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STUDYING THE DOME OF PISA CATHEDRAL VIA A MODERN REINTERPRETATION OF DURAND-CLAYE'S METHOD

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The dome of the Pisa cathedral is a masonry structure of great interest for many aspects related to history, architecture, building techniques as well as geometry and mechanical behaviour. This Romanesque dome presents a peculiar shape, with an oval base and a pointed profile. Scaffolding has recently been erected in order to perform the restoration operations in preparation for the 900th anniversary of the cathedral's dedication. Thanks to the cooperation of the *Opera della Primaziale Pisana*, this has provided a unique opportunity to carry out a research project aimed at an overall analysis of the dome, including its shape, building details, material properties and the state of preservation of the intrados and extrados surfaces. The present contribution, part of the aforementioned research project, focuses on a modern translation of Durand-Claye's method in order to perform a preliminary study of the mechanical response of the dome. The results obtained can provide an estimate of the geometrical safety factor for the dome under vertical loads. In terms of conventional limit thickness the results suggest that the Pisa dome is a safe structure.

1. Introduction

The present paper represents one of the first outcomes of a multidisciplinary research project that has taken advantage of recent restoration works being carried out on the cathedral in preparation for the 900th anniversary of its dedication (1118). With the cooperation of the *Opera della Primaziale Pisana*, this privileged circumstance has allowed for thorough examination of the dome, which involved its building details, material properties and state of preservation.

The study of the dome's structural response has been performed by means of both analytical and numerical models. Although the study is still currently in progress, a first set of results has already been obtained regarding the effects of the vertical dead loads (a first account of some preliminary results is given in [Bennati et al. 2018]).

The present contribution focuses on a structural assessment of the Pisa cathedral dome made based on an interesting historical method for structural analysis, the *stability area method* of the renowned French scholar Durand-Claye [1867; 1880]. Originally designed for masonry arches, the method was extended (1880) to domes of revolution by subdividing the idealized dome into several lunes by means of meridian planes; each lune is then considered as an independent arch of varying width.

In fact we propose a reinterpretation and enhancement of this method, with the aim of assessing the stability of the Pisa dome. As already proposed by the authors in some previous works on masonry arches [Aita et al. 2015; 2016], the stress distribution in the masonry is considered to be nonlinear, both

Keywords: masonry dome, Pisa cathedral, limit analysis, Durand-Claye's method.

in tension and compression. Furthermore, kinematic compatibility issues at collapse, as well as the influence of hoop forces are addressed [Aita et al. 2017a].

The method has been suitably adapted to the problem at hand: a dome with a peculiar elliptical plan. The solution is pursued by means of an expressly developed, in-house algorithm implemented in Mathematica, which allows for determining statically admissible solutions and defining the dome's level of safety.

2. Durand-Claye's method for assessing the stability of domes of revolution

2.1. ***A brief reference to studies on masonry domes.*** In order to set the work of Durand-Claye within its historical context, it seems worthwhile to briefly recall the main stages of the research conducted on the structural behavior of masonry domes. The first issue regarding to masonry domes was the search for their optimal shape. Early eighteenth-century contributions on the stability of masonry domes extended to domes the results that had already been obtained for the hanging chain problem [Poleni 1748; Bouguer 1734; Bossut 1774; 1778], by showing that in order to attain equilibrium, the meridian must have the same shape as the curve representing the funicular of the loads corresponding to a slice of dome. Hoop forces are thus disregarded.

This approach was subsequently further extended in [Mascheroni 1785] and [Venturoli 1833]. (Some remarks on the historical evolution of these methods are made in [Aita et al. 2017a].) In all of these studies, the masonry dome is divided into lunes (arches of variable width) by meridian planes: if the meridian profile is shaped according to the hanging chain defined by the load conditions, the assumption of zero hoop forces — usually considered different from zero in the context of membrane theory — is justified.

Among the 19th century contributions, the method of stability areas, originally introduced by Durand-Claye [1867; 1880] for masonry arches is central for our purposes. Other contributions in the same century attempted to account for hoop forces, such as [Lamé and Clapeyron 1823; Navier 1839, pp. 155–157; Lévy 1888, pp. 42–52]. In particular, Lévy observed that hoop stresses are compressive in the upper part of the dome, specifically above a meridian point named the *point neutre*, which Lévy determined. Hoop stresses would instead become tensile in the lower part of the dome, under the aforementioned point, and are disregarded, since tensile stresses are considered inadmissible. For our purposes, other interesting methods were aimed at translating the membrane stress solution through graphical statics (Eddy's and Wolfe's procedures [Eddy 1877; Wolfe 1921]). It is interesting to observe that membrane theory allows for hoop forces to be considered so as to determine statically admissible solutions. However, it is unable to determine the actual solution, unless constitutive relations are considered.

In the 20th century assessment methods were introduced and developed based on the new discoveries of structural mechanics and advanced numerical techniques. Without pretending to be exhaustive, we cite the contributions based on linear and nonlinear FEMs, discrete element codes, 3D thrust network analysis, and several other computational procedures [Block and Ochsendorf 2007; Block and Lachauer 2014; D'Ayala and Tomasoni 2011; Simon and Bagi 2016; Varma and Ghosh 2016; Tempesta et al. 2015b; Beatini et al. 2018]. By way of example, in [Tempesta et al. 2015b] the structural analysis of polygonal masonry domes is performed by means of a numerical procedure, according to which the dome is modeled as a discrete system of rigid blocks linked through elastic mortar layers, able to describe the actual behaviour of the material. Another interesting analysis is presented in [Beatini et al. 2018], where the *Project Chrono* software has been employed to show the role of friction on the stability of masonry domes, which are envisioned as composed of blocks in dry contact. The analysis highlights the impact of the bond pattern and the blocks' aspect ratio on their capacity to equilibrate the hoop stress. With reference to these complex models, it seems worth noting that some require knowing a large number of mechanical and geometrical parameters, which are generally uncertain and difficult to determine experimentally.

Because of these operating problems, some authors have revisited simple historical models. In [Heyman 1977; Oppenheim et al. 1989] it is assumed that the dome is composed of a finite number of adjacent lunes. Equilibrium is guaranteed as long as the thrust line lies inside the dome meridian profile. The thrust line does not necessarily have to be coincident with the middle surface, as instead assumed within classical membrane theory. These authors carry out the equilibrium analysis by considering the single "arches" constituting the dome and determining the collapse condition in terms of the minimum dome thickness.

More recent contributions [Bacigalupo et al. 2015; Pavlovic et al. 2016] also adopt the same approach by imposing equilibrium on each dome lune. Furthermore, some graphical methods for assessing dome stability have been reevaluated. For example, the historical contributions [Lévy 1888; Eddy 1877; Wolfe 1921] have been reexamined in the light of numerical calculations to obtain the admissible thrust lines [Rapallini and Tempesta 1997; Lau 2006; Zessin et al. 2010; Tempesta et al. 2015a]. Other works, such as [Foraboschi 2014], underline the fundamental role of geometry, as well as the importance of considering the architectural design of the time in conducting any structural analysis. Some interesting recent contributions focus on oval domes, such as that of the Pisa cathedral. Santiago Huerta [2007] illustrates the origin and application of the oval shape in historical architecture, and addresses the mechanics response of oval domes by using the Rankine's theorem. Simon and Bagi [2016] aim at finding the minimum thickness of oval domes loaded by their own weight, by simulating numerically their collapse mechanisms by means of the discrete element code 3DEC. Finally, Beatini et al. [2018] studied the collapse of oval domes, adopting a nonsmooth contact dynamic approach.

Our contribution falls within that context. Its aim is to discover the application potential of an interesting historical method, namely the Durand-Claye method, which turns out to be quite able to provide effective working tools for structural analysis, as will be explained in the following sections.

2.2. ***A revised version of Durand-Claye's method for masonry domes.*** As already recalled, Durand-Claye extends his stability area method — originally conceived for masonry arches — in order to assess the equilibrium of domes of revolution [Durand-Claye 1880]. In particular, the method illustrated in [Durand-Claye 1867] for masonry arches can also be applied to each single lune composing a dome by determining the stability area under the hypotheses of limited compressive and tensile strength.

As described by one of the authors in [Foce and Aita 2003], Durand-Claye's original method examines the equilibrium of a symmetric arch, by considering an ideal voussoir comprised between the crown joint c_0d_0 and the generic joint cd. The stability area allows for identifying all the admissible crown thrusts with respect to equilibrium and masonry strength [Durand-Claye 1867]. In particular, Durand-Claye assumes a nil tensile strength, and finite values for the compressive strength and friction coefficient. It is worthy to point out that, when the compressive strength is infinite and the tensile strength is nil, the

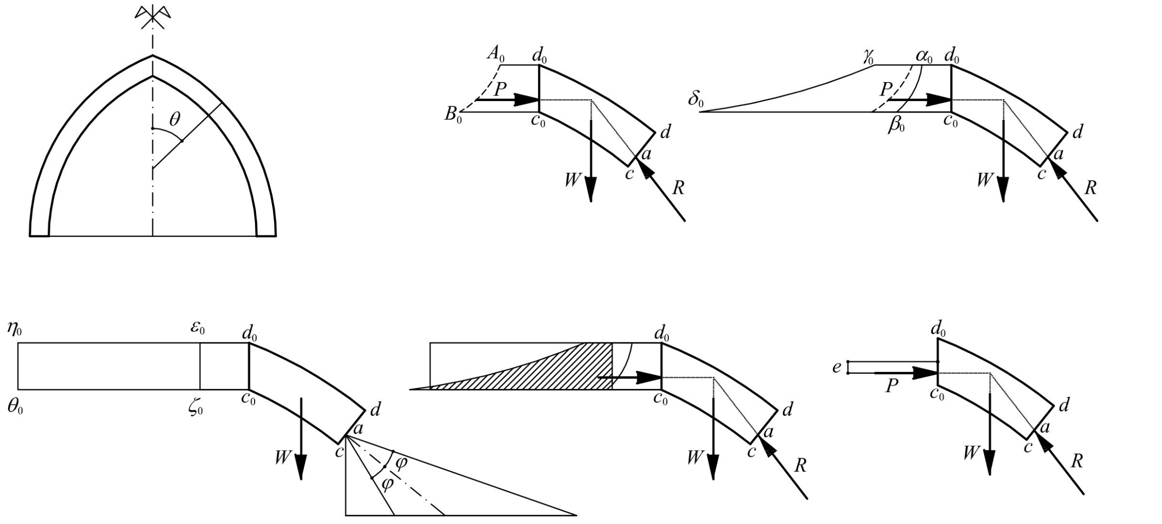

Figure 1. Equilibrium conditions according to the Durand-Claye method.

stability area obtained by Durand-Claye is an effective representation of Coulomb's [1776] method of *maxima and minima*.

The original formulation proposed by Durand-Claye consists in a graphical construction. He considers a symmetric arch (Figure 1, top left) and examines the ideal voussoir comprised between the crown joint c_0d_0, and a generic joint cd (Figure 1, top middle). He assumes that the line of action of the resultant R that makes equilibrium to the horizontal thrust, P, and the weight W of the voussoir passes through the point a of cd. By varying the application point of the crown thrust, P, from c_0 to d_0, whereas point a is fixed, the statically admissible thrusts will then be measured by horizontal segments drawn from the crown joint c_0d_0 to line A_0B_0; for a coinciding with c and d, the two lines $\alpha_0\beta_0$ and $\gamma_0\delta_0$ are similarly obtained (Figure 1, top right), so that the area $\alpha_0\beta_0\gamma_0\delta_0$, contains the extremes of the segments measuring the admissible thrusts compatible with the rotational equilibrium of the voussoir. In order to prevent sliding along cd, it is sufficient to draw at point a the friction cone, defined by the friction angle φ (Figure 1, bottom left). If the weight W is ideally shifted at point a, the *minimum* and *maximum* horizontal thrusts, i.e. the thrusts values which make the resultant of internal actions coincide with the boundaries of the friction cone, can be determined by considering the triangle of forces plotted in Figure 1 (bottom left) starting from point a. Hence, the extremes of the segments representing admissible thrusts at the crown section will lie between the two vertical lines $\varepsilon_0\zeta_0$ and $\eta_0\theta_0$.

To assure both the voussoir rotational and translational equilibrium, the extremes of admissible thrusts must be comprised within the intersection of the area $\alpha_0\beta_0\gamma_0\delta_0$ with the area $\varepsilon_0\zeta_0\eta_0\theta_0$ comprised between two vertical lines, i.e. within the dashed area in Figure 1 (bottom middle). By repeating this procedure for each joint, Durand-Claye finds the area within which the extremes of the statically admissible thrusts must be included. Thus, each point within the stability area corresponds to a statically admissible *line of thrust*, under the assumptions above. When such an area shrinks to a point (or a segment) the arch attains a limit equilibrium condition, i.e., there is only one admissible value left for the crown thrust.

The main innovation of Durand-Claye's method, as pointed out in [Aita et al. 2015; 2016; Foce and Aita 2003], concerns the introduction of finite values for the compressive (and shear) strength. By denoting as e the eccentricity of the crown thrust, P, with respect to the centre of the crown section (Figure 1, bottom right), the area of stability is obtained by considering the region formed by all the points of coordinates (P, e) that fulfil the equilibrium equations of any arch portion, as well as the limitations imposed by the masonry bounded compressive and tensile strength. For the sake of brevity, the description of this final step of the procedure is here omitted; the interested reader is referred, for example, to [Foce and Aita 2003] or some previous works by the authors, where a modern formulation of Durand-Claye's method is used for determining the collapse modes of symmetric masonry arches of different shape [Aita et al. 2015; 2016; 2019a]. In those contributions, the graphical constructions proposed by Durand-Claye are translated in terms of numerical operations on internal forces.

We recall that in the present paper we assume infinite compressive strength and an infinite friction coefficient: for this reason, issues concerning the strength of the material are not considered.

According to Durand-Claye's method when infinite friction is assumed, whenever the stability area reduces to a single point, the arch is in a limit condition and a kinematically admissible mechanism is triggered in the arch. Durand-Claye, in his contribution published in 1880, points out that his method, originally conceived for masonry arches, can be suitably modified to assess the stability of domes of revolution. Durand-Claye starts his reasoning by considering the single lunes composing the dome (Figure 2).

With reference to Figure 2 (left), drawn from the original [Durand-Claye 1880], he considers a lune comprised between two meridional planes forming a small angle, dθ (Figure 2, right). Then, he examines the equilibrium of the voussoir comprised between segments $\alpha\beta$, ab, after expressing its weight by means of Guldin's theorem. The resultant reaction force acting on the arch joint ab (Figure 2, left) is obtained by imposing the equilibrium of the lune upper portion. Furthermore, the crosssectional bending moment and shear capacities are determined as functions of the masonry compressive and tensile strengths and friction coefficient along the joints.

As observed in [Aita et al. 2017a; 2019a; 2019b], the extension of Durand-Claye's method to domes presents some critical issues.

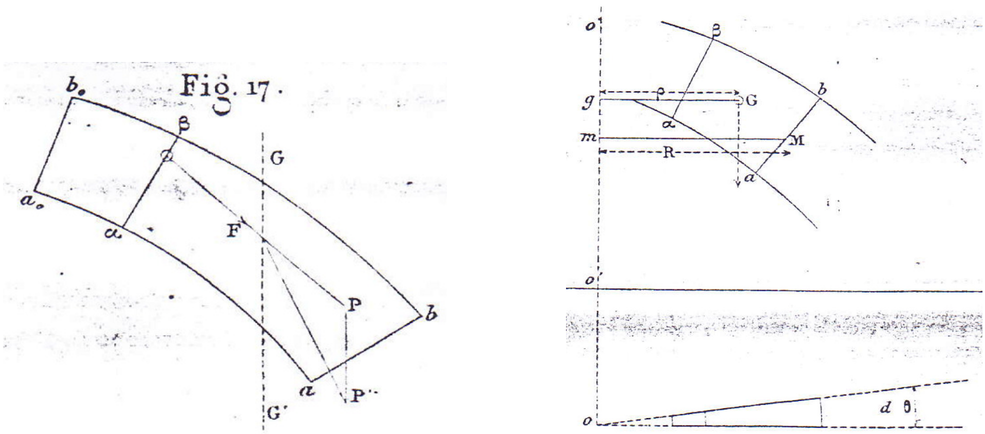

Figure 2. The extension of Durand-Claye's method to the equilibrium of a single lune. Taken from [Durand-Claye 1880].

A first caution concerns the assumption of limited compressive strength: under that assumption the crown thrust would vanish, since the crown's width is nil (see Figure 2, right). A second remark regards the procedure adopted by Durand-Claye. By reasoning in terms of equilibrium of a single lune, the limit condition identified by the vanishing of the stability area might not correspond to a collapse condition in some cases, i.e., when a kinematically admissible collapse mechanism for the entire dome cannot be determined. A third *caveat* regards the analysis of hoop forces proposed by Durand-Claye. He considers only the hoop forces acting within the *anneau supérieur* (the upper part of the dome). However, he assumes that their effect would always be counterproductive by observing that they produce a further reduction of the stability area, since they cannot exceed the limit imposed by the finite compressive strength. As is evident, this approach does not take into consideration the well-known beneficial effects that compressive hoop forces have on system stability, as will be clarified in the following.

Our reinterpretation of Durand-Claye's method aims at assessing the stability of domes by translating the complex graphical construction into a suitable set of equations in terms of the internal forces, namely: the axial force, N, and bending moment, M. Furthermore, the method has been reedited in order to adequately address the abovementioned issues.

This remainder of the section provides some remarks on Durand-Claye's method for the particular case in which an infinite friction coefficient is assumed.

Let us consider a portion of a dome of revolution having the profile shown in Figure 3 (left); it is comprised between two meridian planes forming a "small" angle ζ. A pointed profile with constant thickness, resembling that of the Pisa dome, has been chosen for the example. The meaning of the symbols is as follows: R is the radius of the circular line of axis, β the crown angle, related to the inclination of the line of axis at the crown section, h the constant thickness of the dome profile in the radial direction, l the intrados span, θ the inclination of a generic joint with respect to the vertical axis; moreover, P is the crown thrust, and e its eccentricity with respect to the crown joint's center of gravity.

Firstly, the stability area (Figure 3, right) is determined for a single lune by assuming nil hoop stresses, according to the procedure illustrated in [Aita et al. 2017a]. As in [Aita et al. 2015; 2016], Durand-Claye's method is extended here to domes by accounting for a nonlinear stress distribution in both tension and compression.

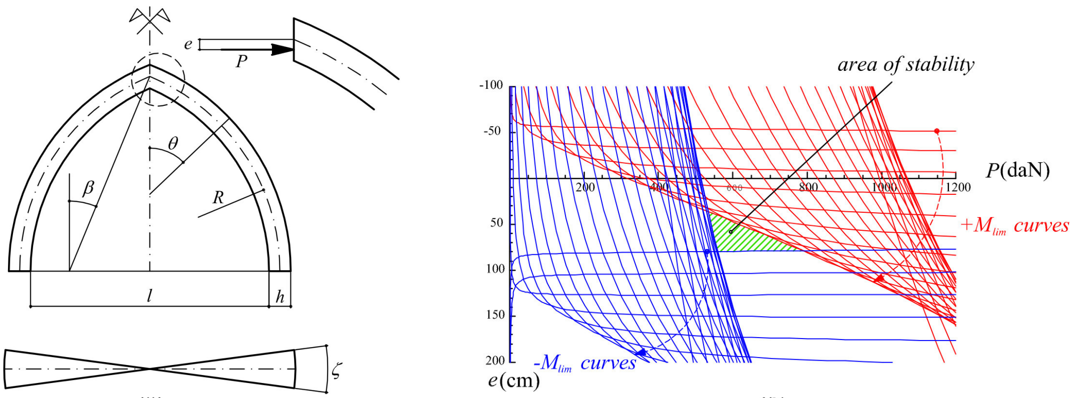

Figure 3. Left: a pointed dome profile. Right: the area of stability associated with a dome's lune.

Without going into detail (the interested reader is referred to [Aita et al. 2017a]), it is enough to recall that a first set of limitations for the crown thrust arises from the bounded bending moment capacity at the cross-sectional level. By indicating with P and e the crown thrust and its eccentricity with respect to the crown joint's center of gravity, respectively, equilibrium conditions regarding the portion between the crown and any given joint θ along the arch enable obtaining the formal expressions for the axial force, $N(P,\theta)$, and bending moment, $M(P,e,\theta)$, for any θ.

At each joint, the bending moment is limited in absolute value by a limit threshold, M_{lim}, which depends on the local value of the axial force N and masonry's bounded compressive and tensile strengths, σ_c and σ_t, as well. This limit bending moment corresponds to a birectangular distribution of stresses along the joint. In the case of nil tensile strength and infinite compressive strength, as assumed in the following applications, this limit value is given by $M_{\text{lim}} = |Nh/2|$, corresponding to the well-known assumptions proposed in [Coulomb 1776] and revisited in [Heyman 1966]. A second set of limitations would follow from the bounded shear capacity at the cross-sectional level. However, these are not taken into consideration herein, as the friction coefficient is assumed to be infinite.

The curves of the (P,e) plane defined implicitly by equations $|M(P,e,\theta)| - M_{\text{lim}}(P,\theta) = 0$, for θ kept constant, correspond to the locus of points where the bending moment reaches its limit value — whether it be positive or negative. Figure 1 (top middle) shows a plot for each joint θ of the red and blue curves corresponding to the attainment of a positive or negative limit bending moment, denoted respectively as $-M_{\text{lim}}$ and $+M_{\text{lim}}$ curves. In the (P,e) plane, the stability area (the green hatched area in Figure 3, right), bounded by the aforementioned limit curves, is defined as the locus of the points (P,e) corresponding to statically admissible solutions under the hypothesis of nil hoop forces, i.e. for the single lune.

The following describes the reinterpretation of the classical Durand-Claye stability area method developed by the authors in order to account for the presence of hoop forces.

Let us choose the dome thickness, h, as a significant parameter in order to define a limit condition. Firstly note that, if all interaction between adjacent lunes is disregarded, by decreasing h, the stability area for each lune would reduce progressively to a single point in the (P,e) plane. Thus, the corresponding values of crown thrust and eccentricity define a collapse condition for the lune itself, since this point would correspond to an equilibrated solution as well as a kinematically admissible mechanism, which could develop according to one of the two modes shown in Figure 4 [Aita et al. 2015; 2016]. As observed

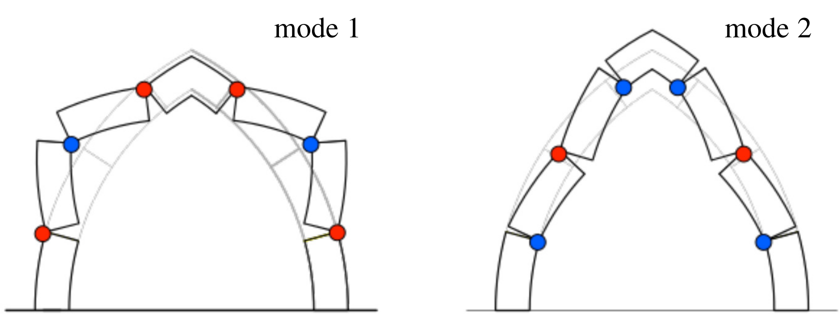

Figure 4. Rotational collapse. Left: mode 1, kinematically admissible for the entire dome. Right: mode 2, not admissible for the entire dome.

in [Heyman 1977], although each lune is considered independently, the collapse mechanism must be a kinematically admissible mechanism for the dome considered as a whole. In this case, only mode 1 is admissible: a central region near the crown of the dome merely descends vertically, while adjacent lunes move apart in the region comprised between the two red hinges (Figure 4, left). On the other hand it is an easy matter to conclude that collapse mode 2 cannot be considered a kinematically compatible mechanism, as it would require some interpenetration between adjacent lunes. In this case, the lateral surfaces of each lune, in particular their portions comprised between the two blue hinges (Figure 4, right) act as constraints, by preventing motion towards inside.

For the reasons reported above, a first significant modification to Durand-Claye's original method is to be made: searching for the limit dome thickness value, h_{lim}, under the assumption that only collapse mode 1 can occur [Heyman 1977; Oppenheim et al. 1989]. To this end, an iterative method has been expressly developed by means of the Mathematica software package. Moreover, as will be explained in more detail in Section 3.2, a second crucial modification to the original Durand-Claye method will be required regarding the action of hoop forces.

3. The case study: the Pisa cathedral dome

The dome of the Pisa cathedral can be dated back as early as the 12th century. It has an elliptical plan and a pointed profile (Figure 5). It is extremely interesting in that it represents quite an original solution, from an architectural as well as a structural point of view [Aita et al. 2017b; Sanpaolesi 1959].

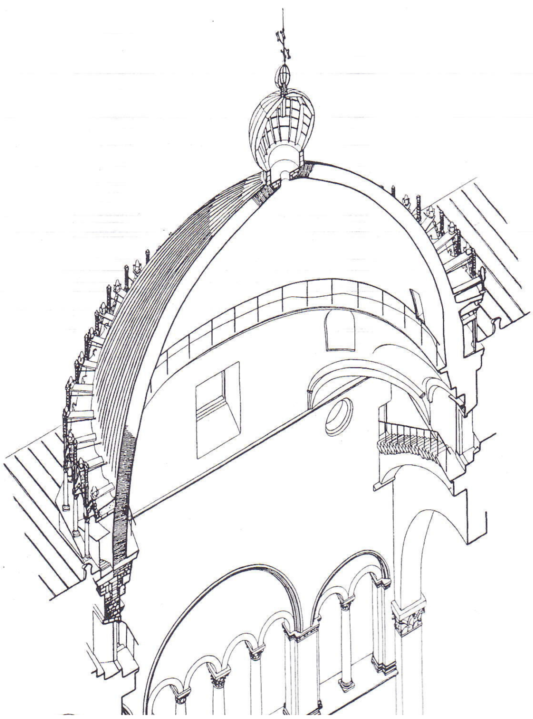

Figure 5. Axonometric view from the survey promoted by the *Istituto di Restauro dei Monumenti* (Florence) [Università di Firenze. Istituto di restauro dei monumenti 1970].

The first issue addressed as preliminary to the structural analysis was to define regular surfaces that allow for a sufficiently simple analytical representation, while at the same time being able to closely approximate the actual shape of the dome intrados and extrados. The results yielded by a geometrical survey of the intrados surface have shown that the meridian sections are approximately circular arcs, and that the oval horizontal sections are well described by ellipses [Aita et al. 2017b]. Hence, the analytical expression for the regular surface chosen to fit the intrados points coordinates was built under the assumption that each horizontal section was an ellipse, while the two vertical sections along the major and minor axes were pointed circular arcs.

The structural analysis of the dome has been performed through formulation of the semianalytical model sketched out in Section 2.2, which will be described in more detail in Sections 3.1 and 3.2. A basic input for this model consists of the representation of the ideal surfaces describing the intrados and extrados geometry, which can be obtained by appropriately processing the output of the survey of the dome (some results of which are reported in [Bennati et al. 2018] and [Aita et al. 2017b]). For the purposes of this study, the analytical representation of the intrados surface proposed in [Barsi 2017] have been taken as a starting point. More in detail, as described in [Bennati et al. 2018; Aita et al. 2017b; Barsi 2017], starting from the laser scanner survey, a postprocessing procedure has been adopted, that yields the dome's intrados and extrados horizontal sections in CAD format from which a representative subset of points has been selected. The coordinates of these points have been used as input data to find the analytically determined best approximating surfaces, described by relatively simple expressions in terms of a few parameters. The search for the parameters' optimal values has been performed by means of a simplified automatic procedure, which makes use of a minimization routine in Mathematica. The results provided by the survey allow for concluding that a fully adequate representation of the extrados surface can be obtained by assuming a conventional, constant dome thickness, h, in the radial direction.

3.1. ***A first assessment of the dome's stability.*** Stability of the Pisa cathedral oval plan dome is addressed through Rankine's theorem of transformation of structures [Rankine 1958; Gross 1913]. In brief, the theorem states that if a masonry structure is in equilibrium, that is, if at least one thrust surface is fully contained within the masonry, an affine projection of the structure will also be in equilibrium, and the new thrust surface will be the affine projection of the original. As pointed out in [Huerta 2010], this theorem can be applied to any masonry structure, provided that its compressive strength and friction coefficient are unbounded.

By taking into account the aforementioned theorem, the degree of safety exhibited by the Pisa cathedral dome has been assessed by studying the stability of two ideal domes of revolution whose profiles coincide with those of the actual dome that lie in the two vertical meridian planes passing respectively through the major and minor axes of its elliptical plan (these are the two symmetry planes of the actual dome).

Figure 6 (top left) shows the first dome profile, namely the one in the meridian plane containing the major axis. It presents a crown angle $\beta = 0.405\,\text{rad}$, and an intrados span $l = 16.30\,\text{m}$. The second dome under examination, corresponding to the profile along the minor axis, presents a crown angle $\beta = 0.550\,\text{rad}$, and an intrados span $l = 14\,\text{m}$. A small amplitude of the lunes ($\zeta = 1^\circ$) and a constant thickness, $h = 0.60\,\text{m}$, in the radial direction have been considered for both domes of revolution.

In the numerical application, we have assumed an infinite friction coefficient, nil tensile strength, infinite compressive strength, and a specific weight $\gamma = 20\,\text{kN/m}^3$. In passing, note that very similar

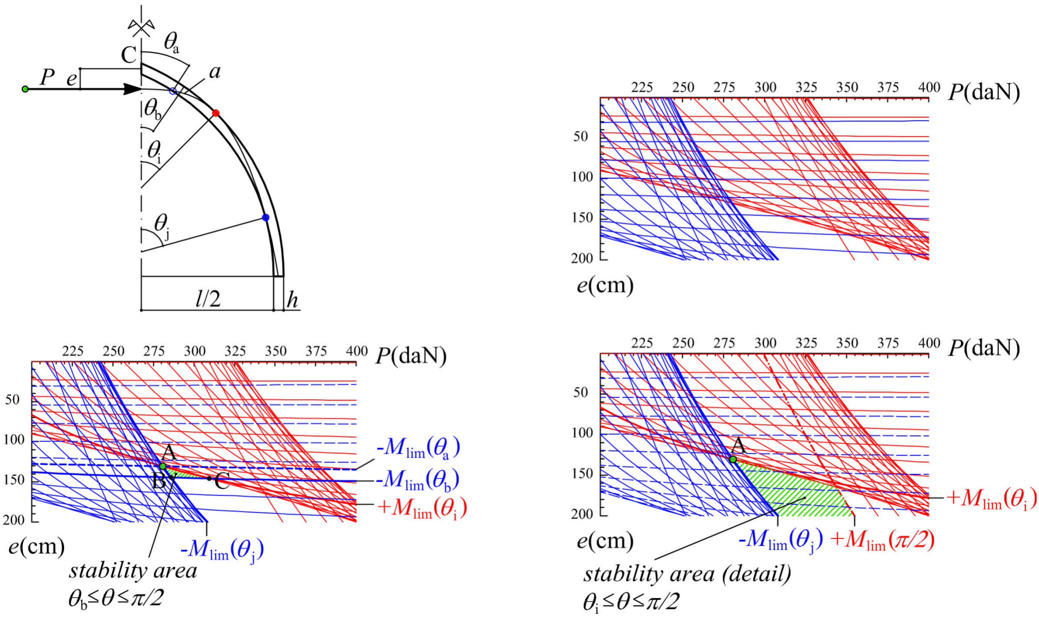

Figure 6. The meridian profile of the Pisa cathedral dome (top left); the limit curves $-M_{\text{lim}}$ and $+M_{\text{lim}}$ in the (P, e) plane, when all (top right) or a selected subset (bottom) of joints are considered.

results would have been obtained by assuming a limited compressive strength for masonry, since this has only a weak influence on the mechanical behaviour of domes subjected to their own weight alone, as the one considered here. In a forthcoming study, the actual masonry mechanical properties will be more accurately taken into account.

As a first step, our analysis begins by assuming that each lune that ideally constitutes the dome can be considered as an independent arch, for which the curves $-M_{\text{lim}}$ and $+M_{\text{lim}}$ in the (P, e) plane are plotted. The problem is solved by means of an expressly developed, in-house algorithm implemented in Mathematica. The limit curves are drawn for a finite number of joints in the radial direction, obtained by subdividing the intrados profile into 60 equal parts. The results thus obtained show that no point in the (P, e) plane could correspond to a statically admissible solution for such a lune (in other terms, the stability area is the empty set, Figure 6, top right). The minimum thickness — corresponding to a unique, statically admissible solution for the lune — would be equal to 1.20 m, which would be twice its actual value. However, what is more important is that at that thickness the lune would undergo collapse according to mode 2 (Figure 4, right), which, as pointed out in the preceding section, is unphysical for the entire dome, since it would require some material interpenetration, as explained in Section 2.2.

Such a clearly unacceptable result shows that it is necessary to account for some distribution of hoop forces to properly assess the dome's stability. To this end, we proceed to the second step of the analysis.

In this stage, the stability area method is effectively modified so as to check for lune stability under the restriction that only collapse mode 1 can occur. Accordingly, we admit the presence of hoop forces in the upper part of the dome only.

The search for a statically admissible solution for the lune is then performed by splitting the problem in two: first, a statically admissible solution is determined for the lower part of the lune, where no hoop forces are allowed, and then static admissibility is restored in the upper part by adding a suitable set of hoop forces.

As per the problem regarding the lune lower part, only a selected subset is considered in the (P, e) plane of all the limit curves that would be required by the standard Durand-Claye's method. This is to say, when the stability area for the single lune turns out to be the empty set, as in the present case (Figure 6, top right), the analysis is performed by progressively disregarding the $\pm M_{\mathrm{lim}}$ curves corresponding to the joints near the crown. The procedure enables finding the inclination $\theta = \theta_{\mathrm{a}}$ (Figure 6, top left), in such a way that the stability area related to the joints $\theta_{\mathrm{a}} \leq \theta \leq \pi/2$ is a single point (point A in Figure 6, bottom left). Note that this limit condition would correspond to a hinge at the intrados of joint θ_{a} and the activation of collapse mode 2 for the lune portion between this joint and the springing (Figure 4, right).

The coordinates $(P_{\mathrm{A}}, e_{\mathrm{A}})$ of point A are determined by means of an expressly developed procedure, which operates on the intersection points (P, e) between a suitable set of $+M_{\mathrm{lim}}$ and $-M_{\mathrm{lim}}$ curves. Furthermore, the two joints θ_{i} and θ_{j} also are determined. For the case at hand, the coordinates of point A are $P_A = 2.8\,\mathrm{kN}$, $e_{\mathrm{A}} = 1.302\,\mathrm{m}$, while the two joints where a limit condition is attained are $\theta_{\mathrm{i}} = 45.84°$ (positive limit bending moment) and $\theta_{\mathrm{j}} = 74.54°$ (negative limit bending moment). It is worth observing that the corresponding thrust line (Figure 6, top left) touches the arch extrados and intrados at the same joints θ_{i} and θ_{j}. However, in the upper portion of the lune (above joint θ_{a}) the thrust line exits from the intrados.

After determining point A, as well as joints θ_{i} and θ_{j}, an extended stability area is found by limiting the analysis to the arch portion that extends from the joint at inclination θ_{i} to the springing ($\theta = \pi/2$), in order to check if a collapse mechanism corresponding to mode 1 can occur. This stability area is shown in Figure 6 (bottom right) for the dome under examination (the green hatched area): the dashed blue $-M_{\mathrm{lim}}$ curves corresponding to the joints above θ_{i} are excluded from the analysis. In the present case, the area is extended. Hence, an infinite number of statically admissible solutions become available for the lune portion in the range $\theta_{\mathrm{i}} \leq \theta \leq \pi/2$, and the actual value of h is larger than that allowing for activation of a type 1 collapse mechanism.

By progressively decreasing the value of the thickness, h, a limit value, h^*, can be found, so as to shrink this new stability area back to a single point again. With reference to Figure 6 (bottom right), in the limit case in which the dash-dot red $+M_{\mathrm{lim}}$ curve corresponding to $\theta = \pi/2$ passes through point A, three hinges are formed, consistent with collapse mode 1. Thus, the limit value h^* is compatible with a mode 1 collapse mechanism that is kinematically admissible for the entire dome.

The procedure described above assures that, although the set of internal forces is statically admissible for the bottom part of the lune (defined by $\theta_{\mathrm{i}} \leq \theta \leq \pi/2$), static admissibility is not verified in the upper portion. Some hoop forces are then needed above joint θ_{i} to obtain fully statically admissible solutions for the entire dome, as will be described in the following section.

3.2. ***Assessment of the dome's stability in the presence of hoop forces.*** In order to find statically admissible solutions for the entire lune, the action of hoop forces in the upper part of the dome will now be accounted for. Hoop forces have been introduced in many contributions, both historical and recent (by way of example, see [Lévy 1888; Eddy 1877; Wolfe 1921; Lau 2006; Zessin et al. 2010]). Roughly

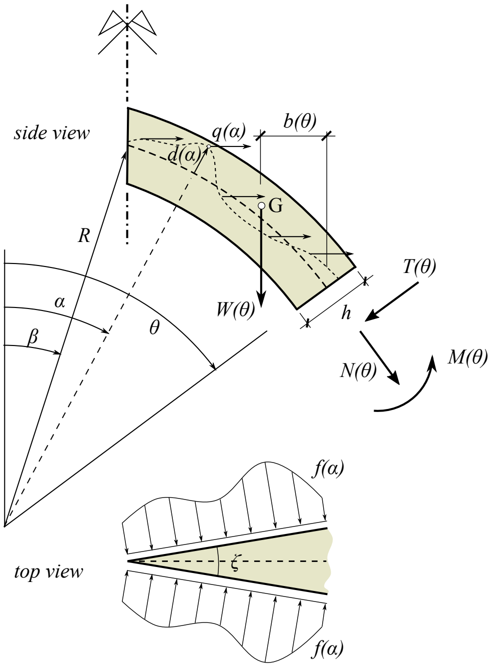

Figure 7. The hoop forces per unit length along the meridian that are exerted on both sides of the lune.

speaking, the presence of hoop forces may be interpreted, as far as lune equilibrium is concerned, as an additional loading term. As schematically represented in Figure 5, the hoop forces per unit length along the meridian, $f(\alpha)$, that are exerted on both sides of the lune result in the horizontal distributed load $q(\alpha)$, possibly acting at a distance $d(\alpha)$ from the middle surface. This additional loading term changes the shape of the thrust line and possibly brings it within the dome thickness.

In order to illustrate the procedure developed to assess dome stability, it turns out to be useful to start by writing the simple, explicit expressions for the internal forces along the lune. With reference to Figure 7, the equilibrium of the lune upper portion, between the crown and any generic joint at inclination θ, is considered. Simple calculations yield

$$N(\theta) = -W(\theta)\sin\theta - \int_\beta^\theta q(\alpha) R\cos\theta \, d\alpha,$$

$$M(\theta) = -b(\theta)W(\theta) + \int_\beta^\theta q(\alpha)[(R + d(\alpha))\cos\alpha - R\cos\theta] R \, d\alpha,$$

where $q(\alpha) = 2 f(\alpha)\sin(\zeta/2)$ is the horizontal distributed load due to hoop forces, $W(\theta)$ is the weight of the portion above θ, and b is the distance between its center of gravity, G, and the centroid of the end joint, measured along the horizontal.

In order to assure that the lune is in equilibrium and the internal forces are statically admissible, a suitable distribution $q(\alpha)$, $d(\alpha)$ has to be found so that $N(\theta) \le 0$ and $|M(\theta)| \le M_{\lim}$ at any joint.

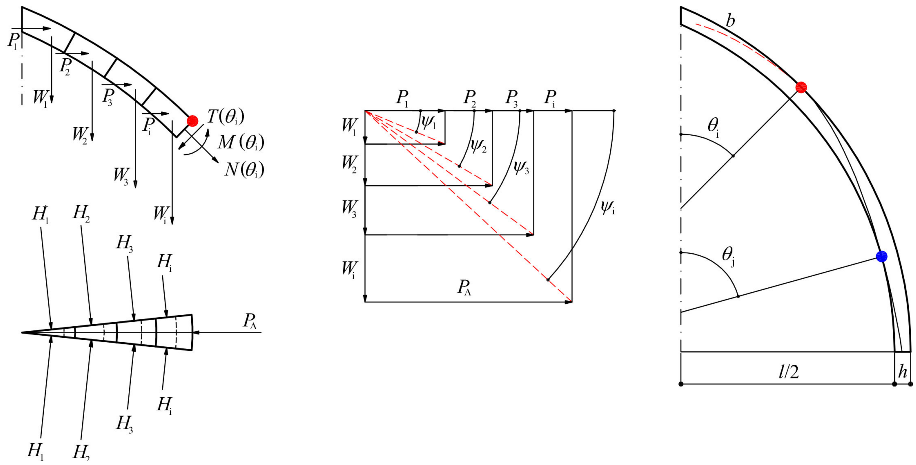

Figure 8. The influence of hoop forces on the portion of the dome's lune above joint θ_i.

Let us reconsider the thrust line a (Figure 6, top left) corresponding to point A of the stability area plotted in Figure 6 (bottom right). As already observed, in the absence of hoop forces, the internal forces would not be statically admissible within the part of the dome's lune above joint θ_a. Hence, we look for some suitable values for the horizontal load q distributed along the upper part of the lune, that is, above joint θ_i, where the positive limit bending moment is attained (see the red hinge in Figure 6, top left). The problem is solved numerically by subdividing this portion into a finite number of voussoirs and by approximating q with a set of concentrated forces (Figure 8). The admissibility conditions, $N(\theta) \leq 0$ and $|M(\theta)| \leq M_{\text{lim}}$, are enforced only at the joints separating adjacent voussoirs. Moreover, we add the two end conditions

$$M(\theta_i) = M_{\text{lim}}, \qquad \int_{\beta}^{\theta_i} q(\alpha) R \, d\alpha = P_A, \tag{1}$$

to ensure continuity of the internal forces between the upper and lower parts of the lune.

This problem is easily seen to have multiple solutions; generally, there are infinitely many distributions of q that yield statically admissible internal forces. For our purposes, it is enough to determine one among these solutions.

The lune upper portion, above joint θ_i, is subdivided into a finite number of voussoirs (in Figure 8, left, for example, four have been plotted). The weights $W_1, W_2, \ldots, W_i$ of each voussoir act, respectively, at the corresponding centers of gravity. A set of pairs of hoop forces, $H_1, H_2, \ldots, H_i$, act on the lateral surfaces of the voussoirs. The resultants of each pair of hoop forces are, respectively, the horizontal forces $P_1, P_2, \ldots, P_i$.

The equilibrium of each portion of the lune above joint θ_i imposes that the resultant's direction at each joint θ_k be given by the angle ψ_k, namely $\tan \psi_k = (W_1 + W_2 + \cdots + W_k)/(P_1 + P_2 + \cdots + P_k)$ (Figure 8, middle). The internal forces N and M can be easily written for each joint θ_k. At joint θ_i the two end conditions (1) are imposed, since N and M are known.

In this way, by assigning a set of arbitrary values to both the horizontal forces $P_1, P_2, \ldots, P_i$ and the coordinates of their application points, the corresponding thrust line can be found by calculating the ratio M/N at each joint. Via a trial-and-error process many (in principle, an infinite number of) different statically admissible internal force distributions can be found for the portion of the dome above joint θ_i, each corresponding to a different thrust surface. Thus, the thrust line a plotted in Figure 6 (top left) can be moved to within the dome thickness (by way of example, line b in Figure 8 (right) represents one such admissible thrust surface).

Via this procedure, statically admissible solutions are determined for each of the two theoretical revolution domes described in Section 3.1, whose profiles correspond to those along the vertical planes containing the major and minor axis of the actual elliptical plan, respectively. This first result suggests that the dome of Pisa cathedral can be considered to be a safe structure.

3.3. ***A safety factor for the Pisa cathedral dome.*** In order to define a geometrical safety factor for the dome of the Pisa cathedral, let us consider once again the two theoretical domes of revolution derived from the actual one (see Section 3.1). By decreasing the value of thickness h, while keeping the intrados profile fixed, the procedure described in sections 2.2 and 3.1 allows for determining the limit thicknesses. In particular, for decreasing h, we check the extension of the stability area for the lune lower part (the hatched green area in Figure 6, bottom right), and we look for the thickness that reduces the stability area to a single point on the (P, e) plane. The limit thicknesses, $h_1^* = 29.8$ cm for the first dome, and $h_2^* = 24.5$ cm for the second one, have been determined in this way. When the thickness is set equal to $h_1^* = 29.8$ cm, at least one statically admissible internal forces distribution can be determined for the lower part of both the lunes under consideration. Moreover, by applying the method described in the preceding section, it can be verified that suitable hoop force distributions are found, each corresponding to statically admissible internal forces within the upper part of the lunes, as well. Hence, the ratio $\nu = h/h_{\text{lim}} = 2$, where $h_{\text{lim}} = h_1^*$, can be taken as an estimate of a geometric safety factor for the dome under its own weight. In this case, by assuming the same notation as in Section 3.1, the limit condition — compatible with a kinematically admissible mechanism — occurs when $P = 1.54$ kN, $e = 1.731$ m. The limit bending moment is attained at three joints: $\theta_i = 50.26°$ (positive limit bending moment), $\theta_j = 74.54°$ (negative limit bending moment), and $\theta_k = 90°$ (positive bending moment).

As already observed at the beginning of Section 3.1, the results obtained for the two domes of revolution considered in this section could be reinterpreted within the framework of Rankine's theorem of transformation of structures [Rankine 1958; Gross 1913], recalled at the beginning of Section 3.1.

For our purposes, the affine transformation to consider is that which maps the middle surface of the larger of the two ideal domes of revolution to that of the actual Pisa cathedral dome. The dome thusly obtained would be of the same shape as the actual one, though its thickness would be slightly variable. In particular, along the major axis of the elliptical plan, the thickness of the projected dome will coincide with that of the actual one, while along the minor axis, the new thickness will be a bit smaller, equal to $60 \times 0.86 = 52$ cm. The projected dome, whose safety factor is still equal to two, would be wholly contained within the actual one, thus suggesting that the dome of Pisa cathedral is a safe masonry structure.

A last remark concerns the role of friction. Since in the present analysis sliding was disregarded, i.e., an infinite friction coefficient was assumed, it is interesting to get a first estimate of the minimum friction coefficient, μ, able to prevent sliding. More precisely, by considering the circular plan dome of limit thickness, $h_{\text{lim}} = h_1^*$ examined above, the ratio $|T/N|$ has been determined between the shear force, T, and the axial force, N, acting at each joint. In the lower part of the dome where hoop forces are set to zero, $\theta_i \leq \theta \leq \theta_k$, $|T/N|$ is easily calculated. Its maximum value is equal to 0.1714 and it is attained at the springing ($\theta = 90°$). In the upper part of the dome, hoop forces are present and an infinite number of different statically admissible distributions of internal forces are allowed at collapse. By considering one of these statically admissible solutions (obtained via a trial and error process, analogous to that reported in Figure 8 and omitted here for the sake of brevity), the maximum value of $|T/N|$ results equal to 0.1723 and it is attained at the joint $\theta = 25.97°$. Hence, if a friction coefficient $\mu > 0.1723$ is assumed, sliding is prevented and only pure rotational collapse modes can occur. This value of μ is low compared the ones likely to be observed in the actual masonry. It is worth observing that the result just obtained concerning the friction coefficient is strictly valid for the circular-plan dome of limit thickness whose intrados surface is circumscribed to the actual one. However, since the ratio between the minor and major axis of the actual elliptic-plan dome is equal to 0.86, not too far from unity, and since the minimum friction coefficient required to prevent sliding is very low, it is reasonable to assume that sliding can be disregarded in the actual dome too.

Further insights on these issues will be provided in a forthcoming paper.

4. Concluding remarks

The present contribution is part of a multidisciplinary research project aimed at studying the structural behaviour of the dome of the Pisa cathedral. In particular, it reports on a preliminary study of the dome's mechanical response by means of a modern translation of Durand-Claye's method.

With reference to domes of revolution, the original Durand-Claye's is suitably modified in order to account for the interaction between the dome's lunes, in terms of kinematic compatibility and hoop force transmission.

The peculiar elliptical plan of the dome of the Pisa cathedral makes it necessary to further modify the method in order to apply it to the present case study. Some first considerations on this issue are advanced by evaluating the stability of two theoretical domes of revolution obtained by considering the profile of the actual dome.

The method allows for determining statically admissible solutions for the two theoretical domes of revolution derived from the Pisa cathedral dome, as well as the limit thickness for both. By considering the ratio between the actual and the limit thicknesses, an estimate of the geometric safety factor for the dome under its own weight has been determined. It turns out to be far larger than the unity, thus suggesting, as expected, that an ample safety factor may be assumed for the dome of the Pisa cathedral under permanent loads.

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