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A NEW ANALYTICAL APPROACH FOR SOLVING EQUATIONS OF ELASTO-HYDRODYNAMICS IN QUASICRYSTALS

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The dynamic equations for quasicrystals are written as time-dependent partial differential equations of the second order relative to phonon and phason displacements. In these equations phonons describe the dynamics of wave propagation and phasons describe diffusion process in quasicrystals. A new approach for deriving a solution (phonon and phason displacements) of the initial value problem is proposed. In this approach the Fourier transform with respect to 3D space variable of the given phonon, phason forces and initial displacements are assumed to be vector functions with components which have finite supports with respect to Fourier parameters for every fixed time variable. The equations for the Fourier images of displacements are reduced to a vector integral equation of the Volterra-type depending on Fourier parameters. The solution of the obtained vector integral equation is solved by successive approximations. Finally, phonon and phason displacements are derived by matrix transformations and the inverse Fourier transform to the solution of the vector integral equation.

1. Introduction

The icosahedral quasicrystal structure was discovered in Al-Mn alloys [Shechtman et al. 1984]. After that quasicrystals (QCs) have become the focus of theoretical and experimental studies [Levine et al. 1985; Wang et al. 1987; De and Pelcovits 1987; Ding et al. 1993; Ovidko 1992; Edagawa and So 2007; Fan and Fan 2008; Li et al. 2009]. The properties of quasicrystalline materials are surprising and could be remarkably useful. Most of these properties combine effectively to give technologically interesting applications which have been protected recently by several patents [van Blaaderen 2009; Dubois 2000]. For instance, the combination of such properties as high hardness, low friction and corrosive resistance of quasicrystals gives almost ideal materials for motor-car engines. The application of QCs in motor-car engines would be undoubtedly result in reduced air pollution and increased engine lifetimes. The same set of associated properties (hardness, low friction, corrosive resistance) combined with biocompatibility is also very promising for introducing QCs in surgical applications as parts used for bone repair and prosthetic applications [van Blaaderen 2009; Dubois 2005]. The description of the dynamical processes in quasicrystals essentially depends on processes which describe phonons and phasons. For example, in [Gao and Zhao 2006; Rochal and Lorman 2002] both phonons and phasons describe the dynamics of the wave propagation in quasicrystals. There is another opinion, given in [Levine et al. 1985; Li and Fan 2016], that phonons describe the dynamics of wave propagation and phasons describe diffusion. The computation of the Green's function and the theoretical study of the existence and uniqueness of solutions of dynamical differential equations, describing the dynamics of the wave propagation in quasicrystals

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with the general structure of anisotropy without diffusion, has been done in [Yakhno and Çerdik Yaslan 2011a; Yakhno and Çerdik Yaslan 2011b; Çerdik Yaslan 2013; Çerdik Yaslan 2019]. Several interesting examples of solving problems for equations of elasto-hydrodynamics when phonon and phason displacements do not depend on one space variable have been described in [Fan et al. 2009; Li and Fan 2016]: the problem of a moving screw dislocation in an icosahedral quasicrystal was studied by a combination of the perturbation method and variational method in [Fan et al. 2009]; a general solution of equations of elasto-hydrodynamics in two-dimensional quasicrystals was derived in [Li 2011]; the dynamic crack problem in three dimensional icosahedral quasicrystals was solved by finite difference method in [Fan 2013; Li and Fan 2016]. We note that the derivation and computation of phonon and phason displacements by solving initial value and initial boundary value problems for dynamic equations in QCs with the general structure of anisotropy, when phonons describe the dynamics of wave propagation and phasons describe diffusion, have not been developed so far.

In the present paper a new analytical approach for derivation of a solution (phonon and phason displacements) of the initial value problem for differential equations of elasto-hydrodynamics in QCs with the general structure of anisotropy is described. In this approach the Fourier transform with respect to 3D space variable of the given phonon and phason forces and initial displacements are assumed to be vector functions with components which have finite supports with respect to Fourier parameters for every fixed time variable. The equations for the Fourier images of displacements are reduced to a vector integral equation of the Volterra type depending on Fourier parameters. The solution of the obtained vector integral equation is solved by successive approximations. Finally, phonon and phason displacements are derived by matrix transformations and the inverse Fourier transform to the solution of the vector integral equation.

2. Basic equations of elasto-hydrodynamics for quasicrystals

2A. ***Hook's law.*** Let us consider a quasicrystal (QC) with three dimensional quasiperiodic structure. Let $x = (x_1, x_2, x_3) \in R^3$ be a space variable, $t \in R$ be a time variable. According to the generalized elasticity theory of 3D QCs [Ding et al. 1993; Hu et al. 2000], the equations of the deformation are

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad \omega_{kl} = \frac{\partial w_k}{\partial x_l}, \quad k, l = 1, 2, 3, \tag{2-1}$$

and the generalized Hooke's law of three dimensional quasicrystals is given by

$$\sigma_{ij} = \sum_{k,l=1}^{3} (C_{ijkl}\varepsilon_{kl} + R_{ijkl}\omega_{kl}), \quad H_{ij} = \sum_{k,l=1}^{3} (R_{klij}\varepsilon_{kl} + K_{ijkl}\omega_{kl}). \tag{2-2}$$

Here $i, j = 1, 2, 3$, u_i and w_i are components of phonon and phason displacements $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$; σ_{ij} and H_{ij} are phonon and phason stresses, $\varepsilon_{ij}(x, t)$, $\omega_{ij}(x, t)$ are phonon and phason strains, respectively.

C_{ijkl} represent the phonon elastic modules (phonon elastic properties of QC), K_{ijkl} are the phason elastic modules (phason elastic properties of CQ), R_{ijkl} are the phonon-phason coupling elastic modules (phonon-phason coupling properties of QC). The following properties for the interchange of indices are satisfied (see, for example, [Hu et al. 2000]):

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \quad K_{ijkl} = K_{klij}, \quad R_{ijkl} = R_{jikl}. \tag{2-3}$$

We assume that these properties are satisfied. Moreover the positivity of the elastic strain energy density implies that for any strains ε_{ij} and ω_{ij} that are not zero entirely, the following inequalities are satisfied (see [Hu et al. 2000] and Appendix A):

$$\sum_{j,l,i,k=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0, \qquad \sum_{j,l,i,k=1}^{3} K_{ijkl}\omega_{ij}\omega_{kl} > 0. \tag{2-4}$$

2B. ***Dynamic equations of elasto-hydrodynamics for quasicrystals.*** According to the arguments of Lubebskey et al. [1985] (see also [Fan et al. 2009; Fan 2013; Li and Fan 2016]) phason modes in quasicrystals correspond to diffusion and for phonon modes Newton's second law must be fulfilled. Hence we have

$$\rho\frac{\partial^2 u_i(x,t)}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial \sigma_{ij}(x,t)}{\partial x_j} + f_i(x,t), \tag{2-5}$$

$$\kappa\frac{\partial w_i(x,t)}{\partial t} = \sum_{j=1}^{3}\frac{\partial H_{ij}(x,t)}{\partial x_j} + g_i(x,t), \quad i=1,2,3,\ x\in R^3,\ t\in R, \tag{2-6}$$

where $\rho > 0$ is the mass density of QC; $\kappa = 1/\Gamma_w$ is the diffusion coefficient, Γ_w is the kinetic coefficient of the phason field, describing the relaxation of the motion; $f_i(x,t)$ and $g_i(x,t)$, $i = 1, 2, 3$ are components of body forces $\boldsymbol{f}$ (phonon) and $\boldsymbol{g}$ (phason), respectively; u_i and w_i are components of phonon and phason displacements $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$; σ_{ij} and H_{ij} are phonon and phason stresses, which used in (2-1) and (2-2). Using (2-2), equations (2-5), (2-6) can be presented in the form

$$\rho\frac{\partial^2 u_i(x,t)}{\partial t^2} = \sum_{j,k,l=1}^{3} C_{ijkl}\frac{\partial^2 u_k(x,t)}{\partial x_j \partial x_l} + \sum_{j,k,l=1}^{3} R_{ijkl}\frac{\partial^2 w_k(x,t)}{\partial x_j \partial x_l} + f_i(x,t), \tag{2-7}$$

$$\kappa\frac{\partial w_i(x,t)}{\partial t} = \sum_{j,k,l=1}^{3} R_{klij}\frac{\partial^2 u_k(x,t)}{\partial x_j \partial x_l} + \sum_{j,k,l=1}^{3} K_{ijkl}\frac{\partial^2 w_k(x,t)}{\partial x_j \partial x_l} + g_i(x,t). \tag{2-8}$$

Equations (2-7), (2-8) can be written as vector equations in the following form:

$$\rho\frac{\partial^2 \boldsymbol{u}(x,t)}{\partial t^2} = \sum_{j,l=1}^{3} \mathcal{C}_{jl}\frac{\partial^2 \boldsymbol{u}(x,t)}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3} \mathcal{R}_{jl}\frac{\partial^2 \boldsymbol{w}(x,t)}{\partial x_j \partial x_l} + \boldsymbol{f}(x,t), \tag{2-9}$$

$$\kappa\frac{\partial \boldsymbol{w}(x,t)}{\partial t} = \sum_{j,l=1}^{3} \mathcal{K}_{jl}\frac{\partial^2 \boldsymbol{w}(x,t)}{\partial x_j \partial x_l} + \sum_{j,l=1}^{3} \mathcal{R}^T_{jl}\frac{\partial^2 \boldsymbol{u}(x,t)}{\partial x_j \partial x_l} + \boldsymbol{g}(x,t), \tag{2-10}$$

where

$$\mathcal{C}_{jl}=\frac{1}{2}\begin{bmatrix}C_{1j1l}+C_{1l1j} & C_{1j2l}+C_{1l2j} & C_{1j3l}+C_{1l3j}\\ C_{2j1l}+C_{2l1j} & C_{2j2l}+C_{2l2j} & C_{2j3l}+C_{2l3j}\\ C_{3j1l}+C_{3l1j} & C_{3j2l}+C_{3l2j} & C_{3j3l}+C_{3l3j}\end{bmatrix}, \tag{2-11}$$

$$\mathcal{R}_{jl}=\frac{1}{2}\begin{bmatrix}R_{1j1l}+R_{1l1j} & R_{1j2l}+R_{1l2j} & R_{1j3l}+R_{1l3j}\\ R_{2j1l}+R_{2l1j} & R_{2j2l}+R_{2l2j} & R_{2j3l}+R_{2l3j}\\ R_{3j1l}+R_{3l1j} & R_{3j2l}+R_{3l2j} & R_{3j3l}+R_{3l3j},\end{bmatrix}, \tag{2-12}$$

$$\mathcal{K}_{jl}=\frac{1}{2}\begin{bmatrix}K_{1j1l}+K_{1l1j} & K_{1j2l}+K_{1l2j} & K_{1j3l}+K_{1l3j}\\ K_{2j1l}+K_{2l1j} & K_{2j2l}+K_{2l2j} & K_{2j3l}+K_{2l3j}\\ K_{3j1l}+K_{3l1j} & K_{3j2l}+K_{3l2j} & K_{3j3l}+K_{3l3j}\end{bmatrix}, \tag{2-13}$$

$\mathcal{R}_{jl}^T$ is the matrix transpose of $\mathcal{R}_{jl}$.

3. Initial value problem (IVP) for dynamic equations of elasto-hydrodynamics for quasicrystals

Let us consider the problem of finding vector functions $\boldsymbol{u}$, $\boldsymbol{w}$ satisfying (2-9), (2-10) and the following initial conditions:

$$\boldsymbol{u}(x,0)=\boldsymbol{u}^0(x),\qquad \frac{\partial}{\partial t}\boldsymbol{u}(x,t)|_{t=0}=\boldsymbol{u}^1(x), \tag{3-1}$$

$$\boldsymbol{w}(x,0)=\boldsymbol{w}^0(x), \tag{3-2}$$

where $\boldsymbol{u}^0(x)$, $\boldsymbol{u}^1(x)$, $\boldsymbol{w}^0(x)$, $\boldsymbol{f}(x,t)$, $\boldsymbol{g}(x,t)$ are given vector functions with three components depending of x and x,t, respectively; $\mathcal{C}_{jl}$, $\mathcal{R}_{jl}$, $\mathcal{R}_{jl}^T$, $\mathcal{K}_{jl}$ are matrices given by (2-11)–(2-13).

We assume that C_{ijkl}, R_{ijkl}, K_{ijkl}, appearing in (2-11)–(2-13), satisfy (2-3) and (2-4).

Remark 3.1. Note that the initial data (3-1), (3-2), where

$$\boldsymbol{u}^0(x)=0,\quad \boldsymbol{u}^1(x)=0,\quad \boldsymbol{w}^0(x)=0,$$

describe the fact that there are no vibrations and there are no sources of perturbations at the initial moment of time. The initial data (3-1), (3-2) have a physical interpretation of the disturbance at the initial instant of time. Moreover these initial data together with phonon and phason forces $\boldsymbol{f}(x,t)$ and $\boldsymbol{g}(x,t)$ are equivalent to the external forces whose densities are described by (see, for example, [Vladimirov 1971, pp. 172–174, 197–198])

$$\vec{F}=\boldsymbol{u}^0(x)\delta'(t)+\boldsymbol{u}^1(x)\delta(t)+\theta(t)\boldsymbol{f}(x,t),\qquad \vec{G}=\boldsymbol{w}^0(x)\delta(t)+\theta(t)\boldsymbol{g}(x,t),$$

where $\delta(t)$ is the Dirac delta function, $\delta'(t)$ is the derivative of $\delta(t)$, $\theta(t)$ is the Heaviside step function (the discontinuous function whose values are zero for negative arguments and are equal to one for nonnegative arguments).

3A. *IVP in terms of the Fourier transform with respect to space variables.*

Let

$$\tilde{\boldsymbol{f}}(\nu,t)=(\tilde f_1(\nu,t),\tilde f_2(\nu,t),\tilde f_3(\nu,t)),\qquad \tilde{\boldsymbol{g}}(\nu,t)=(\tilde g_1(\nu,t),\tilde g_2(\nu,t),\tilde g_3(\nu,t)),$$

be the Fourier images of $\boldsymbol{f}(x,t)$, $\boldsymbol{g}(x,t)$ with respect to $x=(x_1,x_2,x_3)\in R^3$, i.e.,

$$\tilde{f}_j(\nu,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_j(x,t)e^{ix\cdot\nu}\,dx_1\,dx_2\,dx_3,$$
$$\tilde{g}_j(\nu,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_j(x,t)e^{ix\cdot\nu}\,dx_1\,dx_2\,dx_3,$$

where

$$\nu=(\nu_1,\nu_2,\nu_3)\in R^3,\quad x\cdot\nu=x_1\nu_1+x_2\nu_2+x_3\nu_3,\quad i^2=-1,\quad j=1,2,3.$$

Let

$$\tilde{\boldsymbol{u}}^0(\nu)=(\tilde{u}_1^0(\nu),\tilde{u}_2^0(\nu),\tilde{u}_3^0(\nu)),\quad \tilde{\boldsymbol{u}}^1(\nu)=(\tilde{u}_1^1(\nu),\tilde{u}_2^1(\nu),\tilde{u}_3^1(\nu)),\quad \tilde{\boldsymbol{w}}^0(\nu)=(\tilde{w}_1^0(\nu),\tilde{w}_2^0(\nu),\tilde{w}_3^0(\nu))$$

be the Fourier transform of $\boldsymbol{u}^0(x,t)$, $\boldsymbol{u}^1(x,t)$, $\boldsymbol{w}^0(x,t)$ and functions

$$\tilde{u}_j^0(\nu),\,\tilde{u}_j^1(\nu),\,\tilde{w}_j^0(\nu),\,\tilde{f}_j(\nu,t),\,\tilde{g}_j(\nu,t)\quad j=1,2,3$$

have finite supports with respect to ν for every fixed $t>0$.

Problem (2-9), (2-10) can be written in terms of the Fourier transform images as follows

$$\rho\frac{d^2\tilde{\boldsymbol{u}}(\nu,t)}{dt^2}+\mathcal{C}(\nu)\tilde{\boldsymbol{u}}(\nu,t)+\mathcal{R}(\nu)\tilde{\boldsymbol{w}}(\nu,t)=\tilde{\boldsymbol{f}}(\nu,t),\quad t>0, \tag{3-3}$$
$$\kappa\frac{d\tilde{\boldsymbol{w}}(\nu,t)}{dt}+\mathcal{K}(\nu)\tilde{\boldsymbol{w}}(\nu,t)+\mathcal{R}^T(\nu)\tilde{\boldsymbol{u}}(\nu,t)=\tilde{\boldsymbol{g}}(\nu,t),\quad t>0, \tag{3-4}$$
$$\tilde{\boldsymbol{u}}(\nu,0)=\tilde{\boldsymbol{u}}^0(\nu),\quad \frac{d}{dt}\tilde{\boldsymbol{u}}(\nu,t)|_{t=0}=\tilde{\boldsymbol{u}}^1(\nu), \tag{3-5}$$
$$\tilde{\boldsymbol{w}}(\nu,0)=\tilde{\boldsymbol{w}}^0(\nu). \tag{3-6}$$

Here

$$\mathcal{C}(\nu)=\sum_{j,l=1}^{3}\mathcal{C}_{jl}\nu_j\nu_l, \tag{3-7}$$
$$\mathcal{K}(\nu)=\sum_{j,l=1}^{3}\mathcal{K}_{jl}\nu_j\nu_l, \tag{3-8}$$
$$\mathcal{R}(\nu)=\sum_{j,l=1}^{3}\mathcal{R}_{jl}\nu_j\nu_l. \tag{3-9}$$

Remark 3.2. We note that $\mathcal{C}(0)=\mathcal{K}(0)=\mathcal{R}(0)=0$. Taking into account the properties (2-3) and (2-4) we find that $\mathcal{C}(\nu)$, $\mathcal{K}(\nu)$ are symmetric positive definite matrices for $\nu\neq 0$ (see Appendix B). Moreover a solution of (3-3)–(3-6) for $\nu=0$ is given by the following formulas:

$$\tilde{\boldsymbol{u}}(0,t)=\tilde{\boldsymbol{u}}^0(0)+\tilde{\boldsymbol{u}}^1(0)t+\frac{1}{\rho}\int_0^t(t-\tau)\tilde{\boldsymbol{f}}(0,\tau)\,d\tau,\quad \tilde{\boldsymbol{w}}(0,t)=\tilde{\boldsymbol{w}}^0(0)+\frac{1}{\kappa}\int_0^t\tilde{\boldsymbol{g}}(0,\tau)\,d\tau.$$

The construction of the solution of IVP (3-3)–(3-6) for $\nu \neq 0$ are described in the next subsections. Since matrices $\mathcal{C}(\nu)$ and $\mathcal{K}(\nu)$ are real symmetric positive definite for $\nu \neq 0$ (see Appendix B) then $\mathcal{C}(\nu)$ and $\mathcal{K}(\nu)$ are congruent to diagonal matrices of their eigenvalues. That is, there exists orthogonal matrices $\mathcal{T}(\nu)$ and $\mathcal{Z}(\nu)$ such that

$$\mathcal{T}^{-1}(\nu)\mathcal{C}(\nu)\mathcal{T}(\nu) = \mathcal{D}(\nu), \quad \mathcal{Z}^{-1}(\nu)\mathcal{K}(\nu)\mathcal{Z}(\nu) = \mathcal{E}(\nu), \tag{3-10}$$

where $\mathcal{T}^{-1}(\nu)$, $\mathcal{Z}^{-1}(\nu)$ are the inverse matrices to $\mathcal{T}(\nu)$, $\mathcal{Z}(\nu)$; $\mathcal{D}(\nu) = \mathrm{diag}(d_1(\nu), d_2(\nu), d_3(\nu))$, $\mathcal{E}(\nu) = \mathrm{diag}(e_1(\nu), e_2(\nu), e_3(\nu))$ are diagonal matrices with real valued positive diagonal elements for $\nu \neq 0$.

3B. ***Reduction of problem (3-3)–(3-6) to the vector integral equation of the Volterra type.*** Let further $\nu = (\nu_1, \nu_2, \nu_3) \in R^3$ and $\nu \neq 0$. Let us consider vector functions $V(\nu, t)$, $S(\nu, t)$ such that

$$\tilde{\boldsymbol{u}}(\nu, t) = \mathcal{T}(\nu)V(\nu, t), \quad \tilde{\boldsymbol{w}}(\nu, t) = \mathcal{Z}(\nu)S(\nu, t). \tag{3-11}$$

IVP (3-3)–(3-6) can be written in terms of $V(\nu, t)$, $S(\nu, t)$ vector functions as follows:

$$\rho\mathcal{T}(\nu)\frac{d^2V(\nu, t)}{dt^2} + \mathcal{C}(\nu)\mathcal{T}(\nu)V(\nu, t) + \mathcal{R}(\nu)\mathcal{Z}(\nu)S(\nu, t) = \tilde{\boldsymbol{f}}(\nu, t), \quad t > 0, \tag{3-12}$$

$$\kappa\mathcal{Z}(\nu)\frac{dS(\nu, t)}{dt} + \mathcal{K}(\nu)\mathcal{Z}(\nu)S(\nu, t) + \mathcal{R}^T(\nu)\mathcal{T}(\nu)V(\nu, t) = \tilde{\boldsymbol{g}}(\nu, t), \quad t > 0, \tag{3-13}$$

$$\mathcal{T}(\nu)V(\nu, 0) = \tilde{\boldsymbol{u}}^0(\nu), \quad \mathcal{T}(\nu)\frac{d}{dt}V(\nu, t)|_{t=0} = \tilde{\boldsymbol{u}}^1(\nu), \tag{3-14}$$

$$\mathcal{Z}(\nu)S(\nu, 0) = \tilde{\boldsymbol{w}}^0(\nu). \tag{3-15}$$

Multiplying equalities (3-12), (3-14) by $\mathcal{T}^{-1}(\nu)$ and (3-13), (3-15) by $\mathcal{Z}^{-1}(\nu)$ and using (3-10), (3-11) we find

$$\frac{d^2V(\nu, t)}{dt^2} + \frac{1}{\rho}\mathcal{D}(\nu)V(\nu, t) + \mathcal{M}(\nu)S(\nu, t) = \tilde{\boldsymbol{h}}(\nu, t), \quad t > 0, \tag{3-16}$$

$$\frac{dS(\nu, t)}{dt} + \frac{1}{\kappa}\mathcal{E}(\nu)S(\nu, t) + \mathcal{N}(\nu)V(\nu, t) = \tilde{\boldsymbol{p}}(\nu, t), \quad t > 0, \tag{3-17}$$

$$V(\nu, 0) = V^0(\nu), \quad \frac{d}{dt}V(\nu, t)|_{t=0} = V^1(\nu), \tag{3-18}$$

$$S(\nu, 0) = S^0(\nu), \tag{3-19}$$

where matrices $\mathcal{M}(\nu)$, $\mathcal{N}(\nu)$, and vector functions $\tilde{\boldsymbol{h}}(\nu, t)$, $\tilde{\boldsymbol{p}}(\nu, t)$, $V^0(\nu)$, $V^1(\nu)$, $S^0(\nu)$ are defined by

$$\mathcal{M}(\nu) = \frac{1}{\rho}\mathcal{T}^{-1}(\nu)\mathcal{R}(\nu)\mathcal{T}(\nu), \quad \mathcal{N}(\nu) = \frac{1}{\kappa}\mathcal{Z}^{-1}(\nu)\mathcal{R}^T(\nu)\mathcal{Z}(\nu), \tag{3-20}$$

$$\tilde{\boldsymbol{h}}(\nu, t) = \frac{1}{\rho}\mathcal{T}^{-1}(\nu)\tilde{\boldsymbol{f}}(\nu, t), \quad \tilde{\boldsymbol{p}}(\nu, t) = \frac{1}{\kappa}\mathcal{Z}^{-1}(\nu)\tilde{\boldsymbol{g}}(\nu, t), \tag{3-21}$$

$$V^0(\nu) = \mathcal{T}^{-1}(\nu)\tilde{\boldsymbol{u}}^0(\nu), \quad V^1(\nu) = \mathcal{T}^{-1}(\nu)\tilde{\boldsymbol{u}}^1(\nu), \quad S^0(\nu) = \mathcal{T}^{-1}(\nu)\tilde{\boldsymbol{w}}^0(\nu). \tag{3-22}$$

Equations (3-16), (3-17) with conditions (3-18), (3-19) are equivalent to the following system of integral equations of the Volterra-type:

$$V(\nu,t)=\boldsymbol{h}^1(\nu,t)+\int_0^t(\boldsymbol{K}^1S)(\nu,t,\tau)\,d\tau \tag{3-23}$$

$$S(\nu,t)=\boldsymbol{h}^2(\nu,t)+\int_0^t(\boldsymbol{K}^2V)(\nu,t,\tau)\,d\tau, \tag{3-24}$$

where

$$\boldsymbol{h}^1(\nu,t)=\cos\Big(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)\,t\Big)\,V^0(\nu)+\sqrt{\rho}\mathcal{D}^{-1/2}(\nu)\,\sin\Big(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)\,t\Big)\,V^1(\nu)$$
$$+\sqrt{\rho}\int_0^t\mathcal{D}^{-1/2}(\nu)\sin\Big(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)\,(t-\tau)\Big)\tilde{\boldsymbol{h}}(\nu,\tau)\,d\tau, \tag{3-25}$$

$$\boldsymbol{h}^2(\nu,t)=\exp\Big(-\frac{1}{\kappa}\mathcal{E}(\nu)\,t\Big)S^0(\nu)+\int_0^t\exp\Big(-\frac{1}{\kappa}\mathcal{E}(\nu)\,(t-\tau)\Big)\tilde{\boldsymbol{p}}(\nu,\tau)\,d\tau, \tag{3-26}$$

$$(\boldsymbol{K}^1S)(\nu,t,\tau)=-\sqrt{\rho}\mathcal{D}^{-1/2}(\nu)\sin\Big(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)\,(t-\tau)\Big)\mathcal{M}(\nu)S(\nu,\tau), \tag{3-27}$$

$$(\boldsymbol{K}^2V)(\nu,t,\tau)=-\exp\Big(-\frac{\mathcal{E}(\nu)}{\kappa}\,(t-\tau)\Big)\mathcal{N}(\nu)V(\nu,\tau). \tag{3-28}$$

The system of integral equations (3-23), (3-24) can be written in the form of one vector integral equation as follows:

$$\boldsymbol{U}(\nu,t)=\boldsymbol{h}(\nu,t)+\int_0^t(\boldsymbol{K}\boldsymbol{U})(\nu,t,\tau)\,d\tau, \tag{3-29}$$

where $\boldsymbol{h}(\nu,t)$ is the vector function with six components whose the first three components are components of $\boldsymbol{h}^1(\nu,t)$ and the last three components are components of $\boldsymbol{h}^2(\nu,t)$ ($\boldsymbol{h}^1(\nu,t)$ and $\boldsymbol{h}^2(\nu,t)$ are defined by (3-25), (3-26)); $\boldsymbol{U}(\nu,t)=(V(\nu,t),S(\nu,t))$ is the vector function with six components whose the first three components are components of $V(\nu,t)$ and the last three components are components of $S(\nu,t)$; $\boldsymbol{K}$ is the vector operator with 6 components defined by the following formula

$$(\boldsymbol{K}\boldsymbol{U})(\nu,t,\tau)=((\boldsymbol{K}^1S)(\nu,t,\tau),(\boldsymbol{K}^2V)(\nu,t,\tau)) \tag{3-30}$$

for any vector function $\boldsymbol{U}(\nu,t)=(V(\nu,t),S(\nu,t))$ with six components. Here the vector operators $(\boldsymbol{K}^1S)(\nu,t,\tau)$ and $(\boldsymbol{K}^2V)(\nu,t,\tau)$ are determined by (3-27), (3-28).

3C. ***Properties of the kernel of the vector integral equation.*** The operator $\boldsymbol{K}$, defined by (3-27), (3-28), (3-30), satisfies the following properties.

Property 3.3. Let T, ω, Ω be given positive constants ($\omega\le\Omega$);

$$\Delta(T,\omega,\Omega)=\{(\nu,t)\mid \nu=(\nu_1,\nu_2,\nu_3)\in R^3,\ \omega\le|\nu|\le\Omega,\ 0\le t\le T\};$$

$V(\nu,t)=(V_1(\nu,t),V_2(\nu,t),V_3(\nu,t))$, $S(\nu,t)=(S_1(\nu,t),S_2(\nu,t),S_3(\nu,t))$ be vector functions such that $V_j(\nu,t)$, $S_j(\nu,t)$, $j=1,2,3$ are continuous in the region $\Delta(T,\Omega)$; $\boldsymbol{U}(\nu,t)=(V(\nu,t),S(\nu,t))$

be the vector function whose the first three components are components of $V(\nu, t)$, and the last three components are components of $S(\nu, t)$. Then all components of the vector function

$$\int_0^t (\boldsymbol{K}\boldsymbol{U})(\nu, t, \tau)\, d\tau$$

are continuous on $\Delta(T, \omega, \Omega)$ and the first three components of them are twice continuously differentiable on $\Delta(T, \omega, \Omega)$ and the last three components are one time continuously differentiable on $\Delta(T, \omega, \Omega)$.

Property 3.4. Let T, ω, Ω be given positive constants; $\Delta(T, \omega, \Omega)$, $V(\nu, t) = (V_1(\nu, t), V_2(\nu, t), V_3(\nu, t))$, $S(\nu, t) = (S_1(\nu, t), S_2(\nu, t), S_3(\nu, t))$, $\boldsymbol{U}(\nu, t) = (V(\nu, t), S(\nu, t))$ be the region and vector functions defined in Property 3.3; $(\boldsymbol{K}^1 S)_j(\nu, t, \tau)$, $(\boldsymbol{K}^2 S)_j(\nu, t, \tau)$, $j = 1, 2, 3$ be components of vector functions $(\boldsymbol{K}^1 S)(\nu, t, \tau)$ and $(\boldsymbol{K}^2 V)(\nu, t, \tau)$ defined by (3-27), (3-28). Then for any $(\nu, t) \in \Delta(T, \omega, \Omega)$ and $0 \le \tau \le t$ the following inequalities are satisfied:

$$|(\boldsymbol{K}^1 S)_j(\nu, t, \tau)| \le \frac{(t-\tau)}{\rho}\, \Omega^2\, \Upsilon\, \|\boldsymbol{U}(\nu, \tau)\|_2, \tag{3-31}$$

$$|(\boldsymbol{K}^2 V)_j(\nu, t, \tau)| \le \frac{1}{\kappa}\, \Omega^2\, \Upsilon\, \|\boldsymbol{U}(\nu, \tau)\|_2. \tag{3-32}$$

Here the positive constants ρ and κ are introduced in the beginning of Section 2B (after (2-5), (2-6)); the constant Υ is defined by the equality

$$\Upsilon = \sum_{j,l=1}^{3} \|\mathcal{R}_{jl}\|_2, \tag{3-33}$$

where $\|\mathcal{R}_{jl}\|_2$ is the operator norm of the matrix $\mathcal{R}_{jl}$ defined by (2-12),

$$\|\boldsymbol{U}(\nu, \tau)\|_2 = \sqrt{\|V(\nu, \tau)\|_2^2 + \|S(\nu, \tau)\|_2^2},$$
$$\|V(\nu, \tau)\|_2^2 = V_1^2(\nu, \tau) + V_2^2(\nu, \tau) + V_3^2(\nu, \tau),$$
$$\|S(\nu, \tau)\|_2^2 = S_1^2(\nu, \tau) + S_2^2(\nu, \tau) + S_3^2(\nu, \tau).$$

Proof. Let $\mathcal{T}(\nu)$, $\mathcal{Z}(\nu)$ be orthogonal matrices and $\mathcal{D}(\nu)$, $\mathcal{E}(\nu)$ be diagonal matrices with positive diagonal elements for $\nu \ne 0$ defined by (3-10), and $\mathcal{M}(\nu)$, $\mathcal{N}(\nu)$ be matrices defined by (3-20). Then, applying properties of the operator norm of matrix, we find the following inequalities:

$$\left\| \sqrt{\rho}\mathcal{D}^{-1/2}(\nu) \sin\left(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)\,(t-\tau)\right)\right\|_2 \le (t-\tau), \tag{3-34}$$

$$\left\| \exp\left(-\frac{\mathcal{E}(\nu)}{\kappa}(t-\tau)\right)\right\|_2 \le 1, \tag{3-35}$$

$$\|\mathcal{M}(\nu)\|_2 \le \frac{1}{\rho}\|\mathcal{T}^{-1}(\nu)\|_2 \|\mathcal{R}(\nu)\|_2 \|\mathcal{T}(\nu)\|_2 = \frac{1}{\rho}\|\mathcal{R}(\nu)\|_2, \tag{3-36}$$

$$\|\mathcal{N}(\nu)\|_2 \le \frac{1}{\kappa}\|\mathcal{Z}^{-1}(\nu)\|_2 \|\mathcal{R}^T(\nu)\|_2 \|\mathcal{Z}(\nu)\|_2 = \frac{1}{\kappa}\|\mathcal{R}^T(\nu)\|_2. \tag{3-37}$$

Using (3-9) we find

$$\|\mathcal{R}(\nu)\|_2 \le |\nu|^2 \sum_{j,l=1}^{3} \|\mathcal{R}_{jl}\|_2 = |\nu|^2\,\Upsilon, \tag{3-38}$$

$$\|\mathcal{R}^T(\nu)\|_2 \le |\nu|^2 \sum_{j,l=1}^{3} \|\mathcal{R}^T_{jl}\|_2 = |\nu|^2\,\Upsilon, \tag{3-39}$$

where Υ is defined by (3-33). Using (3-27), (3-28) and (3-34)–(3-39) we find (3-31), (3-32). □

3D. ***The solution of the vector integral equation (3-29) by successive approximations.*** Let T, ω, Ω be given positive constants ($\omega \le \Omega$) and $\Delta(T,\omega,\Omega)$ be the region defined in Property 3.3. We assume that vector functions $\tilde{\boldsymbol{u}}^0(\nu)$, $\tilde{\boldsymbol{u}}^1(\nu)$, $\tilde{\boldsymbol{w}}^0(\nu)$, appearing in (3-5), (3-6) are continuous for $|\nu| \le \Omega$ and vector functions $\tilde{\boldsymbol{f}}(\nu,t)$, $\tilde{\boldsymbol{g}}(\nu,t)$, appearing in (3-3), (3-4) are continuous for $|\nu| \le \Omega$, $0 \le t \le T$. Moreover vector functions $\tilde{\boldsymbol{u}}^0(\nu)$, $\tilde{\boldsymbol{u}}^1(\nu)$, $\tilde{\boldsymbol{w}}^0(\nu)$ $\tilde{\boldsymbol{f}}(\nu,t)$, $\tilde{\boldsymbol{g}}(\nu,t)$ are supposed to be zero for $|\nu| > \Omega$, $0 \le t \le T$. It follows from (3-21), (3-22) and (3-25), (3-26) that components of vector function $\boldsymbol{h}(\nu,t)$ defined after (3-29) are continuous on $\Delta(T,\Omega)$ and are equal to zero for any (ν,t) satisfying $\nu > \Omega$, $0 \le t \le T$. Moreover the first three components of $\boldsymbol{h}(\nu,t)$ are twice continuously differentiable with respect to t and the last three components are one time continuously differentiable with respect to t in the region $\Delta(T,\omega,\Omega)$.

To find a solution $\boldsymbol{U}(\nu,t)$ of (3-29) for $0 \le t \le T$ and a fixed $\nu \in R^3$, $\nu \ne 0$, we apply the successive approximations

$$\boldsymbol{U}^{(0)}(\nu,t) = \boldsymbol{h}(\nu,t), \tag{3-40}$$

$$\boldsymbol{U}^{(n)}(\nu,t) = \int_0^t (\boldsymbol{K}\boldsymbol{U}^{(n-1)})(\nu,t,\tau)\,d\tau, \quad n = 1,2,3,\ldots \tag{3-41}$$

Remark 3.5. Equalities (3-40), (3-41) can be written in a component form as follows:

$$\begin{aligned} V_j^{(0)}(\nu,t) &= H_j^1(\nu,t), & S_j^{(0)}(\nu,t) &= H_j^2(\nu,t),\\ V_j^{(n)}(\nu,t) &= \int_0^t (\boldsymbol{K}^1 S_j^{(n-1)}(\nu,\tau))\,d\tau, & S_j^{(n)}(\nu,t) &= \int_0^t (\boldsymbol{K}^2 V_j^{(n-1)}(\nu,\tau))\,d\tau,\end{aligned}$$
$$n = 1,2,3,\ldots;\ j = 1,2,3.$$

Here $\boldsymbol{u}^{(n)}(\nu,t)$ is the vector function with the following six components:

$$V_1^{(n)}(\nu,\tau),\quad V_2^{(n)}(\nu,\tau),\quad V_3^{(n)}(\nu,\tau)(\nu,t),\quad S_1^{(n)}(\nu,\tau),$$
$$S_2^{(n)}(\nu,\tau),\quad S_3^{(n)}(\nu,\tau));\quad H_j^1(\nu,t),\quad H_j^2(\nu,t),\quad j = 1,2,3$$

are components of vector functions $\boldsymbol{h}^1(\nu,t)$, $\boldsymbol{h}^2(\nu,t)$ defined by (3-23), (3-24).

Remark 3.6. Since all components of $\boldsymbol{h}(\nu,t)$ are equal to zero for $|\nu| > \Omega$, $0 \le t \le T$ then $\boldsymbol{U}^{(n)}(\nu,t) = 0$ for $|\nu| > \Omega$, $0 \le t \le T$. Moreover $\boldsymbol{U}(\nu,t) = 0$ is a solution of (3-29) for $|\nu| > \Omega$, $0 \le t \le T$.

The goal of this section is to show that for any $j = 1, 2, 3$ the series

$$\sum_{n=0}^{\infty} V_j^{(n)}(\nu, t), \qquad \sum_{n=0}^{\infty} S_j^{(n)}(\nu, t),$$

converge uniformly on $\Delta(T, \omega, \Omega)$ to some continuous functions $V_j(\nu, \tau)$, $S_j(\nu, \tau)$, $j = 1, 2, 3$, and then, if

$$V(\nu, \tau) = (V_1(\nu, \tau), V_2(\nu, \tau), V_3(\nu, \tau)), \quad S(\nu, \tau) = (S_1(\nu, \tau), S_2(\nu, \tau), S_3(\nu, \tau))$$

are vector functions with found components then the vector function $\boldsymbol{U}(\nu, t) = (V(\nu, \tau), S(\nu, \tau))$ is a solution of (3-29).

Indeed, we find from (3-40), (3-41) and the properties of Section 3C that vector functions $\boldsymbol{U}^{(n)}(\nu, t)$, $n = 0, 1, 2, 3, \dots$ have continuous components on $\Delta(T, \omega, \Omega)$ and

$$|V_j^{(n)}(\nu, t)| = \left| \int_0^t (\boldsymbol{K}^1 S_j^{(n-1)}(\nu, \tau))\, d\tau \right| \le B \int_0^t \|\boldsymbol{U}^{(n-1)}(\nu, \tau)\|_2, \tag{3-42}$$

$$|S_j^{(n)}(\nu, t)| = \left| \int_0^t (\boldsymbol{K}^2 V_j^{(n-1)}(\nu, \tau))\, d\tau \right| \le B \int_0^t \|\boldsymbol{U}^{(n-1)}(\nu, \tau)\|_2, \tag{3-43}$$

$$n = 1, 2, 3, \dots; \qquad j = 1, 2, 3,$$

where B is the constant defined by

$$B = \Omega^2 \gamma \max\left(\frac{1}{\kappa}, \frac{T}{\rho}\right). \tag{3-44}$$

Using (3-42), (3-43) we find the following inequality:

$$\|\boldsymbol{U}^{(n-1)}(\nu, t)\|_2 \le \sqrt{6} \|\boldsymbol{U}^{(n-1)}(\nu, t)\|_\infty \le \sqrt{6} B \int_0^t \|\boldsymbol{U}^{(n-2)}(\nu, \tau)\|_2\, d\tau, \tag{3-45}$$

where

$$\|\boldsymbol{U}^{(n-1)}(\nu, t)\|_\infty = \max\{ \max_{j=1,2,3} |V_j^{(n-1)}(\nu, t)|, \max_{j=1,2,3} |S_j^{(n-1)}(\nu, t)| \}.$$

We have from (3-42), (3-43), (3-45):

$$\|\boldsymbol{U}^{(n-1)}(\nu, t)\|_2 \le \frac{(\sqrt{6}\, B\, t)^{n-1}}{(n-1)!} G, \tag{3-46}$$

$$|V_j^{(n)}(\nu, t)| \le \frac{(\sqrt{6}\, B\, t)^n}{(n)!} \frac{G}{\sqrt{6}}, \tag{3-47}$$

$$|S_j^{(n)}(\nu, t)| \le \frac{(\sqrt{6}\, B\, t)^n}{(n)!} \frac{G}{\sqrt{6}}, \tag{3-48}$$

$$n = 1, 2, 3, \dots; \qquad j = 1, 2, 3,$$

where

$$G = \max_{(\nu,t) \in \Delta(T,\omega,\Omega)} \|\boldsymbol{h}(\nu, t)\|_2.$$

Using (3-46)–(3-48) and the first theorem of Weierstrass [Apostol 1961, p. 425] we find that there are continuous functions $V_j(\nu, t)$, $S_j(\nu, t)$ on the region $\Delta(T, \Omega)$ that the series

$$\sum_{n=0}^{\infty} V_j^{(n)}(\nu, t), \quad \sum_{n=0}^{\infty} S_j^{(n)}(\nu, t)$$

converge uniformly and absolutely on $\Delta(T, \omega, \Omega)$ to $V_j(\nu, t)$, $S_j(\nu, t)$, $j = 1, 2, 3$. Let

$$V(\nu, \tau) = (V_1(\nu, \tau), V_2(\nu, \tau), V_3(\nu, \tau)), \quad S(\nu, \tau) = (S_1(\nu, \tau), S_2(\nu, \tau), S_3(\nu, \tau)),$$

where $V_j(\nu, t)$ and $S_j(\nu, t)$, $j = 1, 2, 3$ are sums of above mentioned series. We want to show now that $\boldsymbol{U}(\nu, t) = (V(\nu, \tau), S(\nu, \tau))$ is a solution of (3-29). Let us consider the vector function $\boldsymbol{U}^{(n)}(\nu, t) = (V^{(n)}(\nu, \tau), S^{(n)}(\nu, \tau))$ defined by (3-40), (3-41). Summing the right and left sides of (3-41) with respect to n from 1 to N we have

$$\sum_{n=1}^{N} \boldsymbol{U}^{(n)}(\nu, t) = \sum_{n=0}^{N-1} \int_0^t (\boldsymbol{K}\boldsymbol{U}^{(n-1)})(\nu, t, \tau)\, d\tau, \tag{3-49}$$

where

$$\sum_{n=1}^{N} \boldsymbol{U}^{(n)}(\nu, t) = \left(\sum_{n=1}^{N} V^{(n)}(\nu, t),\ \sum_{n=1}^{N} S^{(n)}(\nu, t)\right),$$

$$\sum_{n=1}^{N} V^{(n)}(\nu, t) = \left(\sum_{n=1}^{N} V_1^{(n)}(\nu, t),\ \sum_{n=1}^{N} V_2^{(n)}(\nu, t),\ \sum_{n=1}^{N} V_3^{(n)}(\nu, t)\right),$$

$$\sum_{n=1}^{N} S^{(n)}(\nu, t) = \left(\sum_{n=1}^{N} S_1^{(n)}(\nu, t),\ \sum_{n=1}^{N} S_2^{(n)}(\nu, t),\ \sum_{n=1}^{N} S_3^{(n)}(\nu, t)\right).$$

Adding the vector function $\boldsymbol{h}(\nu, t)$ to both sides of (3-49) we find

$$\sum_{n=0}^{N} \boldsymbol{U}^{(n)}(\nu, t) = \boldsymbol{h}(\nu, t) + \int_0^t \left(\boldsymbol{K} \sum_{n=0}^{N-1} \boldsymbol{U}^{(n-1)}\right)(\nu, t, \tau)\, d\tau. \tag{3-50}$$

Letting N tend to infinity and using the second Weierstrass theorem [Apostol 1961, p. 426], we find that the vector function $\boldsymbol{U}(\nu, t) = (V(\nu, t), S(\nu, t))$ satisfies (3-29) for $(\nu, t) \in \Delta(T, \omega, \Omega)$. Since ω is an arbitrary positive number such that $\omega \le \Omega$ then the vector function $\boldsymbol{U}(\nu, t) = (V(\nu, t), S(\nu, t))$ has continuous components for $|\nu| \le \Omega$ $(\nu \ne 0)$, $0 \le t \le T$ and the vector function $\boldsymbol{U}(\nu, t) = (V(\nu, t), S(\nu, t))$ is a solution of (3-29) for $|\nu| \le \Omega$ $(\nu \ne 0)$, $0 \le t \le T$.

3E. ***Properties of the solution of the integral equation (3-29).*** Let T, ω, Ω be given positive constants $(\omega \le \Omega)$ and

$$\Delta(T, \Omega) = \{(\nu, t) \mid \nu = (\nu_1, \nu_2, \nu_3) \in R^3,\ \nu \ne 0,\ |\nu| \le \Omega,\ 0 \le t \le T\}.$$

The vector functions $\tilde{\boldsymbol{u}}^0(\nu)$, $\tilde{\boldsymbol{u}}^1(\nu)$, $\tilde{\boldsymbol{w}}^0(\nu)$, $\tilde{\boldsymbol{f}}(\nu, t)$, $\tilde{\boldsymbol{g}}(\nu, t)$, $\boldsymbol{h}(\nu, t)$ satisfy assumptions described in the beginning of Section 3A.

Property 3.7. Let $\boldsymbol{K}^1$, $\boldsymbol{K}^2$, $\boldsymbol{K}$ be operator defined by (3-27), (3-28), (3-30); $\boldsymbol{u}(\nu,t) = (V(\nu,t), S(\nu,t))$ be a solution of (3-29) whose components are continuous on $\Delta(T,\Omega)$. Then

$$\frac{\partial V(\nu,t)}{\partial t}, \quad \frac{\partial S(\nu,t)}{\partial t}, \quad \frac{\partial^2 V(\nu,t)}{\partial t^2}$$

are continuous on $\Delta(T,\Omega)$.

Proof. Under conditions of Property 3.7 the vector functions

$$\boldsymbol{h}^1(\nu,t), \quad \boldsymbol{h}^2(\nu,t), \quad \frac{\partial \boldsymbol{h}^1(\nu,t)}{\partial t}, \quad \frac{\partial \boldsymbol{h}^2(\nu,t)}{\partial t}, \quad \frac{\partial^2 \boldsymbol{h}^1(\nu,t)}{\partial t^2}$$

are continuous on $\Delta(T,\omega,\Omega)$ and the vector functions $V(\nu,t)$, $S(\nu,t)$ satisfy equalities (3-23), (3-24). Differentiating (3-23), (3-24) with respect to t we find

$$\begin{aligned}
\frac{\partial V(\nu,t)}{\partial t} &= \frac{\partial \boldsymbol{h}^1(\nu,t)}{\partial t} - \int_0^t \cos\Big(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)(t-\tau)\Big)\mathcal{M}(\nu)S(\nu,\tau)\,d\tau,\\
\frac{\partial^2 V(\nu,t)}{\partial t^2} &= \frac{\partial^2 \boldsymbol{h}^1(\nu,t)}{\partial t^2} + \mathcal{M}(\nu)S(\nu,t) + \int_0^t \sqrt{\rho}\mathcal{D}^{1/2}(\nu)\,\sin\Big(\frac{1}{\sqrt{\rho}}\mathcal{D}^{1/2}(\nu)(t-\tau)\Big)\mathcal{M}(\nu)S(\nu,\tau)\,d\tau,\\
\frac{\partial S(\nu,t)}{\partial t} &= \frac{\partial \boldsymbol{h}^2(\nu,t)}{\partial t} - \mathcal{N}(\nu)V(\nu,t) + \int_0^t \frac{\mathcal{E}(\nu)}{\kappa}\exp\Big(-\frac{\mathcal{E}(\nu)}{\kappa}(t-\tau)\Big)\mathcal{N}(\nu)V(\nu,\tau)\,d\tau.
\end{aligned}$$

Since the right sides of the obtained equations are continuous functions on $\Delta(T,\omega,\Omega)$ then the left sides are also continuous on $\Delta(T,\omega,\Omega)$. Therefore

$$\frac{\partial V(\nu,t)}{\partial t}, \quad \frac{\partial S(\nu,t)}{\partial t}, \quad \frac{\partial^2 V(\nu,t)}{\partial t^2}$$

are continuous on $\Delta(T,\omega,\Omega)$. Since ω is an arbitrary positive number such that $\omega \le \Omega$ then the vector functions

$$\frac{\partial V(\nu,t)}{\partial t}, \quad \frac{\partial S(\nu,t)}{\partial t}, \quad \frac{\partial^2 V(\nu,t)}{\partial t^2}$$

are continuous on $\Delta(T,\Omega)$. □

Property 3.8. Let $\boldsymbol{K}^1$, $\boldsymbol{K}^2$, $\boldsymbol{K}$ be operators defined by (3-27), (3-28), (3-30). Then the solution $\boldsymbol{U}(\nu,t) = (V(\nu,t), S(\nu,t))$ of (3-29) is unique in the class of continuous function on $\Delta(T,\Omega)$.

Proof. Let $\boldsymbol{U}(\nu,t) = (V(\nu,t), S(\nu,t))$ and $\boldsymbol{U}^*(\nu,t) = (V^*(\nu,t), S^*(\nu,t))$ be two solutions of (3-29) whose components are continuous on $\Delta(T,\Omega)$. Setting $\hat{V}(\nu,t) = V(\nu,t) - V^*(\nu,t)$, $\hat{S}(\nu,t) = S(\nu,t) - S^*(\nu,t)$, $\hat{\boldsymbol{u}}(\nu,t) = (\hat{V}(\nu,t), \hat{S}(\nu,t))$ we find from (3-29)

$$\hat{V}(\nu,t) = \int_0^t (\boldsymbol{K}^1\hat{S}(\nu,\tau))\,d\tau, \tag{3-51}$$

$$\hat{S}(\nu,t) = \int_0^t (\boldsymbol{K}^2\hat{V}(\nu,\tau))\,d\tau. \tag{3-52}$$

Using Property 3.4 we obtain from (3-51), (3-52)

$$\|\hat{\boldsymbol{u}}(\nu,t)\|_2 \le \sqrt{6}B\int_0^t \|\hat{\boldsymbol{u}}(\nu,\tau)\|_2\,d\tau, \tag{3-53}$$

where B is the constant defined by (3-44). Applying Gronwall's lemma to (3-53) we find $\|\hat{\boldsymbol{u}}(\nu,t)\|_2 = 0$ for $(\nu,t)\in\Delta(T,\Omega)$. Using the continuity of $\hat{\boldsymbol{u}}(\nu,t)$ we conclude that

$$\hat{\boldsymbol{u}}(\nu,t) = (\hat{V}(\nu,t),\hat{S}(\nu,t)) = (V(\nu,t)-V^*(\nu,t), S(\nu,t)-S^*(\nu,t)) = 0$$

for $(\nu,t)\in\Delta(T,\Omega)$. This means that $\boldsymbol{U}(\nu,t)=\boldsymbol{U}^*(\nu,t)$ for $(\nu,t)\in\Delta(T,\Omega)$. □

Property 3.9. Let $\boldsymbol{U}(\nu,t) = (V(\nu,t), S(\nu,t))$ be a solution of (3-29) for $(\nu,t)\in\Delta(T,\Omega)$. Then the vector functions $\tilde{\boldsymbol{u}}(\nu,t)$, $\tilde{\boldsymbol{w}}(\nu,t)$, with components given by (3-11), are a unique solution of the initial value problem (3-3)–(3-6) in the class of vector functions for which

$$\tilde{\boldsymbol{u}}(\nu,t),\quad \tilde{\boldsymbol{w}}(\nu,t),\quad \frac{\partial\tilde{\boldsymbol{u}}(\nu,t)}{\partial t},\quad \frac{\partial\tilde{\boldsymbol{w}}(\nu,t)}{\partial t},\quad \frac{\partial^2\tilde{\boldsymbol{u}}(\nu,t)}{\partial t^2}$$

are continuous on $\Delta(T,\Omega)$.

Proof. Using Properties 3.7 and 3.8 we conclude that the solution of integral equations (3-23), (3-24) (or the vector integral Equation (3-29)) is equivalent to the solution of the initial value problem (3-16)–(3-19). Moreover the initial value problem (3-16)–(3-19) can be written in the form of (3-3)–(3-6). Therefore if $\boldsymbol{u}(\nu,t) = (V(\nu,t), S(\nu,t))$ is a solution of (3-29) for $(\nu,t)\in\Delta(T,\Omega)$ then the vector functions $\tilde{\boldsymbol{u}}(\nu,t)$, $\tilde{\boldsymbol{w}}(\nu,t)$, with components given by (3-11), are a unique solution of the initial value problem (3-3)–(3-6) in the class of vector functions whose components

$$V(\nu,t),\quad S(\nu,t),\quad \frac{\partial V(\nu,t)}{\partial t},\quad \frac{\partial S(\nu,t)}{\partial t},\quad \frac{\partial^2 V(\nu,t)}{\partial t^2}$$

are continuous on $\Delta(T,\Omega)$. □

3F. ***The solution of initial value problem (2-9), (2-10), (3-1), (3-2).*** Let T, Ω be given positive constants and

$$\mathcal{D}(\Omega) = \{\nu = (\nu_1,\nu_2,\nu_3)\in R^3 \mid |\nu|\le\Omega\};$$

$C_\Omega(R^3)$ be a class of continuous functions whose components belong to $\mathcal{D}(\Omega)$; $PW_\Omega(R^3)$ be the class which is the inverse Fourier transform image of the class $C_\Omega(R^3)$;

Remark 3.10. $PW_\Omega(R^3)$ is called the Paley–Wiener class (see, for example, [Andersen 2004])

We assume further that conditions of Section 3A on vector functions $\tilde{\boldsymbol{u}}^0(\nu)$, $\tilde{\boldsymbol{u}}^1(\nu)$, $\tilde{\boldsymbol{w}}^0(\nu)$, $\tilde{\boldsymbol{f}}(\nu,t)$, $\tilde{\boldsymbol{g}}(\nu,t)$ are satisfied. Using Properties 3.7, 3.8 and Remark 3.5 we find that the solution $\tilde{\boldsymbol{u}}(\nu,t)$, $\tilde{\boldsymbol{w}}(\nu,t)$ of (3-3)–(3-6) has components which are continuous together with components of

$$\frac{\partial\tilde{\boldsymbol{u}}(\nu,t)}{\partial t},\quad \frac{\partial\tilde{\boldsymbol{w}}(\nu,t)}{\partial t},\quad \frac{\partial^2\tilde{\boldsymbol{u}}(\nu,t)}{\partial t^2}$$

for $(\nu,t)\in\Delta(T,\Omega)$ and equal to zero for $|\nu|>\Omega$, $[0,T]$. As a result of solving the vector integral equation (3-29) and application of Property 3.9 we determine the Fourier images of the phonon and phason displacements $\tilde{\boldsymbol{u}}(\nu,t)$, $\tilde{\boldsymbol{w}}(\nu,t)$ for $(\nu,t)\in\Delta(T,\Omega)$. The vector functions $\tilde{\boldsymbol{u}}(\nu,t)$, $\tilde{\boldsymbol{w}}(\nu,t)$ satisfy

(3-3)–(3-6). To find unknown phonon and phason displacements $\boldsymbol{u}(x,t)$, $\boldsymbol{w}(x,t)$ satisfying (2-9), (2-10), (3-1), (3-2) we need to derive the inverse Fourier transform of $\tilde{\boldsymbol{u}}(\nu,t)$ and $\tilde{\boldsymbol{w}}(\nu,t)$. Indeed, applying the inverse Fourier transform to $\tilde{\boldsymbol{u}}(\nu,t)$ and $\tilde{\boldsymbol{w}}(\nu,t)$ we find vector functions $\boldsymbol{u}(x,t)$, $\boldsymbol{w}(x,t)$. Since the components of

$$\tilde{\boldsymbol{u}}(\nu,t), \quad \tilde{\boldsymbol{w}}(\nu,t), \quad \frac{\partial\tilde{\boldsymbol{u}}(\nu,t)}{\partial t}, \quad \frac{\partial\tilde{\boldsymbol{w}}(\nu,t)}{\partial t}, \quad \frac{\partial^2\tilde{\boldsymbol{u}}(\nu,t)}{\partial t^2}$$

are continuous for $(\nu,t)\in\Delta(T,\Omega)$ and equal to zero for $|\nu|>\Omega$, $[0,T]$ then the components of $\boldsymbol{u}(x,t)$ belong to the class $C^2([0,T];PW_\Omega(R^3))$ and the components of $\boldsymbol{w}(x,t)$ belong to the class $C^1([0,T];PW_\Omega(R^3))$. Here $C^2([0,T];PW_\Omega(R^3))$ is the class of all two time continuously differentiable mappings of $[0,T]$ into $PW_\Omega(R^3))$ and $C^1([0,T];PW_\Omega(R^3))$ is the class of all one time continuously differentiable mappings of $[0,T]$ into $PW_\Omega(R^3))$. Equalities (3-3)–(3-6) can be written in terms of $\boldsymbol{u}(x,t)$ and $\boldsymbol{w}(x,t)$ as equalities (2-9), (2-10), (3-1), (3-2). This means that we find vector functions $\boldsymbol{u}(x,t)$ and $\boldsymbol{w}(x,t)$ in classes $C^2([0,T];PW_\Omega(R^3))$ and $C^1([0,T];PW_\Omega(R^3))$ which are a solution of (2-9), (2-10), (3-1), (3-2). These vector functions are phonon and phason displacements. We note that the numerical implementation of the inverse Fourier transform can be done by the technique described in [Yakhno and Çerdik Yaslan 2012, §3.5].

4. Example of solving equations of elasto-hydrodynamics in icosahedral quasicrystals

For illustration of the approach we consider the equations of elasto-hydrodynamics in 3D icosahedral quasicrystals. We take the simple case when all functions appearing in equations depend on x_3 and t only. We note that for icosahedral quasicrystals phonon elastic constants (see, for example, [Ding et al. 1993; Hu et al. 2000]) are defined by

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

where δ_{ij} is the Kronecker delta (i.e., $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$) λ, μ are given constants such that $\mu>0$, $\lambda+2\mu>0$. All nonzero phason, phonon-phason coupling elastic constants are given by the following expressions [Ding et al. 1993; Hu et al. 2000]:

$$\begin{aligned}
K_{1111}=K_{2222}=K_{1212}=K_{2121}&=K_1,\\
K_{1131}=K_{3111}=K_{1113}=K_{1311}=K_{2213}=K_{1322}=K_{2312}=K_{1223}&=K_2,\\
K_{2231}=K_{3122}=K_{2321}=K_{2123}=K_{1232}=K_{3312}=K_{3221}=K_{2132}&=-K_2,\\
K_{3333}&=K_1+K_2,\\
K_{2323}=K_{3131}=K_{3232}=K_{1313}&=K_1-K_2;\\
R_{1111}=R_{1122}=R_{1133}=R_{1113}=R_{2233}=R_{2332}=R_{3111}=R_{3131}&=R,\\
R_{1221}=R_{3232}=R_{1311}=R_{1331}=R_{2121}&=R,\\
R_{2211}=R_{2222}=R_{2213}=R_{2312}=R_{2321}=R_{3122}=R_{1223}=R_{1212}&=-R,\\
R_{3212}=R_{3221}=R_{1322}=R_{1321}=R_{2123}=R_{2112}&=-R,\\
R_{3333}&=-2R.
\end{aligned}$$

Here K_1, K_2, R are given constants such that $K_1 > 0$, $K_2 > 0$, $K_1 - K_2 > 0$.

Using this presentation of phonon, phason, and phono-phason coupling constants and the fact that components of vector functions $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$ depend on x_3, t only, equations of elasto-hydrodynamics (2-9), (2-10) can be written in the form

$$\rho \frac{\partial^2 \boldsymbol{u}(x_3,t)}{\partial t^2} = \mathcal{C}_{33} \frac{\partial^2 \boldsymbol{u}(x_3,t)}{\partial x_3^2} + \mathcal{R}_{33} \frac{\partial^2 \boldsymbol{w}(x_3,t)}{\partial x_3^2} + \boldsymbol{f}(x_3,t), \tag{4-1}$$

$$\kappa \frac{\partial \boldsymbol{w}(x_3,t)}{\partial t} = \mathcal{K}_{33} \frac{\partial^2 \boldsymbol{w}(x_3,t)}{\partial x_3^2} + \mathcal{R}_{33} \frac{\partial^2 \boldsymbol{u}(x_3,t)}{\partial x_3^2} + \boldsymbol{g}(x_3,t), \tag{4-2}$$

where

$$\mathcal{C}_{33} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda + 2\mu \end{bmatrix}, \quad \mathcal{R}_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2R \end{bmatrix},$$

$$\mathcal{K}_{33} = \begin{bmatrix} K_1 - K_2 & 0 & 0 \\ 0 & K_1 - K_2 & 0 \\ 0 & 0 & K_1 + K_2, \end{bmatrix},$$

Let us consider, for simplicity, the case $\boldsymbol{f}(x_3,t) = 0$, $\boldsymbol{g}(x_3,t) = 0$. The initial value problem for equations (4-1), (4-2) consists of finding the vector functions $\boldsymbol{u}(x_3,t)$ and $\boldsymbol{w}(x_3,t)$ satisfying (4-1), (4-2) and initial data

$$u_j(x,0) = 0, \quad \frac{\partial}{\partial t} u_j(x,t)|_{t=0} = \delta_{j3}\, \delta_\Omega(x_3), \tag{4-3}$$

$$w_j(x,0) = 0, \quad j = 1, 2, 3; \tag{4-4}$$

where for a given positive constant Ω the function $\delta_\Omega(x_3)$ of the variable x_3 is defined by

$$\delta_\Omega(x_3) = \frac{\sin(\Omega x_3)}{\pi x_3} \ (x_3 \neq 0), \quad \delta_\Omega(0) = \frac{\Omega}{\pi}. \tag{4-5}$$

Remark 4.1. We note that the Fourier transform of $\delta_\Omega(x_3)$ is the rectangular function $\Pi_\Omega(\nu_3)$ which is equal to 1 for $\nu_3 \in [-\Omega, \Omega]$ and equal to zero for $|\nu_3| > \Omega$ (see formula (C-4) in Appendix C).

Equations (4-1)–(4-4) are written in terms of Fourier images with respect to the space variable x_3 as follows:

$$\rho \frac{d^2 \tilde{\boldsymbol{u}}(\nu_3,t)}{dt^2} + \mathcal{C}(\nu_3) \tilde{\boldsymbol{u}}(\nu_3,t) + \mathcal{R}(\nu_3) \tilde{\boldsymbol{w}}(\nu_3,t) = 0, \quad t > 0, \tag{4-6}$$

$$\kappa \frac{d \tilde{\boldsymbol{w}}(\nu_3,t)}{dt} + \mathcal{K}(\nu_3) \tilde{\boldsymbol{w}}(\nu_3,t) + \mathcal{R}(\nu_3) \tilde{\boldsymbol{u}}(\nu_3,t) = 0, \quad t > 0, \tag{4-7}$$

$$\tilde{u}_j(\nu,0) = 0, \quad \frac{d}{dt} \tilde{u}_j(\nu,t)|_{t=0} = \delta_{3j} \Pi_\Omega(\nu_3), \tag{4-8}$$

$$\tilde{w}_j(\nu_3,0) = 0, \tag{4-9}$$

where

$$\mathcal{C}(\nu_3) = \mathrm{diag}(\mu\nu_3^2, \mu\nu_3^2, (\lambda+2\mu)\nu_3^2),$$
$$\mathcal{K}(\nu_3) = \mathrm{diag}((K_1-K_2)\nu_3^2, (K_1-K_2)\nu_3^2, (K_1+K_2)\nu_3^2),$$
$$\mathcal{R}(\nu_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2R\nu_3^2 \end{bmatrix},$$

Applying the reasonings of Section 3 we reduce equations (4-6)–(4-7) with data (4-8)–(4-9) to the system of integral equations of the form (3-23), (3-24), where

$$V(\nu_3,t) = (V_1(\nu_3,t), V_2(\nu_3,t), V_3(\nu_3,t)), \quad S(\nu_3,t) = (S_1(\nu_3,t), S_2(\nu_3,t), S_3(\nu_3,t)),$$
$$V_j(\nu_3,t) \equiv \tilde{u}_j(\nu_3,t), \quad S_j(\nu_3,t) \equiv \tilde{w}_j(\nu_3,t), \quad j=1,2,3;$$
$$\boldsymbol{h}^1(\nu_3,t) = \begin{pmatrix} 0 \\ 0 \\ \Pi_\Omega(\nu_3)\sin(c_p|\nu_3|t)/(c_p|\nu_3|) \end{pmatrix}, \quad \boldsymbol{h}^2(\nu_3,t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$
$$(\boldsymbol{K}^1 S)(\nu_3,t,\tau) = \begin{pmatrix} 0 \\ 0 \\ 2R|\nu_3|\sin(c_p|\nu_3|(t-\tau))S_3(\nu_3,\tau)/(\rho c_p) \end{pmatrix},$$
$$(\boldsymbol{K}^2 V)(\nu_3,t,\tau) = \begin{pmatrix} 0 \\ 0 \\ 2R|\nu_3|^2\exp(-a^2|\nu_3|^2(t-\tau))V_3(\nu_3,\tau)/\kappa \end{pmatrix}.$$

Here

$$c_p = \frac{\lambda+2\mu}{\rho}, \quad a = \frac{K_1+K_2}{\kappa}.$$

It follows from the obtained integral equations that

$$V_k(\nu_3,t) \equiv \tilde{u}_k(\nu_3,t) = 0, \quad S_k(\nu_3,t) \equiv \tilde{w}_k(\nu_3,t) = 0, \quad k=1,2;$$

and functions $V_3(\nu_3,t) \equiv \tilde{u}_3(\nu_3,t)$, $S_3(\nu_3,t) \equiv \tilde{w}_3(\nu_3,t)$ satisfy the following integral equations

$$V_3(\nu_3,t) = \frac{1}{c_p|\nu_3|}\sin(c_p|\nu_3|t)\Pi_\Omega(\nu_3) + \frac{2R|\nu_3|}{\rho c_p}\int_0^t \sin(c_p|\nu_3|(t-\tau))S_3(\nu_3,\tau)\,d\tau, \tag{4-10}$$
$$S_3(\nu_3,t) = \frac{2R\,|\nu_3|^2}{\kappa}\int_0^t \exp(-a^2\,|\nu_3|^2\,(t-\tau))V_3(\nu_3,\tau)\,d\tau. \tag{4-11}$$

As a result, we find a solution $V_3(\nu_3,t)$, $S_3(\nu_3,t)$ of (4-10), (4-11) in the form

$$V_3(\nu_3,t) = \sum_{n=0}^{\infty} V_3^{(n)}(\nu_3,t), \quad S_3(\nu_3,t) = \sum_{n=0}^{\infty} S_3^{(n)}(\nu_3,t), \tag{4-12}$$

where

$$V_3^{(0)}(\nu_3,t) = \frac{1}{c_p\,|\nu_3|}\sin(c_p\,|\nu_3|\,t)\Pi_\Omega(\nu_3), \quad S_3^{(0)}(\nu,t)=0,$$

$$V_3^{(n)}(\nu_3,t) = \frac{2R\,|\nu_3|}{\rho\, c_p}\int_0^t \sin(c_p\,|\nu_3|\,(t-\tau))S_3^{(n-1)}(\nu_3,\tau)\,d\tau,$$

$$S_3^{(n)}(\nu_3,t) = \frac{2R\,|\nu_3|^2}{\kappa}\int_0^t \exp(-a^2\,|\nu_3|^2\,(t-\tau))V_3^{(n-1)}(\nu_3,\tau)\,d\tau, \quad n=1,2,3,\dots.$$

Using the reasonings of Section 3 we prove the absolute and uniform convergence of series (4-12) and that functions $V_3(\nu_3,t)$, $S_3(\nu_3,t)$ and their partial derivatives

$$\frac{\partial V_3(\nu_3,t)}{\partial t},\quad \frac{\partial^2 V_3(\nu_3,t)}{\partial t^2},\quad \frac{\partial S_3(\nu_3,t)}{\partial t}$$

are continuous on $\Delta_3(T,\Omega)=\{(\nu_3,t)\mid \nu_3\in R,\ \nu_3\neq 0,\ |\nu_3|\le\Omega,\ 0\le t\le T\}$ and equal to zero for $|\nu_3|>\Omega$, $0\le t\le T$. Moreover we show that vector functions $\tilde{\boldsymbol{u}}(\nu_3,t)=(0,0,V_3(\nu_3,t))$, $\tilde{\boldsymbol{w}}(\nu_3,t)=(0,0,S_3(\nu_3,t))$ are the solution of (4-6)–(4-9). We note that the inverse Fourier transform with respect to ν_3 of $V_3(\nu_3,t)$, $S_3(\nu_3,t)$ exists and the obtained functions are differentiable. We denote these functions by $u_3(x_3,t)$, $w_3(x_3,t)$. Similar to Section 3 we show that $\boldsymbol{u}(x_3,t)=(0,0,u_3(x_3,t))$, $\boldsymbol{w}(x_3,t)=(0,0,w_3(x_3,t))$ are the solution of the initial value problem of elasto-hydrodynamic equations (4-1)–(4-4). Moreover, applying the inverse Fourier transform to (4-12) and using properties of the direct and inverse Fourier transforms (see equalities (C-1)–(C-11) in Appendix C) we find

$$u_3(x_3,t)=\sum_{n=0}^{\infty}u_3^{(n)}(x_3,t),\quad w_3(x_3,t)=\sum_{n=0}^{\infty}w_3^{(n)}(x_3,t), \tag{4-13}$$

where

$$u_3^{(0)}(x_3,t)=\frac{1}{2c_p}\int_{x_3-c_pt}^{x_3+c_pt}\delta_\Omega(\xi)\,d\xi,\quad w_3^{(0)}(x_3,t)=0, \tag{4-14}$$

$$u_3^{(n)}(x_3,t)=-\frac{R}{c_p\rho}\int_0^t\left(\int_{x_3-c_p(t-\tau)}^{x_3+c_p(t-\tau)}\frac{\partial^2 w_3^{(n-1)}(\xi,\tau)}{\partial\xi^2}\,d\xi\right)d\tau, \tag{4-15}$$

$$w_3^{(n)}(x_3,t)=-\frac{R}{a\kappa\sqrt{\pi}}\int_0^t\left(\int_{-\infty}^{+\infty}\exp\left(-\frac{(\xi-x_3)^2}{4a^2(t-\tau)}\right)\frac{\partial^2 u_3^{(n-1)}(\xi,\tau)}{\partial\xi^2}\,d\xi\right)\frac{d\tau}{\sqrt{t-\tau}}, \tag{4-16}$$
$$n=1,2,3,\dots$$

Hence the solution of the initial value problem (4-1)–(4-4) is found by formulas $\boldsymbol{u}(x_3,t)=(0,0,u_3(x_3,t))$, $\boldsymbol{w}(x_3,t)=(0,0,w_3(x_3,t))$, where $u_3(x_3,t)$ and $w_3(x_3,t)$ are defined by (4-13)–(4-16).

5. Conclusion

The new approach to solve the initial value problem for differential equations of elasto-hydrodynamics in 3D quasicrystals with the general structure of anisotropy is described in the paper. This approach consists of derivation of phonon and phason displacements by solving the dynamic equations describing the wave propagation for phonon and diffusion process for phason in quasicrystals with the arbitrary anisotropy. There are the following important steps in this approach. The equations of elasto-hydrodynamics in quasicrystals are written in terms of the Fourier images with respect to 3D space variable. The solution of the obtained initial value problem is reduced to the vector integral equation of the Volterra type depending on 3D Fourier parameter. The vector integral equation is solved by successive approximations. The phonon and phason displacements are found by matrix transformations and the application of the inverse Fourier transform to the solution of the vector integral equation. To make all steps correctly we need the following assumption. The Fourier transform with respect to 3D space variable of the given phonon, phason forces as well as initial displacements have to be vector functions with components which have finite supports with respect to Fourier parameters.

We note that if the image of the Fourier transform of a smooth function with respect to space variables has the finite support $\{\nu = (\nu_1, \nu_2, \nu_3) : |\nu| \le \Omega\}$ then we can apply the inverse Fourier transform to this image and the obtained result is the smooth function which belongs to the Paley–Wiener space (see, for example, [Andersen 2004]). Here $\Omega > 0$ is a given constant. From a physical point of view we make the following constraints on the mathematical model of elasto-hydrodynamics in QCs. We exclude high space frequencies in oscillation which are described by phonon and phason displacements. This means that we exclude some sources of oscillation in our consideration. For example, many mathematical models use a point source (in initial data or inhomogeneous terms of equations) which is described by the Dirac delta function. The Dirac delta function is not a classical function - it is a generalized function (distribution). The Dirac delta function does not belong to the Paley–Wiener space. This means that our approach can not be applied for equations of elasto-hydrodynamics in the case if the Dirac delta function appears in initial data or inhomogeneous terms of equations. However the Dirac delta function can be approximated by functions from Paley–Wiener space. For example, the Dirac delta function of one variable $\delta(x_1)$ can be approximated by

$$\delta_\epsilon(x_1) = \frac{\sin(\epsilon x_1)}{\pi x_1}, \quad x_1 \neq 0; \quad \delta_\epsilon(0) = \frac{\epsilon}{\pi},$$

where $\epsilon > 0$ is the sufficiently large parameter of approximation (regularization). Moreover the inverse Fourier transform of the rectangular function $\Pi_\epsilon(\nu_1)$, which is equal to 1 for $|\nu_1| \le \epsilon$ and equal to zero for $|\nu_1| > \epsilon$, is the function $\delta_\epsilon(x_1)$. Therefore $\delta_\epsilon(x_1)$ belongs to the Paley–Wiener space. Using this remark we can approximate the Dirac delta function of three space variables (if the Dirac delta function appears in the description of the phonon, phason forces or initial displacements) by the function $\delta_\epsilon(x_1)\delta_\epsilon(x_2)\delta_\epsilon(x_3)$ which is from the Paley–Wiener space and then apply our approach to find an approximate solution of the original problem. In Section 4 the approach is illustrated on the example of solving equations of elasto-hydrodynamics for 3D icosahedral quasicrystals in the case when components of phonon and phason displacements depend on x_3 and t variables and one component of phonon initial data $\boldsymbol{u}^1$ contains the function $\delta_\epsilon(x_3)$. Here the image of the Fourier transform of the solution of equations in 3D icosahedral quasicrystals is found by successive approximations and then the application of the inverse Fourier transform has been made analytically. Unfortunately the operation of the inverse Fourier transform can not be made analytically for all quasicrystals with general structure of anisotropy and we need to apply it numerically.

At the end of conclusion we would like to say that the present paper is the first part of the study which is related with the theoretical background of the second part of this study, where we plan to described the computational methods of the initial value problems solving for differential equations of elasto-hydrodynamics of quasicrystals with the general structure of anisotropy.

Appendix A. Consequence of the positivity of elastic energy for Section 2A

With notation of the work [Hu et al. 2000] the elastic energy is written as follows (see [Hu et al. 2000, Formula 4.17])

$$F = \frac{1}{2}[\varepsilon, \omega]\begin{bmatrix} \boldsymbol{C} & \boldsymbol{R} \\ \boldsymbol{R}^T & \boldsymbol{K} \end{bmatrix}\begin{bmatrix} \varepsilon \\ \omega \end{bmatrix},$$

where $[\varepsilon, \omega]$ is the vector row whose elements are phonon and phason strains ε_{ij}, ω_{ij}, respectively; $\left[\begin{smallmatrix} \varepsilon \\ \omega \end{smallmatrix}\right]$ is the vector column whose elements are phonon and phason strains ε_{kl}, ω_{kl}; $\boldsymbol{C} = [C_{ijkl}]$ is the matrix of the phonon elastic modules, $\boldsymbol{K} = [K_{ijkl}]$ is the matrix of the phason elastic modules, $\boldsymbol{R} = [R_{ijkl}]$ is the matrix of the phonon-phason coupling elastic modules, $\boldsymbol{R}^T = [R_{klij}]$ is transpose to $\boldsymbol{R}$.

Since the elastic energy F is positive (see [Hu et al. 2000, (5.40)]) then the symmetric matrix

$$\begin{bmatrix} \boldsymbol{C} & \boldsymbol{R} \\ \boldsymbol{R}^T & \boldsymbol{K} \end{bmatrix}$$

is always positive definite. This means that

$$[\varepsilon,\ \omega]\begin{bmatrix} \boldsymbol{C} & \boldsymbol{R} \\ \boldsymbol{R}^T & \boldsymbol{K} \end{bmatrix}\begin{bmatrix} \varepsilon \\ \omega \end{bmatrix} > 0$$

for any $[\varepsilon, \omega] \neq 0$ and in particular for any $[\varepsilon, 0] \neq 0$ or any $[0, \omega] \neq 0$ the last inequality implies

$$[\varepsilon, 0]\begin{bmatrix} \boldsymbol{C} & \boldsymbol{R} \\ \boldsymbol{R}^T & \boldsymbol{K} \end{bmatrix}\begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} = \varepsilon^T \boldsymbol{C} \varepsilon > 0,$$

and

$$[0, \omega]\begin{bmatrix} \boldsymbol{C} & \boldsymbol{R} \\ \boldsymbol{R}^T & \boldsymbol{K} \end{bmatrix}\begin{bmatrix} 0 \\ \omega \end{bmatrix} = \omega^T \boldsymbol{K} \omega > 0.$$

Therefore the positivity of energy implies the positive definiteness of matrices $\boldsymbol{C}$ and $\boldsymbol{K}$ and the last inequalities can be written in the form of inequalities (2-4) of the present paper.

Appendix B. Positive definiteness of $\mathcal{C}(\nu)$ and $\mathcal{K}(\nu)$

Let

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad \alpha^T = (\alpha_1, \alpha_2, \alpha_3), \quad \nu = (\nu_1, \nu_2, \nu_3), \quad \nu_1^2 + \nu_2^2 + \nu_3^2 \neq 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0;$$

matrices $\mathcal{C}(\nu)$ and $\mathcal{K}(\nu)$ be defined by (3-7), (3-8), (2-11), (2-13). Using (2-3), (2-11), (2-13), (3-7), (3-8) we find

$$\alpha^T \mathcal{C}(\nu)\alpha = \sum_{j,l,i,k=1}^{3} C_{ijkl}(\alpha_i \nu_j)(\alpha_k \nu_l), \quad \alpha^T \mathcal{K}(\nu)\alpha = \sum_{j,l,i,k=1}^{3} K_{ijkl}(\alpha_i \nu_j)(\alpha_k \nu_l).$$

We note that $\varepsilon_{ij} = \alpha_i \nu_j$ (or $\omega_{ij} = \alpha_i \nu_j$) are not zero entirely. Therefore, using (2-4) we find that

$$\alpha^T \mathcal{C}(\nu)\alpha > 0, \quad \alpha^T \mathcal{K}(\nu)\alpha > 0$$

for $\nu_1^2 + \nu_2^2 + \nu_3^2 \neq 0$ and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \neq 0$. This means the positive definiteness of matrices $\mathcal{C}(\nu)$ and $\mathcal{K}(\nu)$ for $\nu \neq 0$.

Appendix C. Formulas of the direct and inverse transforms of some functions used in Section 4

In this section we describe some formulas of the Fourier transform which we use in Section 4. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be operators of the direct and inverse transforms defined by

$$\mathcal{F}[\varphi(x_3)](\nu_3) = \int_{-\infty}^{+\infty} \varphi(x_3)\exp(ix_3\nu_3)\,dx_3, \quad \mathcal{F}^{-1}[\tilde{\varphi}(\nu_3)](x_3) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{\varphi}(x_3)\exp(-i\nu_3 x_3)\,d\nu_3$$

for the given piecewise continuous functions $\varphi(x_3)$, $\tilde{\varphi}(\nu_3)$ for which the Fourier transform $\mathcal{F}[\varphi(x_3)](\nu_3)$ and the inverse Fourier transform $\mathcal{F}^{-1}[\tilde{\varphi}(\nu_3)](x_3)$ exist.

Let $c_p > 0$, $t > 0$, $\Omega > 0$ be given constants; $\theta(t)$ be the Heaviside step function (i.e., $\theta(t) = 1$ for $t \geq 0$, $\theta(t) = 0$ for $t < 0$); $\delta_\Omega(x_3)$ be the function defined by (4-5). Then the following formulas take place:

$$\mathcal{F}\Big[\frac{1}{2c_p}\theta(c_p t - |x_3|)\Big](\nu_3) = \frac{1}{c_p|\nu_3|}\sin(c_p|\nu_3|t). \tag{C-1}$$

Proof.

$$\mathcal{F}\Big[\frac{1}{2c_p}\theta(c_p t - |x_3|)\Big](\nu_3) = \frac{1}{2c_p}\int_{-c_p t}^{c_p t}\exp(ix_3\nu_3)\,dx_3 = \frac{1}{c_p|\nu_3|}\sin(c_p|\nu_3|t). \qquad \square$$

$$\mathcal{F}^{-1}\Big[\frac{1}{c_p|\nu_3|}\sin(c_p|\nu_3|t)\Big](x_3) = \frac{1}{2c_p}\theta(c_p t - |x_3|). \tag{C-2}$$

Proof. Applying the operator $\mathcal{F}$ to (C-1) we obtain (C-2). □

$$\mathcal{F}^{-1}[\Pi_\Omega(\nu_3)](x_3) = \frac{\sin(\Omega x_3)}{\pi x_3} \equiv \delta_\Omega(x_3). \tag{C-3}$$

Proof. $\mathcal{F}^{-1}[\Pi_\Omega(\nu_3)](x_3) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\exp(-i\nu_3 x_3)\,d\nu_3 = \delta_\Omega(x_3).$ □

$$\mathcal{F}\Big[\frac{\sin(\Omega x_3)}{\pi x_3}\Big](\nu_3) = \Pi_\Omega(\nu_3). \tag{C-4}$$

Proof. Applying the operator $\mathcal{F}$ to (C-3) we obtain (C-4). □

The next formula and its verification can be found, for example, in [Vladimirov 1971]:

$$\mathcal{F}\Big[\int_{-\infty}^{+\infty}\varphi(\xi)\psi(x_3 - \xi)\,d\xi\Big](\nu_3) = \mathcal{F}[\varphi(x_3)](\nu_3)\cdot\mathcal{F}[\psi(x_3)](\nu_3) \tag{C-5}$$

for any piecewise continuous functions $\varphi(x_3)$ and $\psi(x_3)$ for which the convolution

$$\int_{-\infty}^{+\infty}\varphi(\xi)\psi(x_3 - \xi)\,d\xi$$

exists and absolutely integrable.

Applying the operator $\mathcal{F}^{-1}$ to formula (C-5) we find

$$\mathcal{F}^{-1}\big[\mathcal{F}[\varphi(x_3)](\nu_3)\cdot\mathcal{F}[\psi(x_3)](\nu_3)\big](x_3)=\int_{-\infty}^{+\infty}\varphi(\xi)\psi(x_3-\xi)\,d\xi. \tag{C-6}$$

The formula (C-6) for $\psi(x_3)=(1/2\pi)\theta(c_pt-|x_3|)$ and $\varphi(x_3)=\delta_\Omega(x_3)$ can be written in the form

$$\mathcal{F}^{-1}\bigg[\frac{1}{c_p|\nu_3|}\sin(c_p|\nu_3|t)\cdot\Pi_\Omega(\nu_3)\bigg](x_3)=\int_{-\infty}^{+\infty}\frac{1}{2c_p}\theta(c_pt-|\xi-x_3|)\delta_\Omega(\xi)\,d\xi$$
$$=\frac{1}{2c_p}\int_{x_3-c_pt}^{x_3+c_pt}\delta_\Omega(\xi)\,d\xi. \tag{C-7}$$

$$\mathcal{F}^{-1}[\exp(-a^2\,\nu_3^2\,(t-\tau))](x_3)=\frac{1}{2a\sqrt{\pi(t-\tau)}}\exp\bigg(-\frac{x_3^2}{4a^2\,(t-\tau)}\bigg). \tag{C-8}$$

Proof.

$$\mathcal{F}^{-1}[\exp(-a^2\nu_3^2(t-\tau))](x_3)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-a^2\nu_3^2(t-\tau))\exp(-i\nu_3x_3)\,d\nu_3$$
$$=\frac{1}{\pi}\int_0^{+\infty}\exp(-a^2\,\nu_3^2\,(t-\tau))\cos(\nu_3x_3)\,d\nu_3$$
$$=\frac{1}{2a\sqrt{\pi(t-\tau)}}\exp\bigg(-\frac{x_3^2}{4a^2(t-\tau)}\bigg).$$

□

The next formula and its proof can be found in [Vladimirov 1971]:

$$\mathcal{F}^{-1}[\nu_3^2\tilde{\varphi}(\nu_3)](x_3)=-\frac{\partial^2\varphi(x_3)}{\partial x_3^2}. \tag{C-9}$$

Using formulas (C-6), (C-8) and (C-9) for

$$\psi(x_3)=\frac{1}{2a\sqrt{\pi(t-\tau)}}\exp(-\frac{x_3^2}{4a^2\,(t-\tau)})$$

we find

$$\mathcal{F}^{-1}[\exp(-a^2\nu_3^2(t-\tau))\nu_3^2\tilde{\varphi}(\nu_3)](x_3)=-\frac{1}{2a\sqrt{\pi(t-\tau)}}\int_{-\infty}^{+\infty}\exp\bigg(-\frac{(\xi-x_3)^2}{4a^2(t-\tau)}\bigg)\frac{\partial^2\varphi(\xi)}{\partial\xi^2}\,d\xi. \tag{C-10}$$

Using formulas (C-2), (C-6) and (C-9) for $\psi(x_3)=(1/(2c_p))\theta(c_pt-|x_3|)$ we have

$$\mathcal{F}^{-1}\bigg[\frac{1}{c_p|\nu_3|}\sin(c_p|\nu_3|(t-\tau))\nu_3^2\tilde{\varphi}(\nu_3)\bigg](x_3)=-\frac{1}{2c_p}\int_{x_3-c_p(t-\tau)}^{x_3+c_p(t-\tau)}\frac{\partial^2\varphi(\xi)}{\partial\xi^2}\,d\xi. \tag{C-11}$$

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