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THE EFFECT OF BOUNDARY CONDITIONS ON THE LOWEST VIBRATION MODES OF STRONGLY INHOMOGENEOUS BEAMS

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This paper investigates the influence of the boundary conditions on the lowest vibration modes of strongly inhomogeneous beams. It is observed that the softer component of the composite beams asymptotically contributes to an almost rigid-body motion of the stiffer parts and gives rise to one or two nonzero eigenfrequencies contrary to a single beam with free end conditions. An asymptotic procedure is employed to derive the eigenfrequencies as well as the eigenforms in the case of global low frequency regime. The developed model is adapted for two and three-component beams with different end conditions. It is also shown that all eigenforms corresponding to the stiffer components of the beams perform almost rigid body motions. Comparisons of exact and approximate solutions are presented, demonstrating the validity of the proposed approach.

1. Introduction

Low-frequency vibrations of inhomogeneous structural elements have been actively studied in recent years due to their numerous applications in modern engineering; see, e.g., [Le 1999; Horgan and Chan 1999]. Among the latest technological developments, composite materials, each part of which possesses high contrast mechanical and geometrical properties, have received an increased amount of attention in various fields of civil and mechanical engineering; see [Milton 2002; Elishakoff 2005]. As typical examples we refer to sandwich structures (see, e.g., [Vinson 1999; Reddy 2003; Zenkert 1995]), which are widely used in aerospace, automation, naval architecture, etc., because of their conveniently combined light weight and relatively large flexural stiffness properties; see [Kaplunov et al. 2017; Sorokin 2004]. The composite materials are also highly utilized in smart periodic structures [Ruzzene and Baz 2000]; laminated glass beams and plates [Viverge et al. 2016; Schulze et al. 2012] and photovoltaic panels [Aßmus et al. 2017]. We also mention related problem of homogenisation of periodic media [Cherednichenko et al. 2006; Smyshlyaev 2009] and multiparametric asymptotic approach for inhomogeneous layered plate; see [Kaplunov et al. 2017; Prikazchikova et al. 2018]. Metamaterials, which are employed considerably in engineering with recent technological developments, may be given as another example for multilayered structures; see [Martin et al. 2012]. Another promising application area of composite beam structures is connected with soft robotics using deformable materials to construct compliant systems; see, e.g., [Rus and Tolley 2015; Majidi 2014].

The high frequency vibrations in the multilayered structures are very inviting because they contain high energy. However, the low-frequency vibrations are more attractive due to their omnipresent character [Kudaibergenov et al. 2016]. This paper is devoted to analysis of low frequency vibrations of strongly

Keywords: composite beam, low frequency vibration, contrast, perturbation, rigid body motion.

piecewise inhomogeneous beams. The analysis shows that the low eigenfrequencies of the considered composite beam may be observed only when certain restrictions are imposed on the material properties such as Young moduli, densities and lengths; see [Kaplunov et al. 2016; 2019]. It can also be observed that the lowest natural frequencies tend to zero at high contrast of the material properties. An asymptotic approach relying on the concept of "almost rigid body motion" was also developed in [Kaplunov et al. 2016] for strongly inhomogeneous elastic rods. Unlike a rod that has one rigid body motion, the multicomponent beams may have two rigid body motions including translation and rotation.

The paper is organized as follows. In Section 2, a general concept for rigid body motions of a beam with different end conditions is presented. Then, the governing equations of two and three component beams with four types of end condition are introduced. The exact displacements for each problem and exact natural frequencies of two component beams are also derived. In Section 3, an asymptotic procedure is established with the help of a small parameter emerging due to high contrast of the material properties. The restrictions on the material parameters, allowing low natural frequencies, are determined. Then, the established perturbation procedure is applied to the aforementioned problems and approximate eigenfrequencies and eigenforms are obtained. Section 4 contains numerical illustrations of the approximate solutions and their comparison with the exact solutions. Conclusions are presented in the final section.

2. Statement of the problem

Consider a homogeneous beam with free ends. It is well known that such a beam possesses only double zero eigenfrequencies, corresponding to rigid body translation and rotation; see Figure 1a. Changing one of the free end condition of the beam with simply supported or dashed end condition results in only one nonzero small eigenfrequency corresponding rotation or translation; see Figures 1b and 1c. In the case of contact of stiff and soft components, it may be expected that the stiff component with free ends has two small eigenfrequencies arising from perturbation of zero eigenfrequencies corresponding to the limiting rigid body translation and rotation. A similar interpretation can be made for simply supported and dashed end conditions and three component beam composed of two stiff and one soft parts.

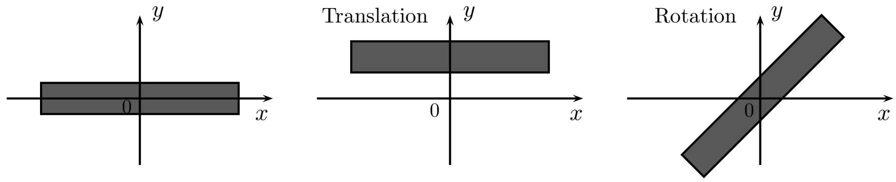
The main purpose of the paper is to investigate the effect of the end conditions on the lowest vibration modes of strongly inhomogeneous beams with the use of an asymptotic approach relying on the concept of almost rigid body motion established in [Kaplunov et al. 2016]. In order to extract the effect of boundary conditions two and three component beams with different boundary conditions will be studied.

Consider time harmonic vibrations of two and three component beams composed of alternating soft and stiff parts of arbitrary lengths with different end conditions. For each problem considered, the beams are supposed to be finite, having conventional continuity conditions between the components and have local coordinates; see Figure 2.

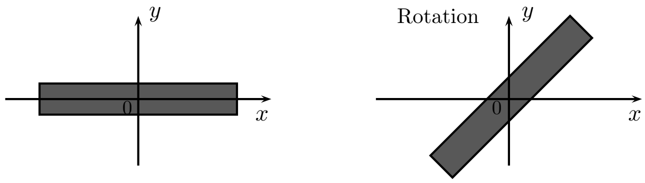
The governing equations for each component of the beam are written as

$$d^4 y_i/dx_i^4 - \omega^2/a_k y_i = 0, \quad i = 1, 2, 3 \quad \text{and} \quad k = st, sf, \tag{2-1}$$

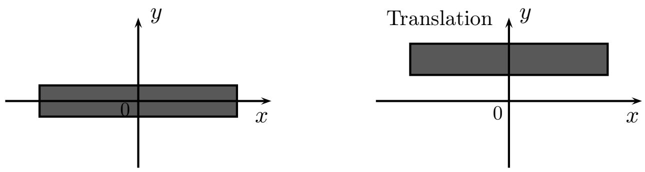
where y_i are displacements, ω is angular frequency, $a_k = a_{st}$ and $a_k = a_{sf}$ correspond to stiff and soft components, respectively, with $a_k = \sqrt{D_k/M_k}$. Here $D_k = E_k I$ is the flexural rigidity and $M_k = \rho_k A$ is linear mass density with I denoting the moment of inertia and A the cross-sectional area. It is clear that all the soft components have the same Young's modulus and density.



(a) free ends



(b) free and simply supported ends



(c) free and dashed ends

Figure 1. Rigid body motion of beam.

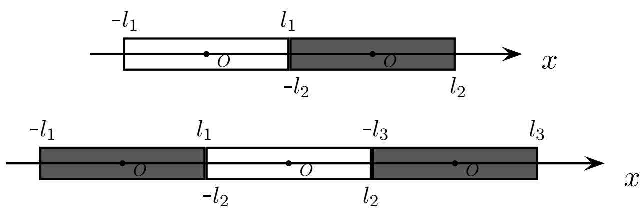


Figure 2. Two and three-component beams. Each 0 represents the origin of its local coordinates.

Let us first the define local dimensionless coordinates and scaled frequencies by

$$\xi_i = x_i/l_i \qquad \text{and} \qquad \Omega_i = l_i\sqrt{\omega/a_k}, \quad i = 1, 2, 3, \quad k = st, sf. \tag{2-2}$$

The equations of motion and the continuity conditions along the interfaces are expressed in terms of the new variables as

$$d^4 y_i/d\xi_i^4 - \Omega_i^4 y_i = 0, \qquad -1 \leqslant \xi_i \leqslant 1, \quad i = 1, 2, 3, \tag{2-3}$$

and

$$y_1(1) = y_2(-1), \qquad y_3(-1) = y_2(1), \tag{2-4}$$

$$y_1'(1) = (l_1/l_2)y_2'(-1), \qquad y_3'(-1) = (l_3/l_2)y_2'(1), \tag{2-5}$$

$$D_{st}y_1''(1) = (l_1/l_2)^2 D_{sf}y_2''(-1), \qquad D_{st}y_3''(-1) = (l_3/l_2)^2 D_{sf}y_2''(1), \tag{2-6}$$

$$D_{st}y_1'''(1) = (l_1/l_2)^3 D_{sf}y_2'''(-1), \qquad D_{st}y_3'''(-1) = (l_3/l_2)^3 D_{sf}y_2'''(1). \tag{2-7}$$

In addition, we introduce the dimensionless quantities

$$D = D_{sf}/D_{st}, \quad M = M_{sf}/M_{st}, \quad a = a_{sf}/a_{st}. \tag{2-8}$$

The displacements for each component of the beam can be written from (2-3) as

$$y_i(\xi_i) = A_i \cos(\Omega_i \xi_i) + B_i \sin(\Omega_i \xi_i) + C_i \cosh(\Omega_i \xi_i) + D_i \sinh(\Omega_i \xi_i), \quad i = 1, 2, 3. \tag{2-9}$$

In the following, we will consider combinations of four types of boundary conditions, namely clamped, free, simply supported and dashed, for the two and three component beams.

In view of the development of new materials including soft robotics, the old problems for multispan beams may take another flavour because of low frequency resonances related to almost rigid body motions which are most harmful for the structures; see [Kaplunov et al. 2016; 2019; Rus and Tolley 2015; Majidi 2014]. In this context, throughout the paper, we assume that Young modulus of the stiff parts is much greater than of the soft part, i.e.,

$$\varepsilon = D_{sf}/D_{st} \ll 1 \tag{2-10}$$

is a small parameter signifying the high contrast material properties.

3. Asymptotic approach

In this section, an asymptotic approach is established for two and three-component beams in case of different boundary conditions. As may be seen from the previous section analytical solutions for the frequency and displacement of such problems cannot be obtained effortlessly. Therefore developing a perturbation scheme which reduces the problem to a simple boundary value problem having solutions in terms of elementary functions is highly important for analysing the frequencies and displacements. In this framework, we develop an asymptotic approach leading to the estimation of the lowest eigenfrequencies and eigenforms of the aforementioned problems.

Let us start by expanding the frequencies and displacements in the asymptotic series on using the small parameter (2-10);

$$\Omega_i^4 = \varepsilon(\Omega_{i,0}^4 + \varepsilon\Omega_{i,1}^4 + \varepsilon^2\Omega_{i,2}^4 + \cdots), \quad y_i = y_{i,0} + \varepsilon y_{i,1} + \varepsilon^2 y_{i,2} + \cdots, \quad i = 1, 2, 3. \tag{3-1}$$

which may also correspond to the low frequency regimes of the problems mentioned at the end of Section 2. Since we consider the global low frequency behaviour, i.e., $\Omega_1^4 \sim \Omega_2^4 \sim \Omega_3^4 \sim \varepsilon$, the ratio of the masses and lengths have the following asymptotic equality

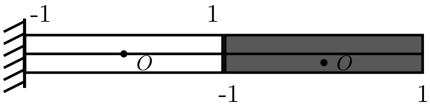
$$\varepsilon \frac{M_{st}}{M_{sf}} = M_*, \quad M_* \sim 1, \tag{3-2}$$

and

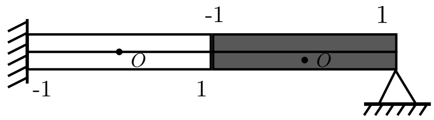
$$(l_1/l_2)^4 = \delta_1^4 \sim 1, \quad (l_3/l_2)^4 = \delta_3^4 \sim 1. \tag{3-3}$$

In case there are no considerable changes in the cross-sectional area A, the assumed contrast in Young's moduli and densities given in (2-10) and (3-2) occurs for photovoltaic panels; see, e.g., [Aßmus et al. 2017; Kaplunov et al. 2017; Schulze et al. 2012]. It can easily be observed from (3-1) that the scaled frequencies are also related to each other by

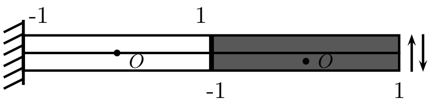
$$\Omega_1^4 = \Omega_2^4 \delta_1^4 M_*, \quad \Omega_3^4 = \Omega_2^4 \delta_3^4 M_*. \tag{3-4}$$



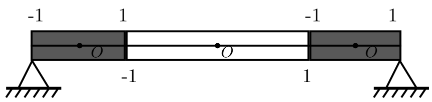
(a) clamped-free ends
$y_1(-1) = y_1'(-1) = 0, \quad y_2''(1) = y_2'''(1) = 0$



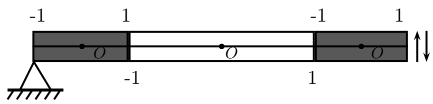
(b) clamped-simply supported ends
$y_1(-1) = y_1'(-1) = 0, \quad y_2(1) = y_2''(1) = 0$



(c) clamped-dashed ends
$y_1(-1) = y_1'(-1) = 0, \quad y_2'(1) = y_2'''(1) = 0$



(d) simply supported at both ends
$y_1(-1) = y_1''(-1) = 0, \quad y_3(1) = y_3''(1) = 0$



(e) simply supported-dashed ends
$y_1(-1) = y_1''(-1) = 0, \quad y_3'(1) = y_3'''(1) = 0$

Figure 3. Types of boundary conditions.

3A. ***Perturbation scheme for a two component beam with clamped and free ends.*** We first consider the two component beam with one end clamped and other free as depicted in Figure 3a. Bearing in mind the definition of the small parameter and the relations between the lengths, see (3-3), the continuity conditions at the interfaces and the boundary conditions take, respectively, form

$$y_1(1) = y_2(-1), \quad y_1'(1) = \delta_1 y_2'(-1), \quad \varepsilon y_1''(1) = \delta_1^2 y_2''(-1), \quad \varepsilon y_1'''(1) = \delta_1^3 y_2'''(-1), \tag{3-5}$$

$$y_1(-1) = 0, \quad y_1'(-1) = 0, \quad y_2''(1) = 0, \quad y_2'''(1) = 0. \tag{3-6}$$

On substituting the asymptotic expansions (3-1) into (2-3), the equations of motion, at the leading order, become

$$\frac{d^4 y_{i,0}}{d\xi_i^4} = 0, \quad i = 1, 2. \tag{3-7}$$

Applying the asymptotic expansion for the displacements in (3-1) into equations (3-5) results in the boundary conditions

$$y_{2,0}''(\pm 1) = y_{2,0}'''(\pm 1) = 0 \tag{3-8}$$

for the stiff component. The solution of the boundary value problem (3-7) and (3-8) for the stiff component may be written as

$$y_{2,0} = A_2 \xi_2 + B_2, \tag{3-9}$$

which corresponds to the rigid body translation and rotation; see Figure 1a.

The soft part of the beam has governing equation (3-7) subject to the boundary condition

$$y_{1,0}(1) = y_{2,0}(-1), \quad y_{1,0}'(1) = \delta_1\, y_{2,0}'(-1), \tag{3-10}$$

and

$$y_{1,0}(-1) = y'_{1,0}(-1) = 0. \tag{3-11}$$

Therefore, it can be easily shown from the solution of the boundary value problem (3-7) and (3-10), (3-11) that the soft component, unlike the stiff component, undergoes an inhomogeneous deformation given by

$$y_{1,0} = (\xi_1^3 - 3\xi_1 - 2)A_1 + (\xi_1^2 + 2\xi_1 + 1)B_1. \tag{3-12}$$

Using equations (3-10), (3-11) and displacements (3-9), (3-12) the coefficients of the soft and stiff displacements are related by

$$A_1 = \tfrac{1}{4}((1+\delta_1)A_2 - B_2), \quad B_1 = \tfrac{1}{4}A_2\delta_1. \tag{3-13}$$

At next order, the problem for the stiff component of the beam is formulated as

$$\frac{d^4 y_{2,1}}{d\xi_2^4} - \Omega_{2,0}^4\, y_{2,0} = 0, \tag{3-14}$$

with the boundary conditions

$$\left.\frac{d^2 y_{2,1}}{d\xi_2^2}\right|_{\xi_2=-1} = \frac{1}{\delta_1^2}\left.\frac{d^2 y_{1,0}}{d\xi_1^2}\right|_{\xi_1=1}, \qquad \left.\frac{d^3 y_{2,1}}{d\xi_2^3}\right|_{\xi_2=-1} = \frac{1}{\delta_1^3}\left.\frac{d^3 y_{1,0}}{d\xi_1^3}\right|_{\xi_1=1}, \tag{3-15}$$

and

$$\left.\frac{d^2 y_{2,1}}{d\xi_2^2}\right|_{\xi_2=1} = 0, \qquad \left.\frac{d^3 y_{2,1}}{d\xi_2^3}\right|_{\xi_2=1} = 0. \tag{3-16}$$

Integrating (3-14) over ξ_2 $(-1 \le \xi_2 \le 1)$ results in

$$3A_1 + \Omega_{2,0}^4\delta_1^3 B_2 = 0. \tag{3-17}$$

Next, multiplying (3-14) by ξ_2 and integrating over the length of the right component gives

$$9A_1 + 3\delta_1(3A_1 + B_1) - \Omega_{2,0}^4\delta_1^3 A_2 = 0. \tag{3-18}$$

Using the relations between the coefficients (3-13), equations (3-17) and (3-18) give the linear system of equations in A_2 and B_2 as

$$\begin{aligned} \tfrac{1}{4}(3+\delta_1)A_2 + \tfrac{1}{4}(4\delta_1^3\Omega_{2,0}^4 - 3)B_2 &= 0, \\ \tfrac{1}{4}(9 + 2(9 + 6\delta_1 - 2\delta_1^2\Omega_{2,0}^4))A_2 - \tfrac{9}{4}(1+\delta_1)B_2 &= 0. \end{aligned} \tag{3-19}$$

Simultaneous equations possess nontrivial solutions provided that the associated determinant vanishes, i.e.,

$$\tfrac{9}{16}\delta_1^2 - \tfrac{3}{2}\delta_1^3(2 + \delta_1(3 + 2\delta_1))\Omega_{2,0}^4 + \delta_1^6\Omega_{2,0}^8 = 0. \tag{3-20}$$

The frequency equation (3-20), corresponding to the second component of the beam, has two nonzero eigenfrequencies given by

$$\Omega_{2,0}^4 = \frac{3\big(2\delta_1^2 + 3\delta_1 + 2 - 2(\delta_1+1)\sqrt{\delta_1^2+\delta_1+1}\big)}{4\delta_1^3},$$
$$\Omega_{2,0}^4 = \frac{3\big(2\delta_1^2 + 3\delta_1 + 2 + 2(\delta_1+1)\sqrt{\delta_1^2+\delta_1+1}\big)}{4\delta_1^3}. \tag{3-21}$$

As might be expected, the contact of a stiff component with a soft one perturbs the double zero eigenfrequencies to two lowest eigenfrequencies associated with the almost rigid translation and rotation.

3B. ***Perturbation scheme for a two component beam with clamped and simply supported ends.*** Let us now study two component beam having clamped and simply supported ends. The continuity conditions (2-4)–(2-7) assume the same forms given by (3-5) and the boundary conditions are as given in Figure 3b.

In this case, the leading order boundary conditions for the stiff component are

$$y''_{2,0}(-1) = y'''_{2,0}(-1) = 0, \qquad y_{2,0}(1) = y''_{2,0}(1) = 0. \tag{3-22}$$

whereas the soft component has, as boundary conditions, (3-10) and (3-11). Therefore, the leading order displacements for the considered problem are written from (3-7) as

$$y_{1,0} = (\xi_1^3 - 3\xi_1 - 2)A_1 + (\xi_1^2 + 2\xi_1 + 1)B_1, \quad y_{2,0} = A_2(\xi_2 - 1). \tag{3-23}$$

The displacements (3-23) together with (3-10) and (3-11) imply

$$A_1 = \tfrac{1}{4}(\delta_1 + 2)A_2, \quad B_1 = \tfrac{1}{4}\delta_1 A_2. \tag{3-24}$$

At next order, the problem is formulated, once again, through equations (3-14), (3-15), and end conditions

$$y_{2,1}(1) = 0, \qquad \left.\frac{d^2 y_{2,1}}{d\xi_2^2}\right|_{\xi_2=1} = 0. \tag{3-25}$$

Integrating (3-14) and using the end conditions (3-15) and (3-25) together with (3-24) result in

$$4\delta_1^3\Omega_{2,0}^4 - 3(\delta_1^2 + 3\delta_1 + 3) = 0. \tag{3-26}$$

3C. ***Perturbation scheme for a two component beam with clamped and dashed ends.*** We now carry our dispersion analysis on to a beam with clamped and dashed ends. Similar to the previous two cases, the boundary value problem for the stiff component at the leading order is formulated by (3-7) and

$$y''_{2,0}(-1) = y'''_{2,0}(-1) = 0, \quad y'_{2,0}(1) = y''_{2,0}(1) = 0, \tag{3-27}$$

from which displacements are obtained as

$$y_{1,0} = (\xi_1^3 - 3\xi_1 - 2)A_1, \quad y_{2,0} = A_2, \tag{3-28}$$

with relation

$$A_1 = -\tfrac{1}{4}A_2. \tag{3-29}$$

The next order problem for the stiff component is again formulated by equations (3-14), (3-15) and

$$\left.\frac{dy_{2,1}}{d\xi_2}\right|_{\xi_2=1}=0,\qquad \left.\frac{d^3y_{2,1}}{d\xi_2^3}\right|_{\xi_2=1}=0. \tag{3-30}$$

the solution of which yields

$$\delta_1^3\Omega_{2,0}^4-\tfrac{3}{4}=0. \tag{3-31}$$

3D. ***Perturbation scheme for a three component simply supported beam.*** We will now present asymptotic formulas for the displacements and frequencies for a three component beam with two stiff outer components simply supported at both ends, shown in Figure 3d. Taking into account the definitions of the small parameter (2-10) and the ratios of the lengths, δ_1 and δ_2, the continuity conditions may be rewritten from equations (2-4)-(2-7) as

$$\begin{aligned} y_1(1)&=y_2(-1), & y_3(-1)&=y_2(1),\\ y_1'(1)&=\delta_1y_2'(-1), & y_3'(-1)&=\delta_3y_2'(1),\\ y_1''(1)&=\varepsilon\delta_1^2y_2''(-1), & y_3''(-1)&=\varepsilon\delta_3^2y_2''(1),\\ y_1'''(1)&=\varepsilon\delta_1^3y_2'''(1), & y_3'''(-1)&=\varepsilon\delta_3^3y_2'''(1), \end{aligned} \tag{3-32}$$

and simply supported boundary conditions at both ends are, also, as presented in Figure 3d.

On substituting the asymptotic expansions (3-1) into equations of motion (2-3) we once again arrive at the leading order (3-7). In this case, the stiff and soft components have leading order boundary conditions given by

$$\begin{aligned} y_{1,0}(-1)&=y_{1,0}''(-1)=0, & y_{1,0}''(1)&=y_{1,0}'''(1)=0,\\ y_{3,0}''(-1)&=y_{3,0}'''(-1)=0, & y_{3,0}(1)&=y_{3,0}''(1)=0, \end{aligned} \tag{3-33}$$

and

$$\begin{aligned} y_{2,0}(-1)&=y_{1,0}(1), & \delta_1y_{2,0}'(-1)&=y_{1,0}'(1),\\ y_{2,0}(1)&=y_{3,0}(-1), & \delta_3y_{2,0}'(1)&=y_{3,0}'(-1), \end{aligned} \tag{3-34}$$

respectively. Thus, the leading order displacements can be written as

$$y_{1,0}=A_1(\xi_1+1),\quad y_{2,0}=A_2\xi_2^3+B_2\xi_2^2+C_2\xi_2+D_2,\quad y_{3,0}=A_3(\xi_3-1), \tag{3-35}$$

in which the coefficients are related as

$$\begin{aligned} A_2&=\tfrac{1}{4}(A_1(2+1/\delta_1)+A_3(2+1/\delta_3)), & B_2&=\tfrac{1}{4}(-A_1/\delta_1+A_3/\delta_3),\\ C_2&=-\tfrac{1}{4}(A_1(6+1/\delta_1)+A_3(6+1/\delta_3)), & D_2&=A_1(1+1/(4\delta_1))-A_3(1+1/(4\delta_3)). \end{aligned} \tag{3-36}$$

Let us now proceed to the next order problem. For the first stiff component we have

$$\frac{d^4y_{1,1}}{d\xi_1^4}-\Omega_{1,0}^4y_{1,0}=0, \tag{3-37}$$

with the boundary conditions

$$\left.\frac{d^2y_{1,1}}{d\xi_1^2}\right|_{\xi_1=1}=\delta_1^2\left.\frac{d^2y_{2,0}}{d\xi_2^2}\right|_{\xi_2=-1},\qquad \left.\frac{d^3y_{1,1}}{d\xi_1^3}\right|_{\xi_1=1}=\delta_1^3\left.\frac{d^3y_{2,0}}{d\xi_2^3}\right|_{\xi_2=-1}, \tag{3-38}$$

and

$$y_{1,1}(-1) = \left.\frac{d^2 y_{1,1}}{d\xi_1^2}\right|_{\xi_1=-1} = 0. \tag{3-39}$$

Multiplying (3-37) by ξ_1 and integrating over $-1 \le \xi_1 \le 1$, taking into account equations (3-38) and (3-39), we arrive at

$$\left.\frac{d^3 y_{1,1}}{d\xi_1^3}\right|_{\xi_1=-1} = -6\delta_1^3 A_2 - \delta_1^2(6A_2 - 2B_2) + \tfrac{2}{3}A_1\Omega_{1,0}^4. \tag{3-40}$$

Integrating (3-38) over $-1 \le \xi_1 \le 1$ and using (3-40) results in

$$9\delta_1^2(2\delta_1+1)A_2 - 3\delta_1^2 B_2 - 4\Omega_{1,0}^4 A_1 = 0. \tag{3-41}$$

Similarly, we derive for the second stiff component, $y_{3,1}$,

$$9\delta_3^2(2\,\delta_3+1)A_2 + 3\delta_3^2 B_2 - 4\Omega_{3,0}^4 A_1 = 0. \tag{3-42}$$

Equations (3-41) and (3-42) with the relations between the coefficients (3-36) lead to a frequency equation given by

$$27\delta_1\delta_3(1+\delta_1+\delta_3)^2 - 48\delta_3(3\delta_3^2+3\delta_3+1)\Omega_{1,0}^4 - 16(3\delta_1(3\delta_1^2+3\delta_1+1) - 4\Omega_{1,0}^4)\Omega_{3,0}^4 = 0. \tag{3-43}$$

Since the frequencies are related together as

$$\Omega_{3,0}^4/\Omega_{1,0}^4 = \delta_3^4/\delta_1^4, \tag{3-44}$$

see (3-4), $\Omega_{1,0}$ may be written as the roots of the frequency equation (3-43), that is

$$\Omega_{1,0}^4 = \frac{3\delta_1}{8\delta_3^3}\Big(\delta_3^3 + 3\delta_1\delta_3^3 + 3\delta_1^2\delta_3^3 + \delta_1^3(1+3\delta_3(1+\delta_3)) - \sqrt{(\delta_3^3(1+3\delta_1(1+\delta_1)) + \delta_1^3(1+3\delta_3(1+\delta_3)))^2 - 3\delta_1^3\delta_3^3(1+\delta_1+\delta_3)^2}\Big), \tag{3-45}$$

and

$$\Omega_{1,0}^4 = \frac{3\delta_1}{8\delta_3^3}\Big(\delta_3^3 + 3\delta_1\delta_3^3 + 3\delta_1^2\delta_3^3 + \delta_1^3(1+3\delta_3(1+\delta_3)) + \sqrt{(\delta_3^3(1+3\delta_1(1+\delta_1)) + \delta_1^3(1+3\delta_3(1+\delta_3)))^2 - 3\delta_1^3\delta_3^3(1+\delta_1+\delta_3)^2}\Big). \tag{3-46}$$

It can easily be seen that the simply supported end conditions for a three component beam do not support pure rigid body motion with zero eigenfrequencies.

3E. ***Perturbation scheme for three component beam with simply supported and dashed ends.*** Finally, consider a three component beam having two stiff outer and one soft central parts. Contrary to the previous problem, in this case we have simply supported and dashed end conditions, stated in Figure 3e. The continuity conditions at the interface given by (3-32) are still valid for this problem. The leading order displacements may be written as

$$y_{1,0} = A_1(\xi_1+1), \quad y_{2,0} = A_2\xi_2^3 + B_2\xi_2^2 + C_2\xi_2 + D_2, \quad y_{3,0} = A_3, \tag{3-47}$$

with

$$A_2=\frac{1}{4\delta_1}(A_1(1+2\,\delta_1)-A_3\delta_1),\qquad B_2=-\frac{A_1}{4\delta_1},$$
$$C_2=-\frac{1}{4\delta_1}(A_1(1+6\delta_1)-3A_3\delta_1),\quad D_2=A_1\Big(1+\frac{1}{4\delta_1}\Big)+\tfrac{1}{2}A_3. \tag{3-48}$$

The next order problem for the first component of the beam is formulated with the equations (3-37), (3-38) and (3-39), which gives (3-41). For the next order problem of the third component of the beam we have boundary conditions (3-38) and

$$\frac{dy_{3,1}}{d\xi_3}\bigg|_{\xi_3=1}=\frac{d^3y_{3,1}}{d\xi_3^3}\bigg|_{\xi_3=1}=0. \tag{3-49}$$

Integrating the next order governing equation for $y_{3,1}$ over $-1\le\xi_3\le 1$ and using the boundary conditions given by (3-38) and (3-49), we get

$$3\delta_3^3A_2+\Omega_{3,0}^4A_3=0. \tag{3-50}$$

The equations (3-41) and (3-50) together with relations (3-48) imply the following frequencies

$$\Omega_{3,0}^4(4\Omega_{1,0}^4-3\delta_1(1+3\delta_1(1+\delta_1)))-3\delta_3^3\Omega_{1,0}^4+\tfrac{9}{16}\delta_1\delta_3^3=0, \tag{3-51}$$

resulting in

$$\Omega_{1,0}^4=\frac{3\delta_1}{8\delta_3}(\delta_1^3+\delta_3+3\delta_1\delta_3+3\delta_1^2\delta_3-\sqrt{(\delta_1^3+\delta_3+3\delta_1\delta_3(1+\delta_1))^2-\delta_1^3\delta_3}), \tag{3-52}$$

and

$$\Omega_{1,0}^4=\frac{3\delta_1}{8\delta_3}(\delta_1^3+\delta_3+3\delta_1\delta_3+3\delta_1^2\delta_3+\sqrt{(\delta_1^3+\delta_3+3\delta_1\delta_3(1+\delta_1))^2-\delta_1^3\delta_3}). \tag{3-53}$$

4. Numerical results

In this section, we illustrate numerically the comparisons of exact and asymptotic results for all of the considered problems. Since all asymptotic estimates for the frequencies and displacements are valid in case of the global low frequency regime, the numerical comparisons are presented when the parameters are asymptotically related as $D_{sf}/D_{st}\sim M_{sf}/M_{st}\sim\varepsilon$ and $\delta_1\sim\delta_2\sim 1$, see (3-2) and (3-3). which is a requirement for the global low frequency regime,

Figures 4, 5 and 6 show the exact and asymptotic curves of the frequency equation for two component beams with clamped-free, clamped-simply supported and clamped-dashed ends. In these figures in order to display more clearly the extent of the validity region of the approximation full agreement regions are chopped off. It can be easily seen from these figures that as the small parameter ε becomes smaller, which corresponds a great contrast between the stiff and soft components, the curves corresponding to the asymptotic solutions become highly compatible with the exact one. In Fig. different length components of the beam are considered, i.e., $l_1=l$ and $l_2=2l$ which gives $\delta_1=0.5$.

Figure 7 demonstrates the comparison of asymptotic curves of the frequency equations of two component beams given by (3-20), (3-26) and (3-31). Thus, it can be observed how the eigenfrequencies of a single beam are affected by the different boundary conditions. It is clear that the clamped-free end condition causes lowest and highest eigenfrequencies compared to other boundary conditions by perturbing the zero eigenfrequency of a beam with free-free end.

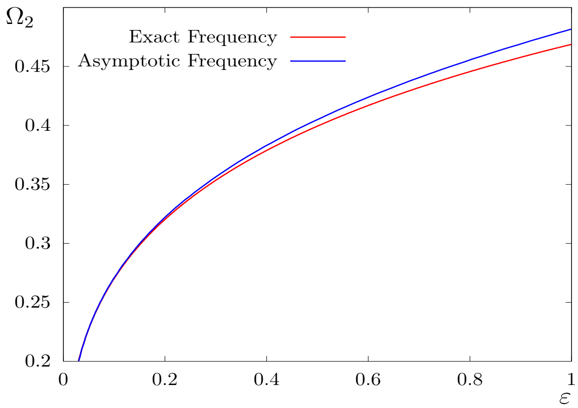


Figure 4. Comparison of asymptotic (3-20) and exact (6-1) frequency equations for two component beam with clamped and free ends at $\delta_1 = 0.99$.

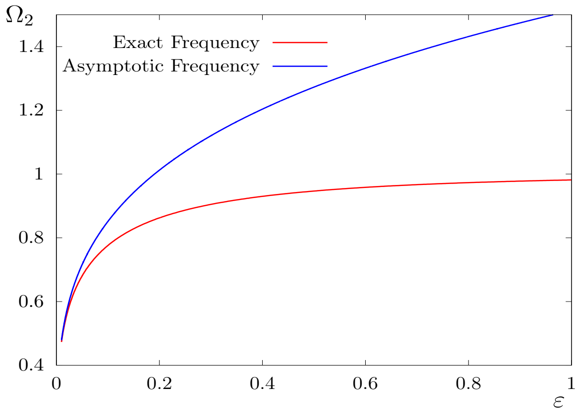


Figure 5. Comparison of asymptotic (3-26) and exact (6-2) frequency equations for two component beam with clamped and simply supported ends at $\delta_1 = 0.99$.

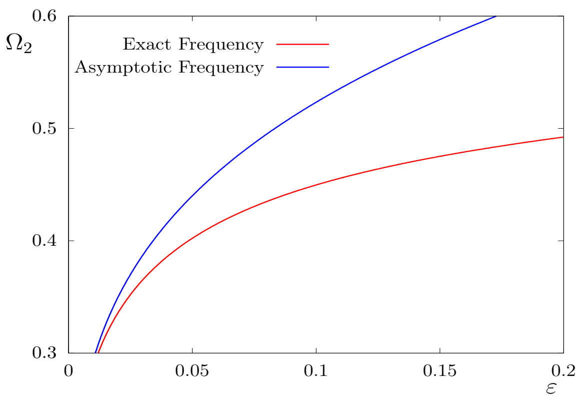


Figure 6. Comparison of asymptotic (3-31) and exact (6-3) frequency equations for two component beam with clamped and dashed ends at $\delta_1 = 0.99$.

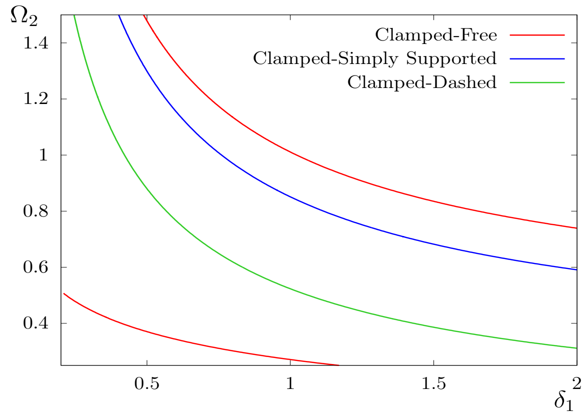


Figure 7. Comparison of asymptotic curves of frequency equations (3-20), (3-26) and (3-31) at $\varepsilon = 0.1$.

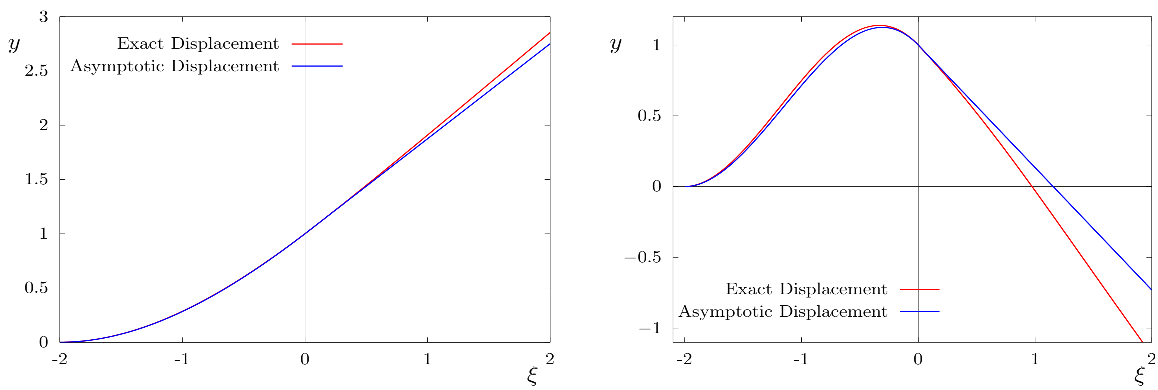


Figure 8. Comparison of exact (2-9), with $i = 2$, and asymptotic (3-9), (3-12) displacements at $\varepsilon = 0.1$, $\delta_1 = 0.99$. Left: eigenfrequency $(3\text{-}21)_1$. Right: eigenfrequency $(3\text{-}21)_2$.

The exact and asymptotic scaled displacements corresponding to eigenfrequencies (3-21); (3-26) and (3-31) for the associated problems are demonstrated in Figures 8, 9 and 10. It can be observed from all these figures that even for not very small value of $\varepsilon = 0.1$, the asymptotic formula (3-9), (3-12); (3-23) and (3-28) present an excellent approximation to the exact displacements (2-9) with $i = 2$.

In Figure 11 comparisons of exact and asymptotic displacements of three component beam with simply supported at both ends are presented when the length of the components are not equal to each other, e.g., $\delta_1 = 1.5$ and $\delta_2 = 1.25$. As can be easily seen from this figure that the approximate displacement (3-35) is, again, in quite a good agreement with the exact one (2-9), with $i = 3$, for $\varepsilon = 0.01$. Similar results follow for three component beam with simply supported and dashed ends illustrated in Figure 12.

5. Concluding remarks

The low-frequency vibrations in two and three-component beams with high contrast properties have been studied. It is known that a stiff single component beam with free ends has double zero eigenfrequencies

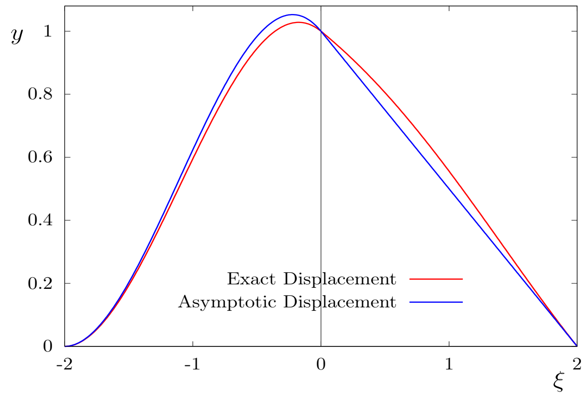


Figure 9. Comparison of exact (2-9), with $i = 2$, and asymptotic (3-23) displacements at $\varepsilon = 0.1$, $\delta_1 = 0.99$.

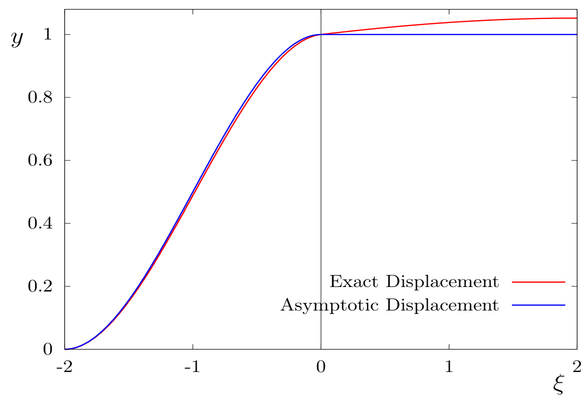


Figure 10. Comparison of exact (2-9), with $i = 2$, and asymptotic (3-28) displacements at $\varepsilon = 0.1$, $\delta_1 = 0.99$.

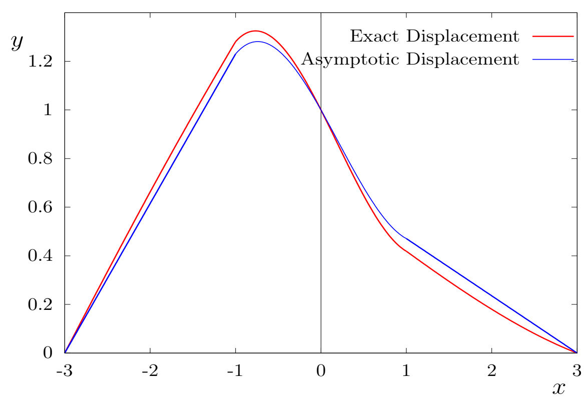


Figure 11. Comparison of exact (2-9), with $i = 3$, and asymptotic (3-35) displacements for the eigenfrequency (3-45) at $\varepsilon = 0.01$, $\delta_1 = 1.5$ and $\delta_2 = 1.25$.

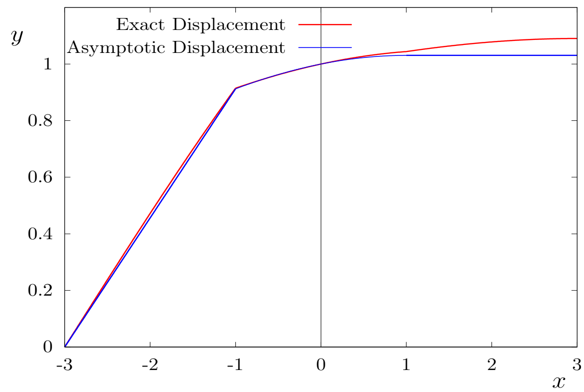


Figure 12. Comparison of exact (2-9), with $i = 3$, and asymptotic (3-47) displacements for the eigenfrequency (3-51) at $\varepsilon = 0.01$, $\delta_1 = 4$ and $\delta_2 = 2$.

corresponding to limiting rigid body translation and rotation. It is seen that the contact of a soft component with a stiffer one and change in the other end condition cause one or two small eigenfrequencies. It is shown that if the stiff component has a free end condition the beam possesses two nonzero lowest eigenfrequencies corresponding to almost rigid body translation and rotation of the stiff part. On the other hand, if the end condition of the stiff component is simply supported or dashed then the beam has only one lowest eigenfrequency arising from perturbation of zero eigenfrequencies, which correspond to rotation or translation. It is also demonstrated that among these three end conditions, free end condition gives the lowest and highest eigenfrequencies.

The exact formulation of eigenfrequencies and eigenforms of such problems are generally given by a sophisticated transcendental relation. Therefore an asymptotic procedure is established in order to investigate near zero frequencies and corresponding displacements. The derived asymptotic formulae are valid for certain conditions on the ratios of material and geometrical properties, which reveal the low frequency vibrations. It may also be emphasized that the approximate formulation for the leading order displacements of the stiffer parts carry out almost rigid body motion whereas the softer parts undergo almost homogeneous deformation. Another point to note is that eigenfrequencies and eigenmodes might have been calculated through a numerical procedure (such as FE method) however, in this case there is a limited potential for physical insight. In particular, there is a risk to miss low frequency eigenforms with polynomial but not expected sinusoidal behaviour.

The proposed perturbation approach may be generalised to multicomponent high-contrast structures with different end conditions including an n component strongly inhomogeneous beam. The approach may also be adapted for 2D and 3D eigenvalue problems for multilayered, high-contrast structures such as plates and shells. Multiparametric structures, in which material parameters are dependent on coordinate axis, are another promising research area which the established model can be applied.

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Appendix

The frequency equations for two component beams with end conditions given in Figures 3a, 3b and 3c is given, respectively, by

$$\begin{aligned}
&\cosh^2(\Omega_1)\big(-4\sinh^2(\Omega_2)(\sqrt{a}\sin(\Omega_1)\cos(\Omega_2)+D\sin(\Omega_2)\cos(\Omega_1))\\
&\qquad\times(a^{3/2}\sin(\Omega_1)\cos(\Omega_2)+D\sin(\Omega_2)\cos(\Omega_1))\\
&\quad+4\cosh^2(\Omega_2)(D\cos(\Omega_1)\cos(\Omega_2)-\sqrt{a}\sin(\Omega_1)\sin(\Omega_2))\\
&\qquad\times(D\cos(\Omega_1)\cos(\Omega_2)-a^{3/2}\sin(\Omega_1)\sin(\Omega_2))\\
&\qquad-(a-1)\sqrt{a}D\sin(2\Omega_1)\cos(2\Omega_2)\sinh(2\Omega_2)\big)\\
&+\sinh^2(\Omega_1)\big(4\sinh^2(\Omega_2)(\sqrt{a}\cos(\Omega_1)\cos(\Omega_2)-D\sin(\Omega_1)\sin(\Omega_2))\\
&\qquad\times(a^{3/2}\cos(\Omega_1)\cos(\Omega_2)-D\sin(\Omega_1)\sin(\Omega_2))\\
&\quad-4\cosh^2(\Omega_2)(\sqrt{a}\sin(\Omega_2)\cos(\Omega_1)+D\sin(\Omega_1)\cos(\Omega_2))\\
&\qquad\times(a^{3/2}\sin(\Omega_2)\cos(\Omega_1)+D\sin(\Omega_1)\cos(\Omega_2))\\
&\qquad-(a-1)\sqrt{a}D\sin(2\Omega_1)\cos(2\Omega_2)\sinh(2\Omega_2)\big)\\
&+\sqrt{a}D\sinh(2\Omega_1)\big(\sinh(2\Omega_2)((a+1)\cos(2\Omega_1)\cos(2\Omega_2)-2\sqrt{a}\sin(2\Omega_1)\sin(2\Omega_2))\\
&\qquad+(a-1)\sin(2\Omega_2)\cos(2\Omega_1)\cosh(2\Omega_2)\big)=0, \qquad (6\text{-}1)
\end{aligned}$$

$$\begin{aligned}
&\sinh(2\Omega_2)\big(\cosh(2\Omega_1)((-a^2-D^2)\cos(2\Omega_1)\cos(2\Omega_2)+2\sqrt{a}D\sin(2\Omega_1)\sin(2\Omega_2))\\
&\quad+(a-D)(a+D)\cos(2\Omega_2)\\
&\quad+2\sinh(2\Omega_1)(aD\sin(2\Omega_1)\cos(2\Omega_2)+\sqrt{a}D\sin(2\Omega_2)\cos(2\Omega_1))\big)\\
&+2\cosh(2\Omega_2)\big(\sinh^2(\Omega_1)(a^{3/2}D\sin(2\Omega_1)\cos(2\Omega_2)\\
&\qquad+\sin(2\Omega_2)(a^2\cos^2(\Omega_1)-D^2\sin^2(\Omega_1)))\\
&\quad+\cosh^2(\Omega_1)(a^{3/2}D\sin(2\Omega_1)\cos(2\Omega_2)\\
&\qquad+\sin(2\Omega_2)(D^2\cos^2(\Omega_1)-a^2\sin^2(\Omega_1)))\\
&\quad+aD\sinh(2\Omega_1)(\sin(2\Omega_1)\sin(2\Omega_2)-\sqrt{a}\cos(2\Omega_1)\cos(2\Omega_2))\big)=0 \qquad (6\text{-}2)
\end{aligned}$$

and

$$\begin{aligned}
&2\cosh^2(\Omega_1)\cosh^2(\Omega_2)\big(2\sin(\Omega_2)\cos(\Omega_2)(D^2\cos^2(\Omega_1)-a^2\sin^2(\Omega_1))\\
&\qquad+\sqrt{a}D\sin(2\Omega_1)\cos(2\Omega_2)\big)\\
&+2\sinh^2(\Omega_1)\cosh^2(\Omega_2)\big(2\sin(\Omega_2)\cos(\Omega_2)(a^2\cos^2(\Omega_1)-D^2\sin^2(\Omega_1))\\
&\qquad+\sqrt{a}D\sin(2\Omega_1)\cos(2\Omega_2)\big)\\
&+\sinh(\Omega_2)\big((a^2+D^2)\cos(2\Omega_1)\cosh(2\Omega_1)(\sin(2\Omega_2)\sinh(\Omega_2)+2\cos(2\Omega_2)\cosh(\Omega_2))\\
&\quad+2\sqrt{a}D\sin(2\Omega_1)\cosh(2\Omega_1)(\cos(2\Omega_2)\sinh(\Omega_2)-2a\sin(2\Omega_2)\cosh(\Omega_2))\\
&\quad-(a-D)(a+D)(\sin(2\Omega_2)\sinh(\Omega_2)+2\cos(2\Omega_2)\cosh(\Omega_2))\big)
\end{aligned}$$

$$+\sqrt{a}D\sinh(2\Omega_1)\big(-2\sqrt{a}\sin(2\Omega_1)\sin(2\Omega_2)\cosh(2\Omega_2)\\ +2a\sin(2\Omega_2)\cos(2\Omega_1)\sinh(2\Omega_2)\\ +2\cos(2\Omega_2)(\sqrt{a}\sin(2\Omega_1)\sinh(2\Omega_2)+\cos(2\Omega_1)\cosh(2\Omega_2))\big)=0. \quad (6\text{-}3)$$

The related frequency equation for three component beam with simply supported both ends can be obtained from the determinant of the following 12×12 matrix which can be obtained by applying the continuity, (2-4)–(2-7), and boundary conditions Figure 3d into the displacements (2-9).

$$\begin{pmatrix}
-c_1 & -s_1 & -ch_1 & -sh_1 & c_2 & -s_2 & ch_2 & sh_2 & 0 & 0 & 0 & 0\\
\sqrt{a}s_1 & -\sqrt{a}c_1 & -\sqrt{a}sh_1 & -\sqrt{a}ch_1 & s_2 & c_2 & -sh_2 & ch_2 & 0 & 0 & 0 & 0\\
ac_1 & as_1 & -ach_1 & -ash_1 & -\varepsilon c_2 & \varepsilon s_2 & \varepsilon ch_2 & -\varepsilon sh_2 & 0 & 0 & 0 & 0\\
-a^{3/2}s_1 & a^{3/2}c_1 & -a^{3/2}sh_1 & -a^{3/2}ch_1 & -\varepsilon s_2 & -\varepsilon c_2 & -\varepsilon sh_2 & \varepsilon ch_2 & 0 & 0 & 0 & 0\\
c_1 & -s_1 & ch_1 & -sh_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-c_1 & s_1 & ch_1 & -sh_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & c_2 & s_2 & ch_2 & sh_2 & -c_3 & s_3 & -ch_3 & sh_3\\
0 & 0 & 0 & 0 & -s_2 & c_2 & sh_2 & ch_2 & -\sqrt{a}s_3 & -\sqrt{a}c_3 & \sqrt{a}sh_3 & -\sqrt{a}ch_3\\
0 & 0 & 0 & 0 & -\varepsilon c_2 & -\varepsilon s_2 & \varepsilon ch_2 & \varepsilon sh_2 & ac_3 & -as_3 & -ach_3 & ash_3\\
0 & 0 & 0 & 0 & \varepsilon s_2 & -\varepsilon c_2 & \varepsilon sh_2 & \varepsilon ch_2 & a^{3/2}s_3 & a^{3/2}c_3 & a^{3/2}sh_3 & -a^{3/2}ch_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & s_3 & ch_3 & sh_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & -s_3 & ch_3 & sh_3
\end{pmatrix} \quad (6\text{-}4)$$

where

$$c_i=\cos(\Omega_i),\quad s_i=\sin(\Omega_i),\quad ch_i=\cosh(\Omega_i),\quad sh_i=\sinh(\Omega_i),\quad i=1,2,3.$$

Since the result of the determinant is quite messy an explicit expression of the frequency equation is not presented.

The frequency equation of three component beam with simply supported and dashed ends, again, can be expressed via the determinant of the coefficient matrix, which cannot be presented straightforwardly.

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