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NONLINEAR FREE VIBRATION OF NANOBEAMS BASED ON NONLOCAL STRAIN GRADIENT THEORY WITH THE CONSIDERATION OF THICKNESS-DEPENDENT SIZE EFFECT

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Although the strain gradient and stress gradient parameters have been widely considered in the frame of nonlocal strain gradient theory, the literature concerned with the additional effect of slender ratio parameter in nonlocal strain gradient beam models is limited. In this paper, a nonlinear dynamical model for nonlocal strain gradient beams is developed and its nonlinear free vibration is analyzed. In the proposed dynamical model, the size-dependent properties associated not only with the nonlocal strain gradient and nonlocal stress gradient parameters but also with the slender ratio parameter are discussed. The effect of slender ratio parameter, which may be also interpreted as the thickness-dependent size effect, is caused by the stress on account of the thickness-direction strain gradient. Based on nonlocal strain gradient theory, the nonlinear governing equation of boundary conditions of the nanobeam are derived first. Then the nonlinear governing equation is simplified for special symmetric boundary conditions and external loadings. In the nonlinear free vibration analysis, an analytical solution for predicting the nonlinear free vibration frequencies is derived via the homotopy analysis method. It is shown that the nonlinear frequencies of the nanobeam display significant size-dependent phenomena for large values of slender ratio parameter and either stiffness-softening or stiffness-hardening behavior may occur. Our results also demonstrate that, besides conventional strain gradient and stress gradient effects, the thickness-dependent size effect can be significant for slender nanobeams and cannot be ignored in many cases.

1. Introduction

In the past years, nanoscience and nanotechnology have developed rapidly. Many nano- and micro-sized devices and structures have been applied in advanced technology, such as biosensors [Pei et al. 2004], nanosensors [Anker et al. 2008; Cui et al. 2001; Patolsky and Lieber 2005], nanoactuators [Shi et al. 2010; Sul and Yang 2009], atomic force microscopy (AFM) [Eaton and West 2010; Eom et al. 2011; Farokhi et al. 2016; Pereira 2001], and nano-/micro-electromechanical systems (NEMS/MEMS) [Li et al. 2003; Li et al. 2007]. In these engineering applications, beams, plates and shells in nano-size are the basic components and have been used widely [Pei et al. 2004; Ç. Demir and Civalek 2017].

For the purpose of better guidance to nanotechnology, more extensive studies of the statics and dynamics of nanobeams, nanoplates and nanoshells are requisite. It has been reported that nano-/micro-scale materials/structures have the properties of size effects observed by both experiments [Lam et al. 2003; McFarland et al. 2005; Kulkarni et al. 2005] and numerical simulations [Agrawal et al. 2008; Duan and Wang 2007]. Some results showed that structures in nano-size may behave either stiffness-hardening or

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stiffness-softening (see, e.g., Li et al. 2015a; Li et al. 2015b; Li 2014). Because of the time-consuming of molecular (atom) dynamics simulations and the difficulties of controlling experiments at micro/nano-scale, many nonclassical continuum theories have been proposed to explain and predict the size effects of structures at small sizes. Among various nonclassical continuum theories, the nonlocal elasticity theory is one of the most popular ones for static and dynamic analysis of nanostructures.

The nonlocal elasticity theory developed by Eringen [2002] thinks that the nonlocal stress at a reference point is influenced by the strain at all points of the body. From this point of view, the nonlocal elasticity theory is quite different from the point-to-point stress-strain relationship in the classical local elasticity theory. According to the nonlocal elasticity theory, by means of an integral with a nonlocal kernel function over the body, the long-range interactions between the atoms are incorporated. Eringen [2002] suggested that the integral constitutive law may be simplified to the form of differential equations when considering specified kind of kernel function. Based on nonlocal elasticity theory, there were a large number of studies on the static and dynamic responses of nanorods [Huang 2012; Lembo 2016; Narendar and Gopalakrishnan 2010; Wang et al. 2006], nanobeams [Aydogdu 2009; Dai et al. 2018; Reddy 2007; Thai 2012; Tuna and Kirca 2016], nanoplates [Assadi and Farshi 2011; Reddy 2010; Murmu and Adhikari 2011; Wang and Zhang 2018] and nanoshells [Shen 2010; Khademolhosseini et al. 2010; Hu et al. 2008; Ghavanloo and Fazelzadeh 2013a]. In many early studies on nonlocal elastic models, it was reported that the nonlocal natural frequency is generally lower than the local one, showing a stiffness-softening effects. For more details on nonlocal elastic models, the interested reader is referred to the comprehensive review by Eltaher et al. [2016].

Other than the nonlocal elastic models, the stiffness-softening effect of which was frequently reported, a stiffness-hardening effect may occur in the strain gradient elasticity theory [Aifantis 1992]. Based on the assumption that small-scale effect is associated with high-order deformation mechanism, additional strain gradient terms were suggested to be considered in the strain gradient elasticity theory. Recently, based on strain gradient elasticity theory, the significant strain gradient effects have been investigated in many studies when considering the static and dynamic behaviors of rods [Rahaeifard 2015], beams [Akgöz and Civalek 2011; Kong et al. 2009; Lazopoulos 2012; Xu and Deng 2016; Wang et al. 2018], plates [Ansari et al. 2015; Movassagh and Mahmoodi 2013; Ieşan 2014; Wang et al. 2011; Zhang et al. 2015], and shells [Ghavanloo and Fazelzadeh 2013b; Zeighampour and Beni 2014; Papargyri-Beskou et al. 2012] in nano-size.

In order to capture both size-dependent stiffness-softening and stiffness-hardening phenomena, Lim et al. [2015] developed a “nonlocal strain gradient theory” and investigated the wave propagation based on nonlocal strain beam models. This theory may be viewed as a combination of nonlocal elasticity theory and strain gradient theory. Because of its more generalized feature, there were fruitful studies on nonlocal strain gradient rods [Li et al. 2016a; Xu et al. 2017b; Zhu and Li 2017], beams [Li et al. 2015c; Li et al. 2016b; Xu et al. 2017a] and plates [Ebrahimi et al. 2016]. It is noted that the boundary conditions in nonlocal strain gradient models are complex due to the high-order stress. In this regard, Xu et al. [2017b] have recommended a weighted residual approach to derive the expressions of high-order forces and boundary conditions.

The aforementioned nonlocal strain gradient beam models assumed that the size effect in the beam’s thickness direction of nanobeams may be neglected for simplification, i.e., the Laplacian operator was supposed to be $\nabla^2 = \partial^2/\partial x^2$ for various beam models. Very recently, however, it was shown by Li et al.

[2018] that the size effect of strain gradient in the thickness direction ($\varepsilon_{xx,z}$) may be important for statics analysis and should be accounted for. Although there were several previous studies on the nonlinear vibration of nonlocal strain gradient beams (see, e.g., Li et al. 2016b; Şimşek 2016), these studies have not considered the strain gradient in the beam's thickness direction. In this work, we initiate to investigate the nonlinear free vibration of nonlocal strain gradient beams incorporating the strain gradient effect in the beam's thickness direction. It will be shown that the thickness-dependent size effect associated with the nanobeam's slender ratio on the nonlinear free vibration of nanobeams may be remarkable.

2. Nonlocal strain gradient theory

According to the nonlocal strain gradient theory developed by Lim et al. [2015], the total stress tensor accounts for not only the nonlocal stress tensor but also the strain gradient stress tensor, i.e.

$$t_{ij} = \sigma_{ij} - \nabla\sigma^{(1)}_{ijm}, \tag{1}$$

where ∇ is the Laplacian operator, and the nonlocal stress tensor σ_{ij} and the higher-order nonlocal stress tensor $\sigma^{(1)}_{ijm}$ are defined by

$$\sigma_{ij} = C_{ijkl}\int_V \alpha_0(|\chi' - \chi|, e_0a)\,\varepsilon'_{kl}\,dV, \tag{2}$$

$$\sigma^{(1)}_{ijm} = l^2 C_{ijkl}\int_V \alpha_1(|\chi' - \chi|, e_1a)\,\varepsilon'_{kl,m}\,dV, \tag{3}$$

where ε_{kl} is the classical strain tensor, $\varepsilon_{kl,m}$ is the strain gradient tensor, C_{ijkl} is the fourth-order elasticity tensor, l is the material length scale parameter introduced to consider the significance of strain gradient stress field, e_0a and e_1a, which are nonlocal parameters, are introduced to consider the significance of nonlocal elastic stress field.

As solving the integral constitutive equations of (1) is very difficult, a simplified form of differential equations will be used in this study. Let $\alpha_0(\chi', \chi, e_0a)$ and $\alpha_1(\chi', \chi, e_1a)$ be the nonlocal functions for the classical stress tensor and the strain gradient stress tensor, respectively. We suppose that α_0 and α_1 can satisfy the conditions given by Eringen [1983]. The linear nonlocal differential operator is used in the nonlocal functions, i.e., $\mathscr{L}_i = 1 - (e_ia)^2\nabla^2$ for $i = 0, 1$. Furthermore, it is assumed that $e = e_0 = e_1$; thus one obtains

$$\big(1 - (ea)^2\nabla^2\big)\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}, \tag{4}$$

$$\big(1 - (ea)^2\nabla^2\big)\sigma^{(1)}_{ijm} = l^2 C_{ijkl}\,\varepsilon_{kl,m}. \tag{5}$$

The general constitutive equations for size-dependent beams can be simplified as [Lim et al. 2015]

$$[1 - (ea)^2\nabla^2]\,t_{ij} = C_{ijkl}\,\varepsilon_{kl} - l^2\nabla^2 C_{ijkl}\,\varepsilon_{kl}, \tag{6}$$

where ea is a stress-gradient parameter introduced to involve stress gradient effect, while l is a strain-gradient parameter introduced to involve strain gradient effect.

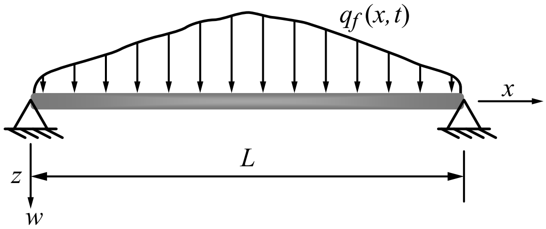

Figure 1. Schematic of a nanobeam.

For an Euler–Bernoulli beam-type structure, the size-dependent behavior may be neglected in the width directions. Thus, the general constitutive relation can be further simplified to [Li et al. 2018]

$$\left[1-(ea)^2\frac{\partial^2}{\partial x^2}\right]t_{xx}=\left[1-l^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\right]E\varepsilon_{xx}, \tag{7}$$

where E denotes the elasticity modulus, t_{xx} denotes the axial normal stress, and ε_{xx} denotes the axial strain. It should be noted that the formulations for the nonlocal elasticity theory [Eringen 1983] or the strain gradient theory [Aifantis 1992; Mindlin 1965; Aifantis and Willis 2005; Polizzotto 2012] can be obtained by setting $ea=0$ or $l=0$.

3. Formulation

The system under consideration consists of a nanobeam of length L between two immovable supports, internal cross-sectional area A, mass density ρ, and flexural rigidity EI, as shown in Figure 1. The beam is uniform along its length and the cross-section is symmetric. In this section, we will derive the equations of motion based on the nonlocal strain gradient theory by accounting for the geometric nonlinearity associated with the axial extension of the beam.

3.1. ***General governing equations.*** The displacements (u_1, u_2, u_3) of an Euler–Bernoulli beam along the (x, z) coordinate directions are given by

$$u_1(x,z)=u(x)-zw', \quad u_2(x,z)=0, \quad u_3(x,z)=w(x), \tag{8}$$

where u is the longitudinal displacement, w is the transverse displacement of the mid-plane, and $(\ )'=\partial/\partial x$.

According to the von-kármán nonlinear strain expression, the nonzero strain for a beam under large displacements can be written as

$$\varepsilon_{xx}=u'+\tfrac{1}{2}w'^2-zw'', \tag{9}$$

where ε_{xx} is the axial strain.

We will derive the nonlinear equations of motion and boundary conditions by utilizing Hamilton's principle. Based on the nonlocal strain gradient theory, the virtual work of the strain energy is given by

[Li et al. 2018]

$$
\begin{aligned}
\delta U &= \int_V \big(\sigma_{xx}\,\delta\varepsilon_{xx} + \sigma^{(1)}_{xxx}\,\delta\varepsilon_{xx,x} + \sigma^{(1)}_{xxz}\,\delta\varepsilon_{xx,z}\big)\,dV \\
&= \int_0^L\!\int_A \big[\sigma_{xx}\,\delta\big(u' + \tfrac{1}{2}w'^2 - zw''\big) + \sigma^{(1)}_{xxx}\,\delta(u'' + w'w'' - zw''') + \sigma^{(1)}_{xxz}\,\delta(-w'')\big]\,dA\,dx \\
&= \int_0^L N^{(0)}\delta\big(u' + \tfrac{1}{2}w'^2\big)\,dx - \int_0^L M^{(0)}\delta w'' dx + \int_0^L N^{(1)}_x \delta(u'' + w'w'')\,dx \\
&\qquad\qquad - \int_0^L M^{(1)}_x \delta w'''\,dx + \int_0^L N^{(1)}_z \delta(-w'')\,dx \\
&= \int_0^L \big(N^{(1)''}_x - N^{(0)'}\big)\,\delta u\,dx \\
&\qquad + \int_0^L \big[-(N^{(0)}w')' - (N^{(1)}_x w'')' + (N^{(1)}_x w')'' + M^{(1)'''}_x - M^{(0)''} - N^{(1)''}_z\big]\,\delta w\,dx \\
&\qquad + \big(N^{(0)} - N^{(1)'}_x\big)\,\delta u\big|_0^L + N^{(1)}_x \delta u'|_0^L \\
&\qquad + \big(N^{(0)}w' + N^{(1)}_x w'' + M^{(0)'} + N^{(1)'}_z - (N^{(1)}_x w')' - M^{(1)''}_x\big)\,\delta w\big|_0^L \\
&\qquad + \big(-M^{(0)} + N^{(1)}_x w' + M^{(1)'}_x - N^{(1)}_z\big)\,\delta w'\big|_0^L - M^{(1)}_x \delta w''\big|_0^L.
\end{aligned}
\tag{10}
$$

In (10), $N^{(0)}$ and $M^{(0)}$ are the lower-order force and moment for axial and transverse directions, respectively; $N^{(1)}_x$ and $N^{(1)}_z$ are the high-order axial forces due to strain gradient in the axial and thickness directions, respectively; $M^{(1)}_x$ is the high-order moment. These resultants are defined by

$$N^{(0)} = \int_A \sigma_{xx}\,dA, \tag{11a}$$

$$N^{(1)}_x = \int_A \sigma^{(1)}_{xxx}\,dA, \tag{11b}$$

$$N^{(1)}_z = \int_A \sigma^{(1)}_{xxz}\,dA, \tag{11c}$$

$$M^{(0)} = \int_A z\sigma_{xx}\,dA, \tag{11d}$$

$$M^{(1)}_x = \int_A z\sigma^{(1)}_{xxx}\,dA. \tag{11e}$$

The stress resultants can be given by

$$(1 - \mu^2\nabla^2)\,N^{(0)} = EA\big(u' + \tfrac{1}{2}w'^2\big), \tag{12a}$$

$$(1 - \mu^2\nabla)^2\,N^{(1)}_x = l^2 EA(u'' + w'w''), \tag{12b}$$

$$(1 - \mu^2\nabla^2)\,M^{(0)} = -EIw'', \tag{12c}$$

$$(1 - \mu^2\nabla^2)\,M^{(1)}_x = -l^2 EIw''', \tag{12d}$$

$$(1 - \mu^2\nabla^2)\,N^{(1)}_z = -l^2 EAw''. \tag{12e}$$

If body force, body couple and externally imposed tension are either absent or neglected, the virtual work done by an external transverse force q_f can be written as

$$\delta W = -\int_0^L q_f \, \delta w \, dx. \tag{13}$$

Since the considered beam is supported at both ends, the longitudinal displacement and velocity are relatively small. Hence, the kinetic energy of the nanobeam may be approximated as

$$K = \frac{1}{2}\int_0^L \rho A \Big(\frac{\partial w}{\partial t}\Big)^2 dx. \tag{14}$$

In the frame of Hamilton's principle, the dynamic governing equation and boundary conditions of this beam can be derived based on the following variational equation:

$$\delta \int_{t_1}^{t_2} (K - U + W)\, dt = 0. \tag{15}$$

Substituting (10), (13), and (14) into (15), one obtains the equations of motion as

$$\delta u : \big(N^{(0)} - N_x^{(1)'}\big)' = 0, \tag{16a}$$

$$\delta w : \rho A \ddot{w} - (N^{(0)} w')' - (N_x^{(1)} w'')' + (N_x^{(1)} w')'' + M_x^{(1)'''} - M^{(0)''} - N_z^{(1)''} + q_f = 0, \tag{16b}$$

and the boundary conditions as

$$\delta u : N^{(0)} - N_x^{(1)'} = 0 \quad \text{or} \quad u = 0, \tag{17a}$$

$$\delta u' : N_x^{(1)} = 0 \quad \text{or} \quad u' = 0, \tag{17b}$$

$$\delta w : N^{(0)} w' + N_x^{(1)} w'' + M^{(0)'} + N_z^{(1)'} - (N_x^{(1)} w')' - M_x^{(1)''} = 0 \quad \text{or} \quad w = 0, \tag{17c}$$

$$\delta w' : -M^{(0)} + N_x^{(1)} w' + M_x^{(1)'} - N_z^{(1)} = 0 \quad \text{or} \quad w' = 0, \tag{17d}$$

$$\delta w'' : M_x^{(1)} = 0 \quad \text{or} \quad w'' = 0, \tag{17e}$$

where the dot above w denotes the time differentiation with respect to t. By defining

$$N_{0x} = N^{(0)} - N_x^{(1)'}, \tag{18}$$

and combining (12a), (12b), and (17a), we have

$$N_{0x} = EA\big(u' + \tfrac{1}{2} w'^2\big) - EIl^2 (u''' + w' w''' + w''^2) = \text{constant}. \tag{19}$$

Hence (16b) can be rewritten as

$$\rho A \ddot{w} - N_{0x} w'' + \big(M_x^{(1)'} - M^{(0)} - N_z^{(1)}\big)'' + q_f = 0. \tag{20}$$

Upon combining (12c)–(12e), we have

$$(1 - \mu^2 \nabla^2)\big(M_x^{(1)'} - M^{(0)} - N_z^{(1)}\big) = -l^2 EI w^{(IV)} + EI w'' + l^2 EA w'', \tag{21}$$

where $^{(IV)}$ denotes the fourth derivative. With the help of (20) and (21), the final governing equation of the nanobeam can be obtained as

$$\rho A\left(1-\mu^2\frac{\partial}{\partial x^2}\right)\ddot{w}-N_{0x}\left(1-\mu^2\frac{\partial}{\partial x^2}\right)w''+\left(EI-EI\,l^2\frac{\partial}{\partial x^2}+l^2EA\right)w^{(IV)}+\left(1-\mu^2\frac{\partial}{\partial x^2}\right)q_f=0. \quad (22)$$

3.2. ***Governing equations for special boundary conditions.*** In Section 3.1, we have obtained the general equation of motion, equation (22). This governing equation can be finalized once N_{0x} is given. Nevertheless, N_{0x} may have different forms for different boundary conditions. In this subsection, we will discuss the N_{0x} expression for several typical boundary conditions, and then finalize the governing equations.

For free-free boundary conditions, since $N_{0x}(0)=N_{0x}(L)=0$, it is clear that $N_{0x}=0$ based on (19). In this work, the system under consideration is a beam assumed supported at two axially immobile supports, i.e., $u(0)=u(L)=0$. Hence, there are three possible types of boundary conditions for the longitudinal displacement to determine the N_{0x} expression. These three types of boundary conditions are

$$\text{case 1}: u(0)=0,\quad u(L)=0,\quad N_x^{(1)}(0)=0,\quad N_x^{(1)}(L)=0, \quad (23a)$$

$$\text{case 2}: u(0)=0,\quad u(L)=0,\quad u'(0)=0,\quad N_x^{(1)}(L)=0, \quad (23b)$$

$$\text{case 3}: u(0)=0,\quad u(L)=0,\quad u'(0)=0,\quad u'(L)=0. \quad (23c)$$

Based on the weighted residual approaches, the high-order axial forces due to strain gradient in the axial direction is given by Xu et al. [2017b]

$$N_x^{(1)}=EAl^2(u''+w'w''). \quad (24)$$

For pinned-pinned or clamped-clamped supports, case 1 and case 2 can be rewritten as

$$\text{case 1}: u(0)=0,\quad u(L)=0,\quad u''(0)=0,\quad u''(L)=0, \quad (25a)$$

$$\text{case 2}: u(0)=0,\quad u(L)=0,\quad u'(0)=0,\quad u''(L)=0. \quad (25b)$$

3.2.1. *Determination of N_{0x} for Case 1.* With the help of (25a), integrating (19) over the beam length results in

$$N_{0x}=\frac{EA}{2L}\int_0^L w'^2dx-\frac{l^2EA}{L}[w'(L)\,w''(L)-w'(0)\,w''(0)]. \quad (26)$$

For pinned-pinned or clamped-clamped supports, we have

$$N_{0x}=\frac{EA}{2L}\int_0^L w'^2dx. \quad (27)$$

Now the governing equation of the nanobeam can be obtained as

$$\rho A(1-\mu^2\partial/\partial x^2)\ddot{w}-\frac{EA}{2L}\left(\int_0^L w'^2dx\right)(1-\mu^2\partial/\partial x^2)w''+EI(1-l^2\partial/\partial x^2)w^{(IV)}$$
$$+l^2EAw^{(IV)}+(1-\mu^2\partial/\partial x^2)\,q_f=0. \quad (28)$$

3.2.2. *Determination of N_{0x} for Case 2 and Case 3.* For case 2 or case 3, N_{0x} cannot be obtained using a similar treatment as that proposed in Section 3.2.1. A more general method is required to obtain the expression of u in the form of w and its derivatives. We rewrite (19) as

$$u' - l^2 u''' = f(x), \tag{29}$$

where

$$f(x) = -\tfrac{1}{2} w'^2 + l^2 (w'' w'' + w' w''') + \frac{N_{0x}}{EA}. \tag{30}$$

The general solution of the homogeneous part of (29) is

$$u_0(x) = C_1 + C_2\, e^{x/l} + C_3\, e^{-x/l}, \tag{31}$$

where C_1, C_2, and C_3 are constants to be determined by using boundary conditions. By rewriting (29) as

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{B}(x), \tag{32}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} u'' \\ u' \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & 1/l^2 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -f/l^2 \\ 0 \end{bmatrix}, \tag{33}$$

and setting

$$\boldsymbol{Y}(x) = \boldsymbol{P}(x)\, \boldsymbol{Q}(x). \tag{34}$$

Equation (33) can be written as

$$(\boldsymbol{P}' - \boldsymbol{A}\boldsymbol{P})\boldsymbol{Q} + \boldsymbol{P}\boldsymbol{Q}' = \boldsymbol{B}. \tag{35}$$

By using the condition

$$\boldsymbol{P}' - \boldsymbol{A}\boldsymbol{P} = 0, \tag{36}$$

Equation (35) leads to

$$\boldsymbol{Q} = \int \boldsymbol{P}^{-1} \boldsymbol{B}\, dx. \tag{37}$$

By solving (36) we have

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{l}\, e^{x/l} & -\frac{1}{l}\, e^{-x/l} \\ e^{x/l} & e^{-x/l} \end{bmatrix}. \tag{38}$$

Upon combining (33), (34), (37), and (38), a particular solution of (29) can be found, i.e.,

$$u_1(x) = -\frac{1}{2l} \int \left(e^{x/l} \int f e^{-x/l} dx \right) dx + \frac{1}{2l} \int \left(e^{-x/l} \int f e^{x/l} dx \right) dx. \tag{39}$$

Therefore, the general solution of (29) is finally given by

$$u(x) = C_1 + C_2\, e^{x/l} + C_3\, e^{-x/l} - \frac{1}{2l} \int \left(e^{x/l} \int f e^{-x/l} dx \right) dx + \frac{1}{2l} \int \left(e^{-x/l} \int f e^{x/l} dx \right) dx. \tag{40}$$

The four undetermined constants C_1, C_2, C_3, and N_{0x} in the u expression of (40) can be determined for a set of given boundary conditions. However, the solving process for C_1, C_2, C_3, and N_{0x} may be complicated.

3.2.3. *Final governing equations for symmetric boundary conditions.* In Section 3.2.1, we have given the explicit expression of N_{0x} for boundary conditions of case 1. As discussed in Section 3.2.2, however, the general expression of N_{0x} is difficult to obtain for case 2 and case 3. It is also noted that the beam of case 3 has the same longitudinal constraints at both ends. For case 3, actually, the expression of N_{0x} can be obtained only for some special situations.

As a result, when the nanobeam with both ends subjected to the same boundary conditions for either longitudinal or transverse motions (i.e., the equations for boundary conditions at both ends are identical), the explicit expression of N_{0x} can be obtained and hence the governing equations finalized. Now we consider the following possible special type of symmetric boundary conditions:

$$u''(0) = u''(L), \quad w'(0) = w'(L), \quad w''(0) = w''(L). \tag{41}$$

Based on (41), integrating (19) over the beam length yields

$$N_{0x} = \frac{EA}{2L}\int_0^L w'^2\,dx. \tag{42}$$

Since the expression in (42) is identical to the result of (27) for case 1, the governing equation (28) is still valid for the considered symmetric boundary conditions. For asymmetric boundary conditions, equation (28) is not applicable.

3.3. ***Nondimensionalization.*** In order to simplify the form of the governing equation and to reduce the number of parameters, we introduce the following dimensionless quantities:

$$\bar{x} = \frac{x}{L}, \quad \bar{w} = \frac{w}{r}, \quad t = \bar{t}\sqrt{\frac{\rho A L^4}{EI}}, \quad \eta = \frac{L}{r}, \quad \tau = \frac{ea}{L}, \quad \zeta = \frac{l}{L}, \quad \bar{q}_f = \frac{q_f L^4}{EIr}, \tag{43}$$

where

$$r = \sqrt{I/A}, \tag{44}$$

is the turning radius of the nanobeam's cross section, and η is the slenderness ratio of the nanobeam.

With the aid of (43), the dimensionless governing equation of the system is written as

$$\left(1-\tau^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^2\bar{w}}{\partial\bar{t}^2} - \frac{1}{2}\left[\int_0^1\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^2 dx\right]\left(1-\tau^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \left(1-\zeta^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4} + \zeta^2\eta^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4} + \left(1-\tau^2\frac{\partial^2}{\partial\bar{x}^2}\right)\bar{q}_f = 0. \tag{45}$$

Since (45) is represented in dimensionless form, it is more convenient for us to further investigate the free vibrations of the nanobeam in a more general sense.

Equation (45) is the dimensionless equation of motion considering geometric nonlinearities and the strain gradient in the lateral direction. Upon dropping the time varying terms and transverse loading $\bar{q}$, equation (45) may be reduced to the equation developed by Li et al. [2018] for static problems. Further, if the terms associated with the thickness-direction strain gradient and geometric nonlinearities are either neglected or absent, we can obtain the same equation given by Lu et al. [2017]. A new term, $\zeta^2\eta^2\,\partial^4\bar{w}/\partial\bar{x}^4$, has been added in (45) if compared with the mathematical model proposed by Şimşek [2016]. This new term represents the thickness-dependent size effect. It must be noted that the value of slenderness ratio η is always large (the order of 10^1 or more). Therefore, the thickness-dependent size effect can be remarkable in many cases and needs to be considered in the dynamic analysis of nanobeams. It should also be mentioned that (45) is valid when (i) the longitudinal constraints satisfy (25a) of case 1 and the transverse constraints at both ends are either pinned or clamped, or (ii) the longitudinal constraints satisfy case 3 and the transverse constraints at both ends are identical with $\bar{q}_f(\bar{x}) = \bar{q}_{f0}(1-\bar{x})$ when $0 \le \bar{x} \le 1$.

4. Solutions

In this section, the nonlinear free vibration of a nanobeam governed by (45) will be studied. The external transverse loading $q_f(x,t)$ is assumed to be absent. Thus, we have

$$\left(1-\tau^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^2\bar{w}}{\partial\bar{t}^2}-\frac{1}{2}\left[\int_0^1\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^2dx\right]\left(1-\tau^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\left(1-\zeta^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+\zeta^2\eta^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}=0. \quad (46)$$

Based on the Galerkin's approach, equation (46) can be solved analytically by using a single-mode discretization or numerically by using a multi-mode discretization.

4.1. ***Analytical solution based on the homotopy analysis method.*** According to the Galerkin's approach, the approximate expression of $\bar{w}(\bar{x},\bar{t})$ is assumed as

$$\bar{w}(\bar{x},\bar{t}) = \phi(\bar{x})\,q(\bar{t}), \quad (47)$$

where $\phi(\bar{x})$ is the characteristic mode for a set of given boundary conditions, and $q(\bar{t})$ is the corresponding time-dependent generalized coordinate. The substitution of (47) into (46) yields

$$\frac{\partial^2 q}{\partial\bar{t}^2}+(D_{L1}+D_{L2}+D_{L3})\,q+(D_{N1}+D_{N2})\,q^3=0, \quad (48)$$

where

$$D_{L1}=\frac{\int_0^1\phi\phi^{(IV)}d\bar{x}}{S},\qquad D_{L2}=\frac{-\zeta^2\int_0^1\phi\phi^{(VI)}d\bar{x}}{S},\qquad D_{L3}=\frac{\zeta^2\eta^2\int_0^1\phi\phi^{(IV)}d\bar{x}}{S}, \quad (49)$$

$$D_{N1}=\frac{-\frac{1}{2}\int_0^1\phi\phi''d\bar{x}\int_0^1\phi'\phi'd\bar{x}}{S},\qquad D_{N2}=\frac{\frac{1}{2}\tau^2\int_0^1\phi\phi^{(IV)}d\bar{x}\int_0^1\phi'\phi'd\bar{x}}{S}. \quad (50)$$

where

$$S=\int_0^1\phi\phi\,d\bar{x}-\tau^2\int_0^1\phi\phi''d\bar{x}.$$

For pinned-pinned nanobeams, the characteristic modes can be defined as follows [Li et al. 2016b; Şimşek 2016]:

$$\phi_n(\bar{x})=\sin(n\pi\bar{x})\quad(n=1,2,3,\ldots). \quad (51)$$

By combining (48)–(51), one obtains

$$\frac{\partial^2 q}{\partial \bar{t}^2}+\frac{(1+\zeta^2\eta^2+n^2\pi^2\zeta^2)\,n^4\pi^4}{1+n^2\pi^2\tau^2}\,q+\frac{n^4\pi^4}{2}\,q^3=0. \tag{52}$$

One may rewrite (52) as

$$\frac{\partial^2 q}{\partial \hat{t}^2}+q+\gamma q^3=0, \tag{53}$$

where $\hat{t}=\omega\bar{t}$, and

$$\omega=n^2\pi^2\sqrt{\frac{1+\zeta^2\eta^2+n^2\pi^2\zeta^2}{1+n^2\pi^2\tau^2}},\qquad \gamma=\frac{1+n^2\pi^2\tau^2}{2(1+\zeta^2\eta^2+n^2\pi^2\zeta^2)}. \tag{54}$$

Equation $(54)_1$ is the expression of linear natural frequencies for nonlocal strain gradient beams. It is noted that the thickness-dependent effect is associated with the term of $\zeta^2\eta^2$. If the term of $\zeta^2\eta^2$ in $(54)_2$ is neglected, equation $(54)_2$ can be reduced to the expression obtained by Li et al. [2016b]. For calculation purpose, the initial conditions of the nanobeam considered are assumed to be

$$q(0)=a,\qquad \dot{q}(0)=0. \tag{55}$$

The second-order approximate frequency of (53) in the frame of homotopy analysis method can be obtained as [Liao 2003]

$$\Omega\approx\frac{131072+393216\gamma a^2+440832\gamma^2a^4+218880\gamma^3a^6+40599\gamma^4a^8}{1024(4+3\gamma a^2)^{7/2}}. \tag{56}$$

Hence the nonlinear free vibration frequency is given by

$$\omega_{NL}=\omega_L\Omega=\frac{n^2\pi^2}{\sqrt{2\gamma}}\frac{131072+393216\gamma a^2+440832\gamma^2a^4+218880\gamma^3a^6+40599\gamma^4a^8}{1024(4+3\gamma a^2)^{7/2}}. \tag{57}$$

4.2. ***Numerical results based on Galerkin's approach using a multi-mode approximation.*** In Section 4.1, we have obtained the analytical solution of (57) via the homotopy analysis method based on a single-mode discretization. To demonstrate the validity of expression (57), we will numerically solve the governing equation (46) by using a multi-mode discretization. Based on the Galerkin's approach, $\bar{w}$ can be expressed as

$$\bar{w}(\bar{x},\bar{t})=\sum_{j=1}^{N}\phi_j(\bar{x})\,q_j(\bar{t}). \tag{58}$$

Substituting (58) into (46), multiplying by $\phi_i(\bar{x})$ and integrating over $\bar{x}$ from 0 to 1 further lead to

$$m_{ij}\frac{\partial^2 q_j}{\partial \bar{t}^2}+k_{ij}q_j+\alpha_{ijkl}\,q_j\,q_k\,q_l=0, \tag{59}$$

where

$$\begin{aligned} m_{ij}&=\int_0^1\phi_i\phi_j\,d\bar{x}-\tau^2\int_0^1\phi_1\phi_j''\,d\bar{x},\qquad k_{ij}=(1+\zeta^2\eta^2)\int_0^1\phi_1\phi_j^{(IV)}\,d\bar{x}-\zeta^2\int_0^1\phi_1\phi_j^{(VI)}\,d\bar{x},\\ \alpha_{ijkl}&=-\frac{1}{2}\int_0^1\phi_i\phi_j''\int_0^1\phi_k'\phi_l'\,d\bar{x}\,d\bar{x}+\frac{1}{2}\tau^2\int_0^1\phi_i\phi_j^{(IV)}\int_0^1\phi_k'\phi_l'\,d\bar{x}\,d\bar{x}. \end{aligned} \tag{60}$$

For the purpose of numerical calculations, equation (59) is rewritten as its first-order state form, i.e.,

$$\dot{z} = Az + G(z), \tag{61}$$

where

$$p = \frac{\partial p}{\partial \bar{t}}, \quad z = [q;\, p] \quad \text{and} \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ -M^{-1}g \end{bmatrix}. \tag{62}$$

The initial conditions of the beam are assumed to be defined by (55) as well.

Equation (62) is then solved by employing a fourth-order Runge–Kutta integration method with variable step sizes. Hence we can obtain the displacement responses for given values of initial conditions, strain-gradient parameter ζ, stress-gradient parameter τ and slenderness ratio η. Typical results are shown in Figure 2.

Numerical results of nonlinear free vibration frequencies are compared in Figure 3 with the analytical solutions, for various truncated mode number N, strain-gradient and stress-gradient parameters. That

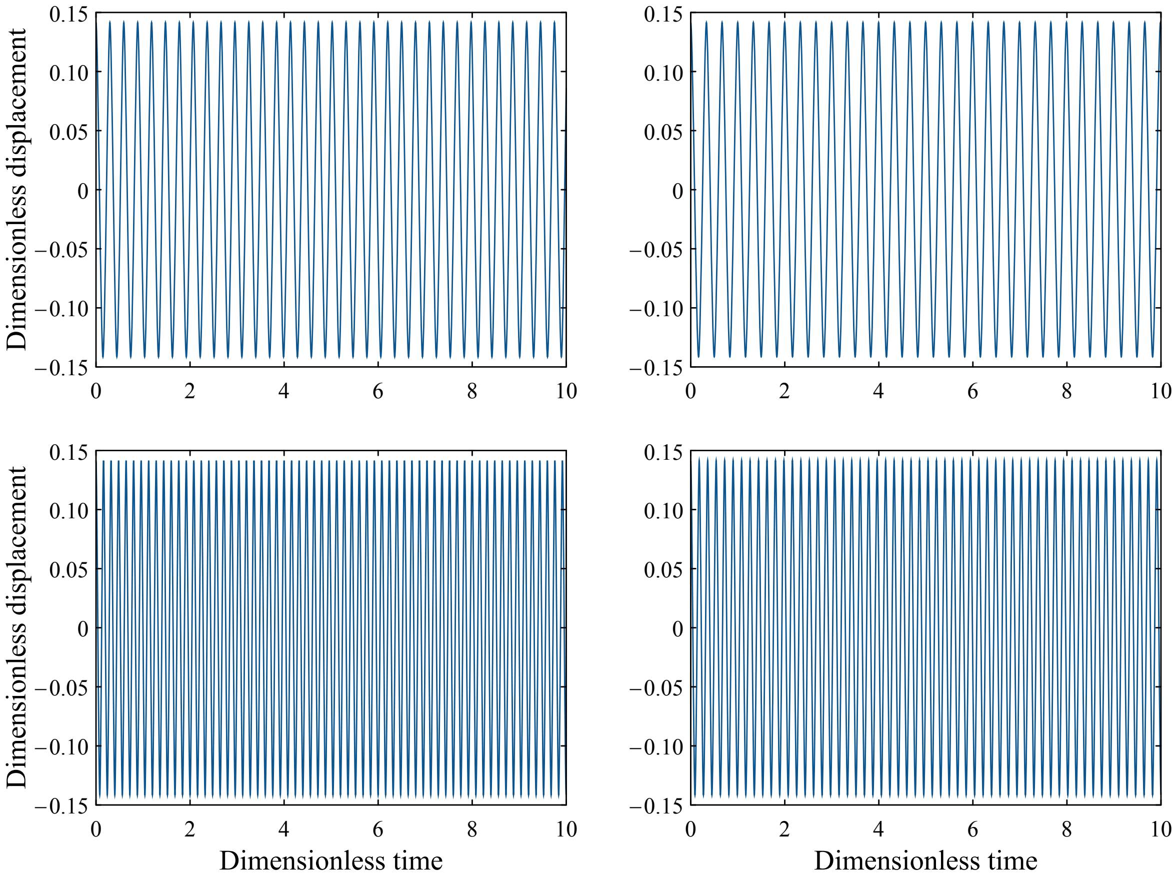

Figure 2. Displacement responses based on high-dimensional Galerkin discretization model. $N = 3$; $a = 0.1$; $\eta = 20$; $\zeta = 0.1$ (top) or $\zeta = 0.2$ (bottom); $\tau = 0.1$ (left) or $\tau = 0.2$ (right).

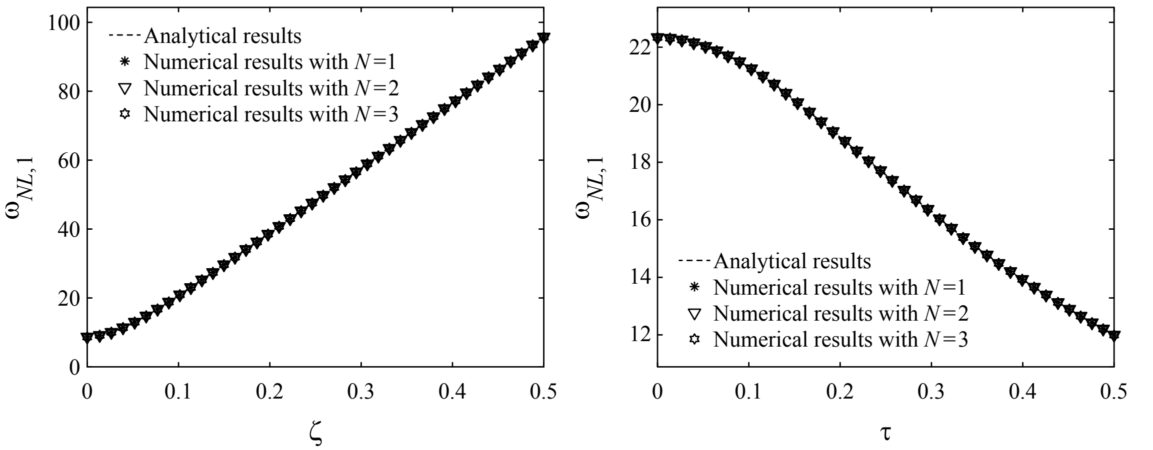

Figure 3. The nonlinear fundamental frequencies $\omega_{NL,1}$ predicted by analytical and numerical methods ($\eta = 20$, $a = 0.1$); $\tau = 0.1$ (left) and $\zeta = 0.1$ (right).

figure suggests that the single-mode analytical results agree well with the high-dimensional numerical ones. Since the single-mode-based analytical result has high precision, we will utilize the analytical expression of (57) to investigate the nonlinear free vibrations of the nanobeam in the following analysis.

4.3. ***Parameter analysis.*** Based on (57), the ratios of nonlinear nonclassical fundamental frequencies $\omega_{NL,1}$ to the linear nonclassical fundamental frequencies $\omega_{L,1}$ are obtained for various γa^2. Typical results are shown in Figure 4 (left). It is seen that the nonlinear frequency ratio increases as the initial amplitude a increases, which is known as a nonlinear "hardening spring" behavior. This is because that the increase of initial amplitude can increase the axial stretching, yielding larger nonlinear frequencies. The size-dependent effects of ζ, τ, and η on the nonlinear frequencies of the nanobeam may be implicit in the nonlinear parameter γ. Thus, there is a "hardening spring" behavior of size effects when γ is larger than the classic one of $\gamma = 0.5$ and a "softening spring" behavior when $\gamma < 0.5$.

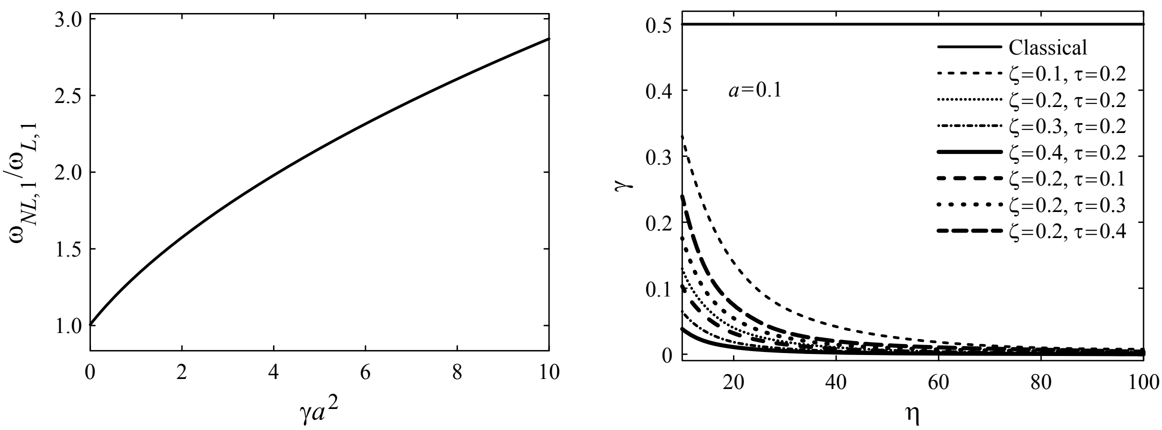

Figure 4. Left: fundamental frequencies ratios $\omega_{NL,1}/\omega_{L,1}$ versus γa^2. Right: nonlinear parameter γ versus slenderness ratio η.

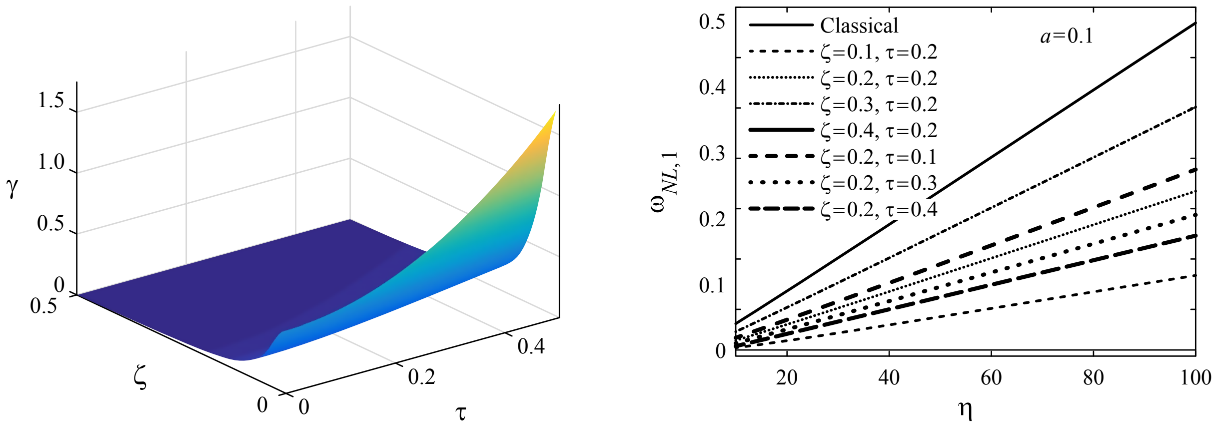

Figure 5. Left: nonlinear parameter γ for various ζ and τ when $a = 0.1$ and $\eta = 20$. Right: the effect of slenderness ratio η on $\omega_{NL,1}$.

Unlike the size-dependent effect of ζ, τ, and η on linear frequencies, the evolution of which shows a trend of monotonous increase (see expression $(54)_1$) when any one of these three key parameters is increased, the evolution of the size-dependent effect of ζ, τ, and η on nonlinear frequencies is quite different. It is observed from Figure 4 (right) and Figure 5 (left) that the parameter γ increases with increasing τ and decreases with increasing ζ and η. This implies that the two key parameters of ζ and η can reduce the size-dependent effect on nonlinear frequencies when considering the geometric nonlinearities quantized by γ. As shown in Figure 5 (left), for $\zeta = 0$ and $\tau = 0$, the value of γ corresponds to the case of a classical beam. Interestingly, the nonlocal strain gradient beams can produce either a lager or a smaller nonlinear frequency ratio than the classical one.

More importantly, the slenderness ratio η has a significant effect on the nonlinearity of the nanobeam, as shown in Figure 4 (right). It is not surprising, therefore, that the influence of η on the nonlinear frequencies is remarkable (see Figure 5, right). It is noted that the nonlinear frequencies increase nearly linearly with increasing slenderness ratio η.

The results shown in Figure 6 indicate that the nonlinear fundamental frequencies $\omega_{NL,1}$ with consideration of the thickness-dependent size effect is much larger than those without that consideration. Indeed, the underlying reason for the thickness-dependent size effect (stiffness-hardening) is associated with the nanobeam's oscillations. During oscillations, the cross section of the nanobeam and the corresponding turning radius become smaller while the slenderness ratio is increased. In this case, the strain gradient in the thickness direction will become larger and the effect of the corresponding high-order stress is amplified, resulting in an added remarkable positive stiffness.

5. Conclusions

We proposed a nonlinear nonlocal strain gradient Euler–Bernoulli beam model for dynamic analysis of nanobeams with two immovable supports and used it to study the nonlinear free vibration of nanobeams. In particular, the effect of strain gradient in the thickness direction, which was usually neglected before, has been accounted for in the current dynamic analysis. The governing equation is derived for all possible boundary conditions and is further simplified for symmetric boundary conditions and lateral loads.

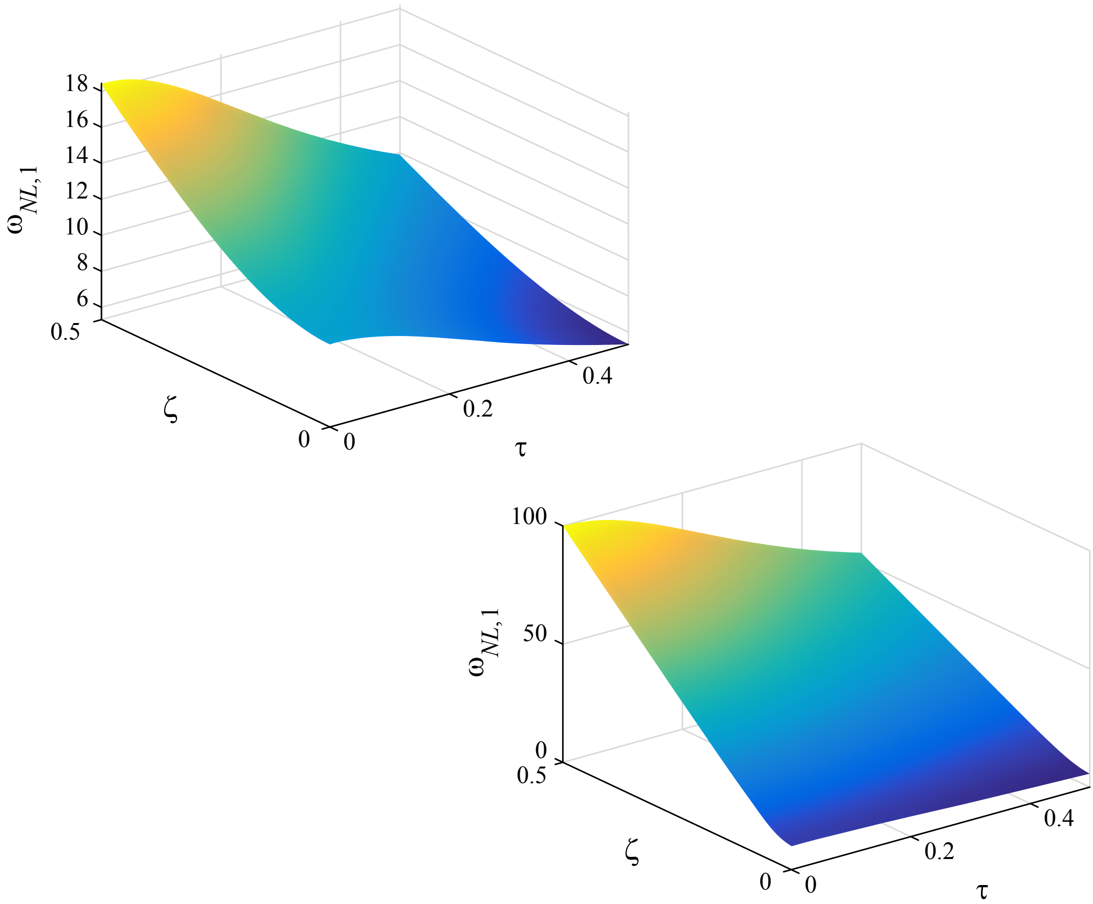

Figure 6. The nonlinear fundamental frequencies $\omega_{NL,1}$ with and without strain gradient effect in the thickness direction: $\eta = 20$, without $\varepsilon_{xx,z}$ effect (left) and $\eta = 20$, with $\varepsilon_{xx,z}$ effect (right).

Using a one-mode Galerkin's discretization, the governing equation is analytically solved using the homotopy analysis method, yielding an approximate analytical formulation of the nonlinear frequencies for pinned-pinned boundary conditions. The governing equation is further numerically solved via a Galerkin approach with a multi-mode discretization. It is found that the numerical results agree well with the analytical one. Our results showed that the thickness-dependent size effect on the nonlinear free vibration of nanobeams may be remarkable, highlighting the importance of thickness-dependent size effects in the design of nanoscale devices and systems.

In this study, we have finalized the nonlinear governing equation for symmetric boundary conditions by deriving the explicit expression of axial stretching forces (N_{0x}). For asymmetric boundary conditions and some other cases with complex kinds of lateral loads, how to determine the expression of N_{0x} is still a challenging question and needs further investigations.

The thickness-dependent size effect is also a kind of strain gradient one. However, some previous studies focused on the strain gradient effect in the lengthwise direction (x direction) only by neglecting the strain gradient effect in the thickness direction (z direction). In this work, we found that the strain gradient in the thickness direction can be very important for slender nanobeams. As can be expected, when a three-dimensional nanobeam is considered, the strain gradient effects in all the x, y, and z directions need to be considered.

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