\epsilon_{22})-\sigma\bar{\sigma}_c[\beta\nu t^4+t^2(1-\beta)-\beta\nu]}{1-\beta t^4}, \tag{4-17}$$

with $\bar{\sigma}_c = \sigma_c/E$, from which

$$t^4+\frac{\beta^2\nu^2(\epsilon_{11}+\bar{\sigma}_c(1-\nu)+\beta[-\epsilon_{11}(1-\nu\phi)+\phi\epsilon_{22}-\sigma\bar{\sigma}_c(1-\nu+\phi\nu^2)+\bar{\sigma}_c\phi]}{\beta\phi\epsilon_{12}}t^3 \\ -\frac{\beta(\nu^2\epsilon_{22}-\sigma\bar{\sigma}_c\nu^3)+\beta[-\epsilon_{22}(1-\nu\phi)-\nu\sigma\bar{\sigma}_c(-1+\nu(\phi-1))]+\phi\epsilon_{11}+\sigma\bar{\sigma}_c(1-\phi)}{\beta\phi\epsilon_{12}}t-\frac{1}{\beta}=0 \tag{4-18}$$

is obtained.

4A. ***Assignment of*** $\mathbb{C}\boldsymbol{E}$ ***to its region.*** Once the strain $\boldsymbol{E}$ (with $\mathbb{C}\boldsymbol{E} \notin \mathcal{K}$) has been assigned, in order to determine the stress it is necessary to establish which region of the partition of $\mathrm{Sym}\setminus\mathcal{K}$ $\mathbb{C}\boldsymbol{E}$ belongs to. For the regions $\mathcal{T}^1$ and $\mathcal{T}^3$ the check is trivial; then, if successful, the inelastic strain and the stress can be obtained by (2-1) and (2-5), respectively.

Otherwise, one can proceed by trial, until the region containing $\mathbb{C}\boldsymbol{E}$ is found. In order to verify whether $\mathbb{C}\boldsymbol{E}$ belongs to region $\mathcal{T}^{12}$, once that t, α, and σ have been determined from (4-13)–(4-15), the relations $\alpha > 0$ and $-\sigma_c < \sigma < \sigma_t$ must be satisfied. In the same way, to verify whether $\mathbb{C}\boldsymbol{E}$ belongs to $\mathcal{T}^{23}$, equations (4-16)–(4-18) can be used. If the check is satisfied, the inelastic strain and the stress are given by (2-2) in the first case and by (2-4) in the second.

In regards to $\mathcal{T}^2$, it is observed that

$$\boldsymbol{T} = -\sigma_c \boldsymbol{f}_1 \otimes \boldsymbol{f}_1 + \sigma_t \boldsymbol{f}_2 \otimes \boldsymbol{f}_2$$

and a generic element belonging to the normal cone to $\partial\mathcal{K}$ at $\mathcal{T}^2$ is of the form

$$\boldsymbol{A} = -\omega \boldsymbol{f}_1 \otimes \boldsymbol{f}_1 + \alpha \boldsymbol{f}_2 \otimes \boldsymbol{f}_2$$

with

$$\alpha > 0 \quad \text{and} \quad \omega > 0. \tag{4-19}$$

Then, with the help of (3-2), (4-3), and (4-4), from the equation $\boldsymbol{E} - \mathbb{C}^{-1}\boldsymbol{T} = \boldsymbol{E}^a$ the system

$$\begin{aligned} \epsilon_{11} - \frac{\sigma_t t^2 - \sigma_c - \nu(\sigma_t - \sigma_c t^2)}{E(1+t^2)} &= \frac{-\omega + \alpha t^2}{1+t^2}, \\ \epsilon_{22} - \frac{\beta\nu(\sigma_c - \sigma_t t^2) + \sigma_t - \sigma_c t^2}{\beta E(1+t^2)} &= \frac{\alpha - \omega t^2}{1+t^2}, \\ \epsilon_{12} + \frac{(1-\beta\nu^2)(\sigma_c + \sigma_t)t}{\phi E(1+t^2)} &= \frac{-(\omega+\alpha)t}{(1+t^2)} \end{aligned}$$

is obtained, whose solution allows us to calculate α, ω, and t, and to verify if the condition (4-19) is satisfied. If so, the inelastic strain and the stress are given by (2-3), with $\boldsymbol{N}^{12}$ and $\boldsymbol{N}^{23}$ given by (4-5) and (4-9), respectively.

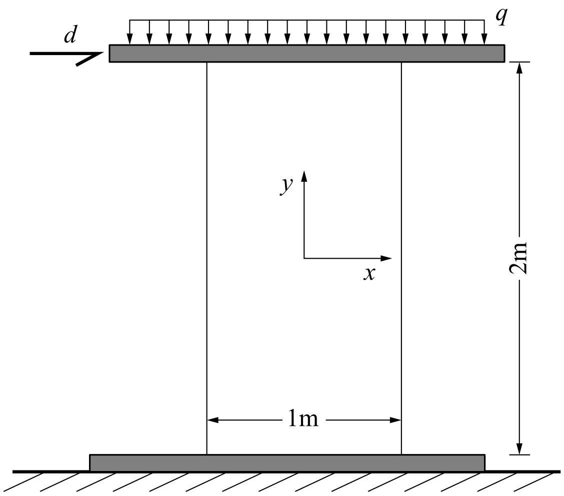

Figure 3. The analyzed masonry panel.

4B. ***Calculation of the stress derivatives.*** For the numerical solution of the equilibrium problem with the Newton–Raphson method, it is necessary to know the derivative of the stress with respect to the strain. To this aim, let D_{ijlm} be the derivative of the stress components T_{ij} with respect to the strain components ϵ_{lm}. It is hypothesized the existence of a differentiable function $\hat{t} : \mathbb{R}^3 \to \mathbb{R}$, such that $t = \hat{t}(\epsilon_{11}, \epsilon_{12}, \epsilon_{22})$, although one does not know its explicit expression. Then

$$D_{ijlm} = \frac{\partial T_{ij}}{\partial \epsilon_{lm}} + \frac{\partial T_{ij}}{\partial t} \frac{\partial \hat{t}}{\partial \epsilon_{lm}}.$$

The derivatives $\partial T_{ij}/\partial \epsilon_{lm}$ and $\partial T_{ij}/\partial t$ can be calculated directly from (4-6)–(4-8) or from (4-10)–(4-12), while in order to calculate $\partial \hat{t}/\partial \epsilon_{lm}$ it is necessary to use the implicit function theorem. For this purpose let $G(\epsilon_{11}, \epsilon_{12}, \epsilon_{22}, t)$ be the left member of (4-15) or (4-18), with $\partial G/\partial t \neq 0$, so that

$$\frac{\partial \hat{t}}{\partial \epsilon_{lm}} = -\frac{\partial G/\partial \epsilon_{lm}}{\partial G/\partial t}.$$

5. Example

The proposed model has been implemented into Mady [Lucchesi et al. 2017], a FEM code developed by the authors. The analysis presented in the following has been conducted by using plane stress four-node isoparametric elements.

Let us consider a masonry wall 2 m height, 1 m wide and 0.1 m thick, which is subjected to a permanent load, consisting of its own weight and a uniform vertical load q, and to a horizontal displacement d incrementally imposed at the top (Figure 3). The mechanical properties assumed for the material are $\nu = 0.1$, $\rho = 1800\,\text{kg/m}^3$, $\sigma_t = 0$, $\sigma_c = 5\,\text{MPa}$, $\Phi = 1 - \nu$. For the vertical load, the values $q_1 = 0.5\,\text{MPa}$ and $q_2 = 1\,\text{MPa}$ have been assumed. For each load case, two further cases have been considered for the Young moduli. In the former, E_x has been assumed equal to 5 GPa while E_y has been varied as shown in Figures 4 and 6; in the second case, E_x has been varied as shown in Figures 5 and 7, whereas $E_y = 5\,\text{GPa}$ has been kept constant.

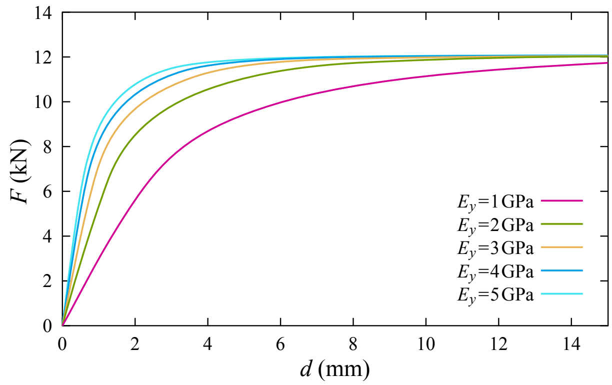

Figure 4. Shear force vs. displacement for $q = 0.5$ MPa, $E_x = 1$ GPa and several values of E_y.

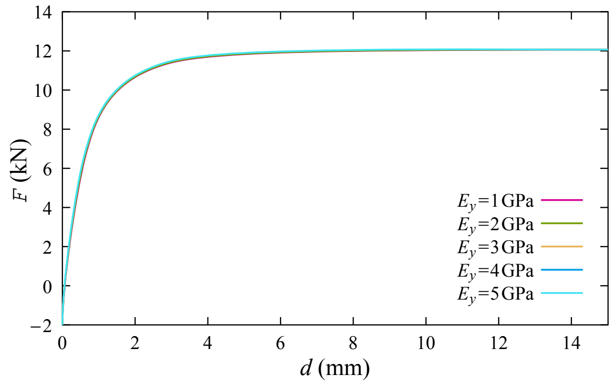

Figure 5. Shear force vs. displacement for $q = 0.5$ MPa, $E_y = 1$ GPa and several values of E_x.

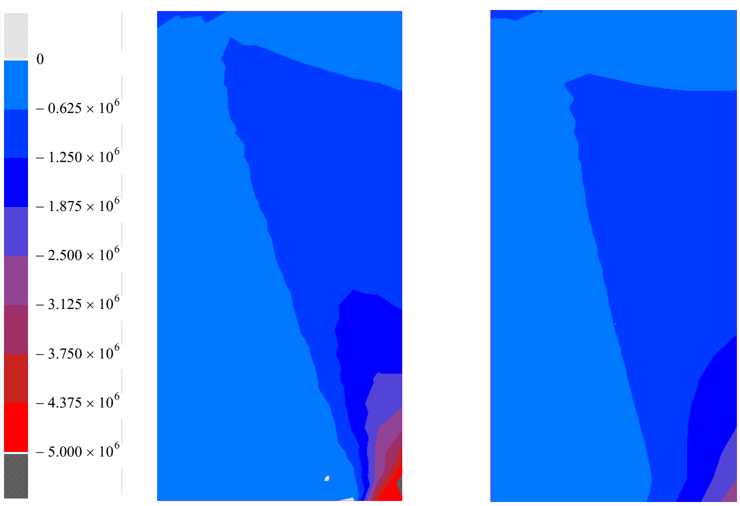

Figure 6. Principal compressive stress for $q = 0.5$ MPa and $d = 5$ mm in two cases. Left: $E_x = E_y = 5$ GPa (isotropic). Right: $E_x = 5$ GPa, $E_y = 1$ GPa.

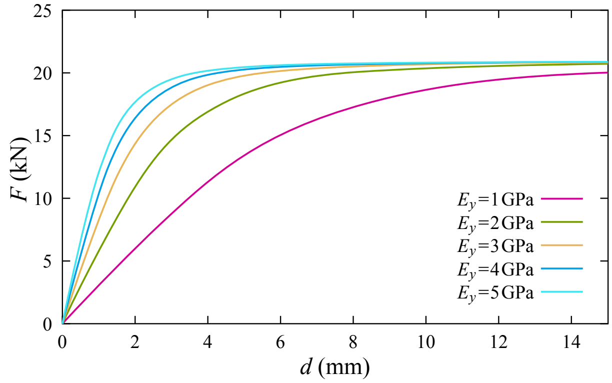

Figure 7. Shear force vs. displacement for $q = 1$ MPa, $E_x = 1$ GPa and several values of E_y.

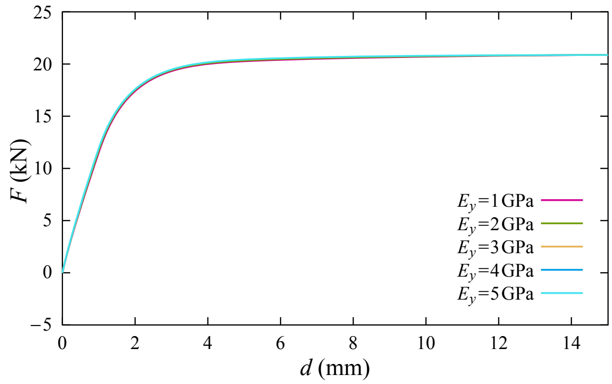

Figure 8. Shear force vs. displacement for $q = 1$ MPa, $E_y = 1$ GPa and several values of E_x.

Figures 4, 5 and Figures 7, 8 show the graphs of the horizontal reaction at the base of the panel as a function of the imposed displacement, for the four cases considered. Their trends confirm a well-known result, i.e., for this load condition, the value of the Young's modulus in the vertical direction affects the stiffness of the panel much more than that in the horizontal direction [Smilovic et al. 2019]. In order to highlight the difference between isotropic and orthotropic behavior, Figure 6 shows the principal compressive stress for $q_1 = 0.5$ MPa and $d = 5$ mm by comparing the case of $E_x = E_y = 5$ MPa (isotropic) and $E_x = 5$ MPa, $E_y = 1$ MPa.

6. Conclusions

The main outcome of the paper is the development of a constitutive model that considers masonry as a nonlinear orthotropic elastic material with bounded tensile and compressive stresses. This approach allows us to describe in a more realistic way the behavior of masonry buildings (which are generally made of a nonisotropic material) and to develop useful nonisotropic damage laws. This latter issue, as well as a more general constitutive equation for orthotropic materials (e.g., those unable to withstand high shear stress) will be the object of a forthcoming paper.

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Received 19 Jul 2018. Revised 6 Feb 2019. Accepted 2 Mar 2019.

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JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2018 is US $615/year for the electronic version, and $775/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

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Journal of Mechanics of Materials and Structures

Volume 13, No. 5 December 2018

Special issue on

Structural Analysis of Real Historic Buildings (part 1)

1559-3959(2018)13:5;1-X