Journal of
Mechanics of Materials and Structures

THE ROLE OF RHEOLOGY IN MODELLING ELASTIC WAVES WITH GAS BUBBLES IN GRANULAR FLUID-SATURATED MEDIA

Adham A. Ali and Dmitry V. Strunin

Volume 14, No. 1 January 2019

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 14, No. 1, 2019
dx.doi.org/10.2140/jomms.2019.14.1

THE ROLE OF RHEOLOGY IN MODELLING ELASTIC WAVES WITH GAS BUBBLES IN GRANULAR FLUID-SATURATED MEDIA

ADHAM A. ALI AND DMITRY V. STRUNIN

Elastic waves in fluid-saturated granular media depend on the rheology which includes elements representing the fluid and, if necessary, gas bubbles. We investigated the effect of different rheological schemes, including and excluding the bubbles, on the linear Frenkel–Biot waves of P1 type. For the wave with the bubbles the scheme consists of three segments representing the solid continuum, fluid continuum, and a bubble surrounded by the fluid. We derived the Nikolaevskiy-type equations describing the velocity of the solid matrix in the moving reference system. The equations are linearized to yield the decay rate λ as a function of the wave number k. We compared the $\lambda(k)$-dependence for the cases with and without the bubbles, using typical values of the input mechanical parameters. For the both cases, the $\lambda(k)$-curve lies entirely below zero, which is in line with the notion of the elastic wave being an essentially passive system. We found that the increase of the radius of the bubbles leads to faster decay, while the increase in the number of the bubbles leads to slower decay of the elastic wave.

A list of symbols can be found on page 16.

1. Introduction

The problem of wave propagation in porous media is of interest in various fields of science and engineering. Over the recent years, researchers studied diverse phenomena of this type in large-scale earthquakes, soil mechanics, acoustics, earthquake engineering, and many other disciplines. The fundamentals of the theory of the wave propagation in porous elastic solids can be found in [Biot 1956a; 1956b] or in a more recent review [Frenkel 2005]. Biot [1956a; 1956b] generalized the first principles of linear elasticity and today, most studies in acoustics, geophysical, and geological mechanics rely on his theory. Biot [1962a; 1962b] also deduced the dynamical equations for the wave propagation in poroelastic media. According to Frenkel–Biot's theory, there are two types of longitudinal waves propagating in a saturated porous medium. The wave of the first type is fast and weakly damped (P1-wave), whereas the wave of the second type is slow and strongly damped (P2-wave). Yang et al. [2014] showed that the dispersion of velocity and attenuation of the P1-wave are greatly affected by the viscoelasticity of the medium.

Liu et al. [1976] demonstrated that rheology based on the scheme often referred to as Generalized Standard Linear Solid (GSLS) helps to better describe measured characteristics of seismic waves in earth continua. The importance of complex multicomponent GSLS was acknowledged by Bohlen [2002] who employed the rheology with many Maxwell bodies connected in parallel. Nikolaevskiy [1989] used complex stress-strain relations in a fluid-saturated grain, where the solid matrix and fluid are in contact.

Keywords: Frenkel–Biot's waves, bubbles, rheology, porous media.

This resulted, in the final analysis, in the nonlinear higher-order partial differential equation of the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \sum_{p=1}^{5} \varepsilon^{p-1} A_{p+1} \frac{\partial^{p+1} v}{\partial x^{p+1}}, \tag{1}$$

where v is the particle velocity of the solid matrix, ε is the small parameter reflecting slow evolution of the wave (this is discussed below), and A_{p+1} are the coefficients linked to mechanical parameters of the system. From the standpoint of wave dynamics, the even derivatives in (1) are responsible for the dissipation and odd derivatives for the dispersion effects. Equation (1) assumes the form of the Korteweg–de Vries–Burgers equation if the index p goes from just 1 and 2. But with the range of p going further as shown, the equation manifests an extension of this classical equation to include high-order spatial derivatives. As explained below, this extension results from the complex rheology of the system.

Experimental evidence indicates that the presence of gas bubbles changes the characteristics of the wave [Dunin and Nikolaevskiy 2005; Van Wijngaarden 1968; Anderson 1980]. Typically, in rocks saturated with fluids, the P1-wave is the only observable wave [Nikolaevskiy 2008]. However, the presence of gas, even in small proportion, can affect the wave type [Nikolaevskiy and Strunin 2012], so that the P2-wave may also be visible. Dunin et al. [2006] studied the effect of gas bubbles on linear P1- and P2-waves, and derived the dispersion relation connecting the frequency and wave number. Of special interest was the transformation of the wave type due to the bubbles. They found that the transformation was due to the change in the motion of the liquid in the porous space. Instead of the overflow between the pores incurring large Darcy friction, which is characteristic of the P2-waves, the liquid may be displaced into the volume released when a bubble is compressed. In this case the oscillations of the porous matrix and of the bubbles occur in phase and, as a result, the decay of the P2-wave diminishes due to the reduction in the Darcy friction. As far as the rheology is concerned, Dunin et al. [2006] used a rather simple stress-strain relation, $\sigma = Ee$, in standard notations. Various aspects of the wave propagation in multifluid and bubbly flows were studied in [Papageorgiou and Chapman 2015; Tisato et al. 2015; Brunner and Spetzler 2001; Collier et al. 2006]. For example, Collier et al. [2006] explored the influence of the gas bubbles on attenuation in volcanic magma, where the bubbles grow not only due to gas expansion, but also due to the exsolution of volatiles, such as water, from the melt into the bubbles. In our present study we do not consider such kind of thermodynamic disequilibrium conditions.

The rheological scheme used in [Nikolaevskiy 1989], despite containing several Maxwell bodies, did not include an element representing gas bubble. Nikolaevskiy and Strunin [2012] pointed out the place in this scheme which the bubble element should occupy; see Figure 1. In the present work we aim to include the bubble element into the rheological scheme and then derive and analyze the equation of the type (1), where the coefficients A_p too depend on the bubble-related parameters. The resulting equation will describe the decay (or attenuation) of the freely propagating seismic wave. We will investigate the influence of the bubble-related parameters, including their radius and concentration, on the decay rate.

During its propagation the seismic wave decays due to the viscous friction both within individual phases, e.g., fluid, and between the phases. The decay may be described in terms of the decay rate in time as in [Nikolaevskiy 1989], or decay rate in space via the attenuation factor as in [Dunin et al. 2006]. These descriptions are closely connected and just correspond to different realizations of the wave. To illustrate this, let us represent Fourier modes of the linear wave as $\exp(i\xi_1)$, where $\xi_1 = wt + kx$, w is

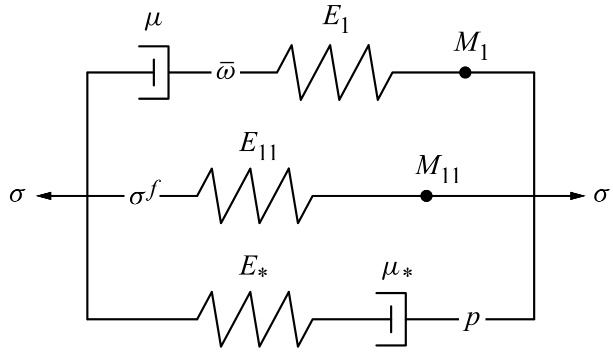

Figure 1. Rheological scheme with the branch ϖ corresponding to a bubble [Nikolaevskiy and Strunin 2012].

the frequency, and k the wave number connected with each other via the dispersion relation. Writing this relation in the form $w = w(k)$ with k being real-valued, we can find the corresponding complex-valued w. Its imaginary part determines how fast the wave decays in time. Physically this situation corresponds to the wave in an unbounded medium, which decays as time goes. Alternatively, one may write down the dispersion relation as $k = k(w)$ and consider the real-valued frequency w as the argument, whereas the wave number k becomes complex-valued. Imaginary part of k governs the decay of the wave in space. From physical standpoint this realization can be associated with the wave which propagates, say, from the surface into underground. The decay of such a wave against the distance is characterized by the attenuation factor.

Importantly, the dynamics of the fluid in porous media may exhibit boundary layers. They form when the frequency of the seismic wave is relatively high. This contrasts the low-frequency waves where the viscous forces dominate throughout the fluid volume so that inertial effects may be neglected. However, at high frequencies the inertial effects dominate in the bulk of the fluid, while the viscous friction concentrates within narrow boundary layers near solid walls due to the no-slip conditions. Allowing for the boundary layers in the analysis significantly affects the frequency dependence of the attenuation of the wave. Namely the low- and high-frequency branches of the attenuation curve become asymmetric. Masson et al. [2006] confirmed this effect by numerical computations of the governing mechanics equations. The model that we use in our present paper is one-dimensional; therefore, it disregards the boundary layers.

In our study of the wave decay we choose to analyze the decay in time, that is, the $w = w(k)$ form of the dispersion relation. We will execute a procedure similar to [Nikolaevskiy 2008], where one-dimensional (x-dependent) dynamics are considered, and all the functions of interest are decomposed into series in small parameter ε characterizing slow evolution of the wave in space and time. Let us denote the phase velocity of the wave by c. Introducing the running variable ξ and using ε to scale the distance x and time t as

$$\xi = \varepsilon(x - ct), \quad \tau = \left(\tfrac{1}{2}\varepsilon^2\right)t, \tag{2}$$

we seek the velocity of the solid matrix in the form

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \cdots$$

We will show, in line with Nikolaevskiy [2008], that the complex rheology generates higher-order time derivatives [Nikolaevskiy 1989]. They, in turn, translate into high-order derivatives in ξ in the resulting equation (1) because, using (2),

$$\frac{\partial}{\partial t} = \left(\tfrac{1}{2}\varepsilon^2\right)\frac{\partial}{\partial \tau} - \varepsilon c \frac{\partial}{\partial \xi} \approx -\varepsilon c \frac{\partial}{\partial \xi},$$

where the quadratic term in the small parameter ε is neglected in comparison with the linear term. Once derived, (1) gives the dispersion relation, which is the main point of interest in this paper. Our focus is on the dissipation controlled by the even x-derivatives. Therefore, we will study a truncated form of (1):

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial \xi} = A_2\frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 A_4 \frac{\partial^4 v}{\partial \xi^4} + \varepsilon^4 A_6 \frac{\partial^6 v}{\partial \xi^6}. \tag{3}$$

Equation (3) can be complemented by spatially periodic boundary conditions, leading to a spatially periodic solution. Substituting the Fourier modes of the solution, $v \sim \exp(\lambda t + ikx)$, into (3) gives

$$\lambda = -A_2 k^2 + \varepsilon^2 A_4 k^4 - \varepsilon^4 A_6 k^6, \tag{4}$$

with k being the wave number associated with the scaled length ξ. When analyzing (4) we remember that k is not allowed to be too large, otherwise the assumption of the slow variation of the wave in space will be violated. As we noted, the slowness is facilitated by the smallness of ε. Therefore, in (4) the term $\varepsilon^2 A_4 k^4$ should be treated just as a correction to the leading term $A_2 k^2$, and the following term $\varepsilon^4 A_6 k^6$ as a correction to the term $\varepsilon^2 A_4 k^4$. Thus, the value of λ remains negative at all plausible values of the mechanical parameters of the system (such as elastic moduli and viscosities). This reflects the essentially dissipative nature of the seismic wave, or, in other words, the impossibility of self-excitation of motion. In view of the crucial presence of the small parameter ε in (3) and (4) we revise our earlier attempt [Strunin 2014] to guarantee this important property of the freely propagating seismic wave in the model. In [Strunin 2014] a popular form of (3) was considered where the small parameter ε was omitted. It was reasoned that the mechanical parameters, of which A_4 and A_6 are composed, should therefore assume special limited values, in order to guarantee that $\lambda < 0$. However, negativity of λ is simply ensured by the smallness of ε, which is the essential part of (3) as explicitly shown.

2. Basic equations of one-dimensional dynamics

2.1. ***Conservation of mass and momentum.*** For a one-dimensional case the momentum and mass balance equations are [Nikolaevskiy 1990]

$$\begin{gathered}
\frac{\partial}{\partial t}(1-m)\rho^{(s)}v + \frac{\partial}{\partial x}(1-m)\rho^{(s)}vv = \frac{\partial}{\partial x}\sigma^{(ef)} - (1-m)\frac{\partial p}{\partial x} - I, \\
\frac{\partial}{\partial t}m\rho^{(f)}u + \frac{\partial}{\partial x}m\rho^{(f)}uu = -m\frac{\partial p}{\partial x} + I, \\
\frac{\partial}{\partial t}(1-m)\rho^{(s)} + \frac{\partial}{\partial x}(1-m)\rho^{(s)}v = 0, \\
\frac{\partial}{\partial t}m\rho^{(f)} + \frac{\partial}{\partial x}m\rho^{(f)}u = 0,
\end{gathered} \tag{5}$$

where the subscripts s and f label the solid and gas-liquid mixture respectively, ρ, v, and u are the corresponding densities and mass velocities, m is the porosity, $\sigma^{(ef)}$ is the effective Terzaghi stress, p is the pore pressure, and I is the interfacial viscous force approximated by

$$I = \delta m(v-u), \quad \delta = \frac{\mu^{(f)} m}{\ell},$$

where $\mu^{(f)}$ is the gas-liquid mixture viscosity and ℓ is the intrinsic permeability.

2.2. ***Dynamics of bubbles.*** The equation of the dynamics of a bubble [Dunin et al. 2006] has the form

$$R\frac{\partial^2}{\partial t^2}R + \frac{3}{2}\Big(\frac{\partial}{\partial t}R\Big)^2 + \frac{4\mu}{\rho^{(L)}}\Big(\frac{1}{R}+\frac{m}{4\ell}R\Big)\frac{\partial}{\partial t}R = (p_g - p)/\rho^{(L)}, \tag{6}$$

where R is the bubble radius, p is the pressure in the liquid, $p_g = p_0(R_0/R)^\chi$ is the gas pressure inside the bubble (here $\chi = 3\zeta$, and ζ is the adiabatic exponent), $\rho^{(L)}$ is the density of the liquid without the bubbles, and μ is the viscosity of the liquid without the bubbles. The density equations for the solid and liquid without gas are

$$\rho^{(s)} = \rho_0^{(s)}(1-\beta^{(s)}\sigma) = \rho_0^{(s)}\Big[1+\beta^{(s)}p - \frac{\beta^{(s)}\sigma^{(ef)}}{1-m}\Big] \approx \rho_0^{(s)}[1+\beta^{(s)}p - \beta^{(s)}\sigma^{(ef)}], \tag{7}$$

$$\rho^{(L)} = \rho_0^{(L)}(1+\beta^{(L)}p). \tag{8}$$

The mean density of the gas-liquid mixture is

$$\rho^{(f)} = (1-\phi)\rho^{(L)} + \phi\rho^{(g)}, \tag{9}$$

where

$$\phi = \tfrac{4\pi}{3}R^3 n_0.$$

Here σ is the true stress, ϕ is the volume gas content and n_0 is the number density of the bubbles per unit volume. In (9) we can neglect the density of the gas $\rho^{(g)}$ due to the low gas content. The change in ϕ is due to the change in the bubble radius R. Then (9) becomes

$$\rho^{(f)} = \rho_0^{(L)}(1+\beta^{(L)}p)\big(1-\tfrac{4\pi}{3}R_0^3 n_0\big). \tag{10}$$

Similarly to [Dunin et al. 2006] we also assume that the pore pressure p is equal to the pressure in the liquid far from the bubble.

2.3. ***Stress-strain relation.*** In this section we derive the stress-strain relation for the viscoelastic medium based on the rheological Maxwell–Voigt model, which includes the gas bubble. The model includes two friction elements with viscosities μ_1 and μ_2, three elastic springs with the elastic moduli E_1, E_2, and E_3, and three oscillating masses M_1, M_2, and M_3. The total stress in denoted σ. We also denote the displacements of the elements of the model by e with respective subscripts as shown in Figure 2. Now we write the second Newton's law for the elements and the kinematic relations:

$$\begin{aligned} M_1\frac{d^2e_1}{dt^2} + M_2\frac{d^2e_2}{dt^2} &= \sigma - E_1e_1 - E_2e_2, \qquad & e = e_2 = e_1 + e_3 + e_4 + e_5, \\ M_3\frac{d^2e_3}{dt^2} &= E_1e_1 - E_3e_3, \qquad & E_3e_3 = \mu_2\frac{de_4}{dt} = \mu_1\frac{de_5}{dt}. \end{aligned} \tag{11}$$

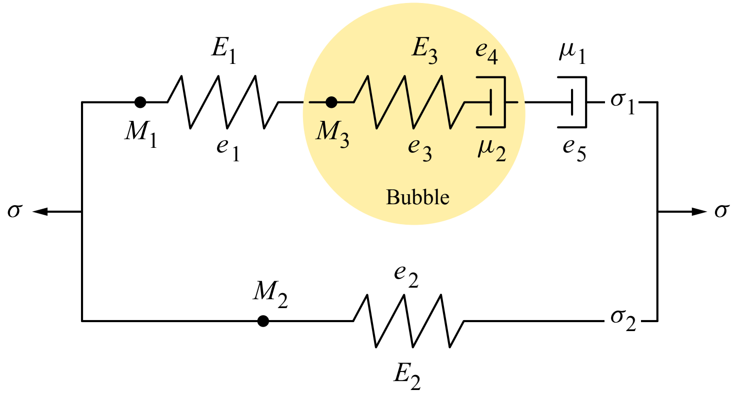

Figure 2. Rheological scheme including a gas bubble.

Equations (11) generate the following relation between the stress and strain:

$$\Big[E_1E_3\Big(\frac{1}{\mu_1}+\frac{1}{\mu_2}\Big)\Big]\sigma+(E_3+E_1)\frac{d\sigma}{dt}+M_3\frac{d^3\sigma}{dt^3}$$
$$=\Big[E_1E_2E_3\Big(\frac{1}{\mu_1}+\frac{1}{\mu_2}\Big)\Big]e+[(E_2+E_1)E_3+E_1E_2]\frac{de}{dt}+\Big[E_1E_3M_2\Big(\frac{1}{\mu_1}+\frac{1}{\mu_2}\Big)\Big]\frac{d^2e}{dt^2}$$
$$+[((E_2+E_1)M_3+(E_3+E_1)M_2)+E_3M_1]\frac{d^3e}{dt^3}+[(M_2+M_1)M_3]\frac{d^5e}{dt^5}. \quad (12)$$

Generalizing (12) using a similar approach to [Nikolaevskiy 2008], we get

$$\sigma^{(ef)}+\eta\sum_{q=1,3}b_q\frac{D^q\sigma^{(ef)}}{Dt^q}=E_2e+\beta^{(s)}k_bp+\eta\sum_{q=1,2,3,5}a_q\frac{D^qe}{Dt^q}, \quad (13)$$

where $\sigma^{(ef)}$ is the effective stress, $\eta=[E_1E_3(1/\mu_1+1/\mu_2)]^{-1}$, k_b is the bulk elastic module of the porous matrix, and the coefficients a_q and b_q are expressed as

$$a_1=[(E_2+E_1)E_3+E_1E_2],\quad a_2=M_2,\quad a_3=[(E_2+E_1)M_3+(E_3+E_1)M_2+E_3M_1],$$
$$a_5=[(M_2+M_1)M_3],\quad b_1=(E_3+E_1),\quad b_3=M_3.$$

Finally, we add the closing relation between the deformation e and the velocity v of the solid:

$$\frac{De}{Dt}\equiv\frac{\partial e}{\partial t}+v\frac{\partial e}{\partial x}=\frac{\partial v}{\partial x}. \quad (14)$$

3. Elastic waves in saturated media including gas bubbles

Following the approach of Nikolaevskiy [2008], we consider the P1-wave in a porous media under the full saturation. Accordingly we assume that the mass velocities v and u have the same sign:

$$v=u+O(\varepsilon v), \quad (15)$$

where ε is the small parameter. The Darcy force has the order as shown:

$$I = \varepsilon^{\gamma}\delta m(v-u) = \varepsilon^{\gamma}\delta m v, \quad \delta = m\mu/k = O(1). \tag{16}$$

Describing a weakly nonlinear wave we use the running coordinate system with simultaneous scale change:

$$\xi = \varepsilon(x-ct), \quad \tau = \tfrac{1}{2}\varepsilon^2 t, \quad \frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = \varepsilon\Big(\tfrac{1}{2}\varepsilon\frac{\partial}{\partial \tau} - c\frac{\partial}{\partial \xi}\Big). \tag{17}$$

Thus, the constitutive law (13) transforms into the form

$$\sigma^{(ef)} + \eta\sum_{q=1,3} b_q\varepsilon^q\Big(\tfrac{1}{2}\varepsilon\frac{\partial}{\partial \tau} + (v-c)\frac{\partial}{\partial \xi}\Big)^q\sigma^{(ef)}$$
$$= E_2 e + \beta^{(s)}k_b p + \eta\sum_{q=1,2,3,5} a_q\varepsilon^q\Big(\tfrac{1}{2}\varepsilon\frac{\partial}{\partial \tau} + (v-c)\frac{\partial}{\partial \xi}\Big)^q e. \tag{18}$$

Now, we seek the unknown functions as power series:

$$\begin{aligned} v &= \varepsilon v_1 + \varepsilon^2 v_2 + \cdots, & u &= \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \\ \sigma^{ef} &= \sigma_0^{(ef)} + \varepsilon\sigma_1^{(ef)} + \varepsilon^2\sigma_2^{(ef)} + \cdots, & p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots, \\ m &= m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \cdots, & e &= e_0 + \varepsilon e_1 + \varepsilon^2 e_2 + \cdots, \\ \phi &= \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots, & R &= R_0(1 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots). \end{aligned} \tag{19}$$

3.1. ***The first approximation.*** Using series (19) in the mass and momentum equations (5), (6), and the stress-strain relation (13) and collecting linear terms $\sim \varepsilon$, we eventually arrive at the system

$$(1-\beta^{(s)}E_2)v_1 + (B - k_b\beta^{(s)}\beta^{(s)})p_0\chi R_1 c = 0, \tag{20}$$
$$(E_2 - \rho_0 c^2)v_1 - (A - k_b\beta^{(s)})p_0\chi R_1 c = 0. \tag{21}$$

We address the reader to Appendix A for details of derivation of (20) and (21).

Equations (20) and (21) must coincide, therefore,

$$\begin{vmatrix} (1-\beta^{(s)}E_2) & (B - k_b\beta^s\beta^s)p_0\chi \\ (E_2 - \rho_0 c^2) & -(A - k_b\beta^{(s)})p_0\chi \end{vmatrix} = 0. \tag{22}$$

Equation (22) gives the velocity of the wave,

$$c^2 = \frac{(A - k_b\beta^{(s)})Z_1 + E_2}{\rho_0}, \tag{23}$$

where

$$Z_1 = \frac{1-\beta^{(s)}E_2}{B - k_b\beta^{(s)}\beta^{(s)}}.$$

Thus, all the variables are expressed through any one selected variable; for example, the velocity v_1:

$$e_1 = -\frac{v_1}{c}, \quad \sigma_1^{(ef)} = -(E_2 - k_b\beta^{(s)}Z_1)\frac{v_1}{c}, \quad p_1 = Z_1\frac{v_1}{c}, \quad R_1 = -\frac{Z_1}{p_0\chi}\frac{v_1}{c},$$

$$m_1 = [((1-m_0) - k_b\beta^{(s)})\beta^{(s)}Z_1 + \beta^{(s)}E_2 - (1-m_0)]\frac{v_1}{c},$$

$$\rho_1^{(f)} = \rho_0^{(L)}Z_1\left(\beta^{(L)}\kappa_1 + \frac{\kappa_2 4\pi n_0 R_0^3}{p_0\chi}\right)\frac{v_1}{c},$$

$$\rho_1^{(s)} = \rho_0^{(s)}\beta^{(s)}[Z_1(1-k_b\beta^{(s)}) + E_2]\frac{v_1}{c}.$$

3.2. ***The second approximation.*** In the second approximation, we collect quadratic terms $\sim \varepsilon^2$ in the mass and momentum equations (5), (6), and stress-strain relation (13) and come to the following system:

$$\frac{\partial}{\partial\xi}[(E_2 - \rho_0 c^2)v_2 - (1-\beta^{(s)}k_b)p_0\chi R_2 c] = E_2 F - c\Sigma + c\frac{\partial T}{\partial\xi} - c(1-\beta^{(s)}k_b)\frac{\partial\Gamma}{\partial\xi}, \tag{24}$$

$$\frac{\partial}{\partial\xi}\left[(1 - E_2\beta^{(s)})v_2 - \left(\omega_1 p_0\chi - \frac{4\pi m_0 n_0 R_0^3}{\kappa_1}\right)R_2 c\right] = \Lambda - \beta^{(s)}E_2F - c\beta^{(s)}\frac{\partial T}{\partial\xi} - c\omega_1\frac{\partial\Gamma}{\partial\xi} + c\omega_2\frac{\partial R_1^2}{\partial\xi}. \tag{25}$$

The details of the derivation of (24) and (25) are provided in Appendix B. The determinant of the left-hand side of the system of equations (24) and (25) coincides with the determinant of (22), which equals zero. A nonzero solution for v_2 exists only if the following compatibility condition takes place:

$$\begin{vmatrix} (E_2 - \rho_0 c^2) & \frac{\partial}{\partial\xi}\left(E_2F - c\Sigma + c\frac{\partial T}{\partial\xi} - (1-\beta^{(s)}k_b)c\frac{\partial\Gamma}{\partial\xi}\right) \\ (1 - E_2\beta^{(s)}) & \frac{\partial}{\partial\xi}\left[\Lambda - \beta^{(s)}E_2F - c\beta^{(s)}\frac{\partial T}{\partial\xi} - c\omega_1\frac{\partial\Gamma}{\partial\xi} + c\omega_2\frac{\partial R_1^2}{\partial\xi}\right] \end{vmatrix} = 0 \tag{26}$$

(see Appendix C). This is the evolution equation with respect to $v \cong v_1$:

$$cM\frac{\partial\Gamma}{\partial\xi} - cN\frac{\partial T}{\partial\xi} + c\omega_2\psi\frac{\partial R_1^2}{\partial\xi} + \Lambda\psi + c\Sigma(1 - E_2\beta^{(s)}) - E_2FN = 0, \tag{27}$$

where

$$\psi = (E_2 - \rho_0c^2), \quad M = (1-\beta^{(s)}k_b)(1-E_2\beta^{(s)}) - \omega_1\psi, \quad N = (1-\beta^{(s)}\rho_0c^2).$$

Now, we rewrite (27) in terms of v and regroup:

$$\frac{1}{2}\left[Y_1 + \psi\left((1-\kappa_1\kappa_2)\bar{m}_1 - E_2\beta^{(s)} - \left(Y_2 + \frac{4\pi n_0R_0^3}{p_0\chi}\right)Z_1\right)\right]\frac{\partial v}{\partial\tau} - \left[N\eta c^2(a_1 - b_1(E_2 - k_b\beta^{(s)}Z_1)) + \frac{MZ_1\mu c^2}{p_0\chi}\left(4 + \frac{m_0R_0^2}{\ell}\right)\right]\frac{\partial^2 v}{\partial\xi^2} + \varepsilon N\eta c^3 a_2\frac{\partial^3 v}{\partial\xi^3} - \varepsilon^2 N\eta c^4[a_3 - b_3(E_2 - k_b\beta^{(s)}Z_1)]\frac{\partial^4 v}{\partial\xi^4} - \varepsilon^4 N\eta a_5 c^6\frac{\partial^6 v}{\partial\xi^6} - [\zeta_1 + \zeta_2]\frac{\partial vv}{\partial\xi} = 0, \tag{28}$$

where

$$\bar{m}_1=((1-m_0)-k_b\beta^{(s)})\beta^{(s)}Z_1+\beta^{(s)}E_2-(1-m_0),$$
$$Y_1=(E_2N+c^2\rho_0(1-E_2\beta^{(s)})),\quad Y_2=m_0(\kappa_1\beta^{(L)}-\beta^{(s)})+\beta^{(s)}(1-\beta^{(s)}k_b),$$
$$\zeta_1=\psi\biggl(\bar{m}_1-(1-m_0)\beta^{(s)}Z_1+\beta^{(s)}(E_2-k_b\beta^{(s)}Z_1)(1-\bar{m}_1)-\beta^{(s)}Z_1\bar{m}_1 \\ +\kappa_1\beta^{(L)}Z_1\bar{m}_1-\kappa_1\kappa_2+4\pi n_0\kappa_2R_0^3\frac{Z_1}{p_0\chi}(\bar{m}_1-m_0)-m_0\kappa_1\beta^{(L)}Z_1+\omega_2\frac{Z_1^2}{(p_0\chi)^2}\biggr),$$
$$\zeta_2=c^2(1-E_2\beta^{(s)})\biggl(\rho_0-\rho_0\kappa_1\beta Z_1-\rho_0^{(s)}\beta^{(s)}(E_2-k_b\beta^{(s)}Z_1)-m_0\kappa_2\rho^{(L)}\frac{Z_1}{p_0\chi}+\bar{m}_1(\rho^{(s)}-\kappa_1\kappa_2\rho^{(L)})\biggr) \\ +\frac{M(\chi+1)}{2p_0\chi}Z_1^2+E_2N.$$

In short, the evolution equation (28) can be written as

$$C_1\frac{\partial v}{\partial\tau}-C_2\frac{\partial^2 v}{\partial\xi^2}+\varepsilon C_3\frac{\partial^3 v}{\partial\xi^3}-\varepsilon^2C_4\frac{\partial^4 v}{\partial\xi^4}-\varepsilon^4C_6\frac{\partial^6 v}{\partial\xi^6}-\zeta\frac{\partial vv}{\partial\xi}=0, \tag{29}$$

where

$$C_1=\frac{1}{2}\biggl[Y_1+\psi\biggl((1-\kappa_1\kappa_2)\bar{m}_1-E_2\beta^{(s)}-\biggl(Y_2+\frac{4\pi n_0R_0^3}{p_0\chi}\biggr)Z_1\biggr)\biggr],$$
$$C_2=\biggl[N\eta c^2(a_1-b_1(E_2-k_b\beta^{(s)}Z_1))+\frac{MZ_1\mu c^2}{p_0\chi}\biggl(4+\frac{m_0R_0^2}{\ell}\biggr)\biggr],\quad C_3=N\eta c^3a_2,$$
$$C_4=N\eta c^4(a_3-b_3(E_2-k_b\beta^{(s)}Z_1)),\quad C_6=N\eta a_5c^6,\quad \zeta=\zeta_1+\zeta_2.$$

4. Elastic waves in saturated media without gas bubbles

Our goal is to study the effect of inclusion of gas bubbles into the rheological scheme on the elastic wave decay. For this purpose we will remove the bubble-representing segment from Figure 2 and rederive the wave equation (note that our rheological scheme consists of only two branches: one for the solid and the other for the bubble-fluid mixture). This differs from the original Nikolaevskiy scheme, which includes three parallel branches [Nikolaevskiy 1989; 2008].

4.1. ***Stress-strain relation.*** By removing the bubble segment from the rheological scheme, we get Figure 3. The second Newton's law for the scheme and the kinematic relation are

$$M_1\frac{d^2e_1}{dt^2}+M_2\frac{d^2e}{dt^2}=\sigma-E_1e_1-E_2e,\quad e=e_1+e_3,\quad E_1e_1=\mu_1\frac{de_3}{dt}. \tag{30}$$

Equations (30) lead to the well-known stress-strain relation [Nikolaevskiy 1985]

$$\sigma+\theta\frac{d\sigma}{dt}=E_2e+(E_1+E_2)\theta\frac{de}{dt}+M_2\frac{d^2e}{dt^2}+(M_1+M_2)\theta\frac{d^3e}{dt^3}, \tag{31}$$

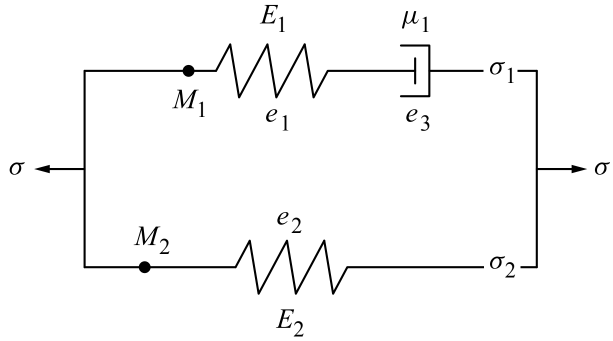

Figure 3. Rheological scheme without gas bubble.

where $\theta = \mu_1/E_1$. Hence, the constitutive law (31) will be written as

$$\sigma^{(ef)} + b_1\varepsilon\Big(\tfrac{1}{2}\varepsilon\frac{\partial}{\partial\tau} + (v-c)\frac{\partial}{\partial\xi}\Big)\sigma^{(ef)} = E_2 e + \beta^{(s)}k_b p + \sum_{q=1}^{3} a_q\varepsilon^q\Big(\tfrac{1}{2}\varepsilon\frac{\partial}{\partial\tau} + (v-c)\frac{\partial}{\partial\xi}\Big)^q e, \tag{32}$$

where

$$a_1 = (E_1+E_2)\theta, \quad a_2 = M_2, \quad a_3 = (M_1+M_2)\theta, \quad b_1 = \theta.$$

4.2. ***First approximation of the system without gas bubbles.*** Following the approach of Section 3.1, in the first approximation for the system without bubbles we arrive at the system

$$(1-\beta^{(s)}E_2)v_1 = (\beta - \beta^{(s)}\beta^{(s)}k_b)cp_1, \tag{33}$$

$$(\rho_0 c^2 - E_2)v_1 = (1-\beta^{(s)}k_b)cp_1. \tag{34}$$

Appendix D provides the derivation details of these equations. The determinant of the system of equations (33) and (34) gives the wave velocity c:

$$\begin{vmatrix} (1-\beta^{(s)}E_2) & -(\beta-\beta^{(s)}\beta^{(s)}k_b) \\ (\rho_0c^2 - E_2) & -(1-\beta^{(s)}k_b) \end{vmatrix} = 0. \tag{35}$$

Thus,

$$c^2 = \frac{E_2 + Z_2(1-\beta^{(s)}k_b)}{\rho_0}, \tag{36}$$

where

$$Z_2 = \frac{(1-\beta^{(s)}E_2)}{(\beta-\beta^{(s)}\beta^{(s)}k_b)}.$$

Again we can express all the variables through the velocity v_1:

$$e_1 = -\frac{v_1}{c}, \quad p_1 = Z_2\frac{v_1}{c}, \quad \rho_1^{(s)} = \rho^{(s)}\beta^{(s)}[Z_2(1-\beta^{(s)}k_b) + E_2]\frac{v_1}{c},$$
$$\rho_1^{(L)} = \rho^{(L)}\beta^{(L)}Z_2\frac{v_1}{c}, \quad m_1 = [(1-m_0)\beta^{(s)}Z_2 + \beta^{(s)}(E_2 - \beta^{(s)}k_bZ_2) - (1-m_0)]\frac{v_1}{c},$$
$$\sigma_1^{(ef)} = -(E_2 - \beta^{(s)}k_bZ_2)\frac{v_1}{c}.$$

4.3. ***Second approximation for the system without gas bubbles.*** Following the approach of Section 3.2, in the second approximation we arrive at the system

$$\frac{\partial}{\partial \xi}[(E_2-\rho_0 c^2)v_2+(1-\beta^{(s)}k_b)cp_2]=E_2F-c\Sigma+c\frac{\partial T}{\partial \xi}, \tag{37}$$

$$\frac{\partial}{\partial \xi}[(1-E_2\beta^{(s)})v_2-(\beta-\beta^{(s)}\beta^{(s)})cp_2]=\Lambda-\beta^{(s)}E_2F-c\beta^{(s)}\frac{\partial T}{\partial \xi}. \tag{38}$$

See Appendix E for the derivation details. In analogy to (26), the compatibility condition for the system of equations (37), (38) has the form

$$\begin{vmatrix} (E_2-\rho_0c^2) & \frac{\partial}{\partial \xi}\Big(E_2F-c\Sigma+c\frac{\partial T}{\partial \xi}\Big) \\ (1-E_2\beta^{(s)}) & \frac{\partial}{\partial \xi}\Big(\Lambda-\beta^{(s)}E_2F-c\beta^{(s)}\frac{\partial T}{\partial \xi}\Big) \end{vmatrix}=0. \tag{39}$$

Then the evolution equation for $v\cong v_1$ is

$$\Lambda\psi-cN\frac{\partial T}{\partial \xi}+c\Sigma(1-E_2\beta^{(s)})-E_2FN=0. \tag{40}$$

Rearranging, we arrive at

$$c^2\rho_0(1-E_2\beta^{(s)})\frac{\partial v}{\partial \tau}-Nc^2(a_1-b_1(E_2-\beta^{(s)}k_bZ_2))\frac{\partial^2 v}{\partial \xi^2} \\ +\varepsilon Na_2c^3\frac{\partial^3 v}{\partial \xi^3}-\varepsilon^2Na_3c^4\frac{\partial^4 v}{\partial \xi^4}+[G_1+G_2]\frac{\partial vv}{\partial \xi}=0, \tag{41}$$

where

$$\hat{m}_1=(1-m_0)\beta^{(s)}Z_2+\beta^{(s)}(E_2-\beta^{(s)}k_bZ_2)-(1-m_0),$$

$$G_1=\psi\bigg(-((1-m_0)+\hat{m}_1)\beta^{(s)}Z_2-\beta^{(s)}(E_2-\beta^{(s)}k_bZ_2)\Big(1+\frac{\hat{m}_1}{(1-m_0)}\Big)+\beta^{(L)}Z_2\hat{m}_1-m_0\beta^{(L)}Z_2\bigg),$$

$$G_2=c^2(1-E_2\beta^{(s)})(\rho_0-\rho_0\beta Z_2-\rho^{(s)}\beta^{(s)}(E_2-\beta^{(s)}k_bZ_2)+\hat{m}_1(\rho^{(s)}-\rho^{(L)}))+E_2(1-\beta^{(s)}\rho_0c^2).$$

Finally, we rewrite the evolution equation (41) as

$$D_1\frac{\partial v}{\partial \tau}-D_2\frac{\partial^2 v}{\partial \xi^2}+\varepsilon D_3\frac{\partial^3 v}{\partial \xi^3}-\varepsilon^2D_4\frac{\partial^4 v}{\partial \xi^4}+G\frac{\partial vv}{\partial \xi}=0, \tag{42}$$

where

$$D_1=c^2\rho_0(1-E_2\beta^{(s)}),\quad D_2=Nc^2(a_1-b_1(E_2-\beta^{(s)}k_bZ_2)),$$
$$D_3=Na_2c^3,\quad D_4=Na_3c^4,\quad G=G_1+G_2.$$

We remark that for the wave propagating to the left, that is, with $\xi=\varepsilon(x+ct)$, one obtains (as we checked) the same equation (42).

5. Linearized model

In this section we investigate the linearized versions of the model with and without the bubbles, that is, (29) and (42). Our particular interest is in its dissipative part responsible for decay of the wave.

5.1. ***Evaluation of the parameters and the wave velocity.*** From [Dunin et al. 2006; Nikolaevskiy 1985; 2016; Sutton and Biblarz 2016; Carcione 1998; Mikhailov 2010; Smeulders 2005], the values of the parameters are: the density $\rho_0^{(L)} = 1000\,\text{kg/m}^3$ for water, $\rho^{(g)} = 2\,\text{kg/m}^3$ for gas, $\rho_0^{(s)} = 2500\,\text{kg/m}^3$ for solid; porosity $m_0 = 0.25$; bulk modulus $k_b = 1.7 \times 10^9$ Pa for the matrix, $k_b = 30 \times 10^9$ Pa for the solid; compressibility $\beta^{(L)} = 2 \times 10^{-9}\,\text{Pa}^{-1}$ for water, $\beta^{(L)} = 2.4 \times 10^{-6}\,\text{Pa}^{-1}$ for gas, $\beta^{(s)} = 2 \times 10^{-10}\,\text{Pa}^{-1}$ for solid; steady pressure $p_0 = 10^5$ Pa; bubble radius $R_0 = 5 \times 10^{-5}$ m; volume gas content $\phi_0 = 10^{-3}$; viscosity $\mu_1 = 10^{-3}$ Pa·s for water, $\mu_2 = 2 \times 10^{-5}$ Pa·s for gas; adiabatic exponent $\zeta = 1.4$, and permeability $\ell = 2 \times 10^{-11}\,\text{m}^2$. Using the data from [Nikolaevskiy and Strunin 2012; Nikolaevskiy 1985; 2016; Nikolaevskiy and Stepanova 2005], the values of the parameters of the rheological scheme in Figure 2 are

$$M_1 = \rho^{(L)} L_s^2 = 10^{-2}\,\text{kg/m}, \quad M_2 = \rho^{(s)} L_s^2 = 0.02\,\text{kg/m}, \quad M_3 = \rho^{(g)} L_s^2 = 2 \times 10^{-6}\,\text{kg/m},$$

and

$$\text{(a)} \qquad E_1 = 1/\beta^{(L)} = 4 \times 10^5\,\text{Pa}, \quad E_2 = c^2 \rho_0 = 2 \times 10^7\,\text{Pa}, \quad E_3 = 3\chi p_0 = 4 \times 10^7\,\text{Pa},$$

where we used, just for the purpose of evaluating of E_i and M_i, the typical velocity $c \sim 100$ m/s and the linear size of the oscillator $L_s = 0.3$ cm from [Nikolaevskiy 1985; Vilchinska et al. 1985]. Note that the above values of E_i are known only approximately. With this in mind, in the present study we also explore other the values of E_i that are considerably different from variant (a):

$$\text{(b)} \qquad E_1 = 5 \times 10^5\,\text{Pa}, \quad E_2 = 5 \times 10^8\,\text{Pa}, \quad E_3 = 5 \times 10^4\,\text{Pa},$$

$$\text{(c)} \qquad E_1 = 6 \times 10^5\,\text{Pa}, \quad E_2 = 2 \times 10^9\,\text{Pa}, \quad E_3 = 5 \times 10^3\,\text{Pa}.$$

The reason for this choice is that the two different rheological schemes that we use (for the wave with and without the bubbles) give close values of λ when we put $R_0 = 0$ and $n_0 = 0$.

Now we apply the formulas for the wave velocity (23) and (36) to show that they give reasonable orders of magnitude. For variant (a) formula (23) for the wave with the bubbles gives $c \approx 577$ m/s, and formula (36) for the wave without the bubbles gives $c \approx 2100$ m/s. For variant (b) the wave with the bubbles has the velocity $c \approx 726$ m/s and the wave without the bubbles the velocity $c \approx 2000$ m/s. For variant (c) the wave with the bubbles has $c \approx 1100$ m/s, and the wave without the bubbles $c \approx 1800$ m/s. This illustrates, in line with the previous studies, that the bubbles may result in a considerable change of the wave velocity. However, our main interest in this study is the dissipation rate of the wave, which we explore in the next section.

5.2. ***Dispersion (dissipation) relation.*** Analyzing the linearized model, we are interested in the influence of the bubbles on the wave dissipation. This effect is controlled by the even derivatives, so we truncate the linearized equations (29) to the form

$$\frac{\partial v}{\partial \tau} = \frac{C_2}{C_1} \frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 \frac{C_4}{C_1} \frac{\partial^4 v}{\partial \xi^4} + \varepsilon^4 \frac{C_6}{C_1} \frac{\partial^6 v}{\partial \xi^6}. \tag{43}$$

For the Fourier modes $v \sim \exp(\lambda t + ikx)$, we get the dispersion (or dissipation) relation

$$\lambda(k) = -\frac{C_2}{C_1} k^2 + \varepsilon^2 \frac{C_4}{C_1} k^4 - \varepsilon^4 \frac{C_6}{C_1} k^6, \tag{44}$$

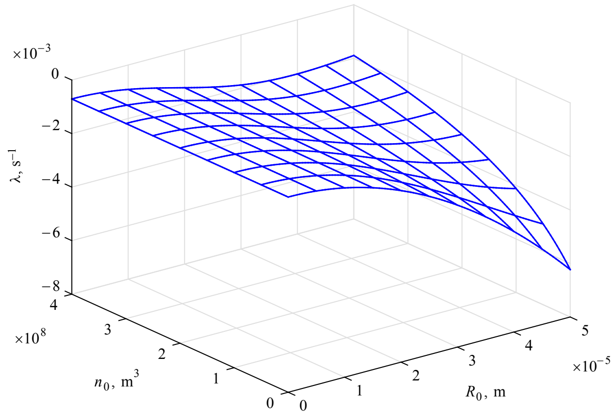

Figure 4. The decay rate by formula (44) for variant (a), $k_* = 0.25\,\mathrm{m}^{-1}$.

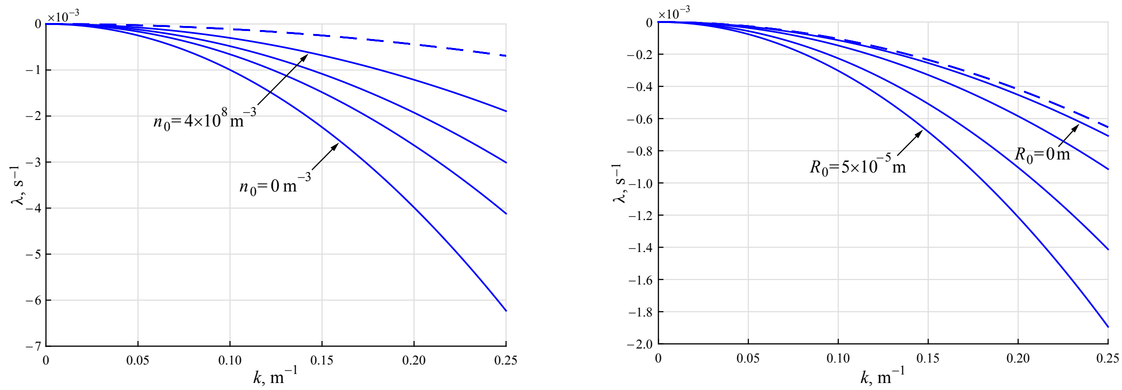

Figure 5. The decay rate by formulas (44) and (46) for variant (a). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

where λ is the decay rate and k is the wave number. For the model without the bubbles, the linearized form of (42) is

$$\frac{\partial v}{\partial \tau} = \frac{D_2}{D_1}\frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 \frac{D_4}{D_1}\frac{\partial^4 v}{\partial \xi^4} \tag{45}$$

(we again consider only even derivatives). Accordingly, the dispersion relation is

$$\lambda(k) = -\frac{D_2}{D_1}k^2 + \varepsilon^2 \frac{D_4}{D_1}k^4. \tag{46}$$

Figure 4 shows the decay rate by formula (44) at fixed $k_* = 0.25\,\mathrm{m}^{-1}$ [Nikolaevskiy 1989] against R_0 and n_0. See that the increase in R_0 significantly affects the decay rate and makes its absolute value larger due to the bubbles increasing their role through the pressure $p_1 = -p_0 \chi R_1$. As for n_0, one should disregard the region of small n_0 in Figure 4 since the equations of continuum mechanics in the form adopted in the model become invalid when there are too few bubbles. This is because one can no longer assume that every fluid particle contains its own bubble (as suggested by (6)) because this would imply that the fluid particles are no longer small and, hence, the continuum mechanics description fails.

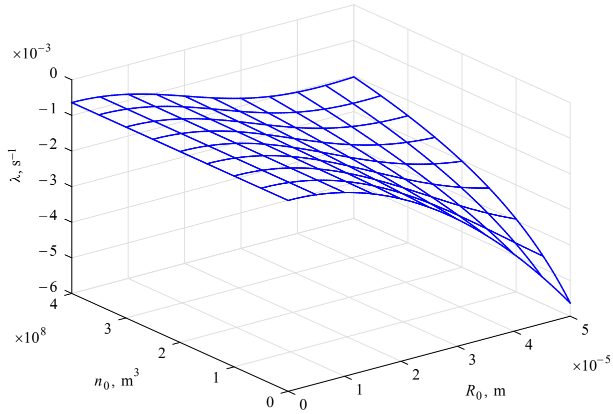

Figure 6. The decay rate by formula (44) for variant (b), $k_* = 0.25\,\mathrm{m}^{-1}$.

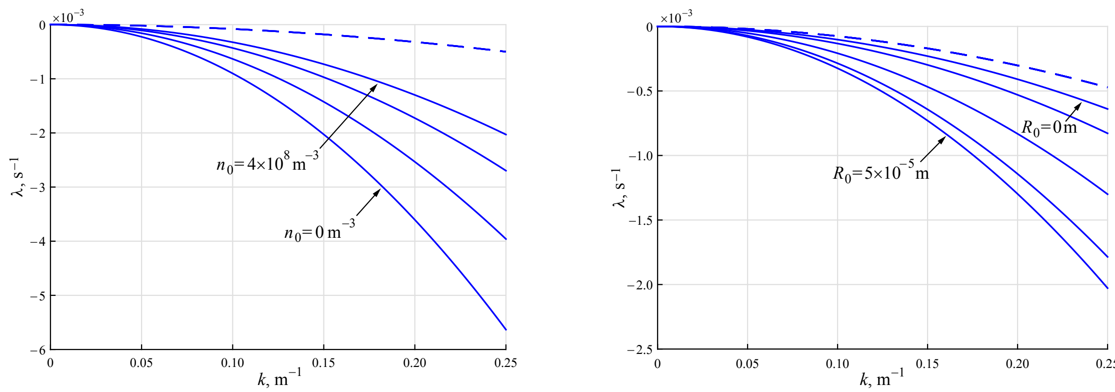

Figure 7. The decay rate by formulas (44) and (46) for variant (b). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

Figure 5 compares the dispersion curves of the wave with the bubbles and the wave without the bubbles. The dashed line describes the case without the bubbles and the solid lines correspond to the wave with the bubbles. The figure on the left is for varying n_0 and fixed R_0; the figure on the right is for varying R_0 and fixed n_0. The decay rate depends on the number and radius of the bubbles. We note that this result agrees with the conception discussed in [Strunin 2014; Strunin and Ali 2016] about the passive nature of the freely propagating elastic wave. Similar results are obtained for variants (b) and (c) as shown in Figures 6–9.

For a different $k_* = 0.52\,\mathrm{m}^{-1}$ [Beresnev and Nikolaevskiy 1993], the results are similar; see Figures 10 and 11.

6. Conclusions

We studied the effect of rheology with and without gas bubbles and of the bubble dynamics on the dissipation of elastic waves in porous solids. The Frenkel–Biot waves of P1 type are analyzed in the

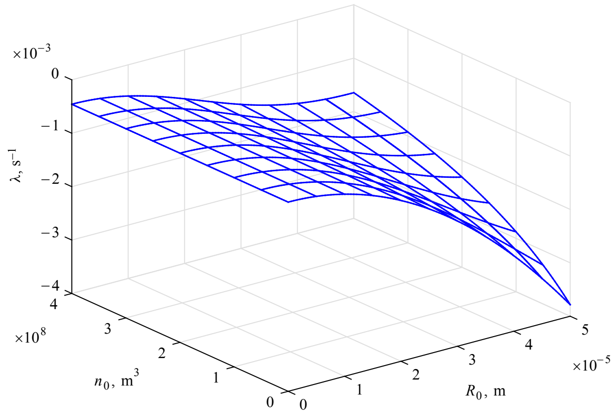

Figure 8. The decay rate by formula (44) for variant (c), $k_* = 0.25\,\mathrm{m}^{-1}$.

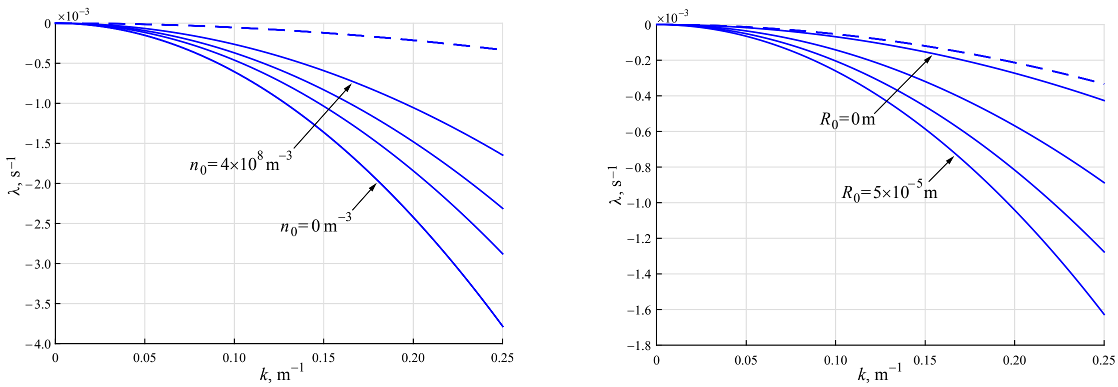

Figure 9. The decay rate by formulas (44) and (46) for variant (c). Left: n_0 varies, $R_0 = 5\times10^{-5}$. Right: R_0 varies, $n_0 = 4\times10^{8}$.

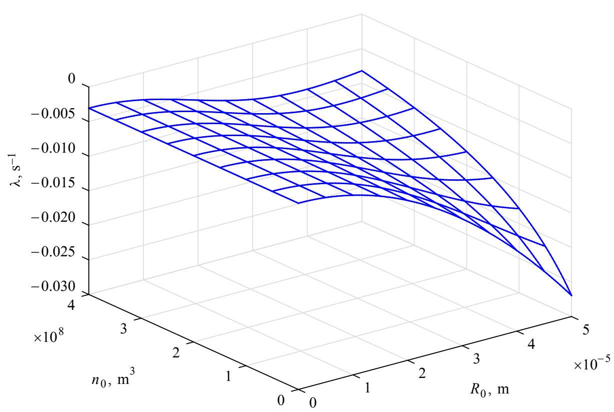

Figure 10. The decay rate by formula (44) for variant (a), $k_* = 0.52\,\mathrm{m}^{-1}$.

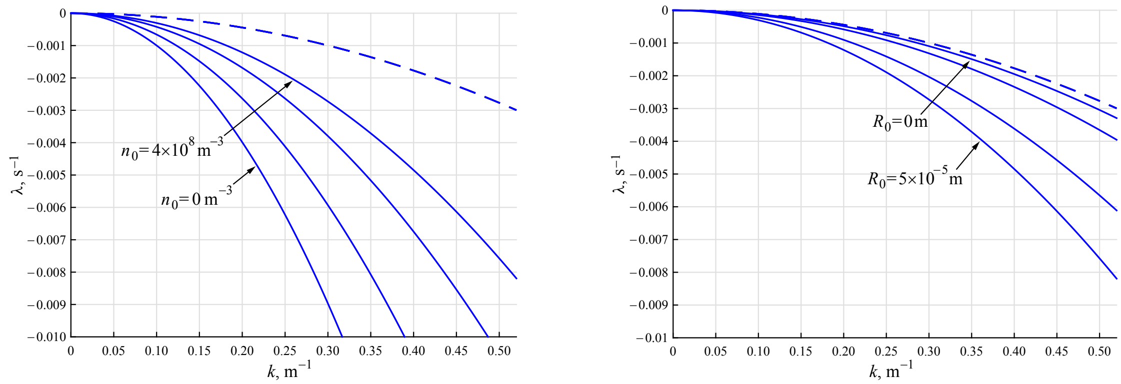

Figure 11. The decay rate by formulas (44) and (46) for variant (a). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

fluid-saturated environment. Using the three-segment rheological scheme (with the bubbles) and two-segment scheme (without the bubbles), we derived the Nikolaevskiy-type equations for the velocity of the solid matrix. The linearized versions of the equations are compared in terms of the decay rate $\lambda(k)$ of the Fourier modes. For the both cases — with and without the bubbles — the $\lambda(k)$-curve lies entirely below the zero. We found out that $|\lambda(k)|$ increases with the increase of the radius of the bubbles but decreases with the increase of the number of the bubbles.

List of symbols

$\beta^{(s)}$	:	compressibility of solid (Pa^{-1})	M_i	:	masses (kg/m)
$\beta^{(L)}$	:	compressibility of water and gas (Pa^{-1})	k_b	:	bulk modulus (Pa)
$\rho_0^{(s)}$	:	density of solid (kg/m^3)	p	:	pressure (Pa)
$\rho_0^{(L)}$	:	density of water (kg/m^3)	$\sigma^{(ef)}$	:	effective stress (Pa)
$\rho_0^{(g)}$	:	density of gas (kg/m^3)	ϕ	:	volume gas content
R_0	:	bubble radius (m)	m	:	porosity
ε	:	small parameter	ζ	:	adiabatic exponent
n_0	:	number of bubbles (m^{-3})	k	:	wave number (m^{-1})
μ	:	viscosity (Pa·s)	c	:	wave velocity (m/s)
ℓ	:	permeability (m^2)	v	:	particle velocity (m/s)
E_i	:	elastic moduli (Pa)	λ	:	decay rate (s^{-1})

Appendix A

Using equations (19), we collect the linear terms $\sim \varepsilon$ in system (5):

$$-(1-m_0)\rho_0^{(s)} c \frac{\partial v_1}{\partial \xi} = \frac{\partial \sigma_1^{(ef)}}{\partial \xi} - (1-m_0)\frac{\partial p_1}{\partial \xi},$$

$$-m_0 \rho_0^{(f)} c \frac{\partial u_1}{\partial \xi} = -m_0 \frac{\partial p_1}{\partial \xi},$$

$$\rho_0^{(s)} c \frac{\partial m_1}{\partial \xi} - (1-m_0) c \frac{\partial \rho_1^{(s)}}{\partial \xi} + (1-m_0)\rho_0^{(s)} \frac{\partial v_1}{\partial \xi} = -\tfrac{1}{2}(1-m_0)\frac{\partial \rho_0^{(s)}}{\partial \tau},$$

$$-m_0 c \frac{\partial \rho_1^{(f)}}{\partial \xi} - \rho_0^{(f)} c \frac{\partial m_1}{\partial \xi} + m_0 \rho_0^{(f)} \frac{\partial u_1}{\partial \xi} = -\tfrac{1}{2} m_0 \frac{\partial \rho_0^{(f)}}{\partial \tau}. \tag{47}$$

The system (47) gives the integrals

$$(1-m_0)\rho_0^{(s)} c v_1 = -\sigma_1^{(ef)} + (1-m_0) p_1, \qquad m_0 \rho_0^{(f)} c u_1 = m_0 p_1,$$
$$(1-m_0)\rho_0^{(s)} v_1 = ((1-m_0)\rho_1^{(s)} - \rho_0^{(s)} m_1) c, \quad m_0 \rho_0^{(f)} u_1 = (\rho_0^{(f)} m_1 + m_0 \rho_1^{(f)}) c. \tag{48}$$

According to (7) and (10) the terms $\sim \varepsilon$ in the density series are

$$\rho_1^{(s)} = \rho_0^{(s)} \left(\beta^{(s)} p_1 - \frac{\beta^{(s)} \sigma_1^{(ef)}}{(1-m_0)} \right), \quad \rho_1^{(f)} = \rho_0^{(L)} (\beta^{(L)} \kappa_1 p_1 - 4\pi n_0 \kappa_2 R_0^3 R_1), \tag{49}$$

and also

$$\rho_0^{(f)} = \kappa_1 \kappa_2 \rho_0^{(L)}, \tag{50}$$

where

$$\kappa_1 = 1 - \tfrac{4\pi}{3} R_0^3 n_0, \quad \kappa_2 = 1 + \beta^{(L)} p.$$

Inserting (49) and (50) into the last two equations in (48) (mass equations), we get

$$(1-m_0) v_1 = [(1-m_0)\beta^{(s)} p_1 - \beta^{(s)} \sigma_1^{(ef)} - m_1] c, \tag{51}$$

$$m_0 u_1 = \left[m_1 + \frac{m_0 \beta^{(L)} p_1}{\kappa_2} - \frac{4\pi n_0 m_0 R_0^3 R_1}{\kappa_1} \right] c. \tag{52}$$

The combination of (51) and (52) gives

$$(1-m_0) v_1 + m_0 u_1 = \left[\frac{(\beta + (1-m_0)\beta^{(s)} \beta^{(L)} p_0) p_1}{\kappa_2} - \beta^{(s)} \sigma_1^{(ef)} - \frac{4\pi n_0 m_0 R_0^3 R_1}{\kappa_1} \right] c, \tag{53}$$

where $\beta = (1-m_0)\beta^{(s)} + m_0 \beta^{(L)}$.

The condition (15) means $v_1 = u_1$, therefore (53) becomes

$$v_1 = \left[\frac{(\beta + (1-m_0)\beta^{(s)} \beta^{(L)} p_0) p_1}{\kappa_2} - \beta^{(s)} \sigma_1^{(ef)} - \frac{4\pi n_0 m_0 R_0^3 R_1}{\kappa_1} \right] c. \tag{54}$$

Due to the conditions $v_1 = u_1$, $\rho_0 = (1-m_0)\rho_0^{(s)} + m_0 \rho_0^{(L)}$ and using (50), the first two of the momentum equations (48) give

$$\rho_0 c v_1 = -\sigma_1^{(ef)} + A p_1, \tag{55}$$

where

$$A = (1-m_0) + \frac{m_0}{\kappa_1 \kappa_2}.$$

Now, the linear terms $\sim \varepsilon$ in relations (14) and (18) give

$$\frac{1}{2}\frac{\partial e_0}{\partial \tau} - c\frac{\partial e_1}{\partial \xi} + v_1\frac{\partial e_0}{\partial \xi} = \frac{\partial v_1}{\partial \xi}, \tag{56}$$

$$\sigma_1^{(ef)} - E_2 e_1 - \beta^{(s)} k_b p_1 = T, \tag{57}$$

where

$$T \equiv \eta\Bigg[\sum_{q=1,2,3,5} a_q(-c)^q \varepsilon^{q-1}\frac{\partial^q e_0}{\partial \xi^q} + \sum_{q=1,3} b_q c^q \varepsilon^{q-1}\frac{\partial^q \sigma_0^{(ef)}}{\partial \xi^q}\Bigg].$$

The linear terms $\sim \varepsilon$ in the bubble (6) give

$$-\frac{\mu c}{\rho_0^{(L)}\kappa_2}\Bigg[\frac{4}{R_0} + \frac{m_0 R_0}{\ell}\Bigg]\frac{\partial R_0}{\partial \xi} = -\frac{1}{\rho_0^{(L)}\kappa_2}(p_0 \chi R_1 + p_1). \tag{58}$$

Equations (56), (57), and (58) lead to the integrals

$$e_1 = -\frac{v_1}{c}, \quad \sigma_1^{(ef)} = E_2 e_1 + \beta^{(s)} k_b p_1, \quad p_1 = -p_0 \chi R_1. \tag{59}$$

The effective stress $\sigma_1^{(ef)}$ in (59) can be rewritten as

$$\sigma_1^{(ef)} = -\Bigg[\frac{E_2 v_1}{c} + p_0 \chi \beta^{(s)} k_b R_1\Bigg]. \tag{60}$$

Substituting (60) and the value of p_1 from (59) into (54) leads to

$$(1-\beta^{(s)}E_2)v_1 + (B - k_b\beta^{(s)}\beta^{(s)})p_0\chi R_1 c = 0, \tag{61}$$

where

$$B = \frac{(\beta + (1-m_0)\beta^{(s)}\beta^{(L)}p_0)}{\kappa_2} + \frac{4\pi n_0 m_0 R_0^3}{\kappa_1 p_0 \chi}.$$

Now, from (55) and using the value of p_1 from (59), we obtain the effective stress as

$$\sigma_1^{(ef)} = -(\rho_0 c v_1 + A)p_0 \chi R_1. \tag{62}$$

The combination of (60) and (62) results in

$$(E_2 - \rho_0 c^2)v_1 - (A - k_b\beta^{(s)})p_0 \chi R_1 c = 0. \tag{63}$$

Appendix B

Collecting the quadratic terms $\sim \varepsilon^2$ in (18), we get

$$\sigma_2^{(ef)} - E_2 e_2 - \beta^{(s)} k_b p_2 = T, \tag{64}$$

where

$$T \equiv \eta\Bigg[\sum_{q=1,2,3,5} a_q(-c)^q \varepsilon^{q-1}\frac{\partial^q e_1}{\partial \xi^q} + \sum_{q=1,3} b_q c^q \varepsilon^{q-1}\frac{\partial^q \sigma_1^{(ef)}}{\partial \xi^q}\Bigg].$$

Note that here we keep (as Nikolaevskiy [2008] did) the higher powers of ε to represent small corrections to the leading terms. These corrections will eventually translate into small corrections in the derived Nikolaevskiy equation further in this paper; they will be the object of our study. Thus,

$$\frac{\partial\sigma_2^{(ef)}}{\partial\xi} - E_2\frac{\partial e_2}{\partial\xi} - \beta^{(s)}k_b\frac{\partial p_2}{\partial\xi} = \frac{\partial T}{\partial\xi}. \tag{65}$$

From (14) in the order $\sim\varepsilon^2$, we get

$$\frac{\partial}{\partial\xi}(ce_2 + v_2) = F, \quad F = -\frac{1}{c}\left(\frac{1}{2}\frac{\partial v_1}{\partial\tau} + \frac{\partial v_1 v_1}{\partial\xi}\right). \tag{66}$$

Therefore,

$$\frac{\partial e_2}{\partial\xi} = \frac{F}{c} - \frac{1}{c}\frac{\partial v_2}{\partial\xi}. \tag{67}$$

Substituting (67) into (65) we obtain

$$\frac{\partial}{\partial\xi}(c\sigma_2^{(ef)} + E_2 v_2 - c\beta^{(s)}k_b p_2) = E_2 F + c\frac{\partial T}{\partial\xi}. \tag{68}$$

From the momentum equations (5) for the solid and liquid, we get

$$(1-m_0)\rho_0^{(s)}c\frac{\partial v_2}{\partial\xi} + \frac{\partial\sigma_2^{(ef)}}{\partial\xi} - (1-m_0)\frac{\partial p_2}{\partial\xi} = \Sigma_1, \tag{69}$$

where

$$\Sigma_1 = (1-m_0)\rho_0^{(s)}\left(\frac{1}{2}\frac{\partial v_1}{\partial\tau} + \frac{\partial v_1 v_1}{\partial\xi}\right) - (1-m_0)\rho_1^{(s)}c\frac{\partial v_1}{\partial\xi} + m_1\rho_0^{(s)}c\frac{\partial v_1}{\partial\xi} - m_1\frac{\partial p_1}{\partial\xi} + \varepsilon^{\gamma-1}\delta m_0 v_1$$

and

$$m_0\rho_0^{(f)}c\frac{\partial u_2}{\partial\xi} - m_0\frac{\partial p_2}{\partial\xi} = \Sigma_2, \tag{70}$$

where

$$\Sigma_2 = m_0\rho_0^{(f)}\left(\frac{1}{2}\frac{\partial u_1}{\partial\tau} + \frac{\partial u_1 u_1}{\partial\xi}\right) - m_0\rho_1^{(f)}c\frac{\partial u_1}{\partial\xi} - m_1\rho_0^{(f)}c\frac{\partial u_1}{\partial\xi} + m_1\frac{\partial p_1}{\partial\xi} - \varepsilon^{\gamma-1}\delta m_0 u_1.$$

Due to the condition (15), the combination of (69) with (70) gives

$$\rho_0 c\frac{\partial v_2}{\partial\xi} + \frac{\partial\sigma_2^{(ef)}}{\partial\xi} - \frac{\partial p_2}{\partial\xi} = \Sigma, \tag{71}$$

where $\Sigma = \Sigma_1 + \Sigma_2$, so that

$$\Sigma = \rho_0\left(\frac{1}{2}\frac{\partial v_1}{\partial\tau} + \frac{\partial v_1 v_1}{\partial\xi}\right) - c((1-m_0)\rho_0^{(s)}\beta^{(s)} + m_0\rho_0^{(L)}\beta^{(L)}\kappa_1)\frac{\partial p_1 v_1}{\partial\xi}$$
$$+ c\rho_0^{(s)}\beta^{(s)}\frac{\partial\sigma_1^{(ef)}v_1}{\partial\xi} + c\rho_0^{(L)}4\pi n_0 m_0\kappa_2 R_0^3\frac{\partial R_1 v_1}{\partial\xi} + c(\rho_0^{(s)} - \kappa_1\kappa_2\rho_0^{(L)})\frac{\partial m_1 v_1}{\partial\xi}.$$

Equations (68) and (71) result in

$$\frac{\partial}{\partial\xi}[(E_2 - \rho_0 c^2)v_2 + (1-\beta^{(s)}k_b)cp_2] = E_2 F - c\Sigma + c\frac{\partial T}{\partial\xi}. \tag{72}$$

From the bubble equation (6), in the order $\sim \varepsilon^2$,

$$\frac{-\mu c}{\rho_0^{(L)}\kappa_2}\left(4+\frac{m_0R_0^2}{\ell}\right)\frac{\partial R_1}{\partial \xi}=\frac{1}{\rho_0^{(L)}\kappa_2}\left[\frac{\beta^{(L)}p_0\chi}{\kappa_2}p_1R_1+\frac{\beta^{(L)}}{\kappa_2}p_1^2+\frac{p_0\chi(\chi+1)}{2}R_1^2-p_0\chi R_2-p_2\right]. \quad (73)$$

We rewrite (73) as

$$p_2=\Gamma-p_0\chi R_2, \quad (74)$$

where

$$\Gamma=\mu c\left(4+\frac{m_0R_0^2}{\ell}\right)\frac{\partial R_1}{\partial \xi}+\frac{\beta^{(L)}p_1}{\kappa_2}(p_0\chi R_1+p_1)+\tfrac{1}{2}p_0\chi(\chi+1)R_1^2.$$

Now we substitute the value of p_2 from (74) into (72) to get

$$\frac{\partial}{\partial \xi}[(E_2-\rho_0c^2)v_2-(1-\beta^{(s)}k_b)p_0\chi R_2c]=E_2F-c\Sigma+c\frac{\partial T}{\partial \xi}-c(1-\beta^{(s)}k_b)\frac{\partial \Gamma}{\partial \xi}. \quad (75)$$

In the second order, the mass balances (5) for the solid and liquid-gas mixture have the form

$$\frac{\partial}{\partial \xi}((1-m_0)v_2-[(1-m_0)\beta^{(s)}p_2-\beta^{(s)}\sigma_2^{(ef)}-m_2]c)=\Lambda^{(s)}/\rho_0^{(s)}, \quad (76)$$

$$\frac{\partial}{\partial \xi}\left(m_0u_2-\left[m_2+\frac{m_0\beta^{(L)}p_2}{\kappa_2}-\frac{4\pi m_0n_0R_0^3(R_2+R_1^2)}{\kappa_1}+\frac{4\pi m_0n_0R_0^3p_0\chi\beta^{(L)}R_1^2}{\kappa_1\kappa_2}\right]c\right)=\frac{\Lambda^{(L)}}{\rho_0^{(L)}}, \quad (77)$$

where

$$\Lambda^{(s)}=\rho_0^{(s)}\frac{1}{2}\frac{\partial}{\partial \tau}[(m_1-(1-m_0)\beta^{(s)}p_1+\beta^{(s)}\sigma_1^{(ef)})]$$
$$+\rho_0^{(s)}\frac{\partial}{\partial \xi}\left[m_1v_1-((1-m_0)p_1+\sigma_1^{(ef)})\beta^{(s)}v_1-c\beta^{(s)}m_1\left(p_1-\frac{\sigma_1^{(ef)}}{(1-m_0)}\right)\right], \quad (78)$$

$$\Lambda^{(L)}=-\rho_0^{(L)}\frac{1}{2}\frac{\partial}{\partial \tau}[\kappa_1(m_1\kappa_2+m_0\beta^{(L)}p_1)-4\pi n_0\kappa_2R_0^3R_1]$$
$$+\rho_0^{(L)}\frac{\partial}{\partial \xi}[(\beta^{(L)}\kappa_1p_1-4\pi n_0\kappa_2R_0^3R_1)(cm_1-m_0u_1)]-\kappa_1\kappa_2\rho_0^{(L)}\frac{\partial m_1u_1}{\partial \xi}. \quad (79)$$

The combination of (76) and (77) gives

$$\frac{\partial}{\partial \xi}\left[v_2-\left(\frac{(\beta+(1-m_0)\beta^{(s)}\beta^{(L)}p_0)p_2}{\kappa_2}-\beta^{(s)}\sigma_2^{(ef)}\right.\right.$$
$$\left.\left.-\frac{4\pi m_0n_0R_0^3(R_2+R_1^2)}{\kappa_1}+\frac{4\pi m_0n_0R_0^3p_0\chi\beta^{(L)}R_1^2}{\kappa_1\kappa_2}\right)c\right]=\Lambda, \quad (80)$$

where

$$\Lambda\equiv\Lambda^{(s)}/\rho_0^{(s)}+\Lambda^{(L)}/\rho_0^{(L)}.$$

From (68) we have

$$\frac{\partial \sigma_2^{(ef)}}{\partial \xi}=\frac{\partial}{\partial \xi}\left(T+k_b\beta^{(s)}\Gamma-k_b\beta^{(s)}p_0\chi R_2-\frac{E_2}{c}v_2\right)+\frac{1}{c}E_2F. \quad (81)$$

Now we insert (81) and the value of p_2 represented by (74) into (80),

$$\frac{\partial}{\partial\xi}\left[(1-E_2\beta^{(s)})v_2-\left(\omega_1 p_0\chi-\frac{4\pi m_0 n_0 R_0^3}{\kappa_1}\right)R_2 c\right]$$
$$=\Lambda-\beta^{(s)}E_2F-c\beta^{(s)}\frac{\partial T}{\partial\xi}-c\omega_1\frac{\partial\Gamma}{\partial\xi}+c\omega_2\frac{\partial R_1^2}{\partial\xi}, \quad (82)$$

where

$$\omega_1=k_b\beta^{(s)}\beta^{(s)}+\frac{\beta+(1-m_0)\beta^{(s)}\beta^{(L)}p_0}{\kappa_2}, \quad \omega_2=\frac{4\pi m_0 n_0 R_0^3\beta^{(L)}p_0\chi}{\kappa_1\kappa_2}-\frac{4\pi m_0 n_0 R_0^3}{\kappa_1}.$$

Appendix C

Equation (26) can be illustrated by the following simple example

$$v_1+cR_1=0, \quad 2v_1+4R_1=0.$$

A nonzero solution of the system exists only if $c=2$ (the eigenvalue of the problem). Here v_1 and R_1 are analogous to the first approximation from our main text. The second approximation, v_2 and R_2, satisfies the system

$$v_2+cR_2=f[v_1], \quad 2v_2+4R_2=g[v_1],$$

which is solvable only if the right-hand sides satisfy the condition $g[v_1]=2f[v_1]$. This solvability condition is the analogy to the Nikolaevskiy-type equation that we aim to derive.

Appendix D

The first approximations for the momentum and mass-balance equations without gas bubbles are the same as for the system (5). As for the density equations, the solid density remains unchanged but for gas-liquid mixture we neglect the volume gas content ϕ in (9):

$$\rho^{(f)}=\rho^{(L)}=\rho_0^{(L)}(1+\beta^{(L)}p). \quad (83)$$

The first approximation of (83) is

$$\rho_1^{(L)}=\rho_0^{(L)}\beta^{(L)}p_1. \quad (84)$$

Inserting this into the mass equation for the fluid (48), we get

$$m_0u_1=[m_1+m_0\beta^{(L)}p_1]c. \quad (85)$$

Now, the combination of (51) and (85) yields

$$(1-m_0)v_1+n_0u_1=[(1-m_0)\beta^{(s)}p_1-\beta^{(s)}\sigma_1^{(ef)}+m_0\beta^{(L)}p_1]c. \quad (86)$$

Due to the condition (15), Equation (86) becomes

$$v_1=[(1-m_0)\beta^{(s)}p_1-\beta^{(s)}\sigma_1^{(ef)}+m_0\beta^{(L)}p_1]c. \quad (87)$$

After we apply the conditions

$$v_1 = u_1, \quad \rho_0 = (1-m_0)\rho_0^{(s)} + m_0\rho_0^{(L)},$$

the first two equations in (48) give

$$\rho_0 c v_1 = -\sigma_1^{(ef)} + p_1. \tag{88}$$

The first approximation of relation (32) is

$$\sigma_1^{(ef)} - E_2 e_1 - \beta^{(s)} k_b p_1 = \sum_{q=1}^{3} a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_0}{\partial \xi^q} + b_1 c \frac{\partial \sigma_0^{(ef)}}{\partial \xi}. \tag{89}$$

Equations (89) and (56) result in the integral

$$\sigma_1^{ef} = -\frac{E_2 v_1}{c} + \beta^{(s)} k_b p_1. \tag{90}$$

Substituting (90) into (87), we get

$$(1-\beta^{(s)}E_2)v_1 = c(1-m_0)\beta^{(s)} p_1 + c m_0 \beta^{(L)} p_1 - \beta^{(s)}\beta^{(s)} k_b c p_1. \tag{91}$$

As $\beta = (1-m_0)\beta^{(s)} + m_0\beta^{(L)}$, (91) becomes

$$(1-\beta^{(s)}E_2)v_1 = (\beta - \beta^{(s)}\beta^{(s)}k_b) c p_1. \tag{92}$$

From (88) we obtain

$$\sigma_1^{(ef)} = p_1 - \rho_0 c v_1. \tag{93}$$

Therefore, the combination of (93) with (90) yields

$$(\rho_0 c^2 - E_2) v_1 = (1-\beta^{(s)}k_b) c p_1. \tag{94}$$

Appendix E

In the second approximation for the system without the bubbles we again arrive at an equation of the form (72), except the formulas for Σ and T are changed:

$$\frac{\partial}{\partial \xi}[(E_2 - \rho_0 c^2) v_2 + (1-\beta^{(s)}k_b) c p_2] = E_2 F - c\Sigma + c\frac{\partial T}{\partial \xi}, \tag{95}$$

where

$$\Sigma = \rho_0\left(\frac{1}{2}\frac{\partial v_1}{\partial \tau} + \frac{\partial v_1 v_1}{\partial \xi}\right) - c\rho_0\beta\frac{\partial p_1 v_1}{\partial \xi} + c\rho_0^{(s)}\beta^{(s)}\frac{\partial \sigma_1^{(ef)} v_1}{\partial \xi} + c(\rho_0^{(s)} - \rho_0^{(L)})\frac{\partial m_1 v_1}{\partial \xi},$$

$$T = \sum_{q=1}^{3} a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_1}{\partial \xi^q} + b_1 c \frac{\partial \sigma_1^{(ef)}}{\partial \xi}.$$

The second approximation of the mass balance for the solid is the same as (76), while for the fluid it takes the form

$$\frac{\partial}{\partial \xi}[m_0 u_2 - (m_2 + m_0\beta^{(L)} p_2) c] = \Lambda^{(L)}/\rho_0^{(L)}, \tag{96}$$

where

$$\Lambda^{(L)} = -\tfrac{1}{2}\rho_0^{(L)}\frac{\partial}{\partial\tau}[m_1 + m_0\beta^{(L)}p_1] + \rho_0^{(L)}\frac{\partial}{\partial\xi}[cm_1\beta^{(L)}p_1 - m_0\beta^{(L)}p_1u_1 - m_1u_1].$$

The combination of (76) and (96) results in

$$\frac{\partial}{\partial\xi}[v_2 - (\beta p_2 - \beta^{(s)}\sigma_2^{(ef)})c] = \Lambda, \tag{97}$$

where

$$\Lambda = (\Lambda^{(s)}/\rho_0^{(s)}) + (\Lambda^{(L)}/\rho_0^{(L)}).$$

From (68) we find

$$\frac{\partial\sigma_2^{(ef)}}{\partial\xi} = \frac{\partial}{\partial\xi}\left(T + k_b\beta^{(s)}p_2 - \frac{E_2}{c}v_2\right) + \frac{1}{c}E_2F. \tag{98}$$

Substituting (98) into (97) we get

$$\frac{\partial}{\partial\xi}[(1 - E_2\beta^{(s)})v_2 - (\beta - \beta^{(s)}\beta^{(s)})cp_2] = \Lambda - \beta^{(s)}E_2F - c\beta^{(s)}\frac{\partial T}{\partial\xi}. \tag{99}$$

References

[Anderson 1980] A. L. Anderson, "Acoustics of gas-bearing sediments, I: Background", *J. Acoust. Soc. Am.* **67**:6 (1980), 1865–1889.

[Beresnev and Nikolaevskiy 1993] I. A. Beresnev and V. N. Nikolaevskiy, "A model for nonlinear seismic waves in a medium with instability", *Physica D* **66**:1-2 (1993), 1–6.

[Biot 1956a] M. A. Biot, "Theory of propagation of elastic waves in a fluid-saturated porous solid, I: Low-frequency range", *J. Acoust. Soc. Am.* **28**:2 (1956), 168–178.

[Biot 1956b] M. A. Biot, "Theory of propagation of elastic waves in a fluid-saturated porous solid, II: Higher frequency range", *J. Acoust. Soc. Am.* **28**:2 (1956), 179–191.

[Biot 1962a] M. A. Biot, "Generalized theory of acoustic propagation in porous dissipative media", *J. Acoust. Soc. Am.* **34**:9A (1962), 1254–1264.

[Biot 1962b] M. A. Biot, "Mechanics of deformation and acoustic propagation in porous media", *J. Appl. Phys.* **33**:4 (1962), 1482–1498.

[Bohlen 2002] T. Bohlen, "Parallel 3-D viscoelastic finite difference seismic modelling", *Comput. Geosci.* **28**:8 (2002), 887–899.

[Brunner and Spetzler 2001] W. Brunner and H. A. Spetzler, "Observations of time-dependent meniscus behavior with implications for seismic attenuation in three-phase systems", *Geophys. Res. Lett.* **28**:9 (2001), 1867–1870.

[Carcione 1998] J. M. Carcione, "Viscoelastic effective rheologies for modelling wave propagation in porous media", *Geophys. Prospecting* **46**:3 (1998), 249–270.

[Collier et al. 2006] L. Collier, J. W. Neuberg, N. Lensky, V. Lyakhovsky, and O. Navon, "Attenuation in gas-charged magma", *J. Volcanology Geotherm. Res.* **153**:1-2 (2006), 21–36.

[Dunin and Nikolaevskiy 2005] S. Z. Dunin and V. N. Nikolaevskii, "Nonlinear waves in porous media saturated with live oil", *Acoust. Phys.* **51**:Suppl. 1 (2005), S61–S66.

[Dunin et al. 2006] S. Z. Dunin, D. N. Mikhailov, and V. N. Nikolayevskii, "Longitudinal waves in partially saturated porous media: the effect of gas bubbles", *J. Appl. Math. Mech.* **70**:2 (2006), 251–263.

[Frenkel 2005] J. Frenkel, "On the theory of seismic and seismoelectric phenomena in a moist soil", *J. Eng. Mech.* (*ASCE*) **131**:9 (2005), 879–887.

[Liu et al. 1976] H.-P. Liu, D. L. Anderson, and H. Kanamori, "Velocity dispersion due to anelasticity: implications for seismology and mantle composition", *Geophys. J. Int.* **47**:1 (1976), 41–58.

[Masson et al. 2006] Y. J. Masson, S. R. Pride, and K. T. Nihei, "Finite difference modeling of Biot's poroelastic equations at seismic frequencies", *J. Geophys. Res. Solid Earth* **111**:B10 (2006), art. id. B10305.

[Mikhailov 2010] D. N. Mikhailov, "The influence of gas saturation and pore pressure on the characteristics of the Frenkel–Biot P waves in partially saturated porous media", *Fiz. Zemli* **2010**:10 (2010), 99–112. In Russian; translated in *Izv. Phys. Solid Earth* **46**:10 (2010), 897–909.

[Nikolaevskiy 1985] V. N. Nikolaevskiy, "Viscoelasticity with internal oscillators as a possible model of seismoactive medium", *Dokl. Akad. Nauk SSSR* **283**:6 (1985), 1321–1324. In Russian.

[Nikolaevskiy 1989] V. N. Nikolaevskii, "Dynamics of viscoelastic media with internal oscillators", pp. 201–221 in *Recent advances in engineering science*, edited by S. L. Koh and C. G. Speziale, Lect. Notes Eng. **39**, Springer, 1989.

[Nikolaevskiy 1990] V. N. Nikolaevskij, *Mechanics of porous and fractured media*, Series Theor. Appl. Mech. **8**, World Sci., Teaneck, NJ, 1990.

[Nikolaevskiy 2008] V. N. Nikolaevskiy, "Non-linear evolution of P-waves in viscous-elastic granular saturated media", *Transp. Porous Media* **73**:2 (2008), 125–140.

[Nikolaevskiy 2016] V. N. Nikolaevskiy, "A real P-wave and its dependence on the presence of gas", *Fiz. Zemli* **2016**:1 (2016), 3–14. In Russian; translated in *Izv. Phys. Solid Earth* **52**:1 (2016), 1–13.

[Nikolaevskiy and Stepanova 2005] V. N. Nikolaevskiĭ and G. S. Stepanova, "Nonlinear seismics and the acoustic action on the oil recovery from an oil pool", *Acoust. Phys.* **51**:Suppl. 1 (2005), S131–S139.

[Nikolaevskiy and Strunin 2012] V. Nikolaevskiy and D. Strunin, "The role of natural gases in seismics of hydrocarbon reservoirs", pp. 25–29 Int. Conf. Elastic Wave Effect on Fluid in Porous Media **3**, 2012.

[Papageorgiou and Chapman 2015] G. Papageorgiou and M. Chapman, "Multifluid squirt flow and hysteresis effects on the bulk modulus-water saturation relationship", *Geophys. J. Int.* **203**:2 (2015), 814–817.

[Smeulders 2005] D. M. Smeulders, "Experimental evidence for slow compressional waves", *J. Eng. Mech.* (*ASCE*) **131**:9 (2005), 908–917.

[Strunin 2014] D. V. Strunin, "On dissipative nature of elastic waves", *J. Coupled Syst. Multiscale Dyn.* **2**:2 (2014), 70–73.

[Strunin and Ali 2016] D. V. Strunin and A. A. Ali, "On nonlinear dynamics of neutral modes in elastic waves in granular media", *J. Coupled Syst. Multiscale Dyn.* **4**:3 (2016), 163–169.

[Sutton and Biblarz 2016] G. P. Sutton and O. Biblarz, *Rocket propulsion elements*, 9th ed., Wiley, Hoboken, NJ, 2016.

[Tisato et al. 2015] N. Tisato, B. Quintal, S. Chapman, Y. Podladchikov, and J.-P. Burg, "Bubbles attenuate elastic waves at seismic frequencies: first experimental evidence", *Geophys. Res. Lett.* **42**:10 (2015), 3880–3887.

[Van Wijngaarden 1968] L. Van Wijngaarden, "On the equations of motion for mixtures of liquid and gas bubbles", *J. Fluid Mech.* **33**:3 (1968), 465–474.

[Vilchinska et al. 1985] N. Vilchinska, V. N. Nikolaevskiy, and V. Lisin, "Slow waves and natural oscillations in sandy marine soils", *Izv. Acad. Nauk SSSR Oceanology* **25**:4 (1985), 656–663. In Russian.

[Yang et al. 2014] L. Yang, D. Yang, and J. Nie, "Wave dispersion and attenuation in viscoelastic isotropic media containing multiphase flow and its application", *Sci. China G Phys. Mech. Astronom.* **57**:6 (2014), 1068–1077.

Received 25 Jan 2018. Revised 31 Dec 2018. Accepted 7 Jan 2019.

ADHAM A. ALI: u1054916@umail.usq.edu.au
Department of Mathematics, Kirkuk University, Kirkuk, Iraqi

and

Computational Engineering and Science Research Centre, Faculty of Health, Engineering and Sciences, University of Southern Queensland, Toowoomba, Australia

DMITRY V. STRUNIN: strunin@usq.edu.au
Computational Engineering and Science Research Centre, Faculty of Health, Engineering and Sciences, University of Southern Queensland, Toowoomba, Australia

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

Cover photo: Ev Shafrir

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2019 is US $635/year for the electronic version, and $795/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 14, No. 1 January 2019

1559-3959(2019)14:1;1-X