Journal of
Mechanics of
Materials and Structures

EXTENDED HIGHER-ORDER SANDWICH PANEL THEORY FOR PLATES WITH ARBITRARY ASPECT RATIOS

Faisal Siddiqui and George A. Kardomateas

Volume 14, No. 4

July 2019

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 14, No. 4, 2019
dx.doi.org/10.2140/jomms.2019.14.449

msp

EXTENDED HIGHER-ORDER SANDWICH PANEL THEORY FOR PLATES WITH ARBITRARY ASPECT RATIOS

FAISAL SIDDIQUI AND GEORGE A. KARDOMATEAS

A new extended higher-order sandwich panel theory (EHSAPT) for orthotropic elastic sandwich plates is formulated. This new theory extends the one-dimensional extended higher-order sandwich panel beam theory to two dimensions and applies it to plate structures. In this theory, the compressibility of the soft core in the transverse direction is taken into consideration. The in-plane displacements are third-order and the transverse displacement is second-order in the transverse coordinate respectively. This arrangement allows the theory to take the axial, shear and transverse normal stresses in the core in consideration. In order to derive the governing equations and associated boundary conditions, eleven generalized coordinates are considered. Each face sheet has three generalized coordinates (two in-plane and one transverse displacement respectively) and the core has five generalized coordinates which include three displacements and two independent rotations. The governing equations and boundary conditions are derived using a variational approach such that all core/face sheet displacement compatibility conditions are satisfied.

1. Introduction

Typical sandwich panels consist of two metallic or composite thin face sheets separated by a honeycomb or foam core. This configuration gives the sandwich panel high stiffness and strength, and enables excellent energy absorption capabilities with little resultant weight penalty. This makes the sandwich structure a preferred material of choice in a lot of applications including aerospace, naval, wind turbines and civil industries. Many of the currently used methods of analysis on sandwich structures assume a noncompressible core and are categorized as the classical and the first-order shear models when shear effects are taken into consideration [Plantema 1966; Allen 1969]. The assumptions on these theories are only adequate if the core is made of a high-strength and stiff material; but in many cases when the core is a more compliant and softer material, the predictions from these theories become more and more inaccurate especially under quasistatic loading [Phan et al. 2012]. Experimental results have also shown that the core can undergo significant transverse deformation under a sudden impulsive load [Gardner et al. 2012; Jackson and Shukla 2011; Nemat-Nasser et al. 2007; Tekalur et al. 2009; Wang et al. 2009]. This implies that in order to get accurate results the transverse deformation and shear stresses in the core must be taken into consideration.

Keeping in view the importance of accurate prediction of failure modes, some of the recent computational models have considered transverse compressibility in the core. Frostig et al. [1992] proposed a theory for sandwich panels in which the resulting shear strain in the core is constant and the resulting

Keywords: Sandwich panel, plate, arbitrary aspect ratios, EHSAPT.

transverse normal strain in the core is linear in z; however, this model was only formulated for a one-dimensional beam (HSAPT). Hohe et al. [2006] developed a model for sandwich plates in which the transverse normal strain is constant along the transverse coordinate z, and the shearing strains are first-order in z. Also, Li and Kardomateas [2008] explored a higher-order theory for plates in which the transverse normal strain in the core is of third-order in z, and the shear strains in the core are of fourth-order in z.

The accuracy of any of these models can be readily assessed because an elasticity solution already exists. Pagano [1970] presented a three-dimensional elasticity solution for laminated rectangular plates for the following cases:

(1) *Orthotropic material:* the cubic characteristic equation has a negative discriminant and results in real and unequal roots.

(2) *Isotropic material:* the cubic characteristic equation has a zero discriminant and results in real and equal roots.

Kardomateas [2008] then presented the solution for the case of positive discriminant, in which two of the roots are complex conjugates. This is actually a case frequently encountered in sandwich construction in which the orthotropic core is stiffer in the transverse direction than the in-plane directions.

In this paper we present an advanced new extended higher-order sandwich panel theory (EHSAPT), which is a two-dimensional extension of the EHSAPT beam model presented in [Phan et al. 2012]. In that reference the authors extended the HSAPT given in [Frostig et al. 1992] for beams, to allow for the transverse shear distribution in the core to acquire the proper distribution as the core stiffness increases as a result of nonnegligible in-plane stresses. The current paper extends the concept of Phan et al. [2012] and applies it to two-dimensional plate structures. The theory assumes a transverse displacement in the core that varies as a second-order equation in z, and in-plane displacements that are of third-order in z. The novelty of this approach is that it allows for five generalized coordinates in the core (the in-plane and transverse displacements and two independent rotations).

The theory is formulated for a sandwich panel with a symmetric layout. The major assumptions of the theory are as follows:

(1) The face sheets satisfy the Kirchhoff assumptions, and their thicknesses are small compared with the overall thickness of the sandwich section; they undergo large displacements with moderate rotations.

(2) The core is compressible in the transverse and axial directions (transverse displacement is second-order in z and in-plane displacements are third-order in z); it has in-plane, transverse and shear rigidities; and it undergoes large displacements.

(3) The bonding between the face sheets and core is assumed to be perfect.

2. Derivation of EHSAPT theory

We consider a sandwich plate with two identical face sheets of thickness f and a core of thickness $2c$. The cartesian coordinate system is placed in the middle plane of the sandwich plate as shown in Figure 1.

The corresponding displacements are denoted by (u, v, w). Subscript t, b and c refer to the top face sheet, bottom face sheet and core. Subscript 0 refers to the middle surface of the respective phase. The total thickness of the plate is given by $h_{\text{tot}} = 2f + 2c$.

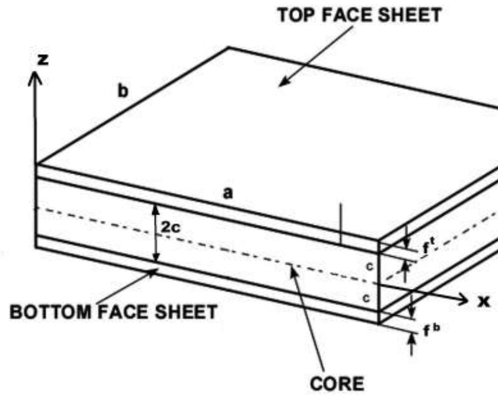

Figure 1. Geometric configuration of the plate.

2A. ***Displacements and strains.*** It is highlighted that the following functions depend on x, y, z and t and this functional dependence will not be explicitly written in the equations that follow in favor of conserving writing space:

$$
\begin{aligned}
&u^{t,b,c} = u^{t,b,c}(x, y, z, t), \quad u_0^{t,b,c} = u_0^{t,b,c}(x, y, t), \quad \psi_0^c = \psi_0^c(x, y, t), \quad u_2^c = u_2^c(x, y, t),\\
&\quad u_3^c = u_3^c(x, y, t),\\
&v^{t,b,c} = v^{t,b,c}(x, y, z, t), \quad v_0^{t,b,c} = v_0^{t,b,c}(x, y, t), \quad \phi_0^c = \phi_0^c(x, y, t), \quad v_2^c = v_2^c(x, y, t),\\
&\quad v_3^c = v_3^c(x, y, t),\\
&w^{t,b,c} = w^{t,b,c}(x, y, z, t), \qquad w_1^c = w_1^c(x, y, t), \qquad w_2^c = w_2^c(x, y, t).
\end{aligned}
$$

2A.1. *Displacements of the face sheets.* The face sheets are assumed to satisfy the Kirchhoff–Love assumptions and their thickness is assumed to be small as compared to the overall thickness of the plate. The displacements are represented as

$$u^t = u_0^t - \zeta^t w^t_{,x}, \tag{2-1a}$$

$$v^t = v_0^t - \zeta^t w^t_{,y}, \tag{2-1b}$$

$$w^t = w^t. \tag{2-1c}$$

Similarly, for the bottom face sheet,

$$u^b = u_0^b - \zeta^b w^b_{,x}, \tag{2-2a}$$

$$v^b = v_0^b - \zeta^b w^b_{,y}, \tag{2-2b}$$

$$w^b = w^b, \tag{2-2c}$$

where $\zeta^{t,b} = z \mp (c + f^{t,b}/2)$.

The nonlinear strain-displacement relations are

$$\left[\epsilon^{t,b}\right] = \begin{bmatrix} \epsilon_{xx}^{t,b} \\ \epsilon_{yy}^{t,b} \\ \gamma_{xy}^{t,b} \end{bmatrix} = \left[\epsilon_0\right] + \zeta\left[\kappa\right] = \begin{bmatrix} \epsilon_{0_x} + \zeta\kappa_x \\ \epsilon_{0_y} + \zeta\kappa_y \\ \gamma_{0_{xy}} + \zeta\kappa_{xy} \end{bmatrix}, \tag{2-3a}$$

$$[\epsilon_0] = \begin{bmatrix} \epsilon_{0_x} \\ \epsilon_{0_y} \\ \gamma_{0_{xy}} \end{bmatrix} = \begin{bmatrix} u_{0,x} + \frac{1}{2}w_{,x}^2 \\ v_{0,y} + \frac{1}{2}w_{,y}^2 \\ u_{0,y} + v_{0,x} + w_{,x}w_{,y} \end{bmatrix}. \tag{2-3b}$$

Moreover, $[\kappa]$ is the curvature matrix and can be given as

$$[\kappa] = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{xy} \end{bmatrix}. \tag{2-3c}$$

2A.2. *Displacements for the higher-order core.* First-order approximation of the classical sandwich panel theory neglects the transverse deformation of the core and leads to erroneous results in many practical cases. However, in many instances it becomes essential to capture the core compressibility effects and thus we use a higher-order definition of the in-plane and transverse deformation of the core in terms of the transverse coordinate:

$$u^c = u_0^c + \psi_0^c z + u_2^c z^2 + u_3^c z^3, \tag{2-4a}$$

$$v^c = v_0^c - \phi_0^c z + v_2^c z^2 + v_3^c z^3, \tag{2-4b}$$

$$w^c = w_0^c + w_1^c z + w_2^c z^2. \tag{2-4c}$$

In these equations u_0^c, v_0^c and w_0^c are the in-plane and transverse displacements and ϕ_0^c and ψ_0^c are the rotations about the x-axis and y-axis, respectively. Also, u_2^c, u_3^c, v_2^c, v_3^c, w_1^c and w_2^c are the in-plane and transverse unknown functions to be determined by enforcing displacement compatability conditions at the core/face sheets interface. We therefore enforce compatability at $z = \pm c$ and after some algebraic calculations, the following core displacement field is obtained:

$$u^c = u_0^c + z\psi_0^c - \frac{z^3}{4c^3}[2u_0^b - 2u_0^t + 4c\psi_0^c - f^b w_{,x}^b - f^t w_{,x}^t] - \frac{z^2}{4c^2}[-2u_0^b - 2u_0^t + 4u_0^c + f^b w_{,x}^b - f^t w_{,x}^t], \tag{2-5a}$$

$$v^c = v_0^c - z\phi_0^c - \frac{z^3}{4c^3}[2v_0^b - 2v_0^t - 4c\phi_0^c - f^b w_{,y}^b - f^t w_{,y}^t] - \frac{z^2}{4c^2}[-2v_0^b - 2v_0^t + 4v_0^c + f^b w_{,y}^b - f^t w_{,y}^t], \tag{2-5b}$$

$$w^c = w_0^c - \frac{z^2}{2c^2}[-w^b - w^t + 2w_0^c] - \frac{z}{2c}[w^b - w^t]. \tag{2-5c}$$

It is highlighted that in developing their higher-order theories, Li and Kardomateas [2008] and Phan et al. [2012] assumed that the core undergoes large rotation with a small displacement and therefore neglected the in-plane strains. However, the current theory does not make any such assumptions and we consider all six strains in the core. This leads to the following six strain-displacement relations for the core:

$$\epsilon^c_{xx} = u^c_{0,x} + z\psi^c_{0,x} - \frac{z^3}{4c^3}[2u^b_{0,x} - 2u^t_{0,x} + 4c\psi^c_{0,x} - f^b w^b{}_{,xx} - f^t w^t{}_{,xx}] - \frac{z^2}{4c^2}[-2u^b_{0,x} + 4u^c_{0,x} - 2u^t_{0,x} + f^b w^b{}_{,xx} - f^t w^t{}_{,xx}]\psi^c_{0,x}, \quad \text{(2-6a)}$$

$$\epsilon^c_{yy} = v^c_{0,y} - z\phi^c_{0,y} - \frac{z^3}{4c^3}[2v^b_{0,y} - 2v^t_{0,y} - 4c\phi^c_{0,y} - f^b w^b{}_{,yy} - f^t w^t{}_{,yy}] - \frac{z^2}{4c^2}[-2v^b_{0,y} + 4v^c_{0,y} - 2v^t_{0,y} + f^b w^b{}_{,yy} - f^t w^t{}_{,yy}], \quad \text{(2-6b)}$$

$$\epsilon^c_{zz} = -\frac{z}{c^2}[2w^c_0 - w^b - w^t] - \frac{1}{2c}[w^b - w^t] \quad \text{(2-6c)}$$

$$\gamma^c_{xy} = u^c_{0,y} + z\psi^c_{0,y} + v^c_{0,x} - z\phi^c_{0,x} - \frac{z^3}{4c^3}[2u^b_{0,y} - 2u^t_{0,y} + 4c\psi^c_{0,y} - f^b w^b{}_{,xy} - f^t w^t{}_{,xy}] - \frac{z^3}{4c^3}[2v^b_{0,x} - 2v^t_{0,x} - 4c\phi^c_{0,x} - f^b w^b{}_{,xy} - f^t w^t{}_{,xy}] - \frac{z^2}{4c^2}[-2u^b_{0,y} + 4u^c_{0,y} - 2u^t_{0,y} + f^b w^b{}_{,xy} - f^t w^t{}_{,xy}] - \frac{z^2}{4c^2}[-2v^b_{0,x} + 4v^c_{0,x} - 2v^t_{0,x} + f^b w^b{}_{,xy} - f^t w^t{}_{,xy}], \quad \text{(2-6d)}$$

$$\gamma^c_{xz} = \psi^c_0 + w^c_{0,x} - \frac{z^2}{2c^2}[2w^c_{0,x} - w^b{}_{,x} - w^t{}_{,x}] - \frac{z}{2c}[w^b{}_{,x} - w^t{}_{,x}] - \frac{3z^2}{4c^3}[2u^b_0 - 2u^t_0 + 4c\psi^c_0 - f^b w^b{}_{,x} - f^t w^t{}_{,x}] - \frac{z}{2c^2}[-2u^b_0 + 4u^c_0 - 2u^t_0 + f^b w^b{}_{,x} - f^t w^t{}_{,x}], \quad \text{(2-6e)}$$

$$\gamma^c_{yz} = -\phi^c_0 + w^c_{0,y} - \frac{z^2}{2c^2}[2w^c_{0,y} - w^b{}_{,y} - w^t{}_{,y}] - \frac{z}{2c}[w^b{}_{,y} - w^t{}_{,y}] - \frac{3z^2}{4c^3}[2v^b_0 - 2v^t_0 - 4c\phi^c_0 - f^b w^b{}_{,y} - f^t w^t{}_{,y}] - \frac{z}{2c^2}[-2v^b_0 + 4v^c_0 - 2v^t_0 + f^b w^b{}_{,y} - f^t w^t{}_{,y}]. \quad \text{(2-6f)}$$

2A.3. *Constitutive relations.* We assume that the face sheets are composite laminates and the core is fully orthotropic. The stress-strain relations for the top and bottom sheets read as

$$\begin{bmatrix} \sigma^{t,b}_{xx} \\ \sigma^{t,b}_{yy} \\ \tau^{t,b}_{xy} \end{bmatrix} = \begin{bmatrix} C^{t,b}_{11} & C^{t,b}_{12} & C^{t,b}_{16} \\ C^{t,b}_{12} & C^{t,b}_{22} & C^{t,b}_{26} \\ C^{t,b}_{16} & C^{t,b}_{26} & C^{t,b}_{66} \end{bmatrix} \begin{bmatrix} \epsilon^{t,b}_{xx} \\ \epsilon^{t,b}_{yy} \\ \gamma^{t,b}_{xy} \end{bmatrix}, \quad \text{(2-7a)}$$

where C_{ij} $(i, j = 1, 2, 6)$ are the plane stress reduced stiffness coefficients. The core is considered to be fully orthotropic:

$$\begin{bmatrix}\sigma^c_{xx}\\ \sigma^c_{yy}\\ \sigma^c_{zz}\\ \tau^c_{yz}\\ \tau^c_{xz}\\ \tau^c_{xy}\end{bmatrix}=\begin{bmatrix}C^c_{11} & C^c_{12} & C^c_{13} & 0 & 0 & 0\\ C^c_{12} & C^c_{22} & C^c_{23} & 0 & 0 & 0\\ C^c_{13} & C^c_{23} & C^c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C^c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C^c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C^c_{66}\end{bmatrix}\begin{bmatrix}\epsilon^c_{xx}\\ \epsilon^c_{yy}\\ \epsilon^c_{zz}\\ \gamma^c_{yz}\\ \gamma^c_{xz}\\ \gamma^c_{xy}\end{bmatrix}. \tag{2-7b}$$

Since the face sheets are laminated composite plates with the face sheets composed of multiple laminas, each fiber angle of an individual lamina can be chosen independently. The following constitutive relations are defined:

$$\chi(\theta)=\begin{bmatrix}1 & 1 & \cos\theta & \cos 4\theta\\ 1 & 1 & -\cos\theta & \cos 4\theta\\ 1 & -1 & 0 & -\cos 4\theta\\ 0 & 1 & 0 & -\cos 4\theta\\ 0 & 0 & \frac{1}{2}\sin 2\theta & \sin 4\theta\\ 0 & 0 & \frac{1}{2}\sin 2\theta & -\sin 4\theta\end{bmatrix}. \tag{2-8}$$

Similarly, the following four material invariants are defined:

$$\alpha_1=\frac{E_1+E_2+2\nu_{12}E_2}{4\alpha_0},\quad \alpha_2=\frac{E_1+E_2-2\nu_{12}E_2}{8\alpha_0}+\frac{G_{12}}{2},$$
$$\alpha_3=\frac{E_1-E_2}{2\alpha_0},\quad \alpha_4=\frac{E_1+E_2-2\nu_{12}E_2}{8\alpha_0}-\frac{G_{12}}{2},$$

where $\alpha_0=1-\nu_{12}E_2/E_1$. Next, we define an array of the lamina stiffness coefficients such that

$$C=\{C_{11},\ C_{22},\ C_{12},\ C_{66},\ C_{16},\ C_{26}\}^T. \tag{2-9}$$

We then define an array of the material invariants as

$$\alpha=\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}^T. \tag{2-10}$$

Therefore

$$[C(\theta)]=[\chi(\theta)]\{\alpha\}. \tag{2-11}$$

Hence depending upon the angle of individual laminas, the material coefficients for the face sheets are defined. Next, the stress and moment resultants for the facesheets are defined as

$$[N^{t,b}]=\begin{bmatrix}N^{t,b}_{xx}\\ N^{t,b}_{yy}\\ N^{t,b}_{xy}\end{bmatrix}=\begin{bmatrix}N^{t,b^1}_{xx}\\ N^{t,b^1}_{yy}\\ N^{t,b^1}_{xy}\end{bmatrix}+\begin{bmatrix}N^{t,b^2}_{xx}\\ N^{t,b^2}_{yy}\\ N^{t,b^2}_{xy}\end{bmatrix}=\int_c^{c+f^{t,b}/2}[\sigma^{t,b^1}]\,dz+\int_{c+f^{t,b}/2}^{c+f^{t,b}}[\sigma^{t,b^2}]\,dz. \tag{2-12}$$

Similarly

$$[M^{t,b}] = \begin{bmatrix} M_{xx}^{t,b} \\ M_{yy}^{t,b} \\ M_{xy}^{t,b} \end{bmatrix} = \begin{bmatrix} M_{xx}^{t,b^1} \\ M_{yy}^{t,b^1} \\ M_{xy}^{t,b^1} \end{bmatrix} + \begin{bmatrix} M_{xx}^{t,b^2} \\ M_{yy}^{t,b^2} \\ M_{xy}^{t,b^2} \end{bmatrix} = \int_c^{c+f^{t,b}/2} [\sigma^{t,b^1}]\, \zeta^{t,b}\, dz + \int_{c+f^{t,b}/2}^{c+f^{t,b}} [\sigma^{t,b^2}]\, \zeta^{t,b}\, dz. \quad (2\text{-}13)$$

For the core the following resultants are defined:

$$[N^c] = \begin{bmatrix} N_{xx}^c \\ N_{yy}^c \\ N_{zz}^c \\ N_{xy}^c \\ Q_x^c \\ Q_y^c \end{bmatrix} = \int_{-c}^{c} \begin{bmatrix} \sigma_{xx}^c \\ \sigma_{yy}^c \\ \sigma_{zz}^c \\ \sigma_{xy}^c \\ \sigma_{xz}^c \\ \sigma_{yz}^c \end{bmatrix} dz, \quad \text{for the core.} \quad (2\text{-}14a)$$

Similarly, the following resultants are also defined for the core:

$$\begin{bmatrix} M_{xx}^c \\ M_{yy}^c \\ M_{zz}^c \\ M_{xy}^c \\ M_{yz}^c \\ M_{xz}^c \end{bmatrix} = \int_{-c}^{c} \begin{bmatrix} \sigma_{xx}^c \\ \sigma_{yy}^c \\ \sigma_{zz}^c \\ \sigma_{xy}^c \\ \sigma_{yz}^c \\ \sigma_{xz}^c \end{bmatrix} z\, dz, \quad \begin{bmatrix} R_{xx}^c \\ R_{yy}^c \\ R_{xy}^c \\ R_{yz}^c \\ R_{xz}^c \end{bmatrix} = \int_{-c}^{c} \begin{bmatrix} \sigma_{xx}^c \\ \sigma_{yy}^c \\ \sigma_{xy}^c \\ \sigma_{yz}^c \\ \sigma_{xz}^c \end{bmatrix} z^2\, dz, \quad \begin{bmatrix} P_{xx}^c \\ P_{yy}^c \\ P_{xy}^c \end{bmatrix} = \int_{-c}^{c} \begin{bmatrix} \sigma_{xx}^c \\ \sigma_{yy}^c \\ \sigma_{xy}^c \end{bmatrix} z^3\, dz. \quad (2\text{-}14b)$$

Also

$$I_i = \int_{-h/2}^{h/2} \rho(z)^i\, dz \quad (i = 0, 1, 2, 3, \dots, 6). \quad (2\text{-}15)$$

2B. ***Governing differential equations.*** The governing differential equations and associated boundary conditions can be derived using the Hamilton's principle. The sandwich panel is subjected to a transverse load $q(x, y, t)$ on the top and bottom face sheets. Let the strain energy be denoted by U, the kinetic energy by K and the external work by W. The variational principle states that

$$\delta[T - (U - W)] = 0, \quad (2\text{-}16)$$

in which the first variation of the energy functionals can be written as

$$\begin{aligned} \delta U = \int_0^t \int_0^b \int_0^a \Bigg[\int_c^{c+f^t} & \big(\sigma_{xx}^t \delta\epsilon_{xx}^t + \sigma_{yy}^t \delta\epsilon_{yy}^t + \tau_{xy}^t \delta\gamma_{xy}^t\big)\, dz \\ & + \int_{-c}^{c} \big(\sigma_{xx}^c \delta\epsilon_{xx}^c + \sigma_{yy}^c \delta\epsilon_{yy}^c + \sigma_{zz}^c \delta\epsilon_{zz}^c + \tau_{xy}^c \delta\gamma_{xy}^c + \tau_{xz}^c \delta\gamma_{xz}^c + \tau_{yz}^c \delta\gamma_{yz}^c\big)\, dz \\ & + \int_{-c-f^b}^{-c} \big(\sigma_{xx}^b \delta\epsilon_{xx}^b + \sigma_{yy}^b \delta\epsilon_{yy}^b + \tau_{xy}^b \delta\gamma_{xy}^b\big)\, dz \Bigg]\, dx\, dy, \quad (2\text{-}17) \end{aligned}$$

$$\delta T=\int_0^t\int_0^b\int_0^a\biggl[\int_c^{c+f^t}\rho^t(\dot u^t\delta\dot u^t+\dot v^t\delta\dot v^t+\dot w^t\delta\dot w^t)\,dz+\int_{-c}^{c}\rho^c(\dot u^c\delta\dot u^c+\dot v^c\delta\dot v^c+\dot w^c\delta\dot w^c)\,dz \\ +\int_{-c-f^b}^{-c}\rho^b(\dot u^b\delta\dot u^b+\dot v^b\delta\dot v^b+\dot w^b\delta\dot w^b)\biggr]\,dz, \tag{2-18}$$

and the work done by external forces is

$$\delta W=\int_0^t\int_0^b\int_0^a q^t(x,y,t)\,\delta w^t+q^b(x,y,t)\,\delta w^b\,dx\,dy\,dt, \tag{2-19}$$

where ρ is the mass density and dot above the variables represents differentiation with respect to time; $q^t(x,y,z)$ and $q^b(x,y,z)$ are the distributed transverse load on top and bottom face sheets, respectively; δw^t and δw^b represents the virtual transverse displacements of top and bottom face sheets, respectively. Equating time derivatives equal to zero would recover the governing differential equations and associated boundary conditions for a static case.

2B.1. *Equations of motion.* The governing equations and associated boundary conditions can be obtained by substituting the strain-displacement relations ((2-3) and (2-6)) and stress-strain relations (2-7) in the first variations of the energy functionals. We make use of the stress and moment resultants defined by using (2-12), (2-13) and (2-14), respectively. We then employ Green's theorem to relieve the primary variables of derivatives. This results in eleven governing equations: three for each face sheet and five for the core. Also, boundary conditions are acquired as a result of the process:

$$\delta u_0^b:\quad 4\alpha_2M^c_{xz}-6\alpha_3R^c_{xz}-N^b_{xy,y}+2\alpha_3P^c_{xy,y}-2\alpha_2R^c_{xy,y}-N^b_{xx,x}+2\alpha_3P^c_{xx,x} \\ -2\alpha_2R^c_{xx,x}+\beta_1\ddot u_0^b-2\beta_2\ddot u_0^c+4\beta_3\ddot u_0^t+2\beta_5\ddot\psi_0^c-\beta_4\ddot w^b_{,x}+2f^t\beta_3\ddot w^t_{,x}=0, \tag{2-20a}$$

$$\delta v_0^b:\quad 4\alpha_2M^c_{yz}-6\alpha_3R^c_{yz}-N^b_{yy,y}+2\alpha_3P^c_{yy,y}-2\alpha_2R^c_{yy,y}-N^b_{xy,x}+2\alpha_3P^c_{xy,x} \\ -2\alpha_2R^c_{xy,x}+\beta_1\ddot v_0^b+2\beta_2\ddot v_0^c+4\beta_3\ddot v_0^t-2\beta_5\ddot\phi_0^c-\beta_4\ddot w^b_{,y}+2f^t\beta_3\ddot w^t_{,y}=0, \tag{2-20b}$$

$$\delta w^b:\quad 4\alpha_2M^c_{zz}-\alpha_1N^c_{zz}+(\alpha_1+2f^b\alpha_2)\bigl(M^c_{xz,x}+M^c_{yz,y}\bigr)-2M^b_{xy,xy}-M^b_{xx,xx} \\ +f^b\alpha_3\bigl(P^c_{xx,xx}+P^c_{yy,yy}+2P^c_{xy,xy}\bigr)-f^b\alpha_2\bigl(R^c_{xx,xx}+R^c_{yy,yy}+2R^c_{xy,xy}\bigr) \\ -R^c_{xz,x}(2\alpha_2+3f^b\alpha_3)+\beta_6\ddot w^b-\beta_7\ddot w_0^c-\beta_8\ddot w^t+\beta_4\bigl(\ddot u^b_{0,x}+\ddot v^b_{0,y}\bigr) \\ +f^b\beta_2\bigl(\ddot u^c_{0,x}+\ddot v^c_{0,y}\bigr)+2f^b\beta_3\bigl(\ddot u^t_{0,x}+\ddot v^t_{0,y}\bigr)+f^b\beta_5\bigl(\ddot\psi^c_{0,x}-\ddot\phi^c_{0,y}\bigr) \\ +f^bf^t\beta_3\bigl(\ddot w^t_{,xx}+\ddot w^t_{,yy}\bigr)-\beta_9\bigl(\ddot w^b_{,xx}+\ddot w^b_{,yy}\bigr)=q^b[x,y,t], \tag{2-20c}$$

$$\delta u_0^t:\quad 4\alpha_2M^c_{xz}+6\alpha_3R^c_{xz}-N^t_{xy,y}-2\alpha_3P^c_{xy,y}-2\alpha_2R^c_{xy,y}-N^t_{xx,x}-2\alpha_3P^c_{xx,x} \\ -2\alpha_2R^c_{xx,x}+4\beta_3\ddot u_0^b+2\xi_2\ddot u_0^c+\xi_1\ddot u_0^t+2\xi_5\ddot\psi_0^c-2f^b\beta_3\ddot w^b_{,x}+\xi_4\ddot w^t_{,x}=0, \tag{2-20d}$$

$$\delta v_0^t:\quad 4\alpha_2M^c_{yz}+6\alpha_3R^c_{yz}-N^t_{yy,y}-2\alpha_3P^c_{yy,y}-2\alpha_2R^c_{yy,y}-N^t_{xy,x}-2\alpha_3P^c_{xy,x} \\ -2\alpha_2R^c_{xy,x}+4\beta_3\ddot v_0^b+2\xi_2\ddot v_0^c+\xi_1\ddot v_0^t-2\xi_5\ddot\phi_0^c-2f^b\beta_3\ddot w^b_{,x}+\xi_4\ddot w^t_{,x}=0, \tag{2-20e}$$

$$\begin{aligned}\delta w^t:\quad & 4\alpha_2 M^c_{zz}+\alpha_1 N^c_{zz}-(\alpha_1+2f^t\alpha_2)\big(M^c_{yz,y}-M^c_{xz,x}\big)-2M^t_{xy,xy}-M^t_{xx,xx}\\ &+f^t\alpha_3\big(P^c_{yy,yy}+P^c_{xx,xx}+2P^c_{xy,xy}\big)+f^t\alpha_2\big(R^c_{yy,yy}+2R^c_{xy,xy}+R^c_{xx,xx}\big)\\ &-R^c_{xz,x}(2\alpha_2+3\alpha_3 f^t)-\beta_8\ddot{w}^b+\xi_7\ddot{w}^c_0+\xi_6\ddot{w}^t+f^bf^t\beta_3\big(\ddot{w}^b_{,yy}+\ddot{w}^b_{,xx}\big)\\ &-2f^t\beta_3\big(\ddot{u}^b_{0,x}+\ddot{v}^b_{0,x}\big)-f^t\xi_2\big(\ddot{u}^c_{0,x}+\ddot{v}^c_{0,y}\big)-\xi_4\big(\ddot{u}^t_{0,x}+\ddot{v}^t_{0,y}\big)\\ &-f^t\xi_5\big(\ddot{\phi}^c_{0,y}+\ddot{\psi}^c_{0,x}\big)-\xi_9\big(\ddot{w}^t_{,yy}+\ddot{w}^b_{,xx}\big)=q^t[x,y,t],\end{aligned}\qquad (2\text{-}20f)$$

$$\begin{aligned}\delta u^c_0:\quad & 8\alpha_2 M^c_{xz}+N^c_{xy,y}-4\alpha_2R^c_{xy,y}+N^c_{xx,x}-4\alpha_2R^c_{xx,x}-2\beta_2\ddot{u}^b_0-\Delta_1\ddot{u}^c_0-2\beta_2\ddot{u}^t_0\\ &-\Delta_2\ddot{\psi}^c_0+\beta_2f^b\ddot{w}^b_{,x}-\beta_2f^t\ddot{w}^t_{,x}=0,\end{aligned}\qquad (2\text{-}20g)$$

$$\begin{aligned}\delta v^c_0:\quad & 8\alpha_2 M^c_{yz}+N^c_{yy,y}-4\alpha_2R^c_{yy,y}+N^c_{xy,x}-4\alpha_2R^c_{xy,x}-2\beta_2\ddot{v}^b_0-\Delta_1\ddot{v}^c_0-2\beta_2\ddot{v}^t_0\\ &+\Delta_2\ddot{\phi}^c_0+\beta_3f^b\ddot{w}^b_{,y}-\beta_3f^t\ddot{w}^t_{,y}=0,\end{aligned}\qquad (2\text{-}20h)$$

$$\delta w^c_0:\quad 8\alpha_2M^c_{zz}+Q^c_{y,y}-4\alpha_2R^c_{yz,y}+Q^c_{x,x}-4\alpha_2R^c_{xz,x}+\beta_7\ddot{w}^b-\Delta_1\ddot{w}^c_0-\xi_7\ddot{w}^t=0,\qquad (2\text{-}20i)$$

$$\begin{aligned}\delta\phi^c_0:\quad & -Q^c_y+12R^c_{yz}-M^c_{xy,x}+M^c_{yy,y}-4\alpha_2P^c_{yy,y}-4\alpha_2P^c_{xy,x}-2\beta_5\ddot{v}^b_0-\Delta_2\ddot{v}^c_0\\ &+2\xi_5\ddot{v}^t_0+\Delta_4\ddot{\phi}^c_0+f^b\beta_5\ddot{w}^b_{,y}+f^t\xi_5\ddot{w}^t_{,y}=0,\end{aligned}\qquad (2\text{-}20j)$$

$$\begin{aligned}\delta\psi^c_0:\quad & Q^c_x-12R^c_{xz}-M^c_{xy,y}-M^c_{xx,x}+4\alpha_2P^c_{xx,x}+4\alpha_2P^c_{xy,y}+2\beta_5\ddot{u}^b_0+\Delta_2\ddot{u}^c_0\\ &+2\xi_5\ddot{u}^t_0+\Delta_4\ddot{\psi}^c_0-f^b\beta_5\ddot{w}^b_{,x}+f^t\xi_5\ddot{w}^t_{,x}=0,\end{aligned}\qquad (2\text{-}20k)$$

where $\alpha_1=1/2c,\ \alpha_2=1/4c^2,\ \alpha_3=1/4c^3$, and

$$\beta_1=\frac{4c^6I^b_0+c^2I^c_4-2cI^c_5+I^c_6}{4c^6},\quad \beta_2=\frac{c^4I^c_2-c^3I^c_3-c^2I^c_4+cI^c_5}{4c^6},\quad \beta_3=\frac{c^2I^c_4-I^c_6}{16c^6},$$

$$\beta_4=\frac{8c^7I^b_0+4c^6f^bI^b_0+8c^6I^b_1+c^2f^bI^c_4-2cf^bI^c_5+f^bI^c_6}{8c^6},\quad \beta_5=\frac{c^3I^c_3-c^2I^c_4-cI^c_5+I^c_6}{4c^5},$$

$$\beta_6=\frac{4c^4I^b_0+c^2I^c_2-2cI^c_3+I^c_4}{4c^4},\quad \beta_7=\frac{c^3I^c_1-c^2I^c_2-cI^c_3+I^c_4}{2c^4},\quad \beta_8=\frac{c^2I^c_2-I^c_4}{4c^4},$$

$$\beta_9=\frac{16c^8I^b_0+16c^7f^bI^b_0+4c^6f^{b2}I^b_0+32c^7I^b_1+16c^6f^bI^b_1+16c^6I^b_2+c^2f^{b2}I^c_4-2cf^{b2}I^c_5+f^{b2}I^c_6}{16c^6},$$

$$\Delta_1=\frac{c^4I^c_0-2c^2I^c_2+I^c_4}{c^4},\quad \Delta_2=\frac{c^4I^c_1-2c^2I^c_3+I^c_5}{c^4},\quad \Delta_4=\frac{c^4I^c_2-2c^2I^c_4+I^c_6}{c^4},$$

$$\xi_1=\frac{4c^6I^t_0=c^2I^c_4+2cI^c_5+I^c_6}{4c^6},\quad \xi_2=\frac{c^4I^c_2+c^3I^c_3-c^2I^c_4-cI^c_5}{4c^6},$$

$$\xi_4=\frac{8c^7I^t_0+4c^6f^tI^t_0-8c^6I^t_1+c^2f^tI^c_4+2cf^tI^c_5+f^tI^c_6}{8c^6},$$

$$\xi_5 = \frac{c^4 I_3^c + c^3 I_4^c - c^2 I_5^c - c I_6^c}{4c^6}, \quad \xi_6 = \frac{4c^4 I_0^t + c^2 I_2^c + 2c I_3^c + I_4^c}{4c^6}, \quad \xi_7 = \frac{c^3 I_1^c + c^2 I_2^c - c I_3^c - I_4^c}{2c^4},$$

$$\xi_9 = \frac{16c^8 I_0^t + 16c^7 f^t I_0^t + 4c^6 f^{t2} I_0^t - 32c^7 I_1^t - 16c^6 I_1^t + 16c^6 I_2^b + c^2 f^{t2} I_4^c + 2c f^{t2} I_5^c + f^{t2} I_6^c}{16c^6}.$$

The associated boundary conditions at $x = 0, a$ read as

$$
\begin{aligned}
u_0^b &= \tilde{u}^b, &&\text{or} \quad \tilde{N}_{xx}^b, = N_{xx}^b - 2\alpha_3 P_{xx}^c + 2\alpha_2 R_{xx}^c,\\
u_0^c &= \tilde{u}^c, &&\text{or} \quad \tilde{N}_{xx}^c, = N_{xx}^c - 4\alpha_2 R_{xx}^c,\\
u_0^t &= \tilde{u}^t, &&\text{or} \quad \tilde{N}_{xx}^t, = N_{xx}^t + 2\alpha_3 P_{xx}^c + 2\alpha_2 R_{xx}^c,\\
v_0^b &= \tilde{v}^b, &&\text{or} \quad \tilde{N}_{yy}^b, = N_{xy}^b - 2\alpha_3 P_{xy}^c + 2\alpha_2 R_{xy}^c,\\
v_0^c &= \tilde{v}^c, &&\text{or} \quad \tilde{N}_{yy}^c, = N_{xy}^c - 4\alpha_2 R_{xy}^c,\\
v_0^t &= \tilde{v}^t, &&\text{or} \quad \tilde{N}_{yy}^t, = N_{xy}^t + 2\alpha_3 P_{xy}^c + 2\alpha_2 R_{xy}^c,\\
w^b &= \tilde{w}^b, &&\text{or} \quad \tilde{Q}_x{}^b = -\alpha_1 M_{xz}^c - 2\alpha_2 f^b M_{xz}^c + 2\alpha_2 R_{xz}^c + 3\alpha_3 f^b R_{xz}^c - \beta_4 \ddot{u}_0^b - f^b \beta_2 \ddot{u}_0^c\\
& && \qquad - 2\beta_3 f^b \ddot{u}_0^t - \beta_5 \ddot{\psi}_0^c + M_{xy,y}^b - f^b \alpha_3 P_{xy,y}^c + \alpha_2 f^b R_{xy,y}^c\\
& && \qquad + M_{xx,x}^b - \alpha_3 f^b P_{xx,x}^c + \alpha_2 f^b R_{xx,x}^c + \beta_9 \ddot{w}_{,x}^b + f^b f^t \beta_3 \ddot{w}_{,x}^t,\\
w^b{}_{,y} &= \tilde{w}_{,y}^b, &&\text{or} \quad \tilde{M}_{xy}{}^b = -M_{xy}^b + \alpha_3 f^b P_{xy}^c - \alpha_2 f^b R_{xy}^c,\\
w_0^c &= \tilde{w}^c, &&\text{or} \quad \tilde{Q}_x{}^c = Q_x^c - 4\alpha_2 R_{xz}^c,\\
\psi_0^c &= \tilde{\psi}_0^{c\,c}, &&\text{or} \quad \tilde{M}_{xx}{}^c = M_{xx}^c - 4\alpha_2 P_{xx}^c,\\
\phi_0^c &= \tilde{\phi}_0^{c\,c}, &&\text{or} \quad \tilde{M}_{xy}{}^c = -M_{xy}^c + 4\alpha_2 P_{xy}^c,\\
w^t &= \tilde{w}^t, &&\text{or} \quad \tilde{Q}_x{}^t = \alpha_1 M_{xz}^c + 2\alpha_2 f^t M_{xz}^c + 2\alpha_2 R_{xz}^c + 3\alpha_3 f^t R_{xz}^c + 2 f^t \beta_3 \ddot{u}_0^b + f^t \xi_2 \ddot{u}_0^c\\
& && \qquad + \xi_4 \ddot{u}_0^t + \xi_5 \ddot{\psi}_0^c + M_{xy,y}^t - f^t \alpha_3 P_{xy,y}^c - f^t \alpha_2 R_{xy,y}^c\\
& && \qquad + M_{xx,x}^t - f^t \alpha_3 P_{xx,x}^c - f^t \alpha_2 R_{xx,x}^c - f^b f^t \beta_3 \ddot{w}_{,x}^b + \xi_9 \ddot{w}_{,x}^t,\\
w^t{}_{,y} &= \tilde{w}_{,y}^t, &&\text{or} \quad \tilde{M}_{xy}{}^t = -M_{xy}^t + \alpha_3 f^t P_{xy}^c + \alpha_2 f^t R_{xy}^c,
\end{aligned}
$$

where the tilde accent denotes the known external boundary values. Similar equations can be written for $y = 0, b$.

3. Conclusion

In this paper, a new higher order sandwich panel plate theory (EHSAPT) is presented. This is a two dimensional extension of the one dimensional theory presented in [Phan et al. 2012]. In this derivation both the core compressibility effects and the core shear stresses are considered, the theory also allows for nonzero axial stresses in the core. In order to capture all these effects, eleven generalized coordinates are defined with five generalized coordinates for the core and three each for the two face sheets. The equations are derived using a variational approach and associated boundary conditions are presented.

References

[Allen 1969] H. J. Allen, *Analysis and design of structural sandwich panels*, Pergamon, Oxford, 1969.

[Frostig et al. 1992] Y. Frostig, M. Baruch, O. Vilnay, and I. Sheinman, "High-order theory for sandwich-beam behavior with transversely flexible core", *J. Eng. Mech.* (*ASCE*) **118**:5 (1992), 1026–1043.

[Gardner et al. 2012] N. Gardner, E. Wang, P. Kumar, and A. Shukla, "Blast mitigation in a sandwich composite using graded core and polyurea interlayer", *Exp. Mech.* **52**:2 (2012), 119–133.

[Hohe et al. 2006] J. Hohe, L. Librescu, and S. Y. Oh, "Dynamic buckling of flat and curved sandwich panels with transversely compressible core", *Compos. Struct.* **74**:1 (2006), 10–24.

[Jackson and Shukla 2011] M. Jackson and A. Shukla, "Performance of sandwich composites subjected to sequential impact and air blast loading", *Compos. B Eng.* **42**:2 (2011), 155–166.

[Kardomateas 2008] G. A. Kardomateas, "Three-dimensional elasticity solution for sandwich plates with orthotropic phases: the positive discriminant case", *J. Appl. Mech.* (*ASME*) **76**:1 (2008), art. id. 014505.

[Li and Kardomateas 2008] R. Li and G. A. Kardomateas, "Nonlinear high-order core theory for sandwich plates with orthotropic phases", *AIAA J.* **46**:11 (2008), 2926–2934.

[Nemat-Nasser et al. 2007] S. Nemat-Nasser, W. J. Kang, J. D. McGee, W.-G. Guo, and J. B. Isaacs, "Experimental investigation of energy-absorption characteristics of components of sandwich structures", *Int. J. Impact Eng.* **34**:6 (2007), 1119–1146.

[Pagano 1970] N. J. Pagano, "Exact solutions for rectangular bidirectional composites and sandwich plates", *J. Compos. Mater.* **4**:1 (1970), 20–34.

[Phan et al. 2012] C. N. Phan, Y. Frostig, and G. A. Kardomateas, "Analysis of sandwich beams with a compliant core and with in-plane rigidity: extended high-order sandwich panel theory versus elasticity", *J. Appl. Mech.* (*ASME*) **79**:4 (2012), art. id. 041001.

[Plantema 1966] F. J. Plantema, *Sandwich construction: the bending and buckling of sandwich beams, plates, and shells*, Wiley, New York, 1966.

[Tekalur et al. 2009] S. A. Tekalur, A. E. Bogdanovich, and A. Shukla, "Shock loading response of sandwich panels with 3-D woven E-glass composite skins and stitched foam core", *Compos. Sci. Technol.* **69**:6 (2009), 736–753.

[Wang et al. 2009] E. Wang, N. Gardner, and A. Shukla, "The blast resistance of sandwich composites with stepwise graded cores", *Int. J. Solids Struct.* **46**:18-19 (2009), 3492–3502.

Received 27 Nov 2018. Revised 25 Apr 2019. Accepted 4 Jun 2019.

FAISAL SIDDIQUI: faisals@gatech.edu
Georgia Institute of Technology, Atlanta, GA, United States

GEORGE A. KARDOMATEAS: george.kardomateas@aerospace.gatech.edu
Georgia Institute of Technology, Atlanta, GA, United States

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

Cover photo: Mando Gomez, www.mandolux.com

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2019 is US $635/year for the electronic version, and $795/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 14, No. 4 July 2019

1559-3959(2019)14:4;1-U