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ANALYTICAL INVESTIGATION OF FREE VIBRATIONS OF A BOUNDED NONLINEAR BULK-ELASTIC MEDIUM IN A FIELD OF MASS FORCES

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Free vibrations of a homogeneous nonlinear bulk-elastic medium, namely a solid with negligible shear stiffness, occupying a bounded domain and nonuniformly deformed under the action of a field of mass forces are investigated through variational methods. The nonlinear constitutive law of bulk elasticity is assumed to be arbitrary, the applied field of mass forces is assumed to be an arbitrary potential field, and conditions of free sliding are prescribed on the whole boundary. The associated problem of free vibrations corresponds to significantly nonuniform distribution of mechanical parameters of the medium, which results in significantly varying coefficients of governing equations — a case where standard methods are inapplicable and results of analysis are almost absent. A crucial element of presented variational analysis is the use of derived by the authors earlier the canonical form for the second variation of the total potential energy. This canonical form enables to state and prove a modified spectral theorem, and additionally a comparison theorem for the free vibration frequencies of different media in different fields of mass forces, provided the media occupy domains possessing the same or similar shapes. For some special shapes, the bilateral bounds for all the free vibration frequencies are obtained. The results are illustrated by clarifying examples.

1. Introduction

"Vibration problems for inhomogeneous structural elements present a challenge to the acoustics and vibration community. In contrast to classical problems for homogeneous strings, rods, membranes or plates for which exact solutions and a variety of approximate methods are available for finding frequencies and mode shapes, analogous results for inhomogeneous bodies are relatively scarce" [Horgan and Chan 1999]. This kind of problems is addressed in the present paper, considering the free vibrations of nonlinear bulk-elastic media occupying bounded domains. The nonuniformity of initial state of the medium is a result of 'stratification', namely, nonuniform strain produced by the action of a mass force field. With the term "bulk-elastic media" we mean nonlinearly elastic media characterized by an elastic potential depending exclusively on the volumetric strain (for instance, compressible fluids and gases), so that bulk-elastic media have a null shear stiffness. Although, strictly speaking, the presented analysis is based on constitutive relations of this very type, one can expect that the results may be extended to media characterized by shear moduli which are small as compared to their bulk moduli (say, gels, foams, etc.).

The considered problem is related to numerous technical devices, for instance, gas storage tanks, distillation columns, and pipeline sections (in the cases when there is a significant variation in altitude along the pipe). In addition to engineering equipment, some mechanical systems of natural origin (for instance atmospheres of planets) can be of interest in geophysics, atmospheric physics, or astrophysics.

Keywords: free vibrations, bulk-elastic medium, field of mass forces, variational methods, bilateral bounds for frequencies.

In technical devices (especially those of large size) the vibrations are an undesirable (but inevitable) phenomenon, which may result even in the failure of the device. To prevent the excitation of resonant vibrations of large amplitude, it is desirable to know at least the ranges for free vibration frequencies, i.e. to find bilateral bounds for them, and also to understand how they depend on the mechanical properties of the bulk-elastic medium and on geometry of the domain. In the case of the gravity field for the domains large in the vertical dimension, the nonuniformity of density and bulk modulus distributions under the action of gravity is significant, which affects free vibrations. For large natural systems, as well as for rotating devices, where mass forces are implemented as the centrifugal forces, the nonuniformity of mass forces (that can manifest itself both as nonuniformity in absolute value and nonuniformity in the direction) can also play an important role.

Free vibrations of bulk-elastic media in bounded domains were not previously explored with regard for significantly nonuniform distribution of mechanical parameters of a medium. The well-known analytical solutions (based on equations with constant coefficients) are applicable only if nonuniformity of distribution of mechanical parameters of a compressible medium due to the action of mass forces is neglected (see e.g. Landau and Lifshitz 1987), that can be justified only in cases of low compressibility, or low mass forces, or small "vertical" dimension of the system. It should be mentioned that the particular case of constant sound velocity in a heavy medium (independent of pressure) and the resulting exponential distribution of density over depth, has been studied analytically and numerically by means of some special method (see e.g. Gaziev and Kopachevsky 2013). Nevertheless, this particular case does not exhaust the general problem, and its relevance motivates the purpose of the present article, namely, the general study of free vibrations of bulk-elastic media in bounded domains with regard for the action of mass forces. The bulk-elastic media are assumed to be constitutively homogeneous with an arbitrary nonlinear bulk-elastic law. The field of mass forces is assumed to be an arbitrary potential field. The domains are assumed to be bounded, simply connected, and having a piecewise smooth boundary (obeying some additional requirements which will be specified in detail in formulation of the lemma proved in Appendix A). The domains of several special shapes are studied separately.

Note that the problem setting is highly general, so that one can hardly expect to obtain any definite analytical results for the characteristics of free vibrations. In particular, the conventional methods of solving the corresponding problems in the presence of significant nonuniformity of coefficients and for rather arbitrary shape of the domain are not effective. The only known method that can be effective in this case is the variational method: the free vibration modes are the extremals of a well-known functional, and the squares of eigenfrequencies are the corresponding extremal values. The functional mentioned is the ratio of two quadratic functionals, namely, the second variation of the total potential energy of the system and its doubled kinetic energy (in the latter the velocity field is replaced by the field of small displacements). *The analytical expression for the second variation of the total potential energy is derived in principle easily and briefly through linearization of the equations of nonlinear elasticity, but this rather cumbersome expression yields almost nothing for investigation of free vibrations. A truly effective research tool is the usage of derived earlier by the authors* [Ryzhak et al. 2017]*: the canonical form of the second variation of the total potential energy. We use the term "canonical form" in the same sense as it is used in linear algebra with regard to quadratic forms: it does reveal the structure (in particular, the sign) of the quadratic functional representing the second variation. In contrast to the*

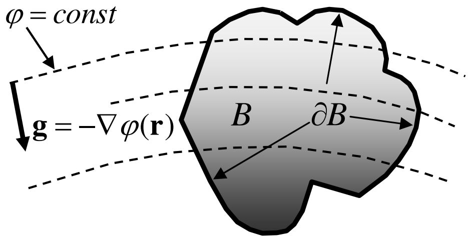

Figure 1. The mechanical system under investigation.

general case, when canonical form of the second variation consists of four terms [Ryzhak et al. 2017]*; in the problem under consideration it is especially simple and consists of a single term.*

In this paper, the investigation of free vibrations is carried out by the variational method, the crucial element of which being the usage of the above-mentioned canonical form of the second variation. Firstly, the canonical form reveals the not evident fact that for a homogeneous bulk-elastic medium the mass forces affect free vibrations only indirectly, specifically via resulting nonuniformity of equilibrium distribution of mechanical parameters of a medium. Secondly, the usage of the canonical form allows to prove for the considered class of problems some appropriate modifications of fundamental theorems of the theory of free vibrations for elastic solids. The modified theorems are applied both to compare the free vibration frequencies for different bulk-elastic media in domains of identical or similar shapes and to obtain analytical bilateral bounds for the frequencies in the case of domains that have the shape of either a rectangular parallelepiped or some different polyhedra, depending on the quantities characterizing nonuniformity of the equilibrium state of a medium and the geometry of a domain.

In Section 7 the general analytical results are illustrated by examples, where for a bulk-elastic medium of some specific type there are studied various situations characterized by different relations between gravity, geometry, and mechanical properties of the medium. The considered examples allow both to compare our results with the classical ones, corresponding to the case of governing equations with constant coefficients, and to clarify the situation where classical methods are inapplicable.

2. The investigated classes of mechanical systems and some elements of their kinematics in reference description

We assume that the potential of mass forces $\varphi(\boldsymbol{r})$, where $\boldsymbol{r}$ is the space position vector, is a continuously differentiable function with nonzero bounded gradient. The mass forces field intensity $\boldsymbol{g}$ is a vector opposite to the gradient of φ:

$$\boldsymbol{g} = -\nabla\varphi(\boldsymbol{r}). \tag{2-1}$$

The mechanical system under investigation consists of a constitutively homogeneous bulk-elastic medium occupying a domain with fixed and perfectly smooth boundary (Figure 1). A law of bulk elasticity is supposed to be arbitrary. In spite of constitutive homogeneity, the equilibrium state of the medium is nonuniform due to stratification caused by the action of mass forces. The boundary conditions of free sliding are posed on the whole of the boundary of the domain. The kinematical part of conditions of sliding specify the class of kinematically admissible motions of the medium at deviations of considered mechanical system from the equilibrium state which is assumed to be stable.

In all subsequent calculations we use the reference description of the continua considered (see for example Truesdell 1972). As a reference configuration κ (which is fixed in principle) it is taken the equilibrium configuration of the system. Material points of a continuum are identified by their position vectors $\boldsymbol{x}$ in the reference configuration; for each material point its identifier $\boldsymbol{x}$ is invariable. The actual position of a material point $\boldsymbol{x}$ is specified by the image of mapping

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{x}, t), \tag{2-2}$$

which will be called the "transformation"; here t is a time, $\boldsymbol{r}(\boldsymbol{x}, t)$ is the position vector of a given material point at a given time instant. Without loss of generality, we assume that the reference configuration coincides with the actual one at $t = 0$:

$$\boldsymbol{r}(\boldsymbol{x}, 0) = \boldsymbol{x}. \tag{2-3}$$

The domain occupied by the medium in the reference equilibrium state will be denoted B (Figure 1). The domain B is supposed to be bounded and simply connected. Its boundary ∂B is supposed to be piecewise smooth (Figure 1). Since the boundary is immovable, the domain occupied by the medium in any actual configuration is the same.

3. Some formulas of coordinateless tensor calculus and the relations associated with the reference and spatial descriptions of continua

We introduce some notations related to the reference description and derive equations permanently used in further analysis.

For any physical field $\boldsymbol{\Psi}(\boldsymbol{x}, t)$ (scalar, vector or tensor) its derivative with respect to t at a constant $\boldsymbol{x}$ will be called the "material" time derivative, conventionally denoted by a dot over a symbol:

$$\dot{\boldsymbol{\Psi}}(\boldsymbol{x}, t) := \left(\frac{\partial \boldsymbol{\Psi}}{\partial t}\right)_{\boldsymbol{x}}. \tag{3-1}$$

Note that the velocity and acceleration are the first and the second material derivatives of the position vector $\boldsymbol{r}(\boldsymbol{x}, t)$:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \dot{\boldsymbol{r}}(\boldsymbol{x}, t), \tag{3-2}$$

$$\dot{\boldsymbol{v}}(\boldsymbol{x}, t) = \ddot{\boldsymbol{r}}(\boldsymbol{x}, t). \tag{3-3}$$

Along with the material derivatives with respect to time, in some cases there will be used the "spatial" derivatives (i.e., the derivatives with respect to time at a constant $\boldsymbol{r}$), which will be denoted $(\partial \boldsymbol{\Psi}/\partial t)_{\boldsymbol{r}}$. The material and spatial derivatives with respect to time are related to each other by the well-known Euler's formula which will be presented below.

Throughout the paper we use the coordinateless tensor calculus in the notation, almost coinciding with the system of tensor notation of J. W. Gibbs. Presented below is a summary of the main formulas used in this work: just the minimum necessary to enable us to avoid citing any additional sources.

The main strain-rotation quantity in the reference description is the transformation gradient:

$$\boldsymbol{F}(\boldsymbol{x}, t) := \nabla_\kappa \otimes \boldsymbol{r}(\boldsymbol{x}, t), \tag{3-4}$$

$$d\boldsymbol{r}(\boldsymbol{x}, t, d\boldsymbol{x}) = d\boldsymbol{x} \cdot \nabla_\kappa \otimes \boldsymbol{r}(\boldsymbol{x}, t). \tag{3-5}$$

The transformation gradient $\boldsymbol{F}(\boldsymbol{x}, t)$ is a tensor of rank two (TR(2)), i.e. the linear operator that specifies the principal linear part of mapping of the vicinity of a material point $\boldsymbol{x}$ in the reference configuration into the vicinity of corresponding point $\boldsymbol{r}(\boldsymbol{x}, t)$ in the actual configuration.

The subscript "κ" referred to the symbol "nabla" in the gradient notation indicates that this is a "reference" gradient, i.e. differentiation is carried out with respect to $\boldsymbol{x}$. The gradient of a tensor field of rank k is defined in a completely similar way:

$$d\boldsymbol{M}(\boldsymbol{x}, t, d\boldsymbol{x}) = d\boldsymbol{x} \cdot \nabla_k \otimes \boldsymbol{M}(\boldsymbol{x}, t), \tag{3-6}$$

i.e. $\nabla_\kappa \otimes \boldsymbol{M}(\boldsymbol{x}, t)$ is a TR$(k+1)$ that specifies the principal linear part of the increment of the rank-k tensor field $\boldsymbol{M}(\boldsymbol{x}, t)$ in the vicinity of a point x in the reference configuration. In the case of a scalar field gradient, the tensor product sign $\otimes$ in the gradient notation is omitted ($\chi(\boldsymbol{x}, t) \leftrightarrow \nabla_\kappa \chi(\boldsymbol{x}, t)$).

Along with the reference description sometimes it is needed to use the spatial description with the independent variables $(\boldsymbol{r}, t)$. For the gradient with respect to $\boldsymbol{r}$ we will use the symbol ∇ (without a subscript):

$$d\boldsymbol{M}(\boldsymbol{r}, t, d\boldsymbol{r}) = d\boldsymbol{r} \cdot \nabla \otimes \boldsymbol{M}(\boldsymbol{r}, t). \tag{3-7}$$

Due to (3-5) we have

$$\nabla_\kappa \otimes \boldsymbol{M} = \boldsymbol{F} \cdot \nabla \otimes \boldsymbol{M}, \quad \nabla \otimes \boldsymbol{M} = \boldsymbol{F}^{-1} \cdot \nabla_\kappa \otimes \boldsymbol{M}. \tag{3-8}$$

Mentioned above Euler's formula is represented by the following equality:

$$\dot{\boldsymbol{M}}(\boldsymbol{r}, t) = \left(\frac{\partial \boldsymbol{M}}{\partial t}\right)_{\boldsymbol{r}}(\boldsymbol{r}, t) + \boldsymbol{v}(\boldsymbol{r}, t) \cdot \nabla \otimes \boldsymbol{M}(\boldsymbol{r}, t). \tag{3-9}$$

The reference and space divergences of a tensor field of rank k are the TR$(k-1)$ specified by the following equalities:

$$\nabla_\kappa \cdot \boldsymbol{M}(\boldsymbol{r}, t) := \boldsymbol{I} : \nabla_\kappa \otimes \boldsymbol{M}(\boldsymbol{x}, t), \tag{3-10}$$

$$\nabla \cdot \boldsymbol{M}(\boldsymbol{r}, t) := \boldsymbol{I} : \nabla \otimes \boldsymbol{M}(\boldsymbol{r}, t), \tag{3-11}$$

where $\boldsymbol{I}$ is the unit TR(2) (the unit linear operator in a vector space).

4. The nonlinear and linearized (incremental) relations of the theory of elasticity used in the work

We present without derivation those known formulas of the theory of elastic constitutive relations, which are used subsequently.

Let $\sigma_\kappa(\boldsymbol{F}, \boldsymbol{x})$ be the reference volume density of the elastic energy of a material at the point $\boldsymbol{x}$ of the reference configuration. Due to the principle of material objectivity [Truesdell 1972], it does not depend on rotations, i.e. for any proper orthogonal TR(2) $\boldsymbol{Q}$ (a rotation) and for any $\boldsymbol{F}$ (having a positive determinant), the following equality holds:

$$\sigma_\kappa(\boldsymbol{F} \cdot \boldsymbol{Q}, \boldsymbol{x}) = \sigma_\kappa(\boldsymbol{F}, \boldsymbol{x}). \tag{4-1}$$

The Piola stress tensor $\boldsymbol{T}_\kappa(\boldsymbol{F},\boldsymbol{x})$ is related to $\sigma_\kappa(\boldsymbol{F},\boldsymbol{x})$ by the following equivalent equalities:

$$\delta\sigma_\kappa(\boldsymbol{F},\boldsymbol{x}) = \delta\boldsymbol{F} : \boldsymbol{T}_\kappa(\boldsymbol{F},\boldsymbol{x}), \tag{4-2}$$

$$\boldsymbol{T}_\kappa(\boldsymbol{F},\boldsymbol{x}) = \frac{\partial\sigma_\kappa}{\partial\boldsymbol{F}}(\boldsymbol{F},\boldsymbol{x}). \tag{4-3}$$

The surface force vector $d\boldsymbol{f}$, which acts on a surface element with the normal $\boldsymbol{n}$ and the area $d\Sigma$ in the actual configuration, whose pre-image in the reference configuration is the surface element with the normal $\boldsymbol{n}_\kappa$ and the area $d\Sigma_\kappa$, can be expressed in terms of the Cauchy stress tensor $\boldsymbol{T}(\boldsymbol{F},\boldsymbol{x})$ and the Piola stress tensor $\boldsymbol{T}_\kappa(\boldsymbol{F},\boldsymbol{x})$ as follows:

$$d\boldsymbol{f} = d\Sigma\,\boldsymbol{n}\cdot\boldsymbol{T} = d\Sigma_\kappa\,\boldsymbol{n}_\kappa\cdot\boldsymbol{T}_\kappa, \tag{4-4}$$

which results in the known relationship between the two stress tensors:

$$\boldsymbol{T}_\kappa = (\det\boldsymbol{F})(\boldsymbol{F}^{-1})^T\cdot\boldsymbol{T}, \quad \boldsymbol{T} = \frac{1}{\det\boldsymbol{F}}\boldsymbol{F}^T\cdot\boldsymbol{T}_\kappa. \tag{4-5}$$

Due to the principle of material objectivity and in accordance with their own specific properties, the Piola and Cauchy stress tensors differently depend on rotations:

$$\boldsymbol{T}_\kappa(\boldsymbol{F}\cdot\boldsymbol{Q}) = \boldsymbol{T}_\kappa(\boldsymbol{F})\cdot\boldsymbol{Q} \tag{4-6}$$

$$\boldsymbol{T}(\boldsymbol{F}\cdot\boldsymbol{Q}) = \boldsymbol{Q}^T\cdot\boldsymbol{T}(\boldsymbol{F})\cdot\boldsymbol{Q} =: \boldsymbol{T}(\boldsymbol{F}) * \boldsymbol{Q}. \tag{4-7}$$

The following incremental (linearized) constitutive relations correspond to nonlinear relations $\boldsymbol{T}(\boldsymbol{F})$ and $\boldsymbol{T}_\kappa(\boldsymbol{F})$ (with regard for the principle of material objectivity and equalities (4-5)), the linearization being carried out in the vicinity of reference configuration (where $t = 0$, $\boldsymbol{r} = \boldsymbol{x}$, $\boldsymbol{F} = \boldsymbol{I}$, $\nabla = \nabla_\kappa$):

$$\delta\boldsymbol{T} = \boldsymbol{L} : \delta\boldsymbol{\varepsilon} + \boldsymbol{T}\cdot\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}\cdot\boldsymbol{T}, \tag{4-8}$$

$$\delta\boldsymbol{T}_\kappa|_{\boldsymbol{F}=\boldsymbol{I}} = \boldsymbol{L} : \delta\boldsymbol{\varepsilon} + \boldsymbol{T}(\boldsymbol{I} : \delta\boldsymbol{\varepsilon}) - \delta\boldsymbol{\varepsilon}\cdot\boldsymbol{T} + \boldsymbol{T}\cdot\delta\boldsymbol{\omega}. \tag{4-9}$$

Here, with regard for coincidence of the actual configuration with the reference one at $t = 0$, the incremental tensors of strain $\delta\boldsymbol{\varepsilon}$ and rotation $\delta\boldsymbol{\omega}$ are related to the small displacement field $\delta\boldsymbol{u}(\boldsymbol{x})$ by the following equalities:

$$\delta\boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla_\kappa\otimes\delta\boldsymbol{u} + \nabla_\kappa\otimes\delta\boldsymbol{u}^T), \quad \delta\boldsymbol{\omega} = \tfrac{1}{2}(\nabla_\kappa\otimes\delta\boldsymbol{u} - \nabla_\kappa\otimes\delta\boldsymbol{u}^T), \quad \delta\boldsymbol{\varepsilon} + \delta\boldsymbol{\omega} = \nabla_\kappa\otimes\delta\boldsymbol{u}. \tag{4-10}$$

A TR(4) $\boldsymbol{L}(\boldsymbol{F})$ is one of conventional elastic moduli tensors, the values of TR(4) $\boldsymbol{L}(\boldsymbol{F})$ being different for different states of one and the same elastic material.

Since $\delta\boldsymbol{\varepsilon}$ is symmetric, whereas $\delta\boldsymbol{\omega}$ is skew-symmetric, we have

$$\boldsymbol{I} : \delta\boldsymbol{\varepsilon} = \boldsymbol{I} : \nabla_\kappa\otimes\delta\boldsymbol{u} = \nabla_\kappa\cdot\delta\boldsymbol{u}. \tag{4-11}$$

If $\boldsymbol{T} = -p\boldsymbol{I}$, the stress state is called hydrostatic with pressure p. In the neighborhood of such a state the equalities (4-8) and (4-9) are simplified and take the form

$$\delta\boldsymbol{T} = \boldsymbol{L} : \delta\boldsymbol{\varepsilon}, \quad \delta\boldsymbol{T}_\kappa|_{\boldsymbol{F}=\boldsymbol{I}} = \boldsymbol{L} : \delta\boldsymbol{\varepsilon} - p(\nabla_\kappa\cdot\delta\boldsymbol{u})\boldsymbol{I} + p\nabla_\kappa\otimes\delta\boldsymbol{u}^T. \tag{4-12}$$

In what follows it will be assumed that the investigated equilibrium (unperturbed) configuration is characterized by a hydrostatic stress state in the entire domain B. The elastic potential of a bulk-elastic medium is characterized by the property that its dependence on the transformation gradient $\boldsymbol{F}$ is reduced to dependence on its determinant (i.e. on the volume change coefficient):

$$\sigma_\kappa(\boldsymbol{F}, \boldsymbol{x}) = \psi(\det \boldsymbol{F}, \boldsymbol{x}), \tag{4-13}$$

$$\boldsymbol{T}_\kappa(\boldsymbol{F}, \boldsymbol{x}) = \frac{d(\det \boldsymbol{F})}{d\boldsymbol{F}} \frac{\partial \psi}{\partial(\det \boldsymbol{F})}(\det \boldsymbol{F}, \boldsymbol{x}) = (\det \boldsymbol{F})(\boldsymbol{F}^{-1})^T \frac{\partial \psi}{\partial(\det \boldsymbol{F})}(\det \boldsymbol{F}, \boldsymbol{x}), \tag{4-14}$$

$$\boldsymbol{T}(\boldsymbol{F}, \boldsymbol{x}) = \frac{\partial \psi}{\partial(\det \boldsymbol{F})}(\det \boldsymbol{F}, \boldsymbol{x})\boldsymbol{I} =: -p(\det \boldsymbol{F}, \boldsymbol{x})\boldsymbol{I}, \tag{4-15}$$

$$\begin{aligned} \delta \boldsymbol{T}(\boldsymbol{F}, \boldsymbol{x})|_{\boldsymbol{F}=\boldsymbol{I}} &= -\frac{\partial p}{\partial(\det \boldsymbol{F})}(1, \boldsymbol{x})\, \delta(\det \boldsymbol{F})|_{\boldsymbol{F}=\boldsymbol{I}} \boldsymbol{I} \\ &= -\frac{\partial p}{\partial(\det \boldsymbol{F})}(1, \boldsymbol{x})\, \boldsymbol{I}(\boldsymbol{I} : \delta\boldsymbol{\varepsilon}) =: K(1, \boldsymbol{x})\, \boldsymbol{I}(\boldsymbol{I} : \delta\boldsymbol{\varepsilon}), \end{aligned} \tag{4-16}$$

$$\Longrightarrow \quad \boldsymbol{L}(1, \boldsymbol{x}) = K(1, \boldsymbol{x})\, \boldsymbol{I} \otimes \boldsymbol{I} \quad \Longleftrightarrow \quad \boldsymbol{L}_\kappa(\boldsymbol{x}) = K_\kappa(\boldsymbol{x})\, \boldsymbol{I} \otimes \boldsymbol{I}, \tag{4-17}$$

where $K_\kappa(\boldsymbol{x})$ is the bulk modulus field of a medium in the reference configuration.

We emphasize that the reference configuration is neither unloaded nor uniform: there is a nonzero pressure distribution $p(1, \boldsymbol{x})$ characterizing the equilibrium state in the field of mass forces.

5. The problem of finding the free vibration frequencies and modes as a variational problem

The concept of free vibrations is known to be related to the linearized problem of motion of an elastic body (in particular, a bulk-elastic medium) in the neighborhood of stable equilibrium state.

To derive the linearized equations of motion and the linearized boundary conditions, we first consider the exact equations and boundary conditions and then linearize them in displacements and their gradients.

The exact equations of motion in a nonuniform potential field of mass forces $\boldsymbol{g}$ and the exact boundary conditions are as follows:

$$\rho_\kappa(\boldsymbol{x})\, \ddot{\boldsymbol{r}}(\boldsymbol{x}, t) = \nabla_\kappa \cdot \boldsymbol{T}_\kappa(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x}, t)) + \rho_\kappa(\boldsymbol{x})\boldsymbol{g}, \quad \boldsymbol{g} = -\nabla\varphi(\boldsymbol{r}(\boldsymbol{x}, t)), \tag{5-1}$$

$$\boldsymbol{r}(\boldsymbol{x}, t)|_{\boldsymbol{x}\in\partial B} \in \partial B, \tag{5-2}$$

$$\boldsymbol{n}_\kappa(\boldsymbol{x}) \cdot \boldsymbol{T}_\kappa(\boldsymbol{x}, \boldsymbol{F}(\boldsymbol{x}, t)) \cdot \big(\boldsymbol{I} - \boldsymbol{n}_\kappa(\boldsymbol{r}(\boldsymbol{x}, t)) \otimes \boldsymbol{n}_\kappa(\boldsymbol{r}(\boldsymbol{x}, t))\big)\big|_{\boldsymbol{x}\in\partial B} = 0. \tag{5-3}$$

We introduce the displacement field that is supposed to be small in what follows:

$$\delta\boldsymbol{u}(\boldsymbol{x}, t) := \boldsymbol{r}(\boldsymbol{x}, t) - \boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{r}(\boldsymbol{x}, t) = \boldsymbol{x} + \delta\boldsymbol{u}(\boldsymbol{x}, t). \tag{5-4}$$

It is obvious that

$$\dot{\boldsymbol{r}}(\boldsymbol{x}, t) = \delta\dot{\boldsymbol{u}}(\boldsymbol{x}, t), \quad \ddot{\boldsymbol{r}}(\boldsymbol{x}, t) = \delta\ddot{\boldsymbol{u}}(\boldsymbol{x}, t), \quad \boldsymbol{F}(\boldsymbol{x}, t) = \boldsymbol{I} + \nabla_\kappa \otimes \delta\boldsymbol{u}(\boldsymbol{x}, t). \tag{5-5}$$

In the equilibrium state we have

$$\delta\boldsymbol{u}(\boldsymbol{x}, 0) = 0 \quad \Longleftrightarrow \quad \boldsymbol{r}(\boldsymbol{x}, 0) = \boldsymbol{x}, \quad \boldsymbol{F}(\boldsymbol{x}, 0) = \boldsymbol{I}, \quad \boldsymbol{n}_\kappa\big(\boldsymbol{r}(\boldsymbol{x}, 0)\big) = \boldsymbol{n}_\kappa(\boldsymbol{x}), \tag{5-6}$$

$$\nabla_\kappa \cdot \boldsymbol{T}_\kappa(\boldsymbol{x}, \boldsymbol{I}) + \rho_\kappa(\boldsymbol{x})\boldsymbol{g} = 0, \tag{5-7}$$

$$\boldsymbol{n}_\kappa(\boldsymbol{x}) \cdot \boldsymbol{T}_\kappa(\boldsymbol{x}, \boldsymbol{I}) \cdot \big(\boldsymbol{I} - \boldsymbol{n}_\kappa(\boldsymbol{x}) \otimes \boldsymbol{n}_\kappa(\boldsymbol{x})\big)\big|_{\boldsymbol{x}\in\partial B} = 0. \tag{5-8}$$

Subtracting the equilibrium values of the quantities from their current values and linearizing the differences in $\delta\boldsymbol{u}(\boldsymbol{x},t)$ and $\nabla_\kappa\otimes\delta\boldsymbol{u}(\boldsymbol{x},t)$, we obtain

$$\delta\boldsymbol{T}_\kappa(\boldsymbol{x},t)=\boldsymbol{T}_\kappa\big(\boldsymbol{x},\boldsymbol{F}(\boldsymbol{x},t)\big)-\boldsymbol{T}_\kappa(\boldsymbol{x},\boldsymbol{I})=\boldsymbol{C}_\kappa(\boldsymbol{x},\boldsymbol{I}):\nabla_\kappa\otimes\delta\boldsymbol{u}(\boldsymbol{x},t), \tag{5-9}$$

$$\begin{gathered}\delta\boldsymbol{n}_\kappa(\boldsymbol{x},t)=\boldsymbol{n}_\kappa\big(\boldsymbol{r}(\boldsymbol{x},t)\big)-\boldsymbol{n}_\kappa(\boldsymbol{x})=\delta\boldsymbol{u}\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa(\boldsymbol{x}),\\ \delta\boldsymbol{n}_\kappa(\boldsymbol{x},t)\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})=0,\quad \delta(\boldsymbol{I}-\boldsymbol{n}_\kappa\otimes\boldsymbol{n}_\kappa)=-\delta\boldsymbol{n}_\kappa\otimes\boldsymbol{n}_\kappa-\boldsymbol{n}_\kappa\otimes\delta\boldsymbol{n}_\kappa.\end{gathered} \tag{5-10}$$

The surface gradient $\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa$ is specified here by the equality

$$\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa=(\boldsymbol{I}-\boldsymbol{n}_\kappa\otimes\boldsymbol{n}_\kappa)\cdot\nabla_\kappa\otimes\boldsymbol{n}_\kappa. \tag{5-11}$$

It is not difficult to demonstrate that the surface gradient (5-11) is a symmetric TR(2) over the subspace tangent to ∂B at the points of smoothness.

Taking into account the equalities (5-6)–(5-11), we obtain the following linearized equations of motion and boundary conditions:

$$\begin{aligned}\rho_\kappa(\boldsymbol{x})\cdot\delta\ddot{\boldsymbol{u}}(\boldsymbol{x},t)&=\nabla_\kappa\cdot\delta\boldsymbol{T}_\kappa(\boldsymbol{x},\boldsymbol{I})-\rho_\kappa(\boldsymbol{x})\,\delta\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla_\kappa\otimes\nabla_\kappa\,\varphi(\boldsymbol{x})\\&=\nabla_\kappa\cdot\big(\boldsymbol{C}_\kappa(\boldsymbol{x},\boldsymbol{I}):\nabla_\kappa\otimes\delta\boldsymbol{u}(\boldsymbol{x},t)\big)-\rho_\kappa(\boldsymbol{x})\,\delta\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla_\kappa\otimes\nabla_\kappa\,\varphi(\boldsymbol{x}),\end{aligned} \tag{5-12}$$

$$\delta\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})=0,\quad \boldsymbol{x}\in\partial B, \tag{5-13}$$

$$\big(\boldsymbol{n}_\kappa(\boldsymbol{x})\cdot\delta\boldsymbol{T}_\kappa(\boldsymbol{x},\boldsymbol{I})\big)\cdot\big(\boldsymbol{I}-\boldsymbol{n}_\kappa(\boldsymbol{x})\otimes\boldsymbol{n}_\kappa(\boldsymbol{x})\big)-\big(\boldsymbol{n}_\kappa(\boldsymbol{x})\cdot\boldsymbol{T}_\kappa(\boldsymbol{x},\boldsymbol{I})\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})\big)\,\delta\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa(\boldsymbol{x})=0,\quad \boldsymbol{x}\in\partial B. \tag{5-14}$$

The solutions of the linearized equations of motion under linearized boundary conditions, having the form

$$\delta\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x})\sin\omega t, \tag{5-15}$$

are called the free vibrations with frequency ω and mode $\boldsymbol{u}(\boldsymbol{x})$. Thus, for the modes the following equations and boundary conditions take place:

$$\omega^2\rho_\kappa(\boldsymbol{x})\,\boldsymbol{u}(\boldsymbol{x})=-\nabla_\kappa\cdot\big(\boldsymbol{C}_\kappa(\boldsymbol{x},\boldsymbol{I}):\nabla_\kappa\otimes\boldsymbol{u}(\boldsymbol{x})\big)+\rho_\kappa(\boldsymbol{x})\,\boldsymbol{u}(\boldsymbol{x})\cdot\nabla_\kappa\otimes\nabla_\kappa\,\varphi(\boldsymbol{x})=:\mathcal{A}(\boldsymbol{u}), \tag{5-16}$$

$$\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})=0,\quad \boldsymbol{x}\in\partial B, \tag{5-17}$$

$$\boldsymbol{n}_\kappa(\boldsymbol{x})\cdot\big(\boldsymbol{C}_\kappa(\boldsymbol{x},\boldsymbol{I}):\nabla_\kappa\otimes\boldsymbol{u}(\boldsymbol{x})\big)\cdot\big(\boldsymbol{I}-\boldsymbol{n}_\kappa(\boldsymbol{x})\otimes\boldsymbol{n}_\kappa(\boldsymbol{x})\big)=\big(\boldsymbol{n}_\kappa(\boldsymbol{x})\cdot\boldsymbol{T}_\kappa(\boldsymbol{x},\boldsymbol{I})\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})\big)\,\boldsymbol{u}(\boldsymbol{x})\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa(\boldsymbol{x}),\quad \boldsymbol{x}\in\partial B. \tag{5-18}$$

In further calculations, the arguments of related quantities (which remain the same) will not be indicated so detailed.

we rewrite Equation (5-16) in the following operator form:

$$\lambda\rho_\kappa\boldsymbol{u}:=\omega^2\rho_\kappa\boldsymbol{u}=\mathcal{A}(\boldsymbol{u}). \tag{5-19}$$

Thus, the free vibration modes are the generalized eigenvectors of the linear differential operator $\mathcal{A}$, and the squared frequencies are the corresponding generalized eigenvalues (subsequently the term "generalized" will be omitted).

The mechanical system in hand has some fundamental distinctions from the elastic systems for which a rigorous mathematical theory of free vibrations (that is essentially a spectral theory for the corresponding operators under certain boundary conditions) does exist and is presented in the literature. For this reason, when constructing the spectral theory for the system under consideration, it is necessary to substantially modify the well-known fundamental theorems, preserving the general concept of the theory of free vibrations. Although in what follows the distinctions mentioned will be considered in detail, here we characterize them concisely:

(1) The operator $\boldsymbol{\mathcal{A}}$ is *positive semidefinite*, the eigensubspace that corresponds to zero eigenvalue (so-called neutral perturbations not violating the equilibrium of the system) being *infinite-dimensional*.

(2) The boundary conditions are those of free sliding over a curved surface (that in particular, may have flat parts).

Let us derive an integral relationship for the free vibration frequencies and modes with regard for special features of the mechanical system in hand. To this end we multiply both sides of (5-16) by $\boldsymbol{u}(\boldsymbol{x})$ and after integrating over the region B, applying the Gauss theorem and making use of the boundary conditions (5-17), (5-18), we arrive at the equality

$$\begin{aligned}\omega^2\langle\rho_\kappa\boldsymbol{u}\cdot\boldsymbol{u}\rangle_B &= \langle\nabla_\kappa\otimes\boldsymbol{u} : \boldsymbol{C}_\kappa : \nabla_\kappa\otimes\boldsymbol{u}\rangle_B+\langle\rho_\kappa\boldsymbol{u}\cdot\nabla_\kappa\otimes\nabla_\kappa\varphi\cdot\boldsymbol{u}\rangle_B-\langle(\boldsymbol{n}_\kappa\cdot\boldsymbol{T}_\kappa\cdot\boldsymbol{n}_\kappa)(\boldsymbol{u}\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa\cdot\boldsymbol{u})\rangle_{\partial B}\\ &= R\{\boldsymbol{u}\}.\end{aligned} \tag{5-20}$$

Here the angular brackets denote the integral over the reference set indicated as the right subscript. We note that the quadratic functional

$$R\{\boldsymbol{u}\} := \langle\boldsymbol{u}\cdot\boldsymbol{\mathcal{A}}(\boldsymbol{u})\rangle_B, \tag{5-21}$$

is nothing but the second variation of total potential energy of the system on the kinematically admissible displacement fields (with regard for zero work of the boundary tractions). Thus, provided the free vibrations do exist, the following equality for corresponding frequencies and modes takes place:

$$\lambda = \omega^2 = \frac{R\{\boldsymbol{u}\}}{\langle\rho_\kappa\boldsymbol{u}\cdot\boldsymbol{u}\rangle_B} =: \Psi\{\boldsymbol{u}\}. \tag{5-22}$$

We note that the operator $\boldsymbol{\mathcal{A}}$ is symmetric on the displacement fields satisfying boundary conditions (5-17), (5-18) in the sense that the bilinear functional $\langle\tilde{\boldsymbol{u}}\cdot\boldsymbol{\mathcal{A}}(\boldsymbol{u})\rangle_B$ is symmetric on such the fields. Indeed, we reason in exactly the same way as while deriving the equality (5-20), and obtain

$$\begin{aligned}\langle\tilde{\boldsymbol{u}}\cdot\boldsymbol{\mathcal{A}}(\boldsymbol{u})\rangle_B &= \langle\nabla_\kappa\otimes\tilde{\boldsymbol{u}} : \boldsymbol{C}_\kappa : \nabla_\kappa\otimes\boldsymbol{u}\rangle_B+\langle\rho_\kappa\tilde{\boldsymbol{u}}\cdot\nabla_\kappa\otimes\nabla_\kappa\varphi\cdot\boldsymbol{u}\rangle_B-\langle(\boldsymbol{n}_\kappa\cdot\boldsymbol{T}_\kappa\cdot\boldsymbol{n}_\kappa)(\tilde{\boldsymbol{u}}\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa\cdot\boldsymbol{u})\rangle_{\partial B}\\ &=: \mathcal{R}\{\tilde{\boldsymbol{u}},\boldsymbol{u}\},\end{aligned} \tag{5-23}$$

whence, due to the symmetry of $\boldsymbol{C}_\kappa$ (the elastic moduli tensor for the Piola stress) and to the symmetry of TR(2) $\nabla_\kappa\otimes\nabla_\kappa\varphi$ and TR(2) $\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa$, the symmetry of the bilinear functional $\mathcal{R}\{\tilde{\boldsymbol{u}},\boldsymbol{u}\}$ follows:

$$\mathcal{R}\{\tilde{\boldsymbol{u}},\boldsymbol{u}\} = \mathcal{R}\{\boldsymbol{u},\tilde{\boldsymbol{u}}\}. \tag{5-24}$$

In what follows, the symmetric bilinear functional $\mathcal{R}\{\tilde{\boldsymbol{u}},\boldsymbol{u}\}$ will be considered as the functional defined on a broader set, namely, on any kinematically admissible (i.e. tangential at the boundary) fields of small displacements, as well as the quadratic functional

$$R\{\boldsymbol{u}\} = \mathfrak{R}\{\boldsymbol{u}, \boldsymbol{u}\}. \tag{5-25}$$

Kinematically admissible fields $\boldsymbol{u}(\boldsymbol{x})$ form an infinite-dimensional linear space N which will be subsequently supplied with a scalar product that converts it into a Hilbert space.

From equations (5-19) and (5-24) it follows double orthogonality of the free vibration modes $\boldsymbol{u}(\boldsymbol{x})$ and $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ corresponding to different eigenvalues λ and $\tilde{\lambda}$:

$$\left.\begin{aligned} \lambda\rho_\kappa \boldsymbol{u} &= \mathcal{A}(\boldsymbol{u}) \\ \tilde{\lambda}\rho_\kappa \tilde{\boldsymbol{u}} &= \mathcal{A}(\tilde{\boldsymbol{u}}) \\ \lambda &\neq \tilde{\lambda} \end{aligned}\right\} \implies \langle \rho_\kappa \boldsymbol{u} \cdot \tilde{\boldsymbol{u}} \rangle_B = \langle \boldsymbol{u} \cdot \mathcal{A}(\tilde{\boldsymbol{u}}) \rangle_B = 0. \tag{5-26}$$

From the expression (5-20) *for the functional* $R\{\boldsymbol{u}\}$ *it is by no means obvious its positive semidefiniteness, but this is absolutely obvious from its canonical form derived by the authors for the bulk-elastic media in essentially more general assumptions* [Ryzhak et al. 2017]. As for the systems considered here, due to the assumption of constitutive homogeneity of a bulk-elastic medium, the canonical form of the functional is especially simple and reduces to the following:

$$R\{\boldsymbol{u}\} = \left\langle K_\kappa \left(\nabla_\kappa \cdot \boldsymbol{u} + \frac{\rho_\kappa}{K_\kappa} \boldsymbol{u} \cdot \boldsymbol{g} \right)^2 \right\rangle_B, \tag{5-27}$$

where $K_\kappa(\boldsymbol{x})$ and $\rho_\kappa(\boldsymbol{x})$ are the distributions of bulk modulus and density of a bulk-elastic medium in the reference equilibrium state. Taking into account that the constitutive relation for a bulk-elastic medium can be brought to the form

$$\rho = \rho(p), \tag{5-28}$$

its bulk modulus is specified by the equality

$$\frac{1}{K(\rho)} = \frac{1}{\rho(p)} \frac{d\rho}{dp}(p), \tag{5-29}$$

whereas the equilibrium pressure gradient obeys the equation

$$\nabla_\kappa p_\kappa = \rho_\kappa \boldsymbol{g}, \tag{5-30}$$

we reduce $R\{\boldsymbol{u}\}$ to the form

$$R\{\boldsymbol{u}\} = \left\langle K_\kappa \left(\nabla_\kappa \cdot \boldsymbol{u} + \frac{1}{\rho_\kappa} (\boldsymbol{u} \cdot \nabla_\kappa \rho_\kappa) \right)^2 \right\rangle_B = \left\langle \frac{K_\kappa}{\rho_\kappa^2} \big((\nabla_\kappa \cdot (\rho_\kappa \boldsymbol{u}) \big)^2 \right\rangle_B. \tag{5-31}$$

Further spectral analysis will be based on the properties of the functionals entering into (5-22).

From (5-31) it is obvious that positive semidefinite quadratic functional $R\{\boldsymbol{u}\}$ vanishes on the infinite-dimensional subspace $N_0 \subset N$ formed by the fields $\boldsymbol{u}_0(\boldsymbol{x})$ for which

$$\nabla_\kappa \cdot \big(\rho_\kappa(\boldsymbol{x})\, \boldsymbol{u}_0(\boldsymbol{x}) \big) = 0, \qquad \boldsymbol{u}_0 \cdot \boldsymbol{n}_\kappa|_{\partial B} = 0. \tag{5-32}$$

These fields are the "neutral perturbations" that satisfy the linearized equilibrium equations and boundary conditions. In addition to the formal proof of this fact, which will be given in Appendix B, it can be justified also as follows: the fields $\boldsymbol{u}_0(\boldsymbol{x})$ are the incremental displacements of such transformations for which the spatial distribution of density remains unchanged, whence for a homogeneous bulk-elastic medium it follows that the spatial distribution of pressure does not change either.

Let us consider the subspace $N_0^\perp \subset N$ formed by the fields $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ orthogonal to N_0 in the sense of the scalar product $\langle \rho_\kappa \boldsymbol{u} \cdot \tilde{\boldsymbol{u}} \rangle_B$:

$$\tilde{\boldsymbol{u}} \in N_0^\perp \quad \Longleftrightarrow \quad \langle \rho_\kappa \boldsymbol{u}_0 \cdot \tilde{\boldsymbol{u}} \rangle_B = 0, \quad \forall \boldsymbol{u}_0 \in N_0. \tag{5-33}$$

It is easy to demonstrate that the fields $\tilde{\boldsymbol{u}} \in N_0^\perp$ are potential, i.e. they have the form

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \nabla_\kappa \gamma(\boldsymbol{x}), \quad \boldsymbol{n}_\kappa \cdot \nabla_\kappa \gamma|_{\partial B} = 0. \tag{5-34}$$

Moreover, any admissible field $\boldsymbol{u}(\boldsymbol{x})$ can be represented in a unique way as the sum of $\boldsymbol{u}_0(\boldsymbol{x})$ and $\tilde{\boldsymbol{u}}(\boldsymbol{x})$:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) + \tilde{\boldsymbol{u}}(\boldsymbol{x}) \quad \Longleftrightarrow \quad N = N_0 \oplus N_0^\perp, \tag{5-35}$$

meaning that N is the orthogonal direct sum of subspaces N_0 and $N_0^\perp$.

The space of admissible fields N supplied with the above-mentioned scalar product is a Hilbert space. We note that the norm generated by this scalar product, namely

$$\|\boldsymbol{u}\| := (\langle \rho_\kappa \boldsymbol{u} \cdot \boldsymbol{u} \rangle_B)^{1/2}, \tag{5-36}$$

is equivalent to all the norms generated by scalar products of the type $\langle \beta(\boldsymbol{x})\, \boldsymbol{u}(\boldsymbol{x}) \cdot \tilde{\boldsymbol{u}}(\boldsymbol{x}) \rangle_B$ with weight functions $\beta(\boldsymbol{x})$ that are continuous and strictly positive in the closure of the domain B.

The subsets of the subspace $N_0^\perp$, bounded both in the value of the functional $R\{\boldsymbol{u}\}$ and in the norm $\|\boldsymbol{u}\|$ are compact in the sense of convergence with respect to this norm (and with respect to every norm equivalent to it). The presence of such a compactness is proved in Appendix A under additional supposition of piecewise convexity of the boundary ∂B, basing on classical Rellich's lemma together with a specific inequality of Korn's type, not occurred in the literature (cf. Horgan 1995).

Due to the presence of this compactness, a standard spectral theorem can be proved for the operator $\mathcal{A}$ on the subspace $N_0^\perp$, that implies that the standard properties of eigenvalues and eigenvectors (i.e. the free vibration modes) also take place.

We state the spectral theorem on a subspace $N_0^\perp$. An outline of its proof (containing some nonstandard elements resulting from the features of the mechanical system under consideration) is presented in Appendix B.

Spectral theorem. The linear operator $\mathcal{A}$ (5-16) under boundary conditions (5-17) on the subspace $N_0^\perp$ has a positive discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \ldots$, $\lim_{m\to\infty} \lambda_m = +\infty$, with corresponding eigenvectors $\boldsymbol{u}_m(\boldsymbol{x}) \in N_0^\perp$ satisfying boundary conditions (5-18). Herein

$$\lambda_m = \inf_{\substack{\boldsymbol{u}\neq 0 \\ \boldsymbol{u}\in N_{m-1}^\perp}} \Psi\{\boldsymbol{u}\} = \Psi\{\boldsymbol{u}_m\}, \quad m = 1, 2, \ldots, \tag{5-37}$$

where the subspace $N_{m-1}^\perp$ for $m > 1$ is the orthogonal complement to $\operatorname{span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{m-1}) =: N_{m-1}$ in the subspace $N_0^\perp$. Eigenvectors $\boldsymbol{u}_m(\boldsymbol{x})$ are the extremals of the functional $\Psi\{\boldsymbol{u}\}$ on subspaces $N_{m-1}^\perp$; they are mutually orthogonal and can be normalized. The corresponding orthonormal system of eigenvectors is complete in the sense of convergence in the norm (5-36) (and every norm equivalent to it).

A consequence of the spectral theorem in combination with certain fundamental properties of linear spaces is the comparison theorem for eigenvalues corresponding to different operators with different weight functions and related different scalar products. The latter theorem is an effective tool for investigating the free vibration frequencies in the cases when it is impossible to obtain their exact values, since it allows one to obtain upper and lower bounds for them by finding the corresponding frequencies for suitable simple "comparison systems."

The statement and proof of the comparison theorem for the mechanical systems under consideration have a number of distinctions from those for the standard theorem.

Having in mind the comparison theorem, we reformulate in an equivalent way both the spectral problem and the spectral theorem in terms of a new unknown quantity $\boldsymbol{w}(\boldsymbol{x})$:

$$\boldsymbol{w}(\boldsymbol{x}) := \rho_\kappa(\boldsymbol{x})\,\boldsymbol{u}(\boldsymbol{x}) \quad\iff\quad \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{\rho_\kappa(\boldsymbol{x})}\boldsymbol{w}(\boldsymbol{x}), \quad \boldsymbol{w}(\boldsymbol{x})\cdot\boldsymbol{n}_\kappa(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in\partial B, \tag{5-38}$$

$$\begin{gathered} \widetilde{R}\{\boldsymbol{w}\} := R\Big\{\frac{\boldsymbol{w}}{\rho_\kappa}\Big\} = \Big\langle \frac{K_\kappa}{\rho_\kappa^2}(\nabla_\kappa\cdot\boldsymbol{w})^2\Big\rangle_B, \\ \langle\rho_\kappa\boldsymbol{u}\cdot\boldsymbol{u}\rangle_B = \Big\langle\frac{1}{\rho_\kappa}\boldsymbol{w}\cdot\boldsymbol{w}\Big\rangle_B, \quad \widetilde{\Psi}\{\boldsymbol{w}\} := \Psi\Big\{\frac{\boldsymbol{w}}{\rho_\kappa}\Big\}, \quad \widetilde{\mathcal{A}}(\boldsymbol{w}) := \mathcal{A}\Big(\frac{\boldsymbol{w}}{\rho_\kappa}\Big). \end{gathered} \tag{5-39}$$

The equalities (5-19) and (5-22) take the following form:

$$\lambda\boldsymbol{w} = \omega^2\boldsymbol{w} = \widetilde{\mathcal{A}}(\boldsymbol{w}), \tag{5-40}$$

$$\lambda = \omega^2 = \frac{\widetilde{R}\{\boldsymbol{w}\}}{\langle(1/\rho_\kappa)\boldsymbol{w}\cdot\boldsymbol{w}\rangle_B} =: \widetilde{\Psi}\{\boldsymbol{w}\}. \tag{5-41}$$

The space of all $\boldsymbol{w}(\boldsymbol{x})$ tangential at the boundary ∂B and supplied with a scalar product $\langle(1/\rho_\kappa)\boldsymbol{w}\cdot\boldsymbol{w}'\rangle_B$ is denoted $\widetilde{N}$, whereas the infinite-dimensional subspace of divergenceless fields $\boldsymbol{w}_0(\boldsymbol{x})$ corresponding to the zero eigenvalue, is denoted $\widetilde{N}_0$. We introduce also the subspaces into which the subspaces $N_0^\perp$, N_{m-1}, $N_{m-1}^\perp$ will be converted: $\widetilde{N}_0^\perp$ is the subspace orthogonal to $\widetilde{N}_0$ (in the sense of the scalar product of space $\widetilde{N}$); $\widetilde{N}_{m-1} = \operatorname{span}(\boldsymbol{w}_1, \dots, \boldsymbol{w}_{m-1})$, where $\boldsymbol{w}_i =: \rho_\kappa\boldsymbol{u}_i$ is the eigenvector corresponding to the eigenvalue λ_i; $\widetilde{N}_{m-1}^\perp$ is the orthogonal complement to the subspace $\widetilde{N}_{m-1}$ in the subspace $\widetilde{N}_0^\perp$ (or, equivalently, the orthogonal complement to the subspace $\operatorname{span}(\widetilde{N}_0, \widetilde{N}_{m-1})$ in $\widetilde{N}$). It is obvious that $\widetilde{N}$ is the direct orthogonal sum of $\widetilde{N}_0$, $\widetilde{N}_{m-1}$, and $\widetilde{N}_{m-1}^\perp$:

$$\widetilde{N} = \widetilde{N}_0 \oplus \widetilde{N}_{m-1} \oplus \widetilde{N}_{m-1}^\perp. \tag{5-42}$$

The assertion of the spectral theorem remains the same except for the fact that equality (5-37) is converted into the following equality:

$$\lambda_m = \inf_{\boldsymbol{w}\neq 0,\ \boldsymbol{w}\in\widetilde{N}_{m-1}^\perp} \widetilde{\Psi}\{\boldsymbol{w}\} = \widetilde{\Psi}\{\boldsymbol{w}_m\}, \quad m = 1, 2, \dots, \tag{5-43}$$

The eigenvectors $\boldsymbol{w}_m(\boldsymbol{x})$ are the extremals of functional $\widetilde{\Psi}\{\boldsymbol{w}\}$ on the subspaces $\widetilde{N}_{m-1}^\perp$, they are mutually orthogonal ($\langle(1/\rho_\kappa)\boldsymbol{w}_{m_1}\cdot\boldsymbol{w}_{m_2}\rangle_B = 0$, $m_1 \neq m_2$) and can be normalized with respect to the norm of space $\widetilde{N}$:

$$|||\boldsymbol{w}||| := \sqrt{\Big\langle\frac{1}{\rho_\kappa}\boldsymbol{w}\cdot\boldsymbol{w}\Big\rangle_B}. \tag{5-44}$$

The corresponding orthonormal system of eigenvectors is complete in the sense of convergence in the norm (5-44) (and in every norm equivalent to it).

We now assume that there are two different continuous mechanical systems of a similar type occupying identical domains B with identical boundary conditions, but characterized by their own density distributions $\rho_\kappa^{(1)}(\boldsymbol{x})$, $\rho_\kappa^{(2)}(\boldsymbol{x})$ and bulk moduli distributions $K_\kappa^{(1)}(\boldsymbol{x})$, $K_\kappa^{(2)}(\boldsymbol{x})$, which allows one to define for these systems in a completely similar manner the functionals:

$$\widetilde{R}^{(j)}\{\boldsymbol{w}\}, \quad \left\langle \frac{1}{\rho_\kappa^{(j)}} \boldsymbol{w}\cdot\boldsymbol{w} \right\rangle_B, \quad \widetilde{\Psi}^{(j)}\{\boldsymbol{w}\}, \quad j=1,2.$$

It is obvious that for each of the systems there is its own set of eigenvalues $\lambda_0^{(j)}=0$, $0<\lambda_1^{(j)}\le\lambda_2^{(j)}\le\dots$, $j=1,2$ and its own set of eigenvectors whose existence and properties result from the corresponding spectral theorem. Note that the subspaces $\widetilde{N}_0^{(1)}$, $\widetilde{N}_0^{(2)}$ (corresponding to the zero eigenvalue and being formed by the divergenceless fields $\boldsymbol{w}_0^{(j)}(\boldsymbol{x})$ tangential on ∂B) coincide with each other.

The Hilbert spaces $\widetilde{N}^{(j)}$ are characterized by different scalar products $\langle(1/\rho_\kappa^{(j)})\boldsymbol{w}\cdot\boldsymbol{w}'\rangle_B$. Hence, the eigenvectors $\boldsymbol{w}_0^{(j)}(\boldsymbol{x})$, $\boldsymbol{w}_1^{(j)}(\boldsymbol{x}),\dots$ are mutually orthogonal in the sense of the corresponding scalar products, as well as the mutually orthogonal subspaces.

Now we state the comparison theorem.

Comparison theorem. Assume that the functional $\widetilde{\Psi}^{(2)}\{\boldsymbol{w}\}$ majorizes the functional $\widetilde{\Psi}^{(1)}\{\boldsymbol{w}\}$:

$$\widetilde{\Psi}^{(1)}\{\boldsymbol{w}\} \le \widetilde{\Psi}^{(2)}\{\boldsymbol{w}\}, \quad \forall \boldsymbol{w}\in\widetilde{N}, \quad \boldsymbol{w}\neq\boldsymbol{0}. \tag{5-45}$$

Then for the corresponding eigenvalues (numbered in the order of nondecreasing) the following inequalities hold:

$$\lambda_m^{(1)} \le \lambda_m^{(2)}, \quad m=1,2,\dots. \tag{5-46}$$

The proof of the theorem, which differs from the standard proofs (due to the presence of an infinite-dimensional subspace corresponding to the zero eigenvalue) is presented in Appendix B.

The functionals $\widetilde{\Psi}^{(1)}\{\boldsymbol{w}\}$ and $\widetilde{\Psi}^{(2)}\{\boldsymbol{w}\}$ have a form that allows to obtain completely obvious minorizing and majorizing functionals with constant coefficients. To this end we introduce for each of the bulk-elastic media the fields of sound velocities $c_\kappa^{(1)}(\boldsymbol{x})$ and $c_\kappa^{(2)}(\boldsymbol{x})$:

$$c_\kappa^{(1)}(\boldsymbol{x}) = \sqrt{\frac{K_\kappa^{(1)}}{\rho_\kappa^{(1)}}}, \quad c_\kappa^{(2)}(\boldsymbol{x}) = \sqrt{\frac{K_\kappa^{(2)}}{\rho_\kappa^{(2)}}}. \tag{5-47}$$

In addition, we introduce notations for the minimal (line below the symbol) and maximal (line above the symbol) values of a quantity. Then we have

$$\widetilde{\Psi}^{(1)}\{\boldsymbol{w}\} \le (\bar{c}_\kappa^{(1)})^2\,\frac{\bar{\rho}_\kappa^{(1)}}{\underline{\rho}_\kappa^{(1)}}\,\frac{\langle(\nabla_\kappa\cdot\boldsymbol{w})^2\rangle_B}{\langle\boldsymbol{w}\cdot\boldsymbol{w}\rangle_B}, \quad (\underline{c}_\kappa^{(2)})^2\,\frac{\underline{\rho}_\kappa^{(2)}}{\bar{\rho}_\kappa^{(2)}}\,\frac{\langle(\nabla_\kappa\cdot\boldsymbol{w})^2\rangle_B}{\langle\boldsymbol{w}\cdot\boldsymbol{w}\rangle_B} \le \widetilde{\Psi}^{(2)}\{\boldsymbol{w}\}. \tag{5-48}$$

If for two different bulk-elastic media occupying identical domains, the following inequality holds:

$$\sqrt{\frac{\bar{\rho}_\kappa^{(1)}}{\underline{\rho}_\kappa^{(1)}}\,\frac{\bar{\rho}_\kappa^{(2)}}{\underline{\rho}_\kappa^{(2)}}} \le \frac{\underline{c}_\kappa^{(2)}}{\bar{c}_\kappa^{(1)}}, \tag{5-49}$$

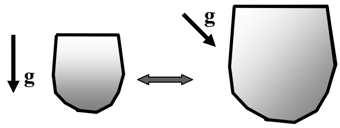

Figure 2. Comparison of the free vibration frequencies for different media occupying similar domains in different fields.

then the functional $\widetilde{\Psi}^{(2)}\{\boldsymbol{w}\}$ obviously majorizes the functional $\widetilde{\Psi}^{(1)}\{\boldsymbol{w}\}$, from which it follows that each of the nonzero free vibration frequencies of the first medium does not exceed the corresponding frequency of the second medium:

$$\omega_m^{(1)} \le \omega_m^{(2)}, \quad m = 1, 2, \ldots . \tag{5-50}$$

Remark. The relations obtained can easily be extended to the case when the domains occupied by different bulk-elastic media are similar (Figure 2).

In that case the functionals $\langle(\nabla_\kappa \cdot \boldsymbol{w})^2\rangle_{B^{(j)}} / \langle \boldsymbol{w}\cdot\boldsymbol{w}\rangle_{B^{(j)}}$ entering into each of the inequalities (5-48), for different (similar) domains are to be reduced to one and the same domain B of some characteristic (unit) size. Then for the functionals mentioned it results in

$$\frac{\langle(\nabla_\kappa \cdot \boldsymbol{w})^2\rangle_{B^{(j)}}}{\langle \boldsymbol{w}\cdot\boldsymbol{w}\rangle_{B^{(j)}}} = \frac{1}{(l^{(j)})^2}\,\frac{\langle(\nabla_\kappa \cdot \boldsymbol{w})^2\rangle_{B}}{\langle \boldsymbol{w}\cdot\boldsymbol{w}\rangle_{B}}, \tag{5-51}$$

where $l^{(j)}$ is the characteristic size (say, the diameter) of the domain $B^{(j)}$. Hence, the inequality (5-49), from which the inequality for frequencies (5-50) follows, takes the following form:

$$\sqrt{\frac{\bar{\rho}_\kappa^{(1)}}{\underline{\rho}_\kappa^{(1)}}\,\frac{\bar{\rho}_\kappa^{(2)}}{\underline{\rho}_\kappa^{(2)}}} \le \frac{\underline{c}_\kappa^{(2)}/l^{(2)}}{\bar{c}_\kappa^{(1)}/l^{(1)}}. \tag{5-52}$$

6. Explicit bounds for the free vibration frequencies for domains of some special shapes

As for domains of arbitrary shape, the comparison theorem, in accordance with its statement, makes it possible only to compare the free vibration frequencies of different bulk-elastic media in the same (or similar) domains, but it fails to yield the explicit frequency bounds. However, for domains of the shape of a rectangular parallelepiped and for a certain set of other shapes the comparison theorem allows to obtain concrete analytical upper and lower bounds for the corresponding free vibration frequencies depending on the limit values of mechanical parameters of a medium and on the geometric parameters of a domain.

We note that the force field enters into the above bounds only via the quantities $\underline{\rho}_\kappa$, $\bar{\rho}_\kappa$, $\underline{c}_\kappa$, $\bar{c}_\kappa$. For this reason orientation of the domain with respect to the field can be absolutely arbitrary, and its influence on the resulting bounds for frequencies is related to its influence on the above-mentioned limit values of the mechanical parameters of a bulk-elastic medium that depend on the orientation.

The derivation of the bounds for frequencies is based on investigation of the functional

$$\Phi\{\boldsymbol{w}\} := \frac{\langle(\nabla_\kappa \cdot \boldsymbol{w})^2\rangle_B}{\langle \boldsymbol{w}\cdot\boldsymbol{w}\rangle_B}, \tag{6-1}$$

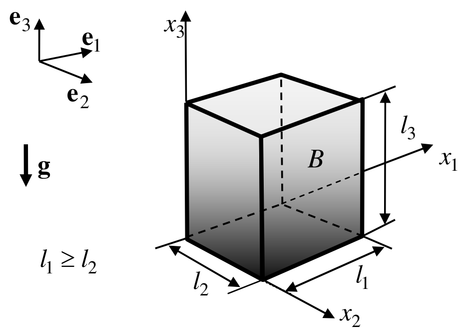

Figure 3. A domain B of the shape of a rectangular parallelepiped.

by means of expansion of the fields $\boldsymbol{w}(\boldsymbol{x})$ (previously converted into spatially periodic fields) into a spatial Fourier series. Various aspects of this method are presented in [Ryzhak 1993, 1994; 1997; 1999].

We consider first the domain B having the shape of a rectangular parallelepiped (Figure 3).

Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an orthonormal basis, and the corresponding Cartesian rectangular coordinates are specified by

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3. \tag{6-2}$$

The rectangular parallelepiped B is specified by the following system of inequalities:

$$B = \{\boldsymbol{x} | 0 \leq x_i \leq l_i, \quad i = 1, 2, 3\}. \tag{6-3}$$

To convert the field $\boldsymbol{w}(\boldsymbol{x})$ tangential at the boundary ∂B into a spatially periodic field, we successively reflect the parallelepiped B together with the field defined on it with respect to the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. Thus the parallelepiped B is converted into the parallelepiped B':

$$B' = \{\boldsymbol{x} | -l_i \leq x_i \leq l_i, \quad i = 1, 2, 3\}, \tag{6-4}$$

whereas the field $\boldsymbol{w}(\boldsymbol{x})$ is converted into a spatially periodic field with three planes of symmetry in the B'. Using for the converted field the same notation $\boldsymbol{w}(\boldsymbol{x})$, we determine it by the following equalities:

$$\boldsymbol{w}(\boldsymbol{x} \cdot \boldsymbol{Q}_i) = \boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{Q}_i, \quad \boldsymbol{x} \in B', \quad i = 1, 2, 3, \tag{6-5}$$

where $\boldsymbol{Q}_i$ is the TR(2) specifying reflection with respect to the plane $x_i = 0$ (Figure 4, plane scheme).

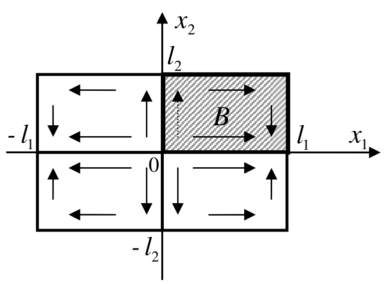

Figure 4. Conversion of the field into a periodic one by means of reflections.

We note that in mirror-symmetric points the following equalities take place:

$$|\boldsymbol{w}(\boldsymbol{x}\cdot\boldsymbol{Q}_i)| = |\boldsymbol{w}(\boldsymbol{x})|, \quad \nabla_\kappa\cdot\boldsymbol{w}(\boldsymbol{x}\cdot\boldsymbol{Q}_i) = \nabla_\kappa\cdot\boldsymbol{w}(\boldsymbol{x}). \tag{6-6}$$

For this reason

$$\langle(\nabla_\kappa\cdot\boldsymbol{w})^2\rangle_{B'} = 8\langle(\nabla_\kappa\cdot\boldsymbol{w})^2\rangle_B, \quad \langle\boldsymbol{w}\cdot\boldsymbol{w}\rangle_{B'} = 8\langle\boldsymbol{w}\cdot\boldsymbol{w}\rangle_B \tag{6-7}$$

$$\Longrightarrow \quad \Phi\{\boldsymbol{w}\} = \frac{\langle(\nabla_\kappa\cdot\boldsymbol{w})^2\rangle_{B'}}{\langle\boldsymbol{w}\cdot\boldsymbol{w}\rangle_{B'}}. \tag{6-8}$$

A well-known consequence of the spatial periodicity of the field $\boldsymbol{w}(\boldsymbol{x})$ is that the Fourier series for the field $\nabla_\kappa\otimes\boldsymbol{w}(\boldsymbol{x})$ is equal to the formally differentiated Fourier series for the field $\boldsymbol{w}(\boldsymbol{x})$. In combination with the Parseval theorem, this allows to replace the integrals in the numerator and denominator of expression (6-8) for the functional $\Phi\{\boldsymbol{w}\}$ by the sums of corresponding series (containing only the Fourier coefficients for the field $\boldsymbol{w}(\boldsymbol{x})$), and then investigate the ratio of the sums.

We introduce the notation for used hereinafter version of the triple Fourier series in the parallelepiped B'. The integer vectors $\boldsymbol{\mu}$ are specified by the following equalities and inequalities:

$$\boldsymbol{\mu} = \mu_1\boldsymbol{e}_1 + \mu_2\boldsymbol{e}_2 + \mu_3\boldsymbol{e}_3, \quad \mu_i \text{ are integer,}$$
$$\mu_1 > 0, \quad \text{or} \quad \mu_1 = 0, \quad \mu_2 > 0, \quad \text{or} \quad \mu_1 = 0, \quad \mu_2 = 0, \quad \mu_3 > 0. \tag{6-9}$$

We define the discrete vectors $\boldsymbol{q}_{\boldsymbol{\mu}}$:

$$\boldsymbol{q}_{\boldsymbol{\mu}} = \sum_{i=1}^{3}\frac{\pi\mu_i}{l_i}\boldsymbol{e}_i, \tag{6-10}$$

and the following system of functions:

$$\varphi_0(\boldsymbol{x}) = (8l_1l_2l_3)^{-1/2}, \quad \varphi_{\boldsymbol{\mu}}(\boldsymbol{x}) = (4l_1l_2l_3)^{-1/2}\cos\boldsymbol{q}_{\boldsymbol{\mu}}\cdot\boldsymbol{x}, \quad \psi_{\boldsymbol{\mu}}(\boldsymbol{x}) = (4l_1l_2l_3)^{-1/2}\sin\boldsymbol{q}_{\boldsymbol{\mu}}\cdot\boldsymbol{x}. \tag{6-11}$$

This system is complete and orthonormal in $L_2(B')$, the convergence in the norm of the corresponding space being independent of the order of summation, which allows to use triple summation (i.e. the summation over $\boldsymbol{\mu}$) without specifying the order of summation. We will use the symbol $\sim$ to denote the correspondence between the field and its Fourier series. Then for the fields $\boldsymbol{w}(\boldsymbol{x})$ and $\nabla_\kappa\otimes\boldsymbol{w}(\boldsymbol{x})$, as well as for $\nabla_\kappa\cdot\boldsymbol{w}(\boldsymbol{x})$ and $\nabla_\kappa\times\boldsymbol{w}(\boldsymbol{x})$, we have

$$\boldsymbol{w} \sim \boldsymbol{a}_0\varphi_0 + \sum_{\boldsymbol{\mu}}(\boldsymbol{a}_{\boldsymbol{\mu}}\varphi_{\boldsymbol{\mu}} + \boldsymbol{b}_{\boldsymbol{\mu}}\psi_{\boldsymbol{\mu}}), \quad \boldsymbol{a}_0 = 0, \tag{6-12}$$

$$\begin{aligned}
\nabla_\kappa\otimes\boldsymbol{w} &\sim \sum_{\boldsymbol{\mu}}(\boldsymbol{q}_{\boldsymbol{\mu}}\otimes\boldsymbol{b}_{\boldsymbol{\mu}}\varphi_{\boldsymbol{\mu}} - \boldsymbol{q}_{\boldsymbol{\mu}}\otimes\boldsymbol{a}_{\boldsymbol{\mu}}\psi_{\boldsymbol{\mu}}),\\
\nabla_\kappa\cdot\boldsymbol{w} &\sim \sum_{\boldsymbol{\mu}}(\boldsymbol{q}_{\boldsymbol{\mu}}\cdot\boldsymbol{b}_{\boldsymbol{\mu}}\varphi_{\boldsymbol{\mu}} - \boldsymbol{q}_{\boldsymbol{\mu}}\cdot\boldsymbol{a}_{\boldsymbol{\mu}}\psi_{\boldsymbol{\mu}}),\\
\nabla_\kappa\times\boldsymbol{w} &\sim \sum_{\boldsymbol{\mu}}(\boldsymbol{q}_{\boldsymbol{\mu}}\times\boldsymbol{b}_{\boldsymbol{\mu}}\varphi_{\boldsymbol{\mu}} - \boldsymbol{q}_{\boldsymbol{\mu}}\times\boldsymbol{a}_{\boldsymbol{\mu}}\psi_{\boldsymbol{\mu}}).
\end{aligned} \tag{6-13}$$

Vanishing of the vector $\boldsymbol{a}_0$ is a consequence of symmetry of the field $\boldsymbol{w}(\boldsymbol{x})$, which is equivalent to tangentiality of the original field on the faces of original parallelepiped B. The presence of these symmetries results in the existence of certain relations between the Fourier coefficients corresponding to integer vectors $\boldsymbol{\mu}$ with the same values of μ_1 and of $|\mu_i|$, $i = 2, 3$. Specifically, basing on a vector $\boldsymbol{\mu}$ corresponding to positive μ_i, we form the remaining three vectors:

$$\begin{aligned} \boldsymbol{\mu}' &= \mu_1\boldsymbol{e}_1 - \mu_2\boldsymbol{e}_2 + \mu_3\boldsymbol{e}_3 = \boldsymbol{\mu}\cdot\boldsymbol{Q}_2, \\ \boldsymbol{\mu}'' &= \mu_1\boldsymbol{e}_1 + \mu_2\boldsymbol{e}_2 - \mu_3\boldsymbol{e}_3 = \boldsymbol{\mu}\cdot\boldsymbol{Q}_3, \\ \boldsymbol{\mu}''' &= \mu_1\boldsymbol{e}_1 - \mu_2\boldsymbol{e}_2 - \mu_3\boldsymbol{e}_3 = -\boldsymbol{\mu}\cdot\boldsymbol{Q}_1 = \boldsymbol{\mu}\cdot\boldsymbol{Q}_2\cdot\boldsymbol{Q}_3. \end{aligned} \tag{6-14}$$

Then for the Fourier coefficients corresponding to vectors $\boldsymbol{\mu}'$, $\boldsymbol{\mu}''$, and $\boldsymbol{\mu}'''$, the following equalities hold:

$$\begin{gathered} \boldsymbol{a}_{\mu} = \boldsymbol{a}_{\mu'} = \boldsymbol{a}_{\mu''} = \boldsymbol{a}_{\mu'''} = \boldsymbol{0}, \\ \boldsymbol{b}_{\mu'} = \boldsymbol{b}_{\mu}\cdot\boldsymbol{Q}_2, \quad \boldsymbol{b}_{\mu''} = \boldsymbol{b}_{\mu}\cdot\boldsymbol{Q}_3, \quad \boldsymbol{b}_{\mu'''} = -\boldsymbol{b}_{\mu}\cdot\boldsymbol{Q}_1. \end{gathered} \tag{6-15}$$

Note that in the Fourier expansion of the divergence for the divergenceless modes all the coefficients vanish, whence by virtue of (6-13) it follows that vectors $\boldsymbol{b}_\mu$ are orthogonal to the corresponding vectors $\boldsymbol{q}_\mu$:

$$\boldsymbol{b}_\mu\cdot\boldsymbol{q}_\mu = 0, \quad \forall\boldsymbol{\mu}. \tag{6-16}$$

For the elements of the orthogonal subspace in hand, formed by the gradients of the scalar functions, their curls vanish, whence it follows that the vectors $\boldsymbol{b}_\mu$ are collinear to the corresponding vectors $\boldsymbol{q}_\mu$:

$$\boldsymbol{b}_\mu = \beta_\mu\boldsymbol{q}_\mu, \quad \forall\boldsymbol{\mu}. \tag{6-17}$$

Thus, in the above-mentioned subspace, by the Parseval theorem the following equalities hold:

$$\begin{aligned} \Phi\{\boldsymbol{w}\} = \frac{\sum_\mu(\boldsymbol{q}_\mu\cdot\boldsymbol{b}_\mu)^2}{\sum_\mu(\boldsymbol{b}_\mu\cdot\boldsymbol{b}_\mu)} = \frac{\sum_\mu\beta_\mu^2(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu)^2}{\sum_\mu\beta_\mu^2(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu)} &= \sum_\mu\frac{\beta_\mu^2(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu)}{\sum_{\tilde{\mu}}\beta_{\tilde{\mu}}^2(\boldsymbol{q}_{\tilde{\mu}}\cdot\boldsymbol{q}_{\tilde{\mu}})}(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu) \\ &\geq \min_\mu(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu) = (\boldsymbol{q}_{\mu^1}\cdot\boldsymbol{q}_{\mu^1}) \end{aligned} \tag{6-18}$$

$$\Longrightarrow \quad \theta_1 := \inf_{\boldsymbol{w}\in N_0^\perp}\Phi\{\boldsymbol{w}\} = \boldsymbol{q}_{\mu^1}\cdot\boldsymbol{q}_{\mu^1} = \Phi\{\boldsymbol{q}_{\mu^1}\psi_{\mu^1}\} > 0. \tag{6-19}$$

The value θ_1 defined by (6-19) is the smallest positive eigenvalue of the self-adjoint operator corresponding to the functional $\Phi\{\boldsymbol{w}\}$. In accordance with the spectral theorem, the next eigenvalue $\theta_2 \geq \theta_1$ is equal to the minimal value of the functional on the orthogonal complement of the "vector" $\boldsymbol{q}_{\mu^1}\psi_{\mu^1}$ in the subspace $N_0^\perp$. In the Fourier expansions for the elements of such an orthogonal complement the term $\boldsymbol{b}_{\mu^1}\psi_{\mu^1}$ is absent. Therefore on this orthogonal complement we have

$$\Phi\{\boldsymbol{w}\} = \frac{\sum_{\mu\neq\mu^1}\beta_\mu^2(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu)^2}{\sum_{\mu\neq\mu^1}\beta_\mu^2(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu)} \geq \min_{\mu\neq\mu^1}(\boldsymbol{q}_\mu\cdot\boldsymbol{q}_\mu) = \boldsymbol{q}_{\mu^2}\cdot\boldsymbol{q}_{\mu^2} \tag{6-20}$$

$$\Longrightarrow \theta_2 := \inf_{\substack{\boldsymbol{w}\in N_0^\perp \\ \boldsymbol{w}\perp\boldsymbol{q}_{\mu^1}\psi_{\mu^1}}}\Phi\{\boldsymbol{w}\} = \Phi\{\boldsymbol{q}_{\mu^2}\psi_{\mu^2}\} = \boldsymbol{q}_{\mu^2}\cdot\boldsymbol{q}_{\mu^2} \geq \theta_1. \tag{6-21}$$

It is obvious that the integer vectors $\boldsymbol{\mu}$ form a countable set; we number them in the order of nondecreasing of the absolute values of vectors $\boldsymbol{q}_{\boldsymbol{\mu}}$. Then, by induction

$$\theta_m = \Phi\{\boldsymbol{q}_{\boldsymbol{\mu}^m}\psi_{\boldsymbol{\mu}^m}\} = \boldsymbol{q}_{\boldsymbol{\mu}^m}\cdot\boldsymbol{q}_{\boldsymbol{\mu}^m} \geq \theta_{m-1}, \quad m = 1, 2, \dots . \tag{6-22}$$

Thus, for a bulk-elastic medium in a domain having the shape and dimensions of the rectangular parallelepiped B, in accordance with inequalities (5-48), we obtain the following upper and lower bounds for the free vibration frequencies:

$$\underline{c}_\kappa\sqrt{\frac{\underline{\rho}_\kappa}{\bar{\rho}_\kappa}}\sqrt{\theta_m} \leq \omega_m \leq \bar{c}_\kappa\sqrt{\frac{\bar{\rho}_\kappa}{\underline{\rho}_\kappa}}\sqrt{\theta_m}\,, \quad \sqrt{\theta_m} = |\boldsymbol{q}_{\boldsymbol{\mu}^m}| = \sqrt{\sum_{i=1}^{3}\frac{\pi^2}{l_i^2}(\mu_i^m)^2}\,. \tag{6-23}$$

The bounds for the first frequency are determined by the greatest edge length:

$$\sqrt{\theta_1} = |\boldsymbol{q}_{\boldsymbol{\mu}^1}| = \frac{\pi}{\max_i l_i} =: \frac{\pi}{\bar{l}}, \quad \sqrt{\frac{\underline{\rho}_\kappa}{\bar{\rho}_\kappa}}\,\pi\frac{\underline{c}_\kappa}{\bar{l}} \leq \omega_1 \leq \sqrt{\frac{\bar{\rho}_\kappa}{\underline{\rho}_\kappa}}\,\pi\frac{\bar{c}_\kappa}{\bar{l}}. \tag{6-24}$$

The further order of nondecreasing in absolute value vectors $\boldsymbol{q}_{\boldsymbol{\mu}^m}$ and the corresponding eigenvalues θ_m depends on the "proportions" of the rectangular parallelepiped (i.e. on the ratios of the lengths of edges) and can vary greatly. As a simple example, we present the values of the first few eigenvalues θ_m for a cube with an edge l:

$$\begin{aligned} &\theta_1 = \theta_2 = \theta_3 = \frac{\pi^2}{l^2}, &&\theta_4 = \theta_5 = \theta_6 = \frac{2\pi^2}{l^2},\\ &\theta_7 = \frac{3\pi^2}{l^2}, &&\theta_8 = \theta_9 = \theta_{10} = \frac{4\pi^2}{l^2}. \end{aligned} \tag{6-25}$$

Remark. In the absence of a force field, the equilibrium state of a bulk-elastic medium is uniform; consequently $\bar{\rho}_\kappa = \underline{\rho}_\kappa$, $\underline{c}_\kappa = \bar{c}_\kappa = c_\kappa$ and every pair of inequalities (6-24) converts into the equality

$$\omega_1 = \frac{\pi c_\kappa}{\bar{l}}.$$

In exactly the same way, the inequalities (6-23) convert into the following equalities:

$$\omega_m = c_\kappa\sqrt{\theta_m} = c_\kappa|\boldsymbol{q}_{\boldsymbol{\mu}^m}| = c_\kappa\sqrt{\sum_{i=1}^{3}\frac{\pi^2}{l_i^2}(\mu_i^m)^2}\,. \tag{6-26}$$

Thus, the bounds (6-23) and (6-24) give exact values of the free vibration frequencies in the case of a uniform equilibrium state and provide a good approximation in the case when the nonuniformity of the equilibrium state is not significant. The stated remark applies equally to other shapes of the domain considered below.

Remark. In the case of a significant value of the acceleration of gravity (or if the system under consideration moves with a high acceleration), and also in the case of a significant value of one of the dimensions of a system, the quantities entering into the upper and lower bounds (6-23), (6-24) for one and the same domain and the same bulk-elastic medium will be different for different orientations of the domain with respect to the direction of the force field.

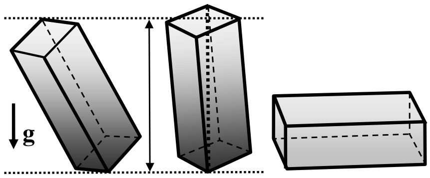

Figure 5. The influence of orientation with respect to the field direction.

Indeed, if the direction of the longest edge is close to the direction orthogonal to the field, and the dimension in the direction parallel to the field is small, then the values $\bar{\rho}_\kappa$, $\bar{c}_\kappa$ differ little from the values $\underline{\rho}_\kappa$, $\underline{c}_\kappa$. If, on the contrary, the greatest dimension corresponds to the direction parallel to the field, then these values differ greatly from each other (Figure 5). Obviously, this remark applies equally to any other shapes of the domain.

The above-described method is straightforward for a domain of the shape of a rectangular parallelepiped, but it can also be applied (in a somewhat more complicated way) to some other polyhedra. Let us consider three more shapes of the domain, which by means of similar techniques (reflection of the domain together with the vector field defined on it) can be converted into parallelepipeds with a spatially periodic vector field generated by the initial one.

The periodic field by construction will have a number of additional symmetries, which will affect the set of extremals of the functional $\Phi\{\boldsymbol{w}\}$ (i.e. eigenvectors of the corresponding operator): only a part of the eigenvectors for the parallelepiped (namely, those that possess the needed symmetries) will correspond to the eigenvectors for the original domain.

(a) A straight trihedral prism having a rectangular isosceles triangle as its base, which, when reflected with respect to the hypotenuse, turns into a square. Thus, the prism itself, when reflected with respect to the corresponding lateral face, is converted into a rectangular parallelepiped with a pair of square faces. Herewith the field $\boldsymbol{w}(\boldsymbol{x})$ reflected together with the original prism, is converted into a field in the rectangular parallelepiped, tangential at its boundary, but additionally possessing the symmetry with respect to one of the "diagonal" planes (Figure 6).

In what follows, as before, the parallelepiped together with the field $\boldsymbol{w}(\boldsymbol{x})$ is reflected with respect to the planes of faces. Finally we obtain a spatially periodic field for which the functional $\Phi\{\boldsymbol{w}\}$ takes the

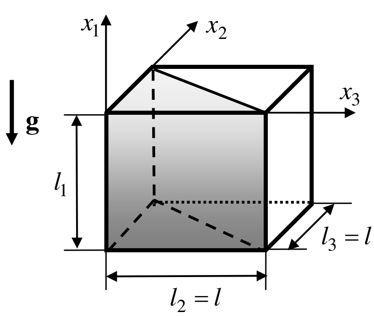

Figure 6. A straight trihedral prism with a rectangular isosceles triangle in the base.

same value as the original functional on the initial field $\boldsymbol{w}(\boldsymbol{x})$, whereas the Fourier series for $\nabla_\kappa \otimes \boldsymbol{w}(\boldsymbol{x})$ appears to be equal to formally differentiated Fourier series for $\boldsymbol{w}(\boldsymbol{x})$.

The presence of an additional symmetry of the field $\boldsymbol{w}(\boldsymbol{x})$ as compared to an arbitrary (tangential at the boundary) field in the rectangular parallelepiped results in certain additional properties of the Fourier coefficients in the expansion of the field $\boldsymbol{w}(\boldsymbol{x})$. Specifically, the pairs of terms of the Fourier series symmetric with respect to the diagonal plane should be equal in amplitude (while for the rectangular parallelepiped with square faces of one of the pairs, the amplitudes could be arbitrary). This results in less multiplicity of corresponding eigenvalues.

To describe the corresponding modes, we introduce some notation: the axis x_1 is the one directed along the edge orthogonal to the planes of the square faces, the axes x_2 and x_3 are directed along the catheti of an isosceles rectangular triangle, l_1 is the length of the edge directed along the axis x_1, $l = l_2 = l_3$ is the length of the edges directed along the axes x_2 and x_3, μ_1, μ_2, $\mu_3 \le \mu_2$ are the nonnegative integers not equal to zero simultaneously. Then the mentioned pair of terms of the Fourier series has the following form (up to an arbitrary factor):

$$\boldsymbol{w}_{\mu_1\mu_2\mu_3}(\boldsymbol{x}) = \nabla_\kappa \biggl(\cos\Bigl(\frac{\pi}{l_1}\mu_1 x_1\Bigr) \cos\Bigl(\frac{\pi}{l}\mu_2 x_2\Bigr) \cos\Bigl(\frac{\pi}{l}\mu_3 x_3\Bigr) + \cos\Bigl(\frac{\pi}{l_1}\mu_1 x_1\Bigr) \cos\Bigl(\frac{\pi}{l}\mu_3 x_2\Bigr) \cos\Bigl(\pi\mu_2 \frac{x_3}{l}\Bigr) \biggr). \tag{6-27}$$

The field (6-27) is one of the extremals of functional $\Phi\{\boldsymbol{w}\}$ for the considered triangle prism, i.e. the eigenvector of corresponding self-adjoint operator, with related eigenvalue

$$\theta_{\mu_1\mu_2\mu_3} = \pi^2 \biggl(\frac{\mu_1^2}{l_1^2} + \frac{\mu_2^2 + \mu_3^2}{l^2} \biggr) = \boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{q}_{\boldsymbol{\mu}}, \quad \boldsymbol{q}_{\boldsymbol{\mu}} = \frac{\pi\mu_1}{l_1}\boldsymbol{e}_1 + \frac{\pi\mu_2}{l}\boldsymbol{e}_2 + \frac{\pi\mu_3}{l}\boldsymbol{e}_3. \tag{6-28}$$

Numbering the integer vectors $\boldsymbol{\mu} = \boldsymbol{\mu}^m$ in the order of nondecreasing of $|\boldsymbol{q}_{\boldsymbol{\mu}^m}|$, we obtain a nondecreasing sequence of eigenvalues:

$$\theta_{\boldsymbol{\mu}^m} = \boldsymbol{q}_{\boldsymbol{\mu}^m} \cdot \boldsymbol{q}_{\boldsymbol{\mu}^m}, \quad m = 1, 2, \dots . \tag{6-29}$$

It is obvious that

$$\theta_{\boldsymbol{\mu}^1} = \min_{\boldsymbol{\mu}} \boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{q}_{\boldsymbol{\mu}} = \Bigl(\frac{\pi}{\max(l_1, l)} \Bigr)^2. \tag{6-30}$$

Bounds for the free vibration frequencies ω_m, as before, are given by inequalities (6-23).

(b) A triangular pyramid, the base of which is an isosceles rectangular triangle; the planes of the lateral faces passing through its catheti are orthogonal to the plane of the base, and the plane of lateral face passing through the hypotenuse makes with the plane of the base an angle equal to $\pi/4$ (Figure 7). Such a pyramid by a number of reflections with respect to the planes of the faces is converted into a cube.

Specifically, first, reflections are made with respect to the planes of the faces orthogonal to the base; hereby the original pyramid turns into a quadrangular pyramid, the base of which is a square with a side equal to the hypotenuse of the base of the original pyramid. Then the quadrangular pyramid is reflected with respect to the planes of the side faces and four quadrangular pyramids are formed with bases orthogonal to the base plane of the original pyramid and passing through the sides of the square mentioned. At the last stage, one (any) of the reflected pyramids is reflected with respect to one of its lateral faces in such a way that the plane of the base of reflected pyramid be parallel to the plane of the

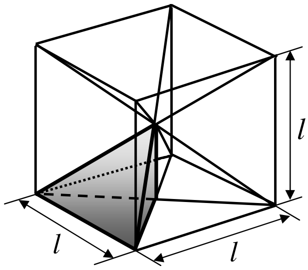

Figure 7. A triangular pyramid with rectangular isosceles triangle in the base.

base of the original pyramid. The totality of six quadrangular pyramids forms a cube with an edge equal to the hypotenuse of the base of the original triangular pyramid.

As for the vector field $\boldsymbol{w}(\boldsymbol{x})$ in the original pyramid, it also undergoes all the listed reflections and eventually turns into a field in the cube, which we still denote $\boldsymbol{w}(\boldsymbol{x})$. Although the cube is a particular case of the rectangular parallelepiped, considered in (a), in this case an important role in finding the extremals of functional $\Phi\{\boldsymbol{w}\}$ is played by the symmetries that the field $\boldsymbol{w}(\boldsymbol{x})$ gets by construction, namely: the resulting field $\boldsymbol{w}(\boldsymbol{x})$ is symmetric with respect to all diagonal planes of the cube (i.e. the planes orthogonal to the planes of a pair of faces and passing through the diagonals of the faces of this pair).

We introduce the Cartesian rectangular coordinates with the origin at one of vertices of the cube and axes directed along its edges. The edge length (the length of the hypotenuse of an isosceles triangle) will be denoted l. It can be shown that in the case under consideration the groups of terms of the Fourier series corresponding to the sets of integers μ_1, μ_2, μ_3, such that

$$0 \le \mu_1 \le \mu_2 \le \mu_3, \quad \mu_3 > 0, \tag{6-31}$$

are given by the following equality:

$$\boldsymbol{w}_{\mu_1\mu_2\mu_3}(\boldsymbol{x}) = \sum_{(i,j,k)} \nabla_\kappa \left(\cos\Big(\frac{\pi}{l}\mu_i x_1\Big) \cos\Big(\frac{\pi}{l}\mu_j x_2\Big) \cos\Big(\frac{\pi}{l}\mu_k x_3\Big) \right), \tag{6-32}$$

where (i, j, k) are possible permutations of numbers $(1, 2, 3)$, and the numbers μ_1, μ_2, μ_3 additionally possess the property that the sum of any pair of them is an even number. The property is equivalent to that the numbers μ_i either all are odd, or all are even. The fields (6-32) are extremals of the functional $\Phi\{\boldsymbol{w}\}$, i.e. eigenvectors of the corresponding operator with eigenvalues

$$\theta_{\mu_1\mu_2\mu_3} = \frac{\pi^2}{l^2}(\mu_1^2 + \mu_2^2 + \mu_3^2) = \boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{q}_{\boldsymbol{\mu}}, \quad \boldsymbol{q}_{\boldsymbol{\mu}} = \frac{\pi}{l}(\mu_1\boldsymbol{e}_1 + \mu_2\boldsymbol{e}_2 + \mu_3\boldsymbol{e}_3). \tag{6-33}$$

Numbering, as before, integer vectors $\boldsymbol{\mu} = \boldsymbol{\mu}^m$ in order of nondecreasing of $|\boldsymbol{q}_{\boldsymbol{\mu}^m}|$, we obtain a nondecreasing sequence of eigenvalues $\theta_{\boldsymbol{\mu}^m}$, where $m = 1, 2, 3, \ldots$. Obviously, the smallest eigenvalue corresponds to the triplet $(1, 1, 1)$:

$$\theta_{\boldsymbol{\mu}^1} = \min_{\boldsymbol{\mu}} \boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{q}_{\boldsymbol{\mu}} = \frac{3\pi^2}{l^2}. \tag{6-34}$$

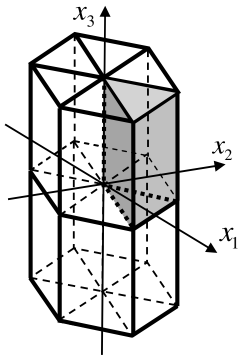

Figure 8. A straight trihedral prism with a regular triangle in the base. Conversion into a hexahedral prism.

It is also obvious that the second eigenvalue corresponds to the triplet (0, 0, 2), and the third one corresponds to the triplet (0, 2, 2):

$$\theta_{\mu^2} = \frac{4\pi^2}{l^2}, \quad \theta_{\mu^3} = \frac{8\pi^2}{l^2}. \tag{6-35}$$

In principle, it is not difficult to find any given number of the first eigenvalues. Bounds for the free vibration frequencies ω_m, as before, are given by inequalities (6-23).

(c) A straight trihedral prism with a height l_3, whose base is a regular triangle with the side l (and corresponding height $h = l\sqrt{3}/2$). We reflect the prism (as before, together with tangential at the boundary vector field defined in it) with respect to the two lateral faces, then the resulting quadrangular prism we reflect with respect to a larger lateral face; the resulting regular straight hexahedral prism is reflected with respect to the base (Figure 8).

As a result, we obtain a regular straight hexahedral prism of double height. Reflecting this hexahedral prism with respect to all the lateral faces, we obtain a polyhedron, the cross-section of which by a plane parallel to the base is shown in Figure 9, left. Continuing such reflections, we obtain a periodic vector field with a period $2h$ in the directions of axes x_1 and x_2 (parallel to the heights of regular triangle) and with a period $2l_3$ in the direction of axis x_3 (orthogonal to the base of the prism). Note that periodicity with a period $2h$ is also present in the direction parallel to the third height of regular triangle, but for the expansion of a vector field into a triple Fourier series only three directions of periodicity should be chosen from the four available; however, the presence of the fourth will result in the structure of terms of the Fourier series.

The elementary cell of periodicity here is an oblique parallelepiped with the edges $(2h, 2h, 2l_3)$, the cross-section of which by a plane parallel to the base of the original prism (that being a parallelogram) is also depicted in Figure 9, left. To represent the corresponding Fourier series taking into account the obliqueness of the coordinate system (x_1, x_2, x_3), we introduce the following notation: $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are the unit basis vectors specifying the axes (x_1, x_2, x_3) (Figures 8 and 9)); the dual basis corresponding to the basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the basis $(\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3 = \boldsymbol{e}_3)$ (Figure 9, right).

The system of basis functions (up to normalizing factors, which play a minor role in this problem) is

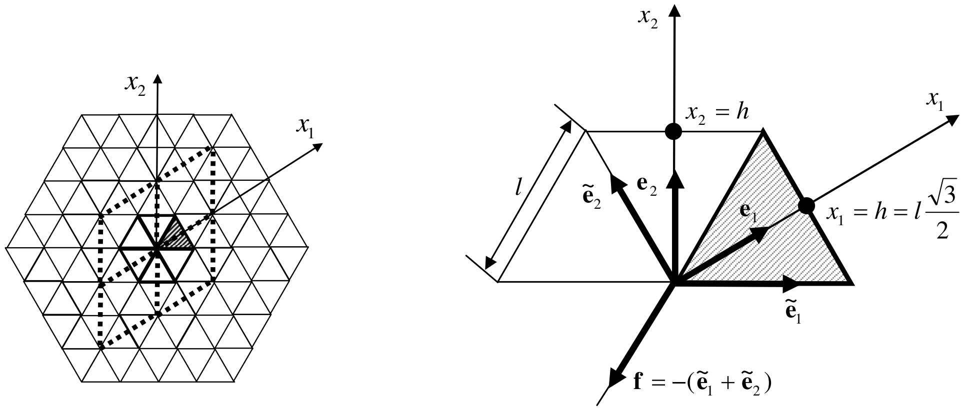

Figure 9. Left: Conversion of the field in the hexahedral prism into a periodic one by means of reflections. Right: Conventions for the Fourier series expansion in an oblique parallelepiped.

taken to be of the following form:

$$\sin(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \sin\Big(\frac{\pi}{l_3}\mu_3 x_3\Big), \quad \sin(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \cos\Big(\frac{\pi}{l_3}\mu_3 x_3\Big),$$
$$\cos(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \sin\Big(\frac{\pi}{l_3}\mu_3 x_3\Big), \quad \cos(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \cos\Big(\frac{\pi}{l_3}\mu_3 x_3\Big),$$
$$\boldsymbol{q}_{\boldsymbol{\mu}} = \frac{\pi}{h}(\mu_1 \tilde{\boldsymbol{e}}_1 + \mu_2 \tilde{\boldsymbol{e}}_2), \tag{6-36}$$
$$\mu_3 \geq 0, \quad \mu_3 > 0 \Rightarrow \mu_1, \mu_2 = 0, \pm 1, \pm 2, \dots,$$
$$\mu_3 = 0 \Rightarrow \mu_1 \geq 0, \quad \mu_1 > 0 \Rightarrow \mu_2 = 0, \pm 1, \pm 2, \dots,$$
$$\mu_1 = 0 \Rightarrow \mu_2 = 1, 2, \dots .$$

Taking into account that the subspace of vector fields of interest is formed by the gradients of scalar fields, as well as the presence of symmetry with respect to the plane $x_3 = 0$, we arrive at the following expressions for two types of terms of the Fourier series corresponding to a set of integers (μ_1, μ_2, μ_3):

$$\begin{aligned} \boldsymbol{w}^c_{\mu_1\mu_2\mu_3} &= \boldsymbol{q}_{\boldsymbol{\mu}} \cos(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \cos\Big(\frac{\pi}{l_3}\mu_3 x_3\Big) - \frac{\pi}{l_3}\mu_3 \boldsymbol{e}_3 \sin(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \sin\Big(\frac{\pi}{l_3}\mu_3 x_3\Big), \\ \boldsymbol{w}^s_{\mu_1\mu_2\mu_3} &= \boldsymbol{q}_{\boldsymbol{\mu}} \sin(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \cos\Big(\frac{\pi}{l_3}\mu_3 x_3\Big) + \frac{\pi}{l_3}\mu_3 \boldsymbol{e}_3 \cos(\boldsymbol{q}_{\boldsymbol{\mu}} \cdot \boldsymbol{x}) \sin\Big(\frac{\pi}{l_3}\mu_3 x_3\Big). \end{aligned} \tag{6-37}$$

Each of the modes (6-37) would enter into an expansion with its own coefficient, these coefficients being arbitrary and independent, if not for the presence of symmetry of the vector field with respect to the planes of the lateral faces of original prism (due to sequence of reflections having led to periodicity): this results in definite relations between the coefficients. To specify these relations, we introduce, apart from the vectors $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_2$, an additional vector $\boldsymbol{f}$ (Figure 9, right):

$$\boldsymbol{f} = -(\tilde{\boldsymbol{e}}_1 + \tilde{\boldsymbol{e}}_2). \tag{6-38}$$

The vectors $(\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \boldsymbol{f})$ under these reflections pass into each other and are equivalent to each other in the sense of the structure and symmetries of the constructed periodic field. The symmetry of a field, when the latter is expanded into the Fourier series, results in the symmetries of groups of its terms corresponding to the same value of μ_3 and the values of $\boldsymbol{q}_{\boldsymbol{\mu}}$ that pass into each other under the reflections. In general case, there are six such values, but in particular, they can coincide in pairs, and then there will be three of them.

We describe both cases. In the first case, we set $\mu_2 > 0$, $\mu_1 > \mu_2$. Then

$$\begin{aligned}
\boldsymbol{q}^{(1)}_{\mu_1\mu_2} &:= \frac{\pi}{h}(\mu_1\tilde{\boldsymbol{e}}_1 + \mu_2\tilde{\boldsymbol{e}}_2), \quad \boldsymbol{q}^{(2)}_{\mu_1\mu_2} := \frac{\pi}{h}(\mu_1\tilde{\boldsymbol{e}}_2 + \mu_2\tilde{\boldsymbol{e}}_1), \quad \boldsymbol{q}^{(3)}_{\mu_1\mu_2} := \frac{\pi}{h}(\mu_1\tilde{\boldsymbol{e}}_2 + \mu_2\boldsymbol{f}), \\
\boldsymbol{q}^{(4)}_{\mu_1\mu_2} &:= \frac{\pi}{h}(\mu_1\boldsymbol{f} + \mu_2\tilde{\boldsymbol{e}}_2), \quad \boldsymbol{q}^{(5)}_{\mu_1\mu_2} := \frac{\pi}{h}(\mu_1\boldsymbol{f} + \mu_2\tilde{\boldsymbol{e}}_1), \quad \boldsymbol{q}^{(6)}_{\mu_1\mu_2} := \frac{\pi}{h}(\mu_1\tilde{\boldsymbol{e}}_1 + \mu_2\boldsymbol{f}).
\end{aligned} \tag{6-39}$$

In the second case, we set $\mu_2 = 0$, $\mu_1 > 0$. Then

$$\boldsymbol{q}^{(1)}_{\mu_1} = \frac{\pi}{h}\mu_1\tilde{\boldsymbol{e}}_1, \quad \boldsymbol{q}^{(2)}_{\mu_1} = \frac{\pi}{h}\mu_1\tilde{\boldsymbol{e}}_2, \quad \boldsymbol{q}^{(3)}_{\mu_1} = \frac{\pi}{h}\mu_1\boldsymbol{f}. \tag{6-40}$$

Modes of each type (i.e. $\boldsymbol{w}^c$ and $\boldsymbol{w}^s$), corresponding to the same set (6-40) or to the same set (6-39) of vectors $\boldsymbol{q}$, should have the same coefficients. It is almost obvious that when such vectors belong to the same set invariant with respect to reflections, that is equivalent to their absolute values being equal to each other.

Taking all the foregoing into account, we represent the Fourier series for the constructed periodic field $\boldsymbol{w}$, which possesses all the necessary symmetries, in the following form:

$$\begin{aligned}
\boldsymbol{w} \sim &\sum_{\mu_3>0} a^s_{\mu_3}\frac{\pi}{l_3}\mu_3\boldsymbol{e}_3 \sin\Big(\frac{\pi}{l_3}\mu_3 x_3\Big) \\
&+ \sum_{\mu_1>0}\sum_{\mu_3\ge 0}\sum_{i=1}^{3}\bigg(a^c_{\mu_1\mu_3}\Big(\boldsymbol{q}^{(i)}_{\mu_1}\cos(\boldsymbol{q}^{(i)}_{\mu_1}\cdot\boldsymbol{x})\cos\Big(\frac{\pi}{l_3}\mu_3x_3\Big) - \frac{\pi}{l_3}\mu_3\boldsymbol{e}_3\sin(\boldsymbol{q}^{(i)}_{\mu_1}\cdot\boldsymbol{x})\sin\Big(\frac{\pi}{l_3}\mu_3x_3\Big)\Big) \\
&\qquad + a^s_{\mu_1\mu_3}\Big(q^{(i)}_{\mu_1}\sin(\boldsymbol{q}^{(i)}_{\mu_1}\cdot\boldsymbol{x})\cos\Big(\frac{\pi}{l_3}\mu_3x_3\Big) + \frac{\pi}{l_3}\mu_3\boldsymbol{e}_3\cos(\boldsymbol{q}^{(i)}_{\mu_1}\cdot\boldsymbol{x})\sin\Big(\frac{\pi}{l_3}\mu_3x_3\Big)\Big)\bigg) \\
&+ \sum_{\substack{\mu_2>0\\ \mu_1>\mu_2}}\sum_{\mu_3\ge 0}\sum_{i=1}^{6}\bigg(a^c_{\mu_1\mu_2\mu_3}\Big(\boldsymbol{q}^{(i)}_{\mu_1\mu_2}\cos(\boldsymbol{q}^{(i)}_{\mu_1\mu_2}\cdot\boldsymbol{x})\cos\Big(\frac{\pi}{l_3}\mu_3x_3\Big) - \frac{\pi}{l_3}\mu_3\boldsymbol{e}_3\sin(\boldsymbol{q}^{(i)}_{\mu_1\mu_2}\cdot\boldsymbol{x})\sin\Big(\frac{\pi}{l_3}\mu_3x_3\Big)\Big) \\
&\qquad + a^s_{\mu_1\mu_2\mu_3}\Big(\boldsymbol{q}^{(i)}_{\mu_1\mu_2}\sin(\boldsymbol{q}^{(i)}_{\mu_1\mu_2}\cdot\boldsymbol{x})\cos\Big(\frac{\pi}{l_3}\mu_3x_3\Big) + \frac{\pi}{l_3}\mu_3\boldsymbol{e}_3\cos(\boldsymbol{q}^{(i)}_{\mu_1\mu_2}\cdot\boldsymbol{x})\sin\Big(\frac{\pi}{l_3}\mu_3x_3\Big)\Big)\bigg).
\end{aligned} \tag{6-41}$$

Due to periodicity of the "constructed" field $\boldsymbol{w}$, the Fourier expansions for $\nabla_\kappa \otimes \boldsymbol{w}$ and $\nabla_\kappa \cdot \boldsymbol{w} = \boldsymbol{I} : \nabla_\kappa \otimes \boldsymbol{w}$ are the result of applying of the corresponding operators to the terms of the Fourier series for the field itself. We note that multiplicity of the terms corresponding to $\mu_1 = \mu_2 = 0,\ \mu_3 > 0$ is equal to one, and multiplicity of the remaining terms is equal to two.

Each group of terms of the Fourier series forms either one-dimensional or two-dimensional subspace, these subspaces being mutually orthogonal both in the sense of the scalar product $\langle \boldsymbol{w} \cdot \widetilde{\boldsymbol{w}}\rangle_{B'}$ corresponding to the denominator of the functional $\Phi\{\boldsymbol{w}\}$ (6-8) and in the sense of the bilinear functional $\langle(\nabla_\kappa \cdot \boldsymbol{w})(\nabla_\kappa \cdot \tilde{\boldsymbol{w}})\rangle_{B'}$ corresponding to its numerator (here the domain B' is an oblique parallelepiped specifying the cell of periodicity of the constructed field). By arguments almost coinciding with those in derivation of (6-18)–(6-22), we obtain a set of values of the functional $\Phi\{\boldsymbol{w}\}$ on the Fourier series terms that correspond to all possible sets of numbers (μ_1, μ_2, μ_3):

$$\theta_{00\mu_3} = \left(\frac{\pi}{l_3}\right)^2 \mu_3^2,$$

$$\theta^c_{\mu_1 0\mu_3} = \theta^s_{\mu_1 0\mu_3} = \boldsymbol{q}^{(i)}_{\mu_1} \cdot \boldsymbol{q}^{(i)}_{\mu_1} + \left(\frac{\pi}{l_3}\right)^2 \mu_3^2 = \pi^2\left(\frac{\mu_1^2}{(\frac{3}{4}l)^2} + \frac{\mu_3^2}{l_3^2}\right), \tag{6-42}$$

$$\theta^c_{\mu_1\mu_2\mu_3} = \theta^s_{\mu_1\mu_2\mu_3} = \boldsymbol{q}^{(i)}_{\mu_1\mu_21} \cdot \boldsymbol{q}^{(i)}_{\mu_1\mu_2} + \left(\frac{\pi}{l_3}\right)^2 \mu_3^2 = \pi^2\left(\frac{\mu_1^2+\mu_2^2-\mu_1\mu_2}{(\frac{3}{4}l)^2} + \frac{\mu_3^2}{l_3^2}\right).$$

From the same arguments it follows that these are the eigenvalues of a self-adjoint operator corresponding to the functional $\Phi\{\boldsymbol{w}\}$. Numbering them in the order of nondecreasing and using the comparison theorem, we obtain upper and lower bounds for numbered in the same way the free vibration frequencies of the bulk-elastic medium in the domain of the shape under consideration:

$$\underline{c}_\kappa \sqrt{\frac{\underline{\rho}_\kappa}{\bar{\rho}_\kappa}}\,\sqrt{\theta} \le \omega \le \bar{c}_\kappa \sqrt{\frac{\bar{\rho}_\kappa}{\underline{\rho}_\kappa}}\,\sqrt{\theta}\,. \tag{6-43}$$

In deriving the equalities (6-42), the following equations are used:

$$\tilde{\boldsymbol{e}}_1 \cdot \tilde{\boldsymbol{e}}_1 = \tilde{\boldsymbol{e}}_2 \cdot \tilde{\boldsymbol{e}}_2 = \boldsymbol{f} \cdot \boldsymbol{f} = \tfrac{4}{3}, \quad \tilde{\boldsymbol{e}}_1 \cdot \tilde{\boldsymbol{e}}_2 = \tilde{\boldsymbol{e}}_1 \cdot \boldsymbol{f} = \tilde{\boldsymbol{e}}_2 \cdot \boldsymbol{f} = -\tfrac{2}{3}. \tag{6-44}$$

From equalities (6-42) we obtain the value of the smallest eigenvalue:

$$\theta_{\min} = \left(\frac{\pi}{\max(\frac{3}{4}l, l_3)}\right)^2, \tag{6-45}$$

and the corresponding bounds (6-43) for the lowest frequency.

7. Examples of bilateral bounds for free vibration frequencies of the Mooney–Rivlin bulk-elastic media in rectangular parallelepipeds

The examples given below are based on the nonlinear bulk-elastic constitutive relation having some essential features of the well-known Mooney–Rivlin elastic solid. In what follows the former will be called the Mooney–Rivlin bulk-elastic medium. We remark that this type of constitutive relation is meaningful for any values of strain up to arbitrarily large values. This property is very important for the problems considered in the work. We remind that unlike the cases presented in the literature, in the problems investigated here the unperturbed state may be deformed greatly and nonuniformly with respect to the homogeneous unloaded configuration and this entails a great difference in possible methods of analysis.

In the cases presented in the literature, substantial restrictions are assumed regarding the unperturbed state of the medium, because the equations of motion with variable coefficients in general case cannot be solved analytically. In each particular case certain simplifications in the formulation of the problem are employed. For example, there may be equations of motion with variable coefficients of some specific type. In other cases, conditions of low compressibility of the medium are assumed, that allows at further analytical study to consider the equilibrium state of the medium as uniform. Sometimes nonuniformity can be neglected due to a certain geometrical configuration of the system, for example, if the medium under consideration occupies a layer which is very thin in direction perpendicular to the direction of mass force field. All those assumptions allow to reduce the problem under consideration to an analytical study of equations with either constant coefficients or variable coefficients of some specific type, whereas further analysis is based on exact solutions of the equations.

As for our method, it is based on variational approach together with usage of some essential analytical findings of our own [Ryzhak et al. 2017; Ryzhak 1993; 1994; 1997], giving the opportunity of obtaining rigorous upper and lower bounds for the free vibration frequencies for a broad class of mechanical systems whose behavior is governed by equations with variable coefficients.

In the examples given below we demonstrate the bilateral bounds in the case of some mechanical systems with great nonuniformity of loaded configuration, and compare them with the well-known exact values for uniform systems.

For simplicity we assume that the domain occupied by the Mooney–Rivlin bulk-elastic medium is a rectangular parallelepiped. We first set the elastic potential of such a medium with respect to unit volume in the unloaded configuration $\tilde{\kappa}$:

$$\begin{aligned}\sigma_{\tilde{\kappa}}(\widetilde{\boldsymbol{F}}) &= \frac{K_0}{2}\left(\det\,\widetilde{\boldsymbol{F}} + \frac{1}{\det\,\widetilde{\boldsymbol{F}}}\right)\\ &= \frac{K_0}{2}\left(\frac{\rho_0}{\rho} + \frac{\rho}{\rho_0}\right),\end{aligned} \tag{7-1}$$

where $\widetilde{\boldsymbol{F}}$ is the transformation gradient with respect to configuration $\tilde{\kappa}$, K_0, and ρ_0 are the bulk modulus and density in the unloaded state.

Let the medium occupying the rectangular parallelepiped be under the action of constant vertical gravity field $\boldsymbol{g}$. We assume that the largest edge of rectangular parallelepiped l_3 is the vertical one (see Figure 3). In this case it is not difficult to obtain the following analytical expressions for sound velocity, density, and bulk modulus as functions of the vertical coordinate x_3 measured from the bottom. On the upper face of parallelepiped, where $x_3 = l_3$, there are the minimal values of density $\rho(l_3) = \underline{\rho}$, bulk modulus $K(l_3) = \underline{K}$, and sound velocity $c(l_3) = \underline{c}$. Then we have

$$\begin{aligned}\rho(l_3 - x_3) &= \underline{\rho}\left(1 + \frac{\underline{\rho}}{\underline{K}}\,g(l_3 - x_3)\right) = \underline{\rho}\left(1 + \frac{g(l_3 - x_2)}{c^2}\right),\\ K(l_3 - x_3) &= \underline{K}\left(1 + \frac{g(l_3 - x_3)}{\underline{c}^2}\right)^2,\\ c(l_3 - x_3) &= \sqrt{\frac{\underline{K}}{\underline{\rho}}\left(1 + \frac{g(l_3 - x_3)}{\underline{c}^2}\right)} = \underline{c}\sqrt{1 + \frac{g(l_3 - x_3)}{\underline{c}^2}}\,.\end{aligned} \tag{7-2}$$

Maximal values of the quantities are attained at the bottom of parallelepiped, where $x_3 = 0$:

$$\rho(l_3) = \bar{\rho} = \underline{\rho}\left(1 + \frac{gl_3}{\underline{c}^2}\right), \quad K(l_3) = \overline{K} = \underline{K}\left(1 + \frac{gl_3}{\underline{c}^2}\right)^2, \quad c(l_3) = \bar{c} = \underline{c}\sqrt{1 + \frac{gl_3}{\underline{c}^2}}\,. \tag{7-3}$$

Substituting the values (7-3) into inequalities (6-23), we obtain the following upper and lower bounds for the free vibration frequencies:

$$\frac{\underline{c}}{\sqrt{1+(gl_3/\underline{c}^2)}}\sqrt{\theta_m} \leq \omega_m \leq \underline{c}\left(1+\frac{gl_3}{\underline{c}^2}\right)\sqrt{\theta_m}\,, \qquad \sqrt{\theta_m} = |\boldsymbol{q}_{\boldsymbol{\mu}^m}| = \sqrt{\sum_{i=1}^{3}\frac{\pi^2}{l_i^2}(\mu_i^m)^2}\,, \tag{7-4}$$

Introducing the dimensionless quantity

$$\tilde{\theta}_m := l_3^2\theta_m, \tag{7-5}$$

we rewrite inequalities (7-4) as follows:

$$\frac{\underline{c}}{l_3}\frac{\sqrt{\tilde{\theta}_m}}{\sqrt{1+(gl_3/\underline{c}^2)}} \leq \omega_m \leq \frac{\underline{c}}{l_3}\sqrt{\tilde{\theta}_m}\left(1+\frac{gl_3}{\underline{c}^2}\right). \tag{7-6}$$

In particular for the first frequency we obtain:

$$\sqrt{\theta_1} = |\boldsymbol{q}_{\boldsymbol{\mu}^1}| = \frac{\pi}{\max_i l_i} =: \frac{\pi}{l_3}, \qquad \frac{\underline{c}}{l_3}\frac{\pi}{\sqrt{1+(gl_3/\underline{c}^2)}} \leq \omega_1 \leq \frac{\underline{c}}{l_3}\pi\left(1+\frac{gl_3}{\underline{c}^2}\right). \tag{7-7}$$

From formulas (7-3) it follows that if the quantity $gl_3/\underline{c}^2$ is small enough, then

$$\rho(l_3-x_3) \approx \underline{\rho} \approx \bar{\rho}, \qquad K(l_3-x_3) \approx \underline{K} \approx \bar{K}, \qquad c(l_3-x_3) \approx \underline{c} \approx \bar{c},$$

and it is possible to consider the loaded unperturbed configuration of the medium as uniform or close to uniform. *In the case of uniform equilibrium state* (when $\underline{c} = \bar{c} = c$) every pair of inequalities (7-7) and (7-6) converts into the equalities and gives the *well-known exact values of free vibration frequencies*:

$$\omega_1 = \frac{\pi c}{l_3}, \qquad \omega_m = \frac{c}{l_3}\sqrt{\tilde{\theta}_m}\,. \tag{7-8}$$

In the case when the nonuniformity of the equilibrium state is not significant, the formulas (7-6) and (7-7) provide a good (and moreover, rigorous) approximation for free vibration frequencies. Clearly it will take place if:

(1) for given l_3 and $\underline{c}$ the mass force g is small enough,

(2) for given g and l_3 the minimal sound velocity $\underline{c}$ is great enough,

(3) for given g and $\underline{c}$ the vertical dimension l_3 is small enough.

Let us now consider the case when the classical methods by no means give any results. For example we set

$$1+\frac{gl_3}{\underline{c}^2} = 2.$$

Then

$$\bar{\rho} = \underline{\rho}\left(1+\frac{gl_3}{\underline{c}^2}\right) = 2\underline{\rho}, \quad \bar{K} = \underline{K}\left(1+\frac{gl_3}{\underline{c}^2}\right)^2 = 4\underline{K}, \quad \bar{c} = \underline{c}\sqrt{1+\frac{gl_3}{\underline{c}^2}} = \sqrt{2}\,\underline{c},$$

i.e. *the equilibrium state of the medium is strongly nonuniform.* Nevertheless our approach makes it possible to obtain the rigorous bilateral bounds for all free vibration frequencies:

$$\frac{c}{l_3}\frac{\sqrt{\tilde{\theta}_m}}{\sqrt{2}} \le \omega_m \le \frac{c}{l_3}\sqrt{\tilde{\theta}_m}\,, \quad \frac{c}{l_3}\frac{\pi}{\sqrt{2}} \le \omega_1 \le \frac{c}{l_3}2\pi. \tag{7-9}$$

Conclusions

In the work the free vibrations of a homogeneous bulk-elastic medium occupying a closed domain with fixed and perfectly smooth boundary and subject to the action of a field of mass forces are investigated analytically by variational methods. The formulation of the problem is highly general: the nonlinear law of bulk elasticity is arbitrary, the shape of the domain is almost arbitrary (see Appendix A), the force field is an arbitrary potential field. Additionally the domains of a number of special shapes are considered.

For investigation of the problem under consideration it has become necessary to prove modifications of the fundamental theorems in the theory of free vibrations for elastic bodies. The main novel research tool is the use of the canonical form of second variation of total potential energy of the system [Ryzhak et al. 2017], which in this case assumes a particularly simple form, namely (5-27) and (5-31).

Proved in the work version of the comparison theorem for the free vibration frequencies results in the inequalities (5-49), (5-52). The inequalities obtained make it possible to compare with each other all the free vibration frequencies of two different bulk-elastic media in domains of identical or similar shapes and for different fields of mass forces as well as for the same field, but for different orientations of the domain with respect to it (Figure 5). The latter can be regarded as a special case of different fields. These results (expressing a certain combination of geometrical and mechanical similarity of the problems) find a number of applications, e.g. enable to model and investigate in laboratory the free vibrations of real large-scale systems (say, atmospheric layers).

Another important result of the work are the analytical bilateral bounds for all the free vibration frequencies of bulk-elastic media in domains of the following shapes: a rectangular parallelepiped with an arbitrary ratio of the lengths of edges, a straight trihedral prism (with either an isosceles rectangular triangle or regular triangle as the base), and a triangular pyramid with an isosceles rectangular triangle as the base, whose lateral faces passing through the catheti of the base are orthogonal to its plane, whereas a face passing through the hypotenuse of the base is inclined to its plane at a $\pi/4$ angle.

The above-mentioned analytical results are also valid for different fields of mass forces and for different orientations of the domains with respect to the same field, which obviously concerns the technical devices whose spatial position can be varied (see Figure 5). We note that the field itself enters into the obtained bilateral bounds indirectly, namely, via the maximal and minimal equilibrium values of the sound velocity in the medium in combination with the ratio of the maximal and minimal equilibrium density values. Obviously, in the absence of the force field, the equilibrium state of the medium is uniform, and hence, the upper and lower bounds coincide with each other and thus give the exact values of all frequencies of the spectrum not only for rectangular parallelepipeds (for which they could be obtained by the method of separation of variables), but also for other polyhedra (for which separation of variables cannot be applied).

If for a domain of some different shape (for example, a circular cylinder, a sphere or a spherical layer) the problem of free vibrations of a homogeneous bulk-elastic medium in the absence of the force field

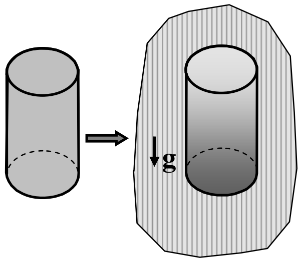

Figure 10. From exact values for the free vibration frequencies in the case of uniform equilibrium state to the bounds for them in the problem with stratification due to mass forces.

can be solved analytically and the values of all quantities θ_m are found, then the comparison theorem via inequalities (6-43) will give upper and lower bounds for all the frequencies ω_m of the medium in that domain with regard for the field and corresponding nonuniform distribution of density and velocity of sound (Figure 10). In the case when the values of quantities θ_m are found analytically only for a part of the free vibration modes in the absence of the field, then for the m-th frequency in the presence of the field only an upper bound will follow from the comparison theorem.

Appendix A. Proof of the compactness of subsets of subspace $N_0^\perp$ bounded both in the value of functional $R\{u\}$ and in the norm $\|u\|$

This compactness is a consequence of classical Rellich's lemma, the condition of which incorporates the boundedness both in the norm $(\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle_B)^{1/2}$ and in the value of the quadratic functional $\langle \nabla_\kappa\otimes\boldsymbol{u} : \nabla_\kappa\otimes\boldsymbol{u}\rangle_B$. Obviously, the norms $\|\boldsymbol{u}\|$ and $(\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle_B)^{1/2}$ are equivalent to each other and to all norms of the same type with positive weights $\beta(\boldsymbol{x})$. Thus, it should be additionally proved that from boundedness of $R\{\boldsymbol{u}\}$ and from boundedness of $\|\boldsymbol{u}\|$ it follows the boundedness of $\langle \nabla_\kappa\otimes\boldsymbol{u} : \nabla_\kappa\otimes\boldsymbol{u}\rangle_B$. Since the fields $\boldsymbol{u}(\boldsymbol{x})$ on the subspace $N_0^\perp$ are the gradients of scalar fields (5-34), we have

$$\boldsymbol{u}(\boldsymbol{x}) = \nabla_\kappa\gamma(\boldsymbol{x}) \implies \nabla_\kappa\otimes\boldsymbol{u}(\boldsymbol{x}) = \nabla_\kappa\otimes\nabla_\kappa\gamma(\boldsymbol{x}) = \big(\nabla_\kappa\otimes\boldsymbol{u}(\boldsymbol{x})\big)^T. \tag{A-1}$$

Thus, the equivalent assertion is the boundedness of the functional

$$\langle \nabla_\kappa\otimes\boldsymbol{u}^T : \nabla_\kappa\otimes\boldsymbol{u}\rangle_B = \langle \nabla_\kappa\otimes\boldsymbol{u} : \nabla_\kappa\otimes\boldsymbol{u}\rangle_B. \tag{A-2}$$

We prove first of all that the integral of the square of divergence of $\boldsymbol{u}(\boldsymbol{x})$ is bounded:

$$\begin{aligned}
\nabla_\kappa\cdot\boldsymbol{u} &= (\nabla_\kappa\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\nabla_\kappa\rho_\kappa/\rho_\kappa) - \boldsymbol{u}\cdot\nabla_\kappa\rho_\kappa/\rho_\kappa \\
&\implies (\nabla_\kappa\cdot\boldsymbol{u})^2 \le 2(\nabla_\kappa\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\nabla_\kappa\rho_\kappa/\rho_\kappa)^2 + 2(\boldsymbol{u}\cdot\nabla_\kappa\rho_\kappa/\rho_\kappa)^2 \\
&\qquad\qquad \le \frac{2}{\underline{K}_\kappa} K_\kappa(\nabla_\kappa\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\nabla_\kappa\rho_\kappa/\rho_\kappa)^2 + 2(|\nabla_\kappa\rho_\kappa|^2/\rho_\kappa^3)\rho_\kappa\boldsymbol{u}\cdot\boldsymbol{u} \\
&\implies \langle(\nabla_\kappa\cdot\boldsymbol{u})^2\rangle_B \le \frac{2}{\underline{K}_\kappa} R\{\boldsymbol{u}\} + 2(|\nabla_\kappa\rho_\kappa|^2/\rho_\kappa^3)_{\max}\|\boldsymbol{u}\|^2.
\end{aligned} \tag{A-3}$$

Further we make use of some formulas of tensor analysis and calculations similar to those in deriving the well-known Kelvin formula:

$$\begin{aligned}\nabla_\kappa\cdot(\nabla_\kappa\otimes\boldsymbol{u}^T\cdot\boldsymbol{u}) &= \nabla_\kappa\otimes\boldsymbol{u}^T:\nabla_\kappa\otimes\boldsymbol{u}+\boldsymbol{u}\cdot\big(\nabla_\kappa\cdot(\nabla_\kappa\otimes\boldsymbol{u}^T)\big)\\ &= \nabla_\kappa\otimes\boldsymbol{u}^T:\nabla_\kappa\otimes\boldsymbol{u}+\boldsymbol{u}\cdot\nabla_\kappa(\nabla_\kappa\cdot\boldsymbol{u})\\ &= \nabla_\kappa\otimes\boldsymbol{u}^T:\nabla_\kappa\otimes\boldsymbol{u}+\nabla_\kappa\cdot\big(\boldsymbol{u}(\nabla_\kappa\cdot\boldsymbol{u})\big)-(\nabla_\kappa\cdot\boldsymbol{u})^2\\ \Longrightarrow\ \langle\nabla_\kappa\otimes\boldsymbol{u}^T:\nabla_\kappa\otimes\boldsymbol{u}\rangle_B &= \langle(\nabla_\kappa\cdot\boldsymbol{u})^2\rangle_B+\langle\boldsymbol{n}_\kappa\cdot\nabla_\kappa\otimes\boldsymbol{u}^T\cdot\boldsymbol{u}\rangle_{\partial B}-\langle(\boldsymbol{n}_\kappa\cdot\boldsymbol{u})\nabla_\kappa\cdot\boldsymbol{u}\rangle_{\partial B}\\ &= \langle(\nabla_\kappa\cdot\boldsymbol{u})^2\rangle_B+\langle\boldsymbol{u}\cdot\nabla_\kappa\otimes\boldsymbol{u}\cdot\boldsymbol{n}_\kappa\rangle_{\partial B}.\end{aligned}\tag{A-4}$$

On smooth pieces of the boundary ∂B

$$\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})=0\ \Longrightarrow\ d\boldsymbol{u}\cdot\boldsymbol{n}_\kappa+\boldsymbol{u}\cdot d\boldsymbol{n}_\kappa=0.\tag{A-5}$$

Hence, for any tangential vector $\boldsymbol{y}$ the following equalities hold:

$$\boldsymbol{y}\cdot\nabla_\kappa\otimes\boldsymbol{u}\cdot\boldsymbol{n}_\kappa=-\boldsymbol{y}\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa\cdot\boldsymbol{u},\quad \boldsymbol{y}\perp\boldsymbol{n}_\kappa\ \Longrightarrow\ \boldsymbol{u}\cdot\nabla_\kappa\otimes\boldsymbol{u}\cdot\boldsymbol{n}_\kappa=-\boldsymbol{u}\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa\cdot\boldsymbol{u}.\tag{A-6}$$

Thus, taking into account the assumption of piecewise convexity of the boundary, we obtain

$$\langle\nabla_\kappa\otimes\boldsymbol{u}^T:\nabla_\kappa\otimes\boldsymbol{u}\rangle_B=\langle(\nabla_\kappa\cdot\boldsymbol{u})^2\rangle_B-\langle\boldsymbol{u}\cdot\nabla_\kappa^\Sigma\otimes\boldsymbol{n}_\kappa\cdot\boldsymbol{u}\rangle_{\partial B}\le\langle(\nabla_\kappa\cdot\boldsymbol{u})^2\rangle_B,\tag{A-7}$$

that, with regard for equalities (A-1), (A-2) and inequality (A-3), proves the assertion.

We remark that condition of piecewise convexity here is only a sufficient condition, not a necessary one.

Appendix B. Outline of the proof of modified spectral theorem

In this appendix there will be stated in detail only the elements of the proof of formulated in the work modified spectral theorem, which differ significantly from the corresponding elements of the proof of classical spectral theorem for elastic bodies (see, for example, Gurtin 1972; Mikhlin 1964). These significant distinctions are related to the presence of an infinite-dimensional subspace of neutral perturbations N_0 (on which $\lambda=\omega^2=0$), and also to nonstandard boundary conditions (free sliding over a nonplane surface).

We consider first of all the variational problem of the infimum of functional $\Psi\{\boldsymbol{u}\}$ (5-22) on nonzero elements of the space N formed by all kinematically admissible fields $\boldsymbol{u}(\boldsymbol{x})$:

$$\boldsymbol{u}(\boldsymbol{x})\in N\quad\Longleftrightarrow\quad\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n}_\kappa(\boldsymbol{x})|_{\partial B}=0.\tag{B-1}$$

From the canonical form (5-31) of the functional $R\{\boldsymbol{u}\}$ it is obvious that the infimum is equal to zero and is attained on nonzero fields $\boldsymbol{u}_0(\boldsymbol{x})$ forming the subspace N_0 (5-32). For what follows it should be proved that the fields $\boldsymbol{u}_0(\boldsymbol{x})$ are the solutions of linearized equilibrium equations (equations (5-16), (5-19) for $\lambda=\omega^2=0$ with boundary conditions (5-18)). We will analyze the inequality

$$R\{\boldsymbol{u}\}\ge R\{\boldsymbol{u}_0\}=0,\quad\forall\boldsymbol{u}\in N,\tag{B-2}$$

not excluding $\boldsymbol{u} = \mathbf{0}$. We represent $\boldsymbol{u}$ as follows:

$$\boldsymbol{u} = \alpha \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}, \quad -\infty < \alpha < \infty, \quad \tilde{\boldsymbol{u}} \in N, \tag{B-3}$$

$$R\{\alpha \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}\} = \alpha^2 R\{\boldsymbol{u}_0\} + 2\alpha \mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_0\} + R\{\tilde{\boldsymbol{u}}\} = 2\alpha \mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_0\} + R\{\tilde{\boldsymbol{u}}\} \geq 0, \quad \forall \alpha$$
$$\Longrightarrow \quad \mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_0\} = 0, \quad \forall \tilde{\boldsymbol{u}} \in N. \tag{B-4}$$

The initial equality (5-23) for the functional $\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_0\}$ will be used:

$$\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_0\} = \langle \nabla_\kappa \otimes \tilde{\boldsymbol{u}} : \boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0 \rangle_B + \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \nabla_\kappa \otimes \nabla_\kappa \varphi \cdot \boldsymbol{u}_0 \rangle_B - \big\langle (\boldsymbol{n}_\kappa \cdot \boldsymbol{T}_\kappa \cdot \boldsymbol{n}_\kappa) \tilde{\boldsymbol{u}} \cdot \nabla_\kappa^\Sigma \otimes \boldsymbol{n}_\kappa \cdot \boldsymbol{u}_0 \big\rangle_{\partial B}$$
$$= 0,$$

$$\nabla_\kappa \otimes \tilde{\boldsymbol{u}} : \boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0 = \nabla_\kappa \cdot \big((\boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0) \cdot \tilde{\boldsymbol{u}} \big) - \big(\nabla_\kappa \cdot (\boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0) \big) \cdot \tilde{\boldsymbol{u}}$$
$$\Longrightarrow \big\langle -\tilde{\boldsymbol{u}} \cdot \big(\nabla_\kappa \cdot (\boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0) - \rho_\kappa \nabla_\kappa \otimes \nabla_\kappa \varphi \cdot \boldsymbol{u}_0 \big) \big\rangle_{\partial B}$$
$$+ \big\langle \big(\boldsymbol{n}_\kappa \cdot \boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0 - (\boldsymbol{n}_\kappa \cdot \boldsymbol{T}_\kappa \cdot \boldsymbol{n}_\kappa) \nabla_\kappa^\Sigma \otimes \boldsymbol{n}_\kappa \cdot \boldsymbol{u}_0 \big) \cdot \tilde{\boldsymbol{u}} \big\rangle_{\partial B} = 0$$
$$\Longrightarrow \mathcal{A}(\boldsymbol{u}_0) = \nabla_\kappa \cdot (\boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_0) - \rho_\kappa \nabla_\kappa \otimes \nabla_\kappa \varphi \cdot \boldsymbol{u}_0 = 0, \quad \boldsymbol{x} \in B,$$
$$(\boldsymbol{n}_\kappa \cdot \boldsymbol{C}_\kappa \cdot \nabla_\kappa \otimes \boldsymbol{u}_0) \cdot (\boldsymbol{I} - \boldsymbol{n}_\kappa \otimes \boldsymbol{n}_\kappa) - (\boldsymbol{n}_\kappa \cdot \boldsymbol{T}_\kappa \cdot \boldsymbol{n}_\kappa) \nabla_\kappa^\Sigma \otimes \boldsymbol{n}_\kappa \cdot \boldsymbol{u}_0 = 0, \quad \boldsymbol{x} \in \partial B.$$

Thus, the fulfillment of the linearized equations of equilibrium and boundary conditions on the subspace N_0 is proved (in other words, it is proved that the elements of the subspace N_0 are the neutral perturbations).

Now we pass to the problem of infimum of the functional $\Psi\{\boldsymbol{u}\}$ (5-22) on the subspace $N_0^\perp$ (5-33). From the canonical form of the functional $R\{\boldsymbol{u}\}$ (5-27), (5-31) it is obvious that the infimum does exist and is nonnegative. Moreover, from the compactness of the subsets of subspace $N_0^\perp$, bounded both in value of $R\{\boldsymbol{u}\}$ and in the norm $\|\boldsymbol{u}\|$ (5-36) (that is proved in Appendix A), it follows that it is attained at an extremal $\boldsymbol{u}_1(\boldsymbol{x}) \in N_0^\perp$:

$$\lambda_1 = \inf_{\substack{\boldsymbol{u} \neq \mathbf{0} \\ \boldsymbol{u} \in N_0^\perp}} \Psi\{\boldsymbol{u}\} = \Psi\{\boldsymbol{u}_1\} \geq 0. \tag{B-5}$$

Making use of the canonical form of $R\{\boldsymbol{u}\}$ together with existence of an extremal, it is not difficult to prove that

$$\lambda_1 > 0. \tag{B-6}$$

Indeed, assume that $\lambda_1 = 0$. Then it follows from (5-31) that

$$\nabla_\kappa \cdot (\rho_\kappa \boldsymbol{u}) = \nabla_\kappa \cdot (\rho_\kappa \nabla_\kappa \gamma) = 0, \quad \boldsymbol{n}_\kappa \cdot \nabla_\kappa \gamma|_{\partial B} = 0$$
$$\Longrightarrow \nabla_\kappa \cdot (\gamma \rho_\kappa \nabla_\kappa \gamma) = \rho_\kappa \nabla_\kappa \gamma \cdot \nabla_\kappa \gamma$$
$$\Longrightarrow \langle \rho_\kappa \nabla_\kappa \gamma \cdot \nabla_\kappa \gamma \rangle_B = \langle \gamma \rho_\kappa \boldsymbol{n}_\kappa \cdot \nabla_\kappa \gamma \rangle_{\partial B} = 0 \Longrightarrow \nabla_\kappa \gamma \equiv 0. \tag{B-7}$$

The contradiction obtained proves the positivity of λ_1 (B-6).

Now it is to be proved that the field $\boldsymbol{u}_1(\boldsymbol{x})$ is a free vibration mode, i.e. an eigenvector of operator $\mathcal{A}$ with eigenvalue $\lambda_1 = \omega_1^2$, satisfying not only the kinematic boundary conditions of the tangentiality at the boundary of the domain ∂B (5-17), but also the free sliding condition, whose linearized form is given by (5-18). In the proof, we again use the initial form of the functional $R\{\boldsymbol{u}\}$ (5-20).

The condition for minimality of the functional $\Psi\{\boldsymbol{u}\}$ (B-1) is equivalent to the following inequality:

$$R\{\boldsymbol{u}\} \geq \lambda_1 \langle \rho_\kappa \boldsymbol{u} \cdot \boldsymbol{u} \rangle_B, \quad \forall \boldsymbol{u} \in N_0^\perp, \tag{B-8}$$

the value $\boldsymbol{u} = \boldsymbol{0}$ being not excluded. We represent $\boldsymbol{u}$ as follows:

$$\boldsymbol{u} = \alpha \boldsymbol{u}_1 + \tilde{\boldsymbol{u}}, \quad -\infty < \alpha < \infty, \quad \tilde{\boldsymbol{u}} \in N_0^\perp. \tag{B-9}$$

Substituting $\boldsymbol{u}$ (B-9) into (B-8), we obtain

$$\alpha^2 (R\{\boldsymbol{u}_1\} - \lambda_1 \langle \rho_\kappa \boldsymbol{u}_1 \cdot \boldsymbol{u}_1 \rangle_B) + 2\alpha (\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_1\} - \lambda_1 \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \boldsymbol{u}_1 \rangle_B) + R\{\tilde{\boldsymbol{u}}\} - \lambda_1 \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} \rangle_B \geq 0, \quad \forall \tilde{\boldsymbol{u}} \in N_0^\perp, \quad \forall \alpha. \tag{B-10}$$

It is obvious that the first term is equal to zero, the third term is nonnegative and does not depend on α. Hence, for satisfaction of the inequality (B-6) due to arbitrariness of α it is necessary and sufficient that the following equality be valid:

$$\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_1\} - \lambda_1 \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \boldsymbol{u}_1 \rangle_B = 0, \quad \forall \tilde{\boldsymbol{u}} \in N_0^\perp.$$

Taking into account that (B-4) implies the equality

$$\mathcal{R}\{\boldsymbol{u}_0, \boldsymbol{u}_1\} = 0, \quad \forall \boldsymbol{u}_0 \in N_0,$$

and also taking into account the orthogonality of the subspaces N_0 (5-32) and $N_0^\perp$ (5-33), we obtain a stronger equality:

$$\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_1\} - \lambda_1 \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \boldsymbol{u}_1 \rangle_B = 0, \quad \forall \tilde{\boldsymbol{u}} \in N. \tag{B-11}$$

Repeating the calculations used earlier in the analysis of the equality (B-4), we obtain from (B-11) both the equality

$$\lambda_1 \rho_\kappa \boldsymbol{u}_1 = \mathcal{A}(\boldsymbol{u}_1), \quad \boldsymbol{x} \in B, \tag{B-12}$$

and boundary conditions (5-18):

$$(\boldsymbol{n}_\kappa \cdot \boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_1) \cdot (\boldsymbol{I} - \boldsymbol{n}_\kappa \otimes \boldsymbol{n}_\kappa) - (\boldsymbol{n}_\kappa \cdot \boldsymbol{T}_\kappa \cdot \boldsymbol{n}_\kappa) \nabla_\kappa^\Sigma \otimes \boldsymbol{n}_\kappa \cdot \boldsymbol{u}_1 = 0, \quad \boldsymbol{x} \in \partial B,$$

which means the following: $\boldsymbol{u}_1(\boldsymbol{x})$ is a free vibration mode of the system with the frequency $\omega_1 = \sqrt{\lambda_1}$. Subsequently, as in the standard proof of the spectral theorem in linear elasticity, it is assumed that the assertion of the spectral theorem is valid for the first $m-1$ eigenvectors and it is proved (by means of the reasoning completely analogous to the preceding one) that the next extremal (with number m) on the subspace $N_{m-1}^\perp$ also satisfies the equality

$$\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_m\} - \lambda_m \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \boldsymbol{u}_m \rangle_B = 0, \quad \forall \tilde{\boldsymbol{u}} \in N_{m-1}^\perp, \quad \lambda_m \geq \lambda_{m-1}, \quad \boldsymbol{u}_m \in N_{m-1}^\perp. \tag{B-13}$$

From the preceding steps it followed that for each $k < m$

$$\begin{aligned} &\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_k\} - \lambda_k \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \boldsymbol{u}_k \rangle_B = 0, \quad \forall \tilde{\boldsymbol{u}} \in N, \\ \Longrightarrow\ &\mathcal{R}\{\boldsymbol{u}_k, \boldsymbol{u}_m\} - \lambda_k \langle \rho_\kappa \boldsymbol{u}_k \cdot \boldsymbol{u}_m \rangle_B = \mathcal{R}\{\boldsymbol{u}_k, \boldsymbol{u}_m\} = 0, \quad k = 0, 1, \dots, m-1, \\ \Longrightarrow\ &\mathcal{R}\{\tilde{\boldsymbol{u}}, \boldsymbol{u}_m\} - \lambda_m \langle \rho_\kappa \tilde{\boldsymbol{u}} \cdot \boldsymbol{u}_m \rangle_B = 0, \quad \forall \tilde{\boldsymbol{u}} \in N. \end{aligned} \tag{B-14}$$

From (B-14) follows the validity of both the equality

$$\lambda_m \rho_\kappa \boldsymbol{u}_m = \mathcal{A}(\boldsymbol{u}_m), \quad \boldsymbol{x} \in B, \tag{B-15}$$

and the boundary conditions (5-18)

$$(\boldsymbol{n}_\kappa \cdot \boldsymbol{C}_\kappa : \nabla_\kappa \otimes \boldsymbol{u}_m) \cdot (\boldsymbol{I} - \boldsymbol{n}_\kappa \otimes \boldsymbol{n}_\kappa) - (\boldsymbol{n}_\kappa \cdot \boldsymbol{T}_\kappa \cdot \boldsymbol{n}_\kappa) \nabla_\kappa^\Sigma \otimes \boldsymbol{n}_\kappa \cdot \boldsymbol{u}_m = 0, \quad \boldsymbol{x} \in \partial B. \tag{B-16}$$

Thus, $\boldsymbol{u}_m(\boldsymbol{x})$ is the free vibration mode with frequency $\omega_m = \sqrt{\lambda_m}$.

The proof of the remaining assertions of the spectral theorem reduces to standard reasoning (see, for example, Gurtin 1972; Mikhlin 1964) with the use of proved in Appendix A the compactness of subsets of space $N_0^\perp$ bounded both in values of $R\{\boldsymbol{u}\}$ and in the norm $\|\boldsymbol{u}\|$ (5-36).

Appendix C. Proof of the comparison theorem

We introduce the scalar products of two Hilbert spaces corresponding to two different bulk-elastic media occupying the identical domains (denoted by the same symbol B):

$$\widetilde{\mathcal{M}}^{(j)}\{\boldsymbol{w}, \boldsymbol{w}'\} := \left\langle \frac{1}{\rho^{(j)}} \boldsymbol{w} \cdot \boldsymbol{w}' \right\rangle_B, \quad j = 1, 2, \tag{C-1}$$

$$\widetilde{M}^{(j)}\{\boldsymbol{w}\} := \widetilde{\mathcal{M}}^{(j)}\{\boldsymbol{w}, \boldsymbol{w}\}, \tag{C-2}$$

$$\widetilde{\Psi}^{(j)}\{\boldsymbol{w}\} = \frac{\widetilde{R}^{(j)}\{\boldsymbol{w}\}}{\widetilde{M}^{(j)}\{\boldsymbol{w}\}}. \tag{C-3}$$

For each of the systems the divergenceless fields $\boldsymbol{w}(\boldsymbol{x})$ satisfying the kinematic boundary conditions $\boldsymbol{w} \cdot \boldsymbol{n}_\kappa|_{\partial B} = 0$ form a common infinite-dimensional subspace $\widetilde{N}_0$ on which $\widetilde{R}^{(j)}\{\boldsymbol{w}\} = 0$ and $\widetilde{\Psi}^{(j)}\{\boldsymbol{w}\} = 0$. From the spectral theorem for each of the systems it follows that on its own orthogonal complement to the subspace $\widetilde{N}_0$ there exists its own discrete set of eigenvectors orthogonal in the sense of its own scalar product $\widetilde{\mathcal{M}}^{(j)}\{\boldsymbol{w}, \boldsymbol{w}'\}$. We denote the orthogonal complements to the subspace $\widetilde{N}_0$ in the sense of scalar products $\widetilde{\mathcal{M}}^{(j)}$ as $(\widetilde{N}_0^{(j)})^\perp$, and the orthogonal complements in $(\widetilde{N}_0^{(j)})^\perp$ to the subspaces $\widetilde{N}_m^{(j)}$ spanned by the first m eigenvectors as $(\widetilde{N}_m^{(j)})^\perp$. The eigenvectors will be denoted $\boldsymbol{w}_k^{(j)}$. Then we have

$$\widetilde{N}_m^{(j)} = \operatorname{span}(\boldsymbol{w}_1^{(j)}, \dots, \boldsymbol{w}_m^{(j)}), \tag{C-4}$$

$$\lambda_{m+1}^{(j)} = \inf_{\substack{\boldsymbol{w} \neq 0 \\ \boldsymbol{w} \in (\widetilde{N}_m^{(j)})^\perp}} \widetilde{\Psi}^{(j)}\{\boldsymbol{w}\} = \widetilde{\Psi}^{(j)}\{\boldsymbol{w}_{m+1}^{(j)}\}. \tag{C-5}$$

The space $\widetilde{N}$ of all kinematically admissible fields can be represented in two ways as a direct sum of subspaces:

$$\widetilde{N} = \widetilde{N}_0 \oplus \widetilde{N}_m^{(j)} \oplus (\widetilde{N}_m^{(j)})^\perp, \quad j = 1, 2, \tag{C-6}$$

where $\widetilde{N}_0$ and $(\widetilde{N}_m^{(j)})^\perp$ are infinite-dimensional subspaces, and $\widetilde{N}_m^{(j)}$ are finite-dimensional ones:

$$\dim \widetilde{N}_m^{(j)} = m. \tag{C-7}$$

Consider the values of the functional $\widetilde{\Psi}^{(2)}\{\boldsymbol{w}\}$ on the “alien” subspace $(\widetilde{N}_m^{(1)})^\perp$, having in mind to prove that there exists a nonzero vector $\boldsymbol{v}$ in it for which

$$\widetilde{\Psi}^{(2)}\{\boldsymbol{v}\} \le \lambda^{(2)}_{m+1}. \tag{C-8}$$

For this purpose we prove that there exists a nonzero vector $\boldsymbol{v}$ such that

$$\boldsymbol{v} \in (\widetilde{N}^{(1)}_m)^{\perp}, \quad \boldsymbol{v} \in \operatorname{span}(\widetilde{N}_0, \widetilde{N}^{(2)}_{m+1}). \tag{C-9}$$

Consider a basis of the subspace $\widetilde{N}^{(2)}_{m+1}$ consisting of orthogonal (in the sense of $\widetilde{\mathcal{M}}^{(2)}$) eigenvectors $\boldsymbol{w}^{(2)}_1, \dots, \boldsymbol{w}^{(2)}_{m+1}$. Normalizing them in the sense of the same scalar product, we denote the corresponding orthonormal basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m+1})$. Due to the expansion (C-6), each of the vectors $\boldsymbol{e}_i$ can be uniquely represented in the form of the following sum:

$$\begin{gathered} \boldsymbol{e}_i = \boldsymbol{g}_i + \boldsymbol{h}_i + \boldsymbol{f}_i = \boldsymbol{g}_i + \boldsymbol{g}'_i, \quad i = 1, \dots, m+1, \\ \boldsymbol{g}_i \in \widetilde{N}_0, \quad \boldsymbol{h}_i \in \widetilde{N}^{(1)}_m, \quad \boldsymbol{f}_i \in (\widetilde{N}^{(1)}_m)^{\perp}. \end{gathered} \tag{C-10}$$

In order to prove that the vectors $(\boldsymbol{g}'_1, \dots, \boldsymbol{g}'_{m+1})$ are linearly independent, we assume that their linear combination is equal to zero:

$$\begin{aligned} \alpha_1 \boldsymbol{g}'_1 + \dots + \alpha_{m+1}\boldsymbol{g}'_{m+1} = 0 &\implies \alpha_1 \boldsymbol{e}_1 + \dots + \alpha_{m+1}\boldsymbol{e}_{m+1} = \alpha_1 \boldsymbol{g}_1 + \dots + \alpha_{m+1}\boldsymbol{g}_{m+1} \\ &\implies \alpha_1 \boldsymbol{e}_1 + \dots + \alpha_{m+1}\boldsymbol{e}_{m+1} = 0 \\ &\implies \alpha_i = 0, \quad i = 1, \dots, m+1. \end{aligned}$$

Thus, the linear independence of the vectors $(\boldsymbol{g}'_1, \dots, \boldsymbol{g}'_{m+1})$ is proved. Consider now once again their linear combination

$$\sum_{i=1}^{m+1} \alpha_i \boldsymbol{g}'_i = \sum_{i=1}^{m+1} \alpha_i \boldsymbol{h}_i + \sum_{i=1}^{m+1} \alpha_i \boldsymbol{f}_i. \tag{C-11}$$

Since $\boldsymbol{h}_i \in \widetilde{N}^{(1)}_m$, which is m-dimensional subspace, the vectors $(\boldsymbol{h}_1, \dots, \boldsymbol{h}_{m+1})$ are linearly dependent; hence, there exists a set of numbers $(\alpha_1, \dots, \alpha_{m+1})$, not all equal to zero, and such that

$$\sum_{i=1}^{m+1} \alpha_i \boldsymbol{h}_i = 0, \tag{C-12}$$

$$\implies \boldsymbol{v} := \sum_{i=1}^{m+1} \alpha_i \boldsymbol{g}'_1 = \sum_{i=1}^{m+1} \alpha_i \boldsymbol{f}_i \ne 0, \tag{C-13}$$

$$\begin{gathered} \implies \sum_{i=1}^{m+1} \alpha_i \boldsymbol{e}_i - \sum_{i=1}^{m+1} \alpha_i \boldsymbol{g}_i = \sum_{i=1}^{m+1} \alpha_i \boldsymbol{f}_i, \\ -\boldsymbol{\gamma} := \sum_{i=1}^{m+1} \alpha_i \boldsymbol{g}_i \in \widetilde{N}_0, \quad \sum_{i=1}^{m+1} \alpha_i \boldsymbol{e}_i \in \widetilde{N}^{(2)}_{m+1}, \quad \sum_{i=1}^{m+1} \alpha_i \boldsymbol{f}_i \in (\widetilde{N}^{(1)}_m)^{\perp}. \end{gathered} \tag{C-14}$$

Thus, it is proved that there exists a nonzero vector $\boldsymbol{v} \in (\widetilde{N}^{(1)}_m)^{\perp}$ such that

$$v = \gamma + \sum_{i=1}^{m+1} \alpha_i e_i \tag{C-15}$$

$$\Longrightarrow \widetilde{\Psi}^{(2)}\{v\} = \frac{\widetilde{R}^{(2)}\{\gamma + \sum_{i=1}^{m+1} \alpha_i e_i\}}{\widetilde{M}^{(2)}\{\gamma + \sum_{i=1}^{m+1} \alpha_i e_i\}} = \frac{\sum_{i=1}^{m+1} \alpha_i^2 \lambda_i^{(2)}}{\widetilde{M}^{(2)}\{\gamma\} + \sum_{i=1}^{m+1} \alpha_i^2} \le \lambda_{m+1}^{(2)}. \tag{C-16}$$

Due to the assumption regarding the functionals $\widetilde{\Psi}^{(i)}\{w\}$ we have

$$\widetilde{\Psi}^{(1)}\{v\} \le \widetilde{\Psi}^{(2)}\{v\} \le \lambda_{m+1}^{(2)} \tag{C-17}$$

$$\Longrightarrow \lambda_{m+1}^{(1)} = \inf_{\substack{w \neq 0 \\ w \in (\widetilde{N}_m^{(1)})^\perp}} \widetilde{\Psi}^{(1)}\{w\} \le \widetilde{\Psi}^{(1)}\{v\} \le \lambda_{m+1}^{(2)}. \tag{C-18}$$

which completes the proof.

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