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SOME GENERAL THEOREMS FOR LOCAL GRADIENT THEORY OF ELECTROTHERMOELASTIC DIELECTRICS

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Using the basic equations of local gradient theory of electrothermoelastic nonferromagnetic polarized solids, which accounts for the local mass displacement and its effect on mechanical, thermal and electromagnetic fields, the governing set of equations is obtained for a linear approximation. On this basis, the coupled initial-boundary-value problems corresponding to this gradient-type theory are formulated. The reciprocity and uniqueness theorems for non-stationary problems of the local gradient electrothermoelasticity are proved.

1. Introduction

As a result of the rapid development of nanotechnologies, the past several decades have been marked by significant scientific attention to the construction of nonlocal theories of the deformation of solids. At a continuous level, such theories account for the long-range effects and the impact of a material's microstructure on its macro-properties. Nonlocal theories have allowed for the description of a range of experimentally-observed phenomena [Liu et al. 2006; Nam et al. 2006; Nysten et al. 2005; Kumikov and Khokonov 1983] that cannot be duly explained by the classical (local) theories. Within the scope of a continuous description, gradient-type theories of dielectrics are constructed by introducing into the space of constitutive parameters of internal variables, or gradients of the strain, polarization, electric field [Hadjigeorgiou et al. 1999; Nowacki 1983; Kafadar 1971; Kalpakidis et al. 1995; Kalpakidis and Agiasofitou 2002; Maugin 1979; 1988; Mindlin 1972; Sahin and Dost 1988; Yang 2006; Yan and Jiang 2007].

In 1987, Burak proposed a new continuum-thermodynamic approach to the construction of a nonlocal theory of the deformation of thermoelastic solids, which consisted in accounting for local mass displacement and its impact on the mechanical and heat fields in the model description [Burak 1987]. In doing so, he linked the local mass displacement to changes in the material structure of a fixed small element of the body. By employing this approach, articles [Burak et al. 2007; 2008] present the foundations of a gradient-type theory of the deformation of electrothermo-elastic nonferromagnetic polarized solids. The mentioned theory is grounded on accounting for the local mass displacement and its effect to mechanical, heat, and electromagnetic fields [Burak et al. 2007; 2008; Kondrat and Hrytsyna 2008; 2012]. The developed theory was called a local gradient theory of dielectrics. This theory enabled us to explain theoretically some observed phenomena, namely, the near-surface and size phenomena [Burak et al. 2007; 2008], high-frequency dispersion of longitudinal elastic waves [Kondrat and Hrytsyna 2010],

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Mead's anomaly [Chapla et al. 2009], piezoelectric effect in high symmetry crystalline dielectrics [Hrytsyna 2012], the existence of anti-plane shear surface SH waves in homogeneous isotropic half-space [Hrytsyna 2017] etc. Note that the above phenomena are not accounted for in the classical theory of dielectrics.

The objective of the proposed paper is to state the boundary value problems of the local gradient theory of dielectrics and to prove the Reciprocity Theorem and the Uniqueness Theorem for the coupled linear problems of this theory. To this end, Section 2 presents a nonlinear complete set of relations of the said theory. Basing on these relations, Section 3 obtains a linearized governing set of equations and shows the possibility of its division into two subsystems of differential equations that can be solved consecutively. Boundary conditions for local gradient theory of dielectrics are presented in Section 4. Section 5 and Section 6 present and prove the Uniqueness and Reciprocity Theorems for linear local gradient theory of electrothermoelastic nonferromagnetic dielectrics.

2. Basic preliminaries

In this Section, we briefly present the basic ideas and equations that describe the coupled fields in the framework of a local gradient theory of electrothermoelastic nonferromagnetic dielectrics, according to Burak et al. [Burak et al. 2007; 2008; Kondrat and Hrytsyna 2008].

We consider a thermoelastic polarized solid body occupying a finite domain (V) bounded by a smooth boundary (Σ). The body is subjected to an action of external forces, thermal and electromagnetic loads. As a result the mechanical, heat, and electromagnetic processes are occurring within a body, potentially followed by changes in the material structure of a fixed small element of the body. Such changes in structure can be observed, for instance, in the near-surface regions of newly-created surfaces. They are caused by a violation of the atom force balance in these regions. In a local gradient theory of electrothermoelastic nonferromagnetic dielectrics, the said changes in material structure are described by mass fluxes $\boldsymbol{J}_{ms}$ of a non-diffusive and non-convective nature [Burak et al. 2007; 2008]. The mentioned changes in material structure are further related to the process of the local mass displacement.

Burak and co-workers [Burak et al. 2007; 2008] use the Cauchy stress tensor $\hat{\boldsymbol{\sigma}}$ and strain tensor $\hat{\boldsymbol{e}}$ to describe the mechanical fields, as well as the density of the heat flux $\boldsymbol{J}_q$, the absolute temperature T, and the entropy S to describe the process of heat conductivity. They characterize the electromagnetic field by the vectors of electric $\boldsymbol{D}$ and magnetic $\boldsymbol{B}$ inductions, electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields, and the polarization vector $\boldsymbol{\Pi}_e$. They introduce the vector of local mass displacement $\boldsymbol{\Pi}_m$, the density of the induced mass $\rho_{m\pi} = -\nabla \cdot \boldsymbol{\Pi}_m$, as well as the potential μ_π to describe the process of local mass displacement [Burak et al. 2008]. Here ∇ is the Hamilton operator; the dot denotes the scalar product. Note that the potential μ_π is defined as an energy measure of the effect of the local mass displacement on internal energy [Burak et al. 2008].

The result of accounting in the model description for the local mass displacement and its coupling to the mechanical, heat, and electromagnetic fields is a modification of the Gibbs equation. Along with the generally-accepted in classic electrothermoelasticity pairs of conjugate parameters of state (stress and strain tensors, temperature and entropy, polarization and electric field), the generalized Gibbs equation contains two additional pairs of parameters. The modified chemical potential $\mu'_\pi = \mu_\pi - \mu$ and specific density of induced mass comprise one pair of the parameters of state, while the specific vector of local mass displacement $\boldsymbol{\pi}_m = \boldsymbol{\Pi}_m/\rho$ and the gradient of the modified chemical potential $\nabla\mu'_\pi$ comprise the other. Here, μ is chemical potential. Thus, within the scope of the developed theory, the Gibbs equation takes the following form [Burak et al. 2008]:

$$df = \frac{1}{\rho}\hat{\boldsymbol{\sigma}}_* : d\hat{\boldsymbol{e}} - s\,dT - \boldsymbol{\pi}_e \cdot d\boldsymbol{E}_* + \mu'_\pi d\rho_m + \boldsymbol{\pi}_m \cdot d\nabla\mu'_\pi. \tag{1}$$

Here, ρ is the mass density, $s = S/\rho$, $\boldsymbol{\pi}_e = \boldsymbol{\Pi}_e/\rho$, f is the free energy, $\boldsymbol{E}_* = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$, $\boldsymbol{v}$ is the velocity vector, and the symbol "$\times$" denotes the vector product.

The consequence of accounting for the processes of polarization and the local mass displacement is the modification of the stress tensor, which is now defined by the formula $\hat{\boldsymbol{\sigma}}_* = \hat{\boldsymbol{\sigma}} - \rho[\boldsymbol{\pi}_e \cdot \boldsymbol{E}_* + \rho_m \mu'_\pi - \boldsymbol{\pi}_m \cdot \nabla\mu'_\pi]\hat{\boldsymbol{I}}$, where $\hat{\boldsymbol{I}}$ is the unit tensor.

Using the differential 1-forms (1) for the generalized theory of dielectrics, we obtain the following constitutive equations:

$$\hat{\boldsymbol{\sigma}}_* = \rho\frac{\partial f}{\partial \hat{\boldsymbol{e}}}, \quad s = -\frac{\partial f}{\partial T}, \quad \boldsymbol{\pi}_e = -\frac{\partial f}{\partial \boldsymbol{E}_*}, \quad \mu'_\pi = \frac{\partial f}{\partial \rho_m}, \quad \boldsymbol{\pi}_m = \frac{\partial f}{\partial(\nabla\mu'_\pi)}. \tag{2}$$

The set of relations of local gradient electrothermomechanics of dielectrics includes the nonlocal constitutive equations (2), as well as the balance equations of mass, induced mass, and induced electric charge

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0, \tag{3}$$

$$\frac{\partial\rho_{m\pi}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J}_{ms} = 0, \tag{4}$$

$$\frac{\partial\rho_{e\pi}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J}_{es} = 0, \tag{5}$$

the momentum equation and the entropy balance equation

$$\rho\frac{d\boldsymbol{v}}{dt} = \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\sigma}}_* + \boldsymbol{F}_e + \rho(\boldsymbol{F} + \boldsymbol{F}'_*), \tag{6}$$

$$T\frac{\partial S}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}_q + \frac{1}{T}\boldsymbol{J}_q\cdot\boldsymbol{\nabla}T - T\boldsymbol{\nabla}\cdot(S\boldsymbol{v}) + T\sigma_s + \rho\mathfrak{R}, \tag{7}$$

the Maxwell equations and the conservation law of induced electric charges

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \quad \nabla\times\boldsymbol{H} = \boldsymbol{J}_e + \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t} + \frac{\partial\boldsymbol{\Pi}_e}{\partial t}, \quad \nabla\cdot\boldsymbol{B} = 0, \quad \nabla\cdot\boldsymbol{D} = \rho_e, \tag{8}$$

$$\frac{\partial\rho_e}{\partial t} + \nabla\cdot\boldsymbol{J}_e = 0, \tag{9}$$

the constitutive relations

$$\boldsymbol{B} = \mu_0\boldsymbol{H}, \qquad \boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{\Pi}_e, \tag{10}$$

$$\boldsymbol{J}_q = \boldsymbol{J}_q\left(-\frac{\boldsymbol{\nabla}T}{T^2}, \frac{\boldsymbol{E}_*}{T}\right), \quad \boldsymbol{J}_{e^*} = \boldsymbol{J}_{e^*}\left(-\frac{\boldsymbol{\nabla}T}{T^2}, \frac{\boldsymbol{E}_*}{T}\right), \tag{11}$$

the geometric relations

$$\boldsymbol{v} = \frac{d\boldsymbol{u}}{dt}, \quad \hat{\boldsymbol{e}} = \tfrac{1}{2}[\nabla \otimes \boldsymbol{u} + (\nabla \otimes \boldsymbol{u})^T], \tag{12}$$

the relations for additional mass force $\boldsymbol{F}'_*$ and ponderomotive force $\boldsymbol{F}_e$

$$\boldsymbol{F}'_* = \rho_m \nabla \mu'_\pi - (\nabla \otimes \nabla \mu'_\pi) \cdot \boldsymbol{\pi}_m, \tag{13}$$

$$\boldsymbol{F}_e = \rho_e \boldsymbol{E}_* + \left[\boldsymbol{J}_{e^*} + \frac{\partial(\rho \boldsymbol{\pi}_e)}{\partial t} \right] \times \boldsymbol{B} + \rho(\nabla \otimes \boldsymbol{E}_*) \cdot \boldsymbol{\pi}_e, \tag{14}$$

the expression for the entropy production

$$\sigma_s = -\boldsymbol{J}_q \cdot \frac{\nabla T}{T^2} + \boldsymbol{J}_{e^*} \cdot \frac{\boldsymbol{E}_*}{T}, \tag{15}$$

the formulae

$$\boldsymbol{J}_{ms} = \frac{\partial \boldsymbol{\Pi}_m}{\partial t}, \quad \boldsymbol{J}_{es} = \frac{\partial \boldsymbol{\Pi}_e}{\partial t}, \tag{16}$$

and a corresponding series for the free energy $f = f(\hat{\boldsymbol{e}}, T, \boldsymbol{E}_*, \rho_m, \nabla \mu'_\pi)$.

Here t denotes the time variable, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{F}$ is the mass force, σ_s is the entropy production per unit of volume and time, $\mathfrak{R}$ denotes the distributed thermal sources, $\boldsymbol{J}_{ms}$ is the density of non-convective and non-diffusive mass flux, ρ_e denotes the density of free electric charges, $\rho_{e\pi} = -\nabla \cdot \boldsymbol{\Pi}_e$ is the density of induced electric charge, $\boldsymbol{J}_e$ is the density of the electric current (convection and conduction currents), $\boldsymbol{J}_{es}$ is the polarization current, ε_0 and μ_0 are electric and magnetic constants, $\boldsymbol{J}_{e^*} = \boldsymbol{J}_e - \rho_e \boldsymbol{v}$, $\otimes$ is the dyadic product, an upper index T denotes a transposed tensor, and $(d \ldots /dt) = (\partial \ldots /\partial t) + \boldsymbol{v} \cdot \nabla \ldots$.

In comparison to the classical theory of elasticity, Burak [1987] introduced into the space of constitutive parameters one additional pair of conjugate constitutive parameters, namely, the vector of the local mass displacement $\boldsymbol{\Pi}_m$ and the gradient of the chemical potential $\nabla \mu$. Note that according to the generalized Gibbs equations (1), the set of conjugate variables for the thermoelastic dielectrics is complemented by two additional pairs of variables (μ'_π, ρ_m) and $(\boldsymbol{\pi}_m, \nabla \mu'_\pi)$, related to the local mass displacement. The equation of motion (6) takes into account the additional stresses

$$\hat{\boldsymbol{\sigma}}'_* = -\rho(\rho_m \mu'_\pi - \boldsymbol{\pi}_m \cdot \nabla \mu'_\pi) \hat{\boldsymbol{I}}$$

and nonlinear mass force $\boldsymbol{F}'$ (see formula (13)), induced within the body by the local mass displacement.

In general, the set of equations (2)–(16) is nonlinear. The number of equations in this set can be reduced by substituting the geometric (12) and physical relations (2), (10), and (11), as well as the expression for the entropy production (15) into the Maxwell equations (8) and the balance equations (3)–(7), and (9).

Below we present a governing set of equations for a linear approximation. To this end, we should write the constitutive equations (2) and the kinetic equations (11) in the explicit form. For isotropic materials, we obtain [Burak et al. 2008]:

$$\hat{\boldsymbol{\sigma}}_* = 2G\hat{\boldsymbol{e}} + \Big[\Big(K - \frac{2}{3}G\Big)e - K(\alpha_T\theta + \alpha_\rho\rho_m)\Big]\hat{\boldsymbol{I}} \tag{17a}$$

$$s = s_o + \frac{G_V}{T_o}\theta + \frac{K\alpha_T}{\rho_o}e + \beta_{T_\rho}\,\rho_m, \tag{17b}$$

$$\mu'_\pi = \mu'_{\pi o} + d_\rho\,\rho_m - \beta_{T_\rho}\,\theta - \frac{K\alpha_\rho}{\rho_o}e, \tag{17c}$$

$$\boldsymbol{\pi}_e = \chi_E\boldsymbol{E} - \chi_{Em}\nabla\mu'_\pi, \tag{17d}$$

$$\boldsymbol{\pi}_m = -\chi_m\nabla\mu'_\pi + \chi_{Em}\boldsymbol{E}, \tag{17e}$$

$$\boldsymbol{J}_q = -\lambda\nabla\theta + \pi_t\boldsymbol{J}_e, \qquad \boldsymbol{J}_e = \sigma_e\boldsymbol{E} - \eta\nabla\theta, \tag{18}$$

where, K, G, α_T, α_ρ, C_V, β_{T_ρ}, d_ρ, χ_E, χ_m, χ_{Em}, λ, σ_e, π_t, and η are the material characteristics, $e = \hat{\boldsymbol{e}} : \hat{\boldsymbol{I}}$, $\theta = T - T_o$, T_o, s_o, and $\mu'_{\pi o}$ are the temperature, entropy and modified chemical potential μ'_π in the reference state. Here, the reference state is considered to be an infinite medium without any disturbances of fields, that is, $\hat{\boldsymbol{e}} = 0$, $\hat{\boldsymbol{\sigma}}_* = 0$, $\boldsymbol{E}_* = 0$, $\boldsymbol{\pi}_e = 0$, $\boldsymbol{\pi}_m = 0$, $\nabla\mu'_\pi = 0$, $T = T_o$, $s = s_o$, $\rho_m = 0$, and $\mu'_\pi = \mu'_{\pi o}$.

3. Governing equations

Note that within the framework of a linear approximation, we have the following formula for the specific density of induced mass

$$\rho_m = -\nabla\cdot\pi_m. \tag{19}$$

Substituting the formulas (10), (12), (16)–(19) into the balance equations (4), (6), (7) and the Maxwell equations (8), we obtain the following governing set of linearized equations to determine the functions $\boldsymbol{u}$, θ, $\tilde{\mu}'_\pi = \mu'_\pi - \mu'_{\mu o}$, $\boldsymbol{E}$, and $\boldsymbol{B}$:

$$\rho_o\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \Big(\bar{K} + \frac{1}{3}G\Big)\nabla(\nabla\cdot\boldsymbol{u}) + G\Delta\boldsymbol{u} - K\bar{\alpha}_T\nabla\theta - K\frac{\alpha_\rho}{d_\rho}\nabla\tilde{\mu}'_\pi + \rho_o\boldsymbol{F}, \tag{20}$$

$$\rho_o\bar{C}_V\frac{\partial\theta}{\partial t} + KT_o\bar{\alpha}_T\frac{\partial(\nabla\cdot\boldsymbol{u})}{\partial t} + \rho_oT_o\frac{\beta_{T_\rho}}{d_\rho}\frac{\partial\tilde{\mu}'_\pi}{\partial t} = (\lambda + \pi_t\eta)\Delta\theta - \sigma_e\,\pi_t\nabla\cdot\boldsymbol{E} + \rho_o\Re, \tag{21}$$

$$\Delta\tilde{\mu}'_\mu - \lambda_\mu^2\tilde{\mu}'_\pi = \lambda_\mu^2\Big(K\frac{\alpha_\rho}{\rho_o}\nabla\cdot\boldsymbol{u} + \beta_{T_\rho}\theta\Big) + \frac{\chi_{Em}}{\chi_m}\nabla\cdot\boldsymbol{E}, \tag{22}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \qquad \nabla\times\boldsymbol{B} = \mu_0\sigma_e\boldsymbol{E} - \mu_0\eta\nabla\theta + \varepsilon\mu_0\frac{\partial}{\partial t}(\boldsymbol{E} - \kappa_E\nabla\tilde{\mu}'_\pi), \tag{23}$$

$$\nabla\cdot\boldsymbol{B} = 0, \qquad \nabla\cdot\boldsymbol{E} - \kappa_E\Delta\tilde{\mu}'_\pi = \frac{\rho_e}{\varepsilon}. \tag{24}$$

Here, Δ is the Laplacian, and coefficients $\bar{K}$, $\bar{\alpha}_T$, $\bar{C}_V$, ε, and κ_E are defined by the formulae

$$\begin{gathered}\bar{K} = K - \frac{K^2\alpha_\rho^2}{\rho_od_\rho}, \quad \bar{\alpha}_T = \alpha_T + \beta_{T_\rho}\frac{\alpha_\rho}{d_\rho}, \quad \bar{C}_V = C_V + T_o\frac{\beta_{T_\rho}^2}{d_\rho}, \\ \lambda_\mu = \big|\sqrt{d_\rho\chi_m}\big|^{-1}, \qquad \varepsilon = \varepsilon_0 + \rho_o\chi_E, \qquad \kappa_E = \rho_o\chi_{Em}/\varepsilon.\end{gathered} \tag{25}$$

The parameter $l_* = 1/\lambda_\mu$ is a material constant, with the dimension of length, and is a characteristic length for near-surface phenomena [Burak et al. 2008]. The appearance of such a constant is related to the consideration in the material model of the local mass displacement. Such a parameter is absent in the classical theory, based on local constitutive relations. The characteristic length l_* may be determined by experiment methods (for example, the electron diffraction measurements), by methods of discrete analysis or the theory of crystal lattice dynamics, etc. Using the methods of lattice dynamics, Mindlin [1972] and Maugin [1988] showed that the characteristic length is the magnitude of the order of distance between the nearest atoms (for example, for Sodium chloride $l_* = 0.73 \cdot 10^{-10}$ m and for Potassium chloride $l_* = 0.93 \cdot 10^{-10}$ m [Mindlin 1972]).

Note that the ponderomotive force is absent in the momentum equation (20), as is Joule heat in the heat conduction equation (21). This is because in the chosen reference state the ponderomotive force and Joule heat are nonlinear functions of the perturbation of fields. We can see that compared to the classical theory of electrothermoelastic dielectrics, as a result of accounting for the process of the local mass displacement, an additional equation (22) appears in the governing set of equations. Another consequence of accounting for local mass displacement is a modification of the equation of motion (20), heat conduction (21), and electrodynamics (23)–(24), which now also contain addends related to the local mass displacement. The consideration of the impact of the gradient of modified chemical potential in the motion equation (20) may be quantitatively interpreted as the emergence of an additional mass force within the body, proportional to $\nabla \tilde{\mu}'_\pi$, while in the equation of heat conduction it may be interpreted as the emergence of a source of heat in the body of the power $-\rho_o T_o(\beta_{T_\rho}/d_\rho)(\partial \tilde{\mu}'_\pi/\partial t)$.

Note that the governing set of equations (20)–(24) can be easily divided into two unrelated subsystems, eliminating the electric field from the second and third equations of this set. Indeed, using the relations (24), equations (21) and (22) can be presented as follows:

$$\rho_o \bar{C}_V \frac{\partial \theta}{\partial t} + K T_o \bar{\alpha}_T \frac{\partial(\nabla \cdot \boldsymbol{u})}{\partial t} + \rho_o T_o \frac{\beta_{T_\rho}}{d_\rho} \frac{\partial \tilde{\mu}'_\pi}{\partial t} = (\lambda + \pi_t \eta)\Delta\theta - \sigma_e \pi_t \kappa_E \left(\Delta \tilde{\mu}'_\pi + \frac{\rho_e}{\rho_o \chi_{Em}} \right) + \rho_o \mathfrak{R}, \tag{26}$$

$$\Delta \tilde{\mu}'_\pi - \lambda^2_{\mu E} \tilde{\mu}'_\pi = \lambda^2_{\mu E} \left(K \frac{\alpha_\rho}{\rho_o} \nabla \cdot \boldsymbol{u} + \beta_{T_\rho} \theta \right) + \lambda^2_{\mu E} \chi_{Em} d_\rho \frac{\rho_e}{\varepsilon}. \tag{27}$$

Here, $\lambda^2_{\mu E} = \lambda^2_\mu (1 - \kappa_E \chi_{Em}/\chi_m)^{-1}$.

Now the formulated problem can be solved consecutively. To determine the functions $\boldsymbol{u}$, θ, and $\tilde{\mu}'_\pi$, we use the related set of equations (20), (26), and (27). The vectors of the electromagnetic field are derived from the equations (23) and (24), where the functions $\boldsymbol{u}$, θ, and $\tilde{\mu}'_\pi$ are known.

For ideal dielectrics, the governing set of equations above is simplified and looks as follows:

$$\rho_o \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \left(\bar{K} + \frac{1}{3} G \right) \nabla(\nabla \cdot \boldsymbol{u}) + G \Delta \boldsymbol{u} - K \bar{\alpha}_T \nabla \theta - K \frac{\alpha_\rho}{d_\rho} \nabla \tilde{\mu}'_\pi + \rho_o \boldsymbol{F}, \tag{28}$$

$$\rho_o \bar{C}_V \frac{\partial \theta}{\partial t} + T_o K \bar{\alpha}_T \frac{\partial(\nabla \cdot \boldsymbol{u})}{\partial t} + \rho_o T_o \frac{\beta_{T_\rho}}{d_\rho} \frac{\partial \tilde{\mu}'_\pi}{\partial t} = \lambda \Delta \theta + \rho_o \mathfrak{R}, \tag{29}$$

$$\Delta \tilde{\mu}'_\pi - \lambda^2_{\mu E} \tilde{\mu}'_\pi = \lambda^2_{\mu E} \left(K \frac{\alpha_\rho}{\rho_o} \nabla \cdot \boldsymbol{u} + \beta_{T_\rho} \theta \right), \tag{30}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = \varepsilon \mu_0 - \frac{\partial}{\partial t} (\boldsymbol{E} - \kappa_E \nabla \tilde{\mu}'_\pi), \tag{31}$$

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} - \kappa_E \Delta \tilde{\mu}'_\pi = 0. \tag{32}$$

4. Boundary conditions

To complete the problems of the local gradient electrothermoelasticity, the boundary (or jump) conditions and initial conditions must be adjoined to the derived set of differential equations. These conditions ensure the uniqueness of the solution of the formulated problem. We proceed to specifying them below.

We assume the polarized solid is subjected to the following boundary conditions:

- mechanical conditions (displacement or traction (force per unit area) are prescribed):

$$\boldsymbol{u} = \boldsymbol{u}_a \quad \text{or} \quad \hat{\boldsymbol{\sigma}}_* \cdot \boldsymbol{n} = \sigma_a, \tag{33}$$

- thermal boundary conditions (temperature, normal heat flux or condition of convective heat exchange are prescribed):

$$T = T_a \quad \text{or} \quad \boldsymbol{J}_q \cdot \boldsymbol{n} = J_{qa}, \quad \text{or} \quad \boldsymbol{J}_q \cdot \boldsymbol{n} - H_*(T - T_c) = 0, \tag{34}$$

- condition for local mass displacement:

$$\mu'_\pi = \mu'_{\pi a}, \tag{35}$$

- electromagnetic boundary conditions are written as a prescribing of a tangential components of vectors of electric and magnetic fields:

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{E}_a, \quad \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{H}_a. \tag{36}$$

In the relations (33)–(36): $\boldsymbol{n}$ is the outward unit normal to the smooth boundary (Σ); $\boldsymbol{u}_a$, $\boldsymbol{\sigma}_a$, $\boldsymbol{E}_a$, $\boldsymbol{H}_a$, J_{qa}, T_a, and $\mu'_{\pi a}$ are given on the surface (Σ) values of the displacement vector, of traction, of the electric and magnetic fields, of the normal component of heat flux, of temperature, and of the modified chemical potential μ'_π; H_* is a heat transfer coefficient from the surface and is the surrounding environment temperature.

In some cases, certain conditions for the body surface can be formulated as boundary, while others — as jump conditions. Indeed, let the body be in contact with a vacuum or an environment with similar properties. In this case, mechanical conditions may be formulated as displacement (kinematic) boundary conditions (if displacements are known on the body surface) or traction boundary conditions (corresponding to a traction-free surface). Thermal boundary conditions should correspond to the prescription of the surface temperature (if we can control it) or the flux from the surface. Now, the equality between the potential μ'_π and zero is a condition for the local mass displacement. Since the perturbation of the electrothermomechanical processes within the body will cause the radiation of the electromagnetic field into the vacuum, the electromagnetic conditions on the body surface should be formulated as contact conditions. Therefore, the Maxwell equations in a vacuum (domain (V_v)) need to be added to the governing set of equations:

$$\nabla \times \boldsymbol{E}_v = -\frac{\partial \boldsymbol{B}_v}{\partial t}, \quad \nabla \times \boldsymbol{H}_v = \frac{\partial \boldsymbol{D}_v}{\partial t}, \quad \nabla \cdot \boldsymbol{B}_v = 0, \quad \nabla \cdot \boldsymbol{D}_v = 0, \tag{37}$$

$$\boldsymbol{D}_v = \varepsilon_0 \boldsymbol{E}_v, \quad \boldsymbol{B}_v = \mu_0 \boldsymbol{H}_v, \tag{38}$$

where $\boldsymbol{E}_v$, $\boldsymbol{H}_v$, $\boldsymbol{D}_v$, and $\boldsymbol{B}_v$ are the electric and magnetic fields, and inductions in vacuum.

The jump conditions take the following form

$$(\boldsymbol{E}-\boldsymbol{E}_v)\times\boldsymbol{n}=0,\qquad (\boldsymbol{H}-\boldsymbol{H}_v)\times\boldsymbol{n}=\boldsymbol{i}_s+\rho_{es}\boldsymbol{v}_s, \tag{39}$$

$$(\boldsymbol{D}-\boldsymbol{D}_v)\cdot\boldsymbol{n}=-\rho_{es},\qquad (\boldsymbol{B}-\boldsymbol{B}_v)\cdot\boldsymbol{n}=0. \tag{40}$$

Here, ρ_{es} and $\boldsymbol{i}_s$ are the surface densities of electric charges and current; $\boldsymbol{v}_s$ is a tangent component of velocity to the body surface.

To solve non-stationary problems, it is necessary to write the corresponding initial conditions. We write them as follows

$$\boldsymbol{u}=\boldsymbol{u}^0,\quad \frac{\partial\boldsymbol{u}}{\partial t}=\boldsymbol{v}^0,\quad \theta=0,\quad \tilde{\mu}'_\pi=0,\quad \boldsymbol{E}=\boldsymbol{E}^0,\quad \boldsymbol{B}=\boldsymbol{B}^0,\quad \text{at}\quad t=0. \tag{41}$$

Note that in conditions (41) it is assumed that the initial time corresponds to the reference equilibrium state of the thermodynamic system.

The coupled initial-boundary-value problem is to determine the displacement vector $\boldsymbol{u}(\boldsymbol{r},t)$, temperature change $\theta(\boldsymbol{r},t)$, modified chemical potential $\tilde{\mu}'_\pi(\boldsymbol{r},t)$, electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and magnetic induction $\boldsymbol{B}(\boldsymbol{r},t)$ of $C^{(2)}$ in the medium, governed by the equations (28)–(32) and subject to the boundary conditions (33)–(35), jump conditions (39), (40), and initial conditions (41). We define the electric field $\boldsymbol{E}_\nu(\boldsymbol{r},t)$ and magnetic induction $\boldsymbol{B}_\nu(\boldsymbol{r},t)$ in a vacuum using the equations (37) and (38).

5. Uniqueness theorem

As shown above, the set of differential equations for local gradient electrothermoelasticity can be divided into two uncoupled subsystems. In view of this, we study the conditions of uniqueness of the solution to the corresponding problems in mathematical physics in two stages: separately for the equations of motion (20), heat conduction (26) and modified chemical potential (27), and separately for the equations of electrodynamics (31) and (32).

Theorem 1. *For a domain (V) bounded by a smooth surface (Σ), and positive G, $K-\frac{2}{3}G-K^2\alpha_\rho^2/(\rho_o d_\rho)$, C_V, d_ρ, χ_m, and H_* there is no more than one set of functions $\boldsymbol{u}(\boldsymbol{r},t)$, $\theta(\boldsymbol{r},t)$, and $\tilde{\mu}'_\pi(\boldsymbol{r},t)$ that*

- $\forall\boldsymbol{r}\in(V)\cup(\Sigma):(\boldsymbol{u},\theta,\tilde{\mu}'_\pi)\in C^{(2)}$;
- $\forall\boldsymbol{r}\in(V)$ *satisfies the set of differential equations* (28)–(30);
- $\forall\boldsymbol{r}\in(V)\cup(\Sigma)$ *satisfies the strain-displacement relation* $(12)_2$, *the constitutive equations* (17) *and* $(18)_1$;
- *satisfies the boundary and initial conditions*:

$$\hat{\boldsymbol{\sigma}}_*\cdot\boldsymbol{n}=\boldsymbol{\sigma}^a,\quad \boldsymbol{J}_q\cdot\boldsymbol{n}-H_*(\theta-\theta_c)=0,\quad \tilde{\mu}'_\pi=\mu'_{\pi a},\quad \forall\boldsymbol{r}\in(\Sigma),$$

$$\boldsymbol{u}=\boldsymbol{u}^0,\quad \frac{\partial\boldsymbol{u}}{\partial t}=\boldsymbol{v}^0,\quad \theta=0,\quad \tilde{\mu}'_\pi=0,\quad \text{at}\quad t=0.$$

Proof. For the linear problems, the material time derivative is equal to Eulerian time derivative. Then, using the Gibbs equation (1) and the Legendre transformation $u=f+Ts+\boldsymbol{E}_*\cdot\boldsymbol{\pi}_e-\nabla\mu'_\pi\cdot\boldsymbol{\pi}_m$, for the specific internal energy u, we can write

$$\rho_o\frac{\partial u}{\partial t}=\rho_o T_o\frac{\partial s}{\partial t}+\rho_o\mu'_{\pi o}\frac{\partial\rho_m}{\partial t}+\hat{\boldsymbol{\sigma}}_*:\frac{\partial\hat{\boldsymbol{e}}}{\partial t}+\rho_o\theta\frac{\partial s}{\partial t}+\rho_o\tilde{\mu}'_\pi\frac{\partial\rho_m}{\partial t}-\rho_o\boldsymbol{\nabla}\tilde{\mu}'_\pi\cdot\frac{\partial\boldsymbol{\pi}_m}{\partial t}. \tag{42}$$

We substitute the constitutive equations (17) into nonlinear summands of the relation (42). After some transformations, we obtain:

$$\rho_o\frac{\partial u}{\partial t}=\rho_o T_o\frac{\partial s}{\partial t}+\rho_o\mu'_{\pi 0}\frac{\partial\rho_m}{\partial t}+\frac{1}{2}\frac{\partial}{\partial t}\Big[\Big(\bar{K}-\frac{2}{3}G\Big)I_1^2+2GI_2+\rho_o\frac{\bar{C}_V}{T_o}\theta^2+\frac{\rho_o}{d_\rho}(\tilde{\mu}'_\pi)^2+\rho_o\chi_m\boldsymbol{\nabla}\tilde{\mu}'_\pi\cdot\boldsymbol{\nabla}\tilde{\mu}'_\pi+2\rho_o\frac{\beta_{T_\rho}}{d_\rho}\theta\tilde{\mu}'_\pi\Big]. \tag{43}$$

The proof of the theorem will be based on the energy balance equation, which for the model of the thermoelastic solid has the form [Kondrat and Hrytsyna 2009]

$$\frac{\partial}{\partial t}\int_{(V)}\rho_o\Big(u+\frac{1}{2}\boldsymbol{v}^2\Big)\,dV=\int_{(V)}\rho_o(\boldsymbol{F}\cdot\boldsymbol{v}+\mathfrak{R})\,dV-\oint_{(\Sigma)}\Big(\boldsymbol{J}_q-\hat{\boldsymbol{\sigma}}_*\cdot\boldsymbol{v}+\mu'_\pi\frac{\partial\boldsymbol{\Pi}_m}{\partial t}\Big)\cdot\boldsymbol{n}\,d\Sigma.$$

Hence, making use of the expression (43), formula (19), kinetic equation $(18)_1$, entropy balance equation (7) and divergence theorem, we can write

$$\frac{\partial\mathscr{E}_*}{\partial t}=\int_{(V)}\Big(\rho_o\boldsymbol{F}\cdot\boldsymbol{v}+\rho_o\mathfrak{R}\frac{\theta}{T}-T_o\sigma_s\Big)\,dV+\oint_{(\Sigma)}\Big[\boldsymbol{\sigma}_n\cdot\boldsymbol{v}-\frac{H_*}{T}\theta^2+\frac{\theta}{T}\Big(\lambda\frac{\partial\theta}{\partial n}+H_*\theta\Big)-\rho_o\tilde{\mu}'_\pi\frac{\partial\pi_{mn}}{\partial t}\Big]\,d\Sigma. \tag{44}$$

Here, $(\partial\theta/\partial n)=\nabla\theta\cdot\boldsymbol{n}$, $\pi_{mn}=\boldsymbol{\pi}_m\cdot\boldsymbol{n}$, and

$$\mathscr{E}_*=\frac{1}{2}\int_{(V)}\Big[\rho_o\boldsymbol{v}^2+\Big(\bar{K}-\frac{2}{3}G\Big)I_1^2+2GI_2+\rho_o\frac{C_V}{T_o}\theta^2+\rho_o\chi_m(\boldsymbol{\nabla}\tilde{\mu}'_\pi)^2+\frac{\rho_o}{d_\rho}(\tilde{\mu}'_\pi+\beta_{T_\rho}\theta)^2\Big]\,dV. \tag{45}$$

The energy balance (44) makes it possible to prove the uniqueness of the solution.

We assume that two distinct solutions $\boldsymbol{u}_1(\boldsymbol{r},t)$, $\theta_1(\boldsymbol{r},t)$, $\tilde{\mu}'_{\pi 1}(\boldsymbol{r},t)$ and $\boldsymbol{u}_2(\boldsymbol{r},t)$, $\theta_2(\boldsymbol{r},t)$, $\tilde{\mu}'_{\pi 2}(\boldsymbol{r},t)$ satisfy the equations (28)–(30) and the appropriate boundary and initial conditions. Their difference $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$, $\theta=\theta_1-\theta_2$, and $\tilde{\mu}'_\pi=\tilde{\mu}'_{\pi 1}-\tilde{\mu}'_{\pi 2}$ therefore satisfies the homogeneous equations (28)–(30) and the homogeneous boundary and initial conditions:

$$\forall\boldsymbol{r}\in(\Sigma):\hat{\boldsymbol{\sigma}}_*\cdot\boldsymbol{n}=0,\quad \lambda\frac{\partial\theta}{\partial n}+H_*\theta=0,\quad \tilde{\mu}'_\pi=0, \tag{46}$$

$$\boldsymbol{u}=0,\quad \frac{\partial\boldsymbol{u}}{\partial t}=0,\quad \theta=0,\quad \tilde{\mu}'_\pi=0,\quad \text{at}\quad t=0. \tag{47}$$

In view of the homogeneity of the equations and boundary conditions (46), from the equation of energy balance (44) we obtain

$$\frac{\partial\mathscr{E}_*}{\partial t}=-\int_{(V)}T_o\sigma_s\,dV-\oint_{(\Sigma)}\frac{H_*}{T}\theta^2\,d\Sigma.$$

Because $\sigma_s\geq 0$ and $(H_*/T)\geq 0$, the following inequality should hold

$$\frac{\partial\mathscr{E}_*}{\partial t}\leq 0. \tag{48}$$

The difference of solutions satisfies the zero initial conditions, and, therefore $\mathscr{E}_*$ equals zero at the initial moment of time. Thus, from inequality (48) it follows that the function $\mathscr{E}_*$ is either negative or zero: $\mathscr{E}_* \le 0$. On the other hand, according to (45) we have that $\mathscr{E}_* > 0$ since G, $K-(2/3)G-(K^2\alpha_\rho^2/\rho_o d_\rho)$, C_V, d_ρ, and χ_m are positive-definite. The above two inequalities can be fulfilled only if $\mathscr{E}_* = 0$. Consequently, taking into account the formula (45), we can write

$$\int_{(V)} \left[\rho_o \boldsymbol{v}^2 + \left(\bar{K} - \frac{2}{3}G\right)I_1^2 + 2GI_2 + \rho_o \frac{C_V}{T_o}\theta^2 + \rho_o \chi_m (\nabla \tilde{\mu}'_\pi)^2 + \frac{\rho_o}{d_\rho}(\tilde{\mu}'_\pi + \beta_{T_\rho}\theta)^2\right] dV = 0.$$

Since $K-(2/3)G-(K^2\alpha_\rho^2/\rho_o d_\rho) > 0$, $G > 0$, $C_V > 0$, $d_\rho > 0$, and $\chi_m > 0$ and the relation in brackets is positive-definite, from last formula we get: $\boldsymbol{v} = 0$, $\hat{\boldsymbol{e}} = 0$, $\theta = 0$, $\tilde{\mu}'_\pi = 0$, and $\nabla \tilde{\mu}'_\pi = 0$. Using the constitutive equation (17), we also obtain that $\rho_m = 0$ and $\boldsymbol{\pi}_m = 0$. So: $\boldsymbol{u}_1 = \boldsymbol{u}_2$, $\theta_1 = \theta_2$, and $\tilde{\mu}'_{\pi 1} = \tilde{\mu}'_{\pi 2}$. Therefore the coupled initial-boundary-value problem of local gradient thermoelasticity has only one solution, which is what we set out to demonstrate. □

Theorem 2. *If ε_0, μ_0, χ_E, σ_e are positive and the functions $\boldsymbol{u}(\boldsymbol{r}, t)$, $\theta(\boldsymbol{r}, t)$, and $\tilde{\mu}'_\pi(\boldsymbol{r}, t)$ are known, then for the body domain (V) and vacuum (V_ν), separated by a smooth surface (Σ), there is not more than one set of functions $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_\nu, \boldsymbol{H}_\nu)$, such that*

- $\forall \boldsymbol{r} \in (V) \cup (\Sigma)$ *and* $\forall \boldsymbol{r}_\nu \in (V_\nu) \cup (\Sigma) : (\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_\nu, \boldsymbol{H}_\nu) \in C^{(2)}$;
- $\forall \boldsymbol{r} \in (V)$ *satisfy the differential equations* (8) *and* $\forall \boldsymbol{r}_\nu \in (V_\nu)$ *satisfy the equation* (37), *respectively*;
- $\forall \boldsymbol{r} \in (V) \cup (\Sigma)$ *satisfy the constitutive relations* (10), (17d) *and the kinetic equation* $(18)_2$, *and* $\forall \boldsymbol{r}_\nu \in (V_\nu) \cup (\Sigma)$ *satisfy the constitutive relations* (38), *respectively*;
- $\forall \boldsymbol{r}, \boldsymbol{r}_\nu \in (\Sigma)$ *fulfils the jump conditions* (39) *and the initial conditions*

$$\boldsymbol{E} = \boldsymbol{E}^0, \quad \boldsymbol{H} = \boldsymbol{H}^0, \quad \boldsymbol{E}_\nu = \boldsymbol{E}^0_\nu, \quad \boldsymbol{H}_\nu = \boldsymbol{H}^0_\nu, \quad \text{at} \quad t = 0.$$

Proof. Suppose that the two sets of fields $(\boldsymbol{E}_1, \boldsymbol{H}_1, \boldsymbol{E}_{\nu 1}, \boldsymbol{H}_{\nu 1})$ and $(\boldsymbol{E}_2, \boldsymbol{H}_2, \boldsymbol{E}_{\nu 2}, \boldsymbol{H}_{\nu 2})$ solve the above problem. The difference fields $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2$, $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2$, $\boldsymbol{E}_\nu = \boldsymbol{E}_{\nu 1} - \boldsymbol{E}_{\nu 2}$, and $\boldsymbol{H}_\nu = \boldsymbol{H}_{1\nu} - \boldsymbol{H}_{2\nu}$ satisfy the relations (8), (37), the trivial initial conditions, the constitutive relations (10), (38), as well as

$$\boldsymbol{\Pi}_e = \rho_o \chi_E \boldsymbol{E}, \tag{49}$$

and the kinetic equation

$$\boldsymbol{J}_e = \sigma_e \boldsymbol{E}. \tag{50}$$

These functions satisfy the following energy balance equations for the electromagnetic field [Burak et al. 2008]:

$$\frac{\partial U_e}{\partial t} + \nabla \cdot \boldsymbol{S}_e + \left(\boldsymbol{J}_e + \frac{\partial \boldsymbol{\Pi}_e}{\partial t}\right) \cdot \boldsymbol{E} = 0, \tag{51}$$

$$\frac{\partial U_{e\nu}}{\partial t} + \nabla \cdot \boldsymbol{S}_{e\nu} = 0, \tag{52}$$

where

$$U_e = \frac{1}{2}(\varepsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2), \qquad \boldsymbol{S}_e = \boldsymbol{E} \times \boldsymbol{H}, \tag{53}$$

$$U_{e\nu} = \frac{1}{2}(\varepsilon_0 \boldsymbol{E}_\nu^2 + \mu_0 \boldsymbol{H}_\nu^2), \quad \boldsymbol{S}_{e\nu} = \boldsymbol{E}_\nu \times \boldsymbol{H}_\nu. \tag{54}$$

Substituting the formulae (53), (49), and (50) into (51), after some manipulations we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}(\varepsilon \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2) + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \sigma_e \boldsymbol{E} \cdot \boldsymbol{E} = 0.$$

Here, ε is defined by the formula $(25)_1$. By integrating the obtained expression over the region (V), and using the divergence theorem, we obtain

$$\frac{1}{2}\int_{(V)} \frac{\partial}{\partial t}(\varepsilon \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2)\, dV = -\int_{(\Sigma)} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \boldsymbol{n}\, d\Sigma - \sigma_e \int_{(V)} \boldsymbol{E}^2\, dV. \tag{55}$$

Substituting the formula (54) into the equation (52) and integrating the obtained result over the domain (V_ν), we obtain

$$\frac{1}{2}\int_{(V_\nu)} \frac{\partial}{\partial t}(\varepsilon_0 \boldsymbol{E}_\nu^2 + \mu_0 \boldsymbol{H}_\nu^2)\, dV = \int_{(\Sigma)} (\boldsymbol{E}_\nu \times \boldsymbol{H}_\nu) \cdot \boldsymbol{n}\, d\Sigma. \tag{56}$$

Combining the expressions (55) and (56), we find that

$$\frac{\partial U_e^t}{\partial t} = -\int_{(\Sigma)} [(\boldsymbol{E} \times \boldsymbol{H}) - (\boldsymbol{E}_\nu \times \boldsymbol{H}_\nu)] \cdot \boldsymbol{n}\, d\Sigma - \sigma_e \int_{(V)} \boldsymbol{E}^2\, dV, \tag{57}$$

where

$$U_e^t = \frac{1}{2}\left[\int_{(V)} (\varepsilon \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2)\, dV + \int_{(V_\nu)} (\varepsilon_0 \boldsymbol{E}_\nu^2 + \mu_0 \boldsymbol{H}_\nu^2)\, dV\right] \geq 0. \tag{58}$$

In view of the jump conditions (39), we can write the expression (57) as follows

$$\frac{\partial U_e^t}{\partial t} = -\int_{(\Sigma)} \boldsymbol{E}_s \cdot \boldsymbol{i}_s\, d\Sigma - \sigma_e \int_{(V)} \boldsymbol{E}^2\, dV.$$

Since $\boldsymbol{i}_s = \sigma_e \boldsymbol{E}_s$, where $\boldsymbol{E}_s$ is the tangential component of vector of the electric field, we have

$$\frac{\partial U_e^t}{\partial t} = -\sigma_e \int_{(\Sigma)} \boldsymbol{E}_s^2\, d\Sigma - \sigma_e \int_{(V)} \boldsymbol{E}^2\, dV. \tag{59}$$

From the formula (59) it follows that $(\partial U_e^t/\partial t) \leq 0$, since σ_e is positive. Thus, U_e^t is either a decreasing function, or a constant. Since at the initial time $t = 0$ the functions $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{E}_\nu$, $\boldsymbol{H}_\nu$ satisfy the trivial initial conditions, then the function U_e^t is equal to zero at the initial moment in time. Hence $U_e^t \leq 0$. At the same time, as follows from (58), the function U_e^t is positive definite or equal to zero: $U_e^t \geq 0$. The last two inequalities hold only if $U_e^t = 0$. Thus

$$U_e^t = \frac{1}{2}\left[\int_{(V)} (\varepsilon \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2)\, dV + \int_{(V_\nu)} (\varepsilon_0 \boldsymbol{E}_\nu^2 + \mu_0 \boldsymbol{H}_\nu^2)\, dV\right] = 0.$$

Since $\varepsilon_0 > 0$, $\mu_0 > 0$, $\varepsilon = \varepsilon_0 + \rho_o \chi_E > 0$, from above equation we obtain that $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2 = 0$, $\boldsymbol{H} = \boldsymbol{H}_1 - \boldsymbol{H}_2 = 0$, $\boldsymbol{E}_\nu = \boldsymbol{E}_{\nu 1} - \boldsymbol{E}_{\nu 2} = 0$, and $\boldsymbol{H}_\nu = \boldsymbol{H}_{1\nu} - \boldsymbol{H}_{2\nu} = 0$. Thus $\boldsymbol{E}_1 = \boldsymbol{E}_2$, $\boldsymbol{H}_1 = \boldsymbol{H}_2$, $\boldsymbol{E}_{\nu 1} = \boldsymbol{E}_{\nu 2}$, and $\boldsymbol{H}_{1\nu} = \boldsymbol{H}_{2\nu}$, which is what had to be proved. This completes the proof. □

6. Reciprocal theorem

We consider two different stress-strain states of dielectric solid, caused by two sets of external loading, namely, the mass force $\boldsymbol{F}_*$ and $\boldsymbol{F}'_*$; the thermal sources $\mathfrak{R}$ and $\mathfrak{R}'$; the surface loadings $\boldsymbol{\sigma}_*$ and $\boldsymbol{\sigma}'_*$ on the surface (Σ_σ); the displacements $\boldsymbol{u}$ and $\boldsymbol{u}'$ on the surface (Σ_u); the surface electric charges $\boldsymbol{\Pi}_e \cdot \boldsymbol{n}$ and $\boldsymbol{\Pi}'_e \cdot \boldsymbol{n}$ on the surface (Σ_p); the electric potentials φ_e and φ'_e on the surface (Σ_φ); the disturbance of the temperature θ and θ' on the surface (Σ_θ); the heat fluxes $\boldsymbol{J}_q$ and $\boldsymbol{J}'_q$ on the surface (Σ_J); the vectors of local mass displacement $\boldsymbol{\pi}_m$ and $\boldsymbol{\pi}'_m$ on the surface (Σ_π) and the potentials $\tilde{\mu}'_\pi$ and $(\tilde{\mu}'_\pi)'$ on the surface (Σ_μ). Here, $(\Sigma_\sigma) \cup (\Sigma_u) = (\Sigma)$, $(\Sigma_\sigma) \cap (\Sigma_u) = \varnothing$, $(\Sigma_\theta) \cup (\Sigma_J) = (\Sigma)$, $(\Sigma_\theta) \cap (\Sigma_J) = \varnothing$, $(\Sigma_\varphi) \cup (\Sigma_p) = (\Sigma)$, $(\Sigma_\varphi) \cap (\Sigma_p) = \varnothing$, $(\Sigma_\pi) \cup (\Sigma_\mu) = (\Sigma)$, $(\Sigma_\pi) \cap (\Sigma_\mu) = \varnothing$. The consequence of such an external action is the two states of the body, which we can be described by the stress tensors $\hat{\boldsymbol{\sigma}}_*$, $\hat{\boldsymbol{\sigma}}'_*$ and strain tensors $\hat{\boldsymbol{e}}$, $\hat{\boldsymbol{e}}'$, by disturbances of temperature θ, and θ' specific entropies s, s', by specific densities of induced mass ρ_m, ρ'_m and modified potentials $\tilde{\mu}'_\pi$, $(\tilde{\mu}'_\pi)'$, by the specific vectors of local mass displacement $\boldsymbol{\pi}_m$, $\boldsymbol{\pi}'_m$ and gradients of potentials $\nabla \tilde{\mu}'_\pi$, $(\nabla \tilde{\mu}'_\pi)'$, as well as by the specific vectors of polarization $\boldsymbol{\pi}_e$, $\boldsymbol{\pi}'_e$ and the electric fields $\boldsymbol{E}$, $\boldsymbol{E}'$ correspondingly.

We apply a one-sided Laplace transform

$$\mathscr{L}[f(\boldsymbol{r}, t)] = f^L(\boldsymbol{r}, \zeta) = \int_0^\infty f(\boldsymbol{r}, t)\, e^{-\zeta t}\, dt,$$

to the equations of the local gradient theory of dielectrics that are provided in Section 2. Here, $f(\boldsymbol{r}, t) = \{\hat{\boldsymbol{\sigma}}_*, \hat{\boldsymbol{e}}, \boldsymbol{F}_*, \boldsymbol{u}, \boldsymbol{B}, \boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{\pi}_e, \boldsymbol{\pi}_m, \theta, \mu'_\pi, \rho_m, \mathfrak{R}\}$, and ζ is a parameter of the Laplace transform.

Assume that all initial conditions for the perturbation of functions are equal to zero. For the considered two systems of external loads, applying a Laplace transform to the linearized momentum equation (6), we obtain

$$\nabla \cdot \hat{\boldsymbol{\sigma}}^L_* + \rho_o \boldsymbol{F}^L_* = \rho_o \zeta^2 \boldsymbol{u}^L, \tag{60}$$

$$\nabla \cdot \hat{\boldsymbol{\sigma}}'^L_* + \rho_o \boldsymbol{F}'^L_* = \rho_o \zeta^2 \boldsymbol{u}'^L. \tag{61}$$

Multiplying the equations (60) and (61) by the displacement vectors $\boldsymbol{u}'^L$ and $\boldsymbol{u}^L$, respectively, taking a difference between the obtained relations and integrating the result over the body volume (V), we obtain the following formula:

$$\int_{(V)} \big[(\nabla \cdot \hat{\boldsymbol{\sigma}}^L_*) \cdot \boldsymbol{u}'^L + \rho_o \boldsymbol{F}^L_* \cdot \boldsymbol{u}'^L - (\nabla \cdot \hat{\boldsymbol{\sigma}}'^L_*) \cdot \boldsymbol{u}^L - \rho_o \boldsymbol{F}'^L_* \cdot \boldsymbol{u}^L\big]\, dV = 0. \tag{62}$$

Making use of the relations

$$(\nabla \cdot \hat{\boldsymbol{\sigma}}^L_*) \cdot \boldsymbol{u}'^L = \nabla \cdot (\hat{\boldsymbol{\sigma}}^L_* \cdot \boldsymbol{u}'^L) - \hat{\boldsymbol{\sigma}}^L_* : \nabla \boldsymbol{u}'^L, \quad (\nabla \cdot \hat{\boldsymbol{\sigma}}'^L_*) \cdot \boldsymbol{u}^L = \nabla \cdot (\hat{\boldsymbol{\sigma}}'^L_* \cdot \boldsymbol{u}^L) - \hat{\boldsymbol{\sigma}}'^L_* : \nabla \boldsymbol{u}^L,$$

the formula (12)$_2$ and the divergence theorem, from the integral equation (62), we arrive at

$$\int_{(\Sigma)}\big(\boldsymbol{\sigma}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{\sigma}_*'^L\cdot\boldsymbol{u}^L\big)\,d\Sigma+\int_{(V)}\rho_o\big(\boldsymbol{F}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{F}_*'^L\cdot\boldsymbol{u}^L\big)\,dV=\int_{(V)}\big(\hat{\boldsymbol{\sigma}}_*^L:\hat{\boldsymbol{e}}'^L-\hat{\boldsymbol{\sigma}}_*'^L:\hat{\boldsymbol{e}}^L\big)\,dV.$$

Here $\boldsymbol{\sigma}_*^L=\hat{\boldsymbol{\sigma}}_*^L\cdot\boldsymbol{n}$, and $\boldsymbol{\sigma}_*'^L=\hat{\boldsymbol{\sigma}}_*'^L\cdot\boldsymbol{n}$.

Substituting the constitutive equation (17a) into the right-hand side of the obtained equation leads to the following result:

$$\int_{(\Sigma)}\big(\boldsymbol{\sigma}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{\sigma}_*'^L\cdot\boldsymbol{u}^L\big)\,d\Sigma+\int_{(V)}\rho_o\big(\boldsymbol{F}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{F}_*'^L\cdot\boldsymbol{u}^L\big)\,dV$$
$$=K\alpha_T\int_{(V)}(\theta'^Le^L-\theta^Le'^L)\,dV-K\alpha_\rho\int_{(V)}\big(\rho_m^Le'^L-\rho_m'^Le^L\big)\,dV. \quad (63)$$

Let us return to the equation of entropy balance (7). Making use of constitutive equations (17b) and (18)$_1$, from this equation in linear approximation we obtain the following heat equation for ideal dielectrics

$$\rho_oC_V\frac{\partial\theta}{\partial t}=\lambda\Delta\theta-T_oK\alpha_T\frac{\partial e}{\partial t}-\rho_oT_o\beta_{T_\rho}\frac{\partial\rho_m}{\partial t}+\rho_o\mathfrak{R}. \quad (64)$$

Applying a Laplace transform to the equation (64), for two systems of external loads, we can write

$$\rho_oC_V\zeta\theta^L=\lambda\Delta\theta^L-T_oK\alpha_T\zeta e^L-\rho_oT_o\beta_{T_\rho}\zeta\rho_m^L+\rho_o\mathfrak{R}^L, \quad (65)$$
$$\rho_oC_V\zeta\theta'^L=\lambda\Delta\theta'^L-T_oK\alpha_T\zeta e'^L-\rho_oT_o\beta_{T_\rho}\zeta\rho_m'^L+\rho_o\mathfrak{R}'^L. \quad (66)$$

Multiplying the equations (65) and (66) by the functions θ'^L and θ^L, respectively, taking a difference between the obtained expressions and integrating the result over the region (V), eventually we find that

$$\lambda\int_{(V)}(\theta'^L\Delta\theta^L-\theta^L\Delta\theta'^L)\,dV-T_oK\alpha_T\zeta\int_{(V)}(\theta'^Le^L-\theta^Le'^L)\,dV$$
$$-\rho_oT_o\beta_{T_\rho}\zeta\int_{(V)}\big(\theta'^L\rho_m^L-\theta^L\rho_m'^L\big)\,dV+\rho_o\int_{(V)}(\theta'^L\mathfrak{R}^L-\theta^L\mathfrak{R}'^L)\,dV=0.$$

In the first integral of the formula obtained above, we take into account the following expressions: $\theta'^L\Delta\theta^L-\theta^L\Delta\theta'^L=\boldsymbol{\nabla}\cdot(\theta'^L\boldsymbol{\nabla}\theta^L-\theta^L\boldsymbol{\nabla}\theta'^L)$. Making use the divergence theorem, we can rewrite this relation as follows

$$\frac{\lambda}{\zeta T_o}\int_{(\Sigma)}(\theta'^L\boldsymbol{\nabla}\theta^L-\theta^L\boldsymbol{\nabla}\theta'^L)\cdot\boldsymbol{n}\,d\Sigma-K\alpha_T\int_{(V)}(\theta'^Le^L-\theta^Le'^L)\,dV$$
$$-\rho_o\beta_{T_\rho}\int_{(V)}\big(\theta'^L\rho_m^L-\theta^L\rho_m'^L\big)\,dV+\frac{\rho_o}{\zeta T_o}\int_{(V)}(\theta'^L\mathfrak{R}^L-\theta^L\mathfrak{R}'^L)\,dV=0. \quad (67)$$

We restrict ourselves to considering a quasi-static electric field and assume that: $\boldsymbol{E}=-\boldsymbol{\nabla}\varphi_e$. In view of constitutive equation (10)$_2$, applying a Laplace transform to the equation (8)$_4$, we obtain

$$-\varepsilon_0\nabla^2\varphi_e^L+\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e^L=0,\quad -\varepsilon_0\nabla^2\varphi_e'^L+\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_e'^L=0. \quad (68)$$

We multiply these equations by the functions $\varphi_e'^L$ and φ_e^L. Proceeding in a similar manner, we obtain

$$-\varepsilon_0\int_{(V)}\big[\nabla\cdot(\nabla\varphi_e^L)\,\varphi_e'^L-\nabla\cdot(\nabla\varphi_e'^L)\,\varphi_e^L\big]\,dV=\int_{(V)}\big[(\nabla\cdot\boldsymbol{\Pi}_e^L)\,\varphi_e'^L-(\nabla\cdot\boldsymbol{\Pi}_e'^L)\,\varphi_e^L\big]\,dV. \tag{69}$$

Further, we take into account the following expressions

$$\begin{aligned}\nabla\cdot(\nabla\varphi_e^L)\,\varphi_e'^L-\nabla\cdot(\nabla\varphi_e'^L)\,\varphi_e^L&=\nabla\cdot\big[(\nabla\varphi_e^L)\,\varphi_e'^L\big]-\nabla\cdot\big[(\nabla\varphi_e'^L)\,\varphi_e^L\big],\\(\nabla\cdot\boldsymbol{\Pi}_e^L)\,\varphi_e'^L-(\nabla\cdot\boldsymbol{\Pi}_e'^L)\,\varphi_e^L&=\nabla\cdot(\boldsymbol{\Pi}_e^L\varphi_e'^L)-\nabla\cdot(\boldsymbol{\Pi}_e'^L\varphi_e^L)+\boldsymbol{\Pi}_e'^L\cdot(\nabla\varphi_e^L)-\boldsymbol{\Pi}_e^L\cdot(\nabla\varphi_e'^L).\end{aligned} \tag{70}$$

In view of the relations (70) and the divergence theorem, the equation (69) may be written as follows:

$$\int_{(\Sigma)}(\varphi_e'^L\boldsymbol{D}^L-\varphi_e^L\boldsymbol{D}'^L)\cdot\boldsymbol{n}\,d\Sigma=\int_{(V)}(\boldsymbol{\Pi}_e^L\cdot\nabla\varphi_e'^L-\boldsymbol{\Pi}_e'^L\cdot\nabla\varphi_e^L)\,dV. \tag{71}$$

Combining the equations (63), (67), and (71) yields

$$\begin{aligned}&\int_{(\Sigma)}\big[\boldsymbol{\sigma}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{\sigma}_*'^L\cdot\boldsymbol{u}^L+(\varphi_e'^L\boldsymbol{D}^L-\varphi_e^L\boldsymbol{D}'^L)\cdot\boldsymbol{n}\big]\,d\Sigma\\&\quad-\frac{\lambda}{\zeta T_o}\int_{(\Sigma)}(\theta'^L\nabla\theta^L-\theta^L\nabla\theta'^L)\cdot\boldsymbol{n}\,d\Sigma\\&\qquad+\rho_o\int_{(V)}(\boldsymbol{F}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{F}_*'^L\cdot\boldsymbol{u}^L)\,dV-\frac{\rho_o}{\zeta T_o}\int_{(V)}(\theta'^L\mathfrak{R}^L-\theta^L\mathfrak{R}'^L)\,dV\\&\qquad\qquad=\int_{(V)}[\boldsymbol{\Pi}_e^L\cdot\nabla\varphi'^L-\boldsymbol{\Pi}_e'^L\cdot\nabla\varphi^L]\,dV\\&\qquad\qquad\quad-\int_{(V)}\big[K\alpha_\rho(\rho_m^Le'^L-\rho_m'^Le^L)+\rho_o\beta_{T_\rho}(\theta'^L\rho_m^L-\theta^L\rho_m'^L)\big]\,dV.\end{aligned} \tag{72}$$

We simplify the integrand in the right-hand side of the equation (72). First, we transform the integrand in the last line of this equation. Using the constitutive relation (17c) we obtain the following formulae

$$K\alpha_\rho e^L=-\rho_o\tilde{\mu}_\pi'^L+\rho_od_\rho\rho_m^L-\rho_o\beta_{T_\rho}\theta^L,\qquad K\alpha_\rho e'^L=-\rho_o(\tilde{\mu}_\pi')'^L+\rho_od_\rho\rho_m'^L-\rho_o\beta_{T_\rho}\theta'^L. \tag{73}$$

Substituting the expressions (73) into the integrand, we can write

$$K\alpha_\rho\big(\rho_m^Le'^L-\rho_m'^Le^L\big)+\rho_o\beta_{T_\rho}\big(\theta'^L\rho_m^L-\theta^L\rho_m'^L\big)=\rho_o\big[\rho_m'^L\tilde{\mu}_\pi'^L-\rho_m^L(\tilde{\mu}_\pi')'^L\big]. \tag{74}$$

In view of the constitutive relations (17), it can be shown that the following expression are true for a quasi-static electric field

$$\boldsymbol{\Pi}_e^L\cdot\nabla\varphi_e'^L-\boldsymbol{\Pi}_e'^L\cdot\nabla\varphi_e^L=\rho_o\big[(\nabla\tilde{\mu}_\pi')^L\cdot\boldsymbol{\pi}_m'^L-(\nabla\tilde{\mu}_\pi')'^L\cdot\boldsymbol{\pi}_m^L\big]. \tag{75}$$

Using the constitutive relations (74) and (75), as well as the formula (19), we transform the right-hand side of the equation (72) to obtain

$$\begin{aligned}&\int_{(V)}\big[\boldsymbol{\Pi}_e^L\cdot\nabla\varphi_e'^L-\boldsymbol{\Pi}_e'^L\cdot\nabla\varphi_e^L-K\alpha_\rho\big(\rho_m^Le'^L-\rho_m'^Le^L\big)+\rho_o\beta_{T_\rho}\big(\theta'^L\rho_m^L-\theta^L\rho_m'^L\big)\big]\,dV\\&\qquad=-\rho_o\int_{(V)}\nabla\cdot\big[\boldsymbol{\pi}_m^L(\tilde{\mu}_\pi')'^L-\boldsymbol{\pi}_m'^L\tilde{\mu}_\pi'^L\big]\,dV.\end{aligned} \tag{76}$$

Finally, substituting the expression (76) into (72) and taking into account the divergence theorem, we obtain the generalized reciprocity theorem in the transformed domain:

$$\zeta T_o\Big\{\int_{(\Sigma)}\big[\boldsymbol{\sigma}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{\sigma}_*'^L\cdot\boldsymbol{u}^L+(\varphi_e'^L\boldsymbol{D}^L-\varphi_e^L\boldsymbol{D}'^L)\cdot\boldsymbol{n} + \rho_o\big(\boldsymbol{\pi}_m^L(\tilde{\mu}_\pi')'^L-\boldsymbol{\pi}_m'^L\tilde{\mu}_\pi'^L\big)\cdot\boldsymbol{n}\big]\,d\Sigma+\rho_o\int_{(V)}(\boldsymbol{F}_*^L\cdot\boldsymbol{u}'^L-\boldsymbol{F}_*'^L\cdot\boldsymbol{u}^L)\,dV\Big\} + \lambda\int_{(\Sigma)}(\theta^L\nabla\theta'^L-\theta'^L\nabla\theta^L)\cdot\boldsymbol{n}\,d\Sigma+\rho_o\int_{(V)}(\theta^L\mathfrak{R}'^L-\theta'^L\mathfrak{R}^L)\,dV=0.$$

Inverting the Laplace transform yields the reciprocity theorem in the desired form

$$T_o\Big\{\int_{(\Sigma)}\big[\boldsymbol{\sigma}_*\odot\boldsymbol{u}'-\boldsymbol{\sigma}_*'\odot\boldsymbol{u}+\varphi_e'\bigcirc(\boldsymbol{D}\cdot\boldsymbol{n})-\varphi_e\bigcirc(\boldsymbol{D}'\cdot\boldsymbol{n}) + \rho_o(\boldsymbol{\pi}_m\cdot\boldsymbol{n})\bigcirc(\tilde{\mu}_\pi')'-\rho_o(\boldsymbol{\pi}_m'\cdot\boldsymbol{n})\bigcirc\tilde{\mu}_\pi'\big]\,d\Sigma+\rho_o\int_{(V)}(\boldsymbol{F}_*\odot\boldsymbol{u}'-\boldsymbol{F}_*'\odot\boldsymbol{u})\,dV\Big\} + \lambda\int_{(\Sigma)}\big[\theta*(\nabla\theta'\cdot\boldsymbol{n})-\theta'*(\nabla\theta\cdot\boldsymbol{n})\big]\,d\Sigma+\rho_o\int_{(V)}(\theta*\mathfrak{R}'-\theta'*\mathfrak{R})\,dV=0. \quad (77)$$

Here we use the following notation to indicate the time convolutions:

$$\boldsymbol{f}\odot\boldsymbol{g}=\int_0^t\boldsymbol{f}(\boldsymbol{r},t-\tau)\cdot\frac{\partial\boldsymbol{g}(\boldsymbol{r},t)}{\partial\tau}\,d\tau,$$
$$f\bigcirc g=\int_0^t f(\boldsymbol{r},t-\tau)\frac{\partial g(\boldsymbol{r},\tau)}{\partial\tau}\,d\tau,$$
$$f*g=\int_0^t f(\boldsymbol{r},t-\tau)\,g(\boldsymbol{r},\tau)\,d\tau.$$

The equation (77) corresponds to the reciprocity theorem generalized to non-stationary problems of the linear theory of local gradient electrothermoelasticity. It is worth noting that the occurrence of convolutions $\rho_o(\boldsymbol{\pi}_m\cdot\boldsymbol{n})\odot(\tilde{\mu}_\pi')'$ and $\rho_o(\boldsymbol{\pi}_m'\cdot\boldsymbol{n})\odot\tilde{\mu}_\pi'$ in (77) is caused by the accounting for local mass displacement. In the absence of the local mass displacement effects, the equation (77) reduces to the reciprocity relation of the classical thermopiezoelectricity obtained by Nowacki [1965; 1983].

For stationary processes, the equations (77) simplifies to the following form

$$\int_{(\Sigma)}\big\{\boldsymbol{\sigma}_*\cdot\boldsymbol{u}'-\boldsymbol{\sigma}_*'\cdot\boldsymbol{u}+(\varphi_e'\boldsymbol{D}-\varphi_e\boldsymbol{D}')\cdot\boldsymbol{n}+\rho_o\big[(\tilde{\mu}_\pi')'\boldsymbol{\pi}_m-\tilde{\mu}_\pi'\boldsymbol{\pi}_m'\big]\cdot\boldsymbol{n}\big\}\,d\Sigma + \rho_o\int_{(V)}\Big[\boldsymbol{F}_*\cdot\boldsymbol{n}'-\boldsymbol{F}_*'\cdot\boldsymbol{u}+\beta_{T_\rho}(\rho_m'\theta-\rho_m\theta')+\frac{K\alpha_T}{\rho_o}(e'\theta-e\theta')\Big]\,dV=0. \quad (78)$$

7. Conclusion

The paper presents a complete set of equations of a continuum-type local gradient model of electrothermoelastic nonferromagnetic solid dielectrics that accounts for the processes of deformation, heat conduction, polarization, and local mass displacement. A governing set of equations and the corresponding boundary conditions are obtained at a linear approximation. It is shown that this set of equations can be divided into two subsets that can be solved consecutively. This allows us to investigate the uniqueness of the solution to the stated linear boundary problems in mathematical physics in two stages: (i) by proving the uniqueness of solution to the problem for a thermoelastic continuum, which accounts for the relationship between thermomechanic processes and the local mass displacement, and (ii) by proving the uniqueness of solution to Maxwell equations with the corresponding jump conditions. Using Laplace transforms, the reciprocity theorem is extended to the linear boundary-value problems of local gradient theory of electrothermoelastic dielectrics. This theorem may be used in the development of analytical methods of computation of the stress-strain state of nonferromagnetic polarized bodies, accounting for the process of local mass displacement.

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References

[Burak 1987] Y. Burak, "Constitutive equations of locally gradient thermomechanics", *Dopovidi Akad. Nauk URSR = Proc. Acad. Sci. Ukrain. SSR* **12** (1987), 19–23. In Ukrainian.

[Burak et al. 2007] Y. I. Burak, V. F. Kondrat, and O. R. Hrytsyna, "Subsurface mechanoelectromagnetic phenomena in thermoelastic polarized bodies in the case of local displacements of mass", *Fiz.-Khim. Mekh. Mater.* **43**:4 (2007), 5–17. In Ukrainian; translated in *Mater. Sci.* **43**:4 (2007), 449–463.

[Burak et al. 2008] Y. Burak, V. Kondrat, and O. Hrytsyna, "An introduction of the local displacements of mass and electric charge phenomena into the model of the mechanics of polarized electromagnetic solids", *J. Mech. Mater. Struct.* **3**:6 (2008), 1037–1046.

[Chapla et al. 2009] Y. Chapla, S. Kondrat, O. Hrytsyna, and V. Kondrat, "On electromechanical phenomena in thin dielectric films", *Task Quart.* **13**:1-2 (2009), 145–154.

[Hadjigeorgiou et al. 1999] E. P. Hadjigeorgiou, V. K. Kalpakides, and C. V. Massalas, "A general theory for elastic dielectrics – Part I: the vectorial approach", *Int. J. Non-Linear Mech.* **34**:5 (1999), 831–841.

[Hrytsyna 2012] O. R. Hrytsyna, "Oscillations of a layer of a crystal with cubic symmetry under the action of harmonic electric field", *Fiz.-Khim. Mekh. Mater.* **48**:5 (2012), 88–96. In Ukrainian; translated in *Mater. Sci.* **48**:5 (2013), 653–663.

[Hrytsyna 2017] O. R. Hrytsyna, "Influence of subsurface inhomogeneity on the propagation of SH waves in isotropic materials", *Fiz.-Khim. Mekh. Mater.* **53**:2 (2017), 128–134. In Ukrainian; translated in *Mater. Sci.* **53**:2 (2017), 272–281.

[Kafadar 1971] C. B. Kafadar, "The theory of multipoles in classical electromagnetism", *Int. J. Eng. Sci.* **9**:9 (1971), 831–853.

[Kalpakidis and Agiasofitou 2002] V. K. Kalpakidis and E. K. Agiasofitou, "On material equations in second order gradient electroelasticity", *J. Elasticity* **67** (2002), 205–227.

[Kalpakidis et al. 1995] V. K. Kalpakidis, E. P. Hadjigeorgiou, and C. V. Massalas, "A variational principle for elastic dielectrics with quadruple polarization", *Int. J. Eng. Sci.* **33**:6 (1995), 793–801.

[Kondrat and Hrytsyna 2008] V. Kondrat and O. Hrytsyna, "The equations of electro-magneto-thermo-mechanics of polarized nonferromagnetic solids taking into account a local displacement of mass", *Fiz.-mat. modelyuvannya ta inform. technologii = Physico-Mathematical Modeling and Information Technologies* **8** (2008), 69–83. In Ukrainian.

[Kondrat and Hrytsyna 2009] V. F. Kondrat and O. R. Hrytsyna, "Equations of thermomechanics of deformable bodies with regard for irreversibility of local displacement of mass", *J. Math. Sci.* **160**:4 (2009), 492–502.

[Kondrat and Hrytsyna 2010] V. F. Kondrat and O. R. Hrytsyna, "Mechanoelectromagnetic interaction in isotropic dielectrics with regard for the local displacement of mass", *J. Math. Sci.* **168**:5 (2010), 688–698.

[Kondrat and Hrytsyna 2012] V. Kondrat and O. Hrytsyna, "Local gradient theory of dielectrics with polarization inertia and irreversibility of local mass displacement", *J. Mech. Mater. Struct.* **7**:3 (2012), 285–296.

[Kumikov and Khokonov 1983] V. K. Kumikov and K. B. Khokonov, "On the measurement of surface free energy and surface tension of solid metals", *J. Appl. Phys.* **54**:3 (1983), 1346–1350.

[Liu et al. 2006] K. H. Liu, W. L. Wang, Z. Xu, L. Liao, X. D. Bai, and E. G. Wang, "*In situ* probing mechanical properties of individual tungsten oxide nanowires directly grown on tungsten tips inside transmission electron microscope", *Appl. Phys. Lett.* **89**:22 (2006), 221908.

[Maugin 1979] G. A. Maugin, "Nonlocal theories or gradient-type theories: a matter of convenience", *Arch. Mech.* **31** (1979), 15–26.

[Maugin 1988] G. A. Maugin, *Continuum mechanics of electromagnetic solids*, Noth Holland, Amsterdam, 1988.

[Mindlin 1972] R. D. Mindlin, "Elasticity, piezoelectricity and crystal lattice dynamics", *J. Elasticity* **2**:4 (1972), 217–282.

[Nam et al. 2006] C.-Y. Nam, P. Jaroenapibal, D. Tham, D. E. Luzzi, S. Evoy, and J. E. Fischer, "Diameter-dependent electromechanical properties of GaN nanowires", *Nano Lett.* **6**:2 (2006), 153–158.

[Nowacki 1965] W. Nowacki, "A reciprocity theorem for coupled mechanical and thermoelectric fields in piezoelectric crystals", *Proc. Vibr. Probl.* **6**:1 (1965), 3–11.

[Nowacki 1983] W. Nowacki, *Efekty elektromagnetyczne w stałych ciałach odkształcalnych*, Państwowe Wydawnictwo Naukowe, Warszawa, 1983.

[Nysten et al. 2005] B. Nysten, C. Fretigny, and S. Cuenot, "Elastic modulus of nanomaterials: resonant contact-AFM measurement and reduced-size effect", *Proc. Soc. Photo-Opt. Ins.* **5766** (2005), 78–88.

[Sahin and Dost 1988] E. Sahin and S. Dost, "A strain-gradient theory of elastic dielectrics with spatial dispersion", *Int. J. Eng. Sci.* **26**:12 (1988), 1231–1245.

[Yan and Jiang 2007] Z. Yan and L. Jiang, "Modified continuum mechanics modeling on size-dependent properties of piezoelectric nanomaterials: a review", *Nanomaterials* **7**:2 (2007), 27p.

[Yang 2006] J. Yang, "Review of a few topics in piezoelectricity", *Appl. Mech. Rev.* (*ASME*) **59**:6 (2006), 335–345.

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