Journal of
Mechanics of Materials and Structures

ELASTIC FIELDS FOR A PARABOLIC HOLE ENDOWED WITH SURFACE EFFECTS

Xu Wang and Peter Schiavone

Volume 15, No. 2 March 2020

msp

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES
Vol. 15, No. 2, 2020
dx.doi.org/10.2140/jomms.2020.15.241

ELASTIC FIELDS FOR A PARABOLIC HOLE ENDOWED WITH SURFACE EFFECTS

XU WANG AND PETER SCHIAVONE

We study the contribution of surface energy to the plane strain deformations of an isotropic elastic material with a parabolic hole. A closed-form full-field solution is derived using Muskhelishvili's complex variable formulation and the technique of conformal mapping. We also obtain the corresponding expressions for the stresses and displacements in the full-field arising solely from the influence of the surface effects. When the radius of curvature at the vertex of the parabola is reduced to the order of the square of the ratio of surface energy to the generalized stress intensity factor, the surface contribution is significant and cannot be disregarded.

1. Introduction

The elastic stress field near the tip of a blunt crack was first obtained in [Creager 1966; Creager and Paris 1967] from the full-field solutions in the case of an elliptical hole and hyperbolic notch. Recent studies [Wang and Schiavone 2019a; 2019b] have shown that the full-field solution for the in-plane deformations of an isotropic material with a traction-free parabolic boundary is simply the asymptotic elastic field near a blunt crack tip presented in [Creager and Paris 1967]. This fact implies that for all intents and purposes, a blunt crack can be represented quite simply by a parabola. On the other hand, surface energies, tension and stresses come into play in nanostructured systems where surface to volume ratios become appreciable [Miller and Shenoy 2000; Shenoy 2002; Sharma and Ganti 2004; Wang and Wang 2006].

In this paper, we use Muskhelishvili's complex variable formulation together with the technique of conformal mapping to derive a closed-form solution to the problem of plane strain deformations of an isotropic elastic material with a parabolic hole. We emphasize that the "hole" studied here is an open hole bounded by an infinitely extended parabolic boundary. Here, the parabola is endowed with a simplified version of the Gurtin–Murdoch surface elasticity model [Gurtin and Murdoch 1975; Gurtin et al. 1998; Shenoy 2002; Ru 2010]. In this simplified version, the surface energy is independent of the surface strain tensor [Yang 2004; Wang and Wang 2006]. The present solution can then be superimposed with the corresponding solution involving a parabola without surface effects in which the material is subjected to far-field external loading characterized by the generalized mode I and mode II stress intensity factors [Creager and Paris 1967; Wang and Schiavone 2019b] to arrive at the solution in the case of a parabolic hole with surface effects in an elastic material subjected to external loading.

Keywords: surface effect, blunt crack, plane elasticity, closed-form solution, complex variable method.

2. Governing equations

In a fixed rectangular coordinate system $\{x_i\}$ $(i = 1, 2, 3)$, let σ_{ij}, ε_{ij} and u_i be respectively, the stresses, strains and displacements in an isotropic elastic material. In the absence of body forces, the equilibrium and constitutive equations describing the deformations of an isotropic elastic (bulk) solid are

$$\sigma_{ij,j} = 0, \quad \sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{1}$$

where λ, μ are the Lamé constants of the material, i, j, k range from 1 to 3, we sum over repeated indices and δ_{ij} is the Kronecker delta.

For the plane strain deformations of the isotropic elastic bulk material, the three in-plane stresses $(\sigma_{11},\ \sigma_{22},\ \sigma_{12})$, two in-plane displacements $(u_1,\ u_2)$ and two stress functions $(\phi_1,\ \phi_2)$ are given in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ of the complex variable $z = x_1 + \mathrm{i}x_2$ as follows [Muskhelishvili 1953]:

$$\sigma_{11} + \sigma_{22} = 2\big[\varphi'(z) + \overline{\varphi'(z)}\big], \qquad \sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} = 2[\bar{z}\varphi''(z) + \psi'(z)], \tag{2}$$

$$2\mu(u_1 + \mathrm{i}u_2) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \qquad \phi_1 + \mathrm{i}\phi_2 = \mathrm{i}\big[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\big]. \tag{3}$$

Here κ is defined as

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\nu, \tag{4}$$

where ν is Poisson's ratio $\big(0 \le \nu \le \frac{1}{2}\big)$.

In addition, the stresses are related to the stress functions through the equalities [Ting 1996]

$$\begin{aligned} \sigma_{11} &= -\phi_{1,2}, & \sigma_{12} &= \phi_{1,1}, \\ \sigma_{21} &= -\phi_{2,2}, & \sigma_{22} &= \phi_{2,1}. \end{aligned} \tag{5}$$

We consider the plane strain deformations of an infinite elastic body containing a parabolic hole as depicted in Figure 1. Let t_1 and t_2 be the surface tractions along the x_1 and x_2 directions on the boundary L. If s is the arc length measured along L such that when traveling in the direction of increasing s, the material remains on the left-hand side, we will have

$$t_1 + \mathrm{i}t_2 = -\frac{d\phi_1}{ds} - \mathrm{i}\frac{d\phi_2}{ds}, \tag{6}$$

from which we can obtain the relation

$$\int_A^z (\sigma_{\rho\rho} + \mathrm{i}\sigma_{\rho\theta})dz = -\mathrm{i}(\phi_1 + \mathrm{i}\phi_2), \tag{7}$$

where $\sigma_{\rho\rho}$ and $\sigma_{\rho\theta}$ are the normal and tangential components of the traction along the boundary L and the integral is taken along L from a fixed point A.

The 2×2 surface stress tensor $\sigma^s_{\alpha\beta}$ is related to the surface energy density γ and the surface strain tensor $\varepsilon^s_{\alpha\beta}$ through the equation [Miller and Shenoy 2000; Shenoy 2002; Wang and Wang 2006]

$$\sigma^s_{\alpha\beta} = \gamma\delta_{\alpha\beta} + \frac{\partial\gamma}{\partial\varepsilon^s_{\alpha\beta}}. \tag{8}$$

We will assume that the surface energy density is independent of the surface strain tensor [Yang 2004;

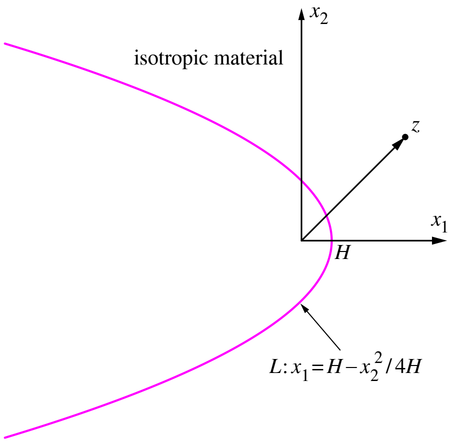

Figure 1. Plane strain deformations of an isotropic elastic material with a parabolic hole endowed with surface effects.

Wang and Wang 2006]. In this case, (8) reduces to

$$\sigma^s_{\alpha\beta} = \gamma\delta_{\alpha\beta}. \tag{9}$$

If the surface is perfectly bonded to the bulk, the equilibrium conditions on the surface are [Gurtin and Murdoch 1975; Gurtin et al. 1998; Shenoy 2002; Ru 2010; Kim et al. 2011]

$$\begin{aligned} \sigma_{\alpha j} n_j \underline{e}_\alpha + \sigma^s_{\alpha\beta,\beta}\underline{e}_\alpha &= 0 \quad \text{(tangential direction)},\\ \sigma_{ij} n_i n_j &= \sigma^s_{\alpha\beta}\kappa_{\alpha\beta} \quad \text{(normal direction)}, \end{aligned} \tag{10}$$

where n_i is the unit normal vector to the surface and $\kappa_{\alpha\beta}$ is the curvature tensor of the surface.

3. A parabolic hole endowed with surface effects

As shown in Figure 1, we consider the plane strain deformations of an isotropic elastic material that occupies the region

$$x_1 \geq H - \frac{x_2^2}{4H}, \qquad H > 0, \tag{11}$$

the boundary of which is a parabola L described by

$$x_1 = H - \frac{x_2^2}{4H}, \qquad z \in L. \tag{12}$$

In this configuration of the problem, the parameter H is the single geometric parameter of the material. It follows from (2), (7), (9) and (10) that when the surface effect is incorporated, the boundary condition along the parabola L can be written as

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \int_A^z (\sigma_{\rho\rho} + \mathrm{i}\sigma_{\rho\theta})\, dz, \qquad z \in L, \tag{13}$$

where

$$\sigma_{\rho\rho} = \frac{\gamma}{\rho}, \quad \sigma_{\rho\theta} = 0, \tag{14}$$

in which ρ is the radius of curvature of the parabola.

For the convenience of the ensuing analysis, we introduce the conformal mapping function

$$z = \omega(\xi) = \xi^2, \quad \xi = \omega^{-1}(z) = \sqrt{z}, \quad \mathrm{Re}\{\xi\} \geq h, \tag{15}$$

where $h = \sqrt{H}$. Using this mapping function, the exterior of the parabolic hole in the z-plane is mapped onto the right half-plane $\mathrm{Re}\{\xi\} \geq h$ in the ξ-plane and the parabola L is mapped onto the vertical straight line $\mathrm{Re}\{\xi\} = h$, $-\infty < \mathrm{Im}\{\xi\} < +\infty$ in the ξ-plane.

Additionally, using the mapping function in (15), the curvature of the parabola can be determined as

$$\kappa = \frac{1}{\rho} = \frac{h}{2(2h\xi - \xi^2)^{3/2}}, \quad \mathrm{Re}\{\xi\} = h. \tag{16}$$

With the aid of (16), the boundary condition in (13) can be further written in the form

$$\varphi(\xi) + \frac{\omega(\xi)\overline{\varphi'(\xi)}}{\overline{\omega'(\xi)}} + \overline{\psi(\xi)} = \frac{\gamma\xi}{\sqrt{2h\xi - \xi^2}}, \quad \mathrm{Re}\{\xi\} = h, \tag{17}$$

where, for convenience, we write $\varphi(\xi) = \varphi(\omega(\xi)) = \varphi(z)$ and $\psi(\xi) = \psi(\omega(\xi)) = \psi(z)$.

In the absence of far-field external loading characterized by the generalized mode I and mode II stress intensity factors K_{I} and K_{II}, the two analytic functions $\varphi(\xi)$ and $\psi(\xi)$ can be derived from (17) as follows:

$$\varphi(\xi) = \gamma\sqrt{\frac{\xi}{2h}}, \qquad \psi(\xi) = \gamma\sqrt{\frac{2h-\xi}{\xi}} - \gamma\sqrt{\frac{2h-\xi}{2h}} - \frac{\gamma(2h-\xi)^2}{4\xi\sqrt{2h\xi}}, \quad \mathrm{Re}\{\xi\} \geq h. \tag{18}$$

Substitution of (18) into (2) and (3) yields the following expressions for the stresses and displacements in the full-field:

$$\begin{aligned}
\sigma_{11} + \sigma_{22} &= \frac{\gamma}{\sqrt{2h}} \mathrm{Re}\{\xi^{-3/2}\}, \\
\sigma_{22} - \sigma_{11} + 2\mathrm{i}\sigma_{12} &= \frac{\gamma}{8\sqrt{2h}}\left[-3\xi^{-7/2}\bar{\xi}^2 + \frac{4\xi^{-5/2}[\xi^{3/2} - (2h)^{3/2}]}{\sqrt{2h-\xi}} + \xi^{-7/2}(2h-\xi)(\xi+6h)\right]; \\
2\mu(u_1 + \mathrm{i}u_2) &= \kappa\gamma\sqrt{\frac{\xi}{2h}} + \frac{\gamma\bar{\xi}^{-3/2}[(2h-\bar{\xi})^2 - \xi^2]}{4\sqrt{2h}} - \gamma\sqrt{\frac{2h-\bar{\xi}}{\bar{\xi}}} + \gamma\sqrt{\frac{2h-\bar{\xi}}{2h}}, \quad \mathrm{Re}\{\xi\} \geq h.
\end{aligned} \tag{19}$$

It is clear from (19) that the stresses and displacements are regular everywhere in the material including at points on the parabola. In particular, the hoop stress $\sigma_{\theta\theta}$ along the parabola can be determined as

$$\sigma_{\theta\theta} = \frac{\gamma}{\sqrt{2h}} \mathrm{Re}\{\xi^{-3/2}\} - \frac{\gamma h}{2(2h\xi - \xi^2)^{3/2}}, \quad \mathrm{Re}\{\xi\} = h, \tag{20}$$

which is illustrated in Figure 2. It is seen from Figure 2 that the hoop stress induced by the surface energy is tensile when $|x_2| < 1.806H$ and compressive when $|x_2| > 1.806H$. In performing the calculations, we have adopted the identity that $x_2 = 2h\,\mathrm{Im}\{\xi\}$ along the parabola with $\mathrm{Re}\{\xi\} = h$.

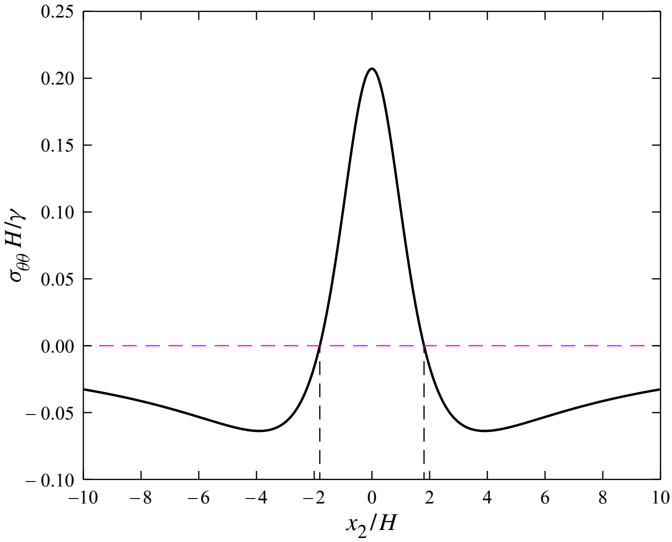

Figure 2. Distribution of the hoop stress along the parabola solely induced by surface energy.

The displacements along the parabola L are given by

$$u_1 + \mathrm{i}u_2 = \frac{\gamma}{2\mu}\left[(\kappa+1)\sqrt{\frac{\xi}{2h}} - \frac{\xi}{\sqrt{2h\xi - \xi^2}}\right], \quad \mathrm{Re}\{\xi\} = h, \tag{21}$$

and are illustrated in Figure 3.

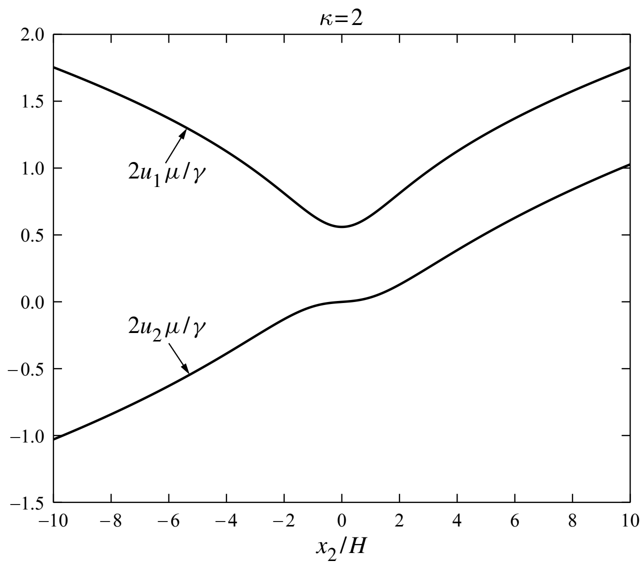

Figure 3. Distributions of the displacement components u_1 and u_2 along the parabola with $\kappa = 2$ solely induced by surface energy.

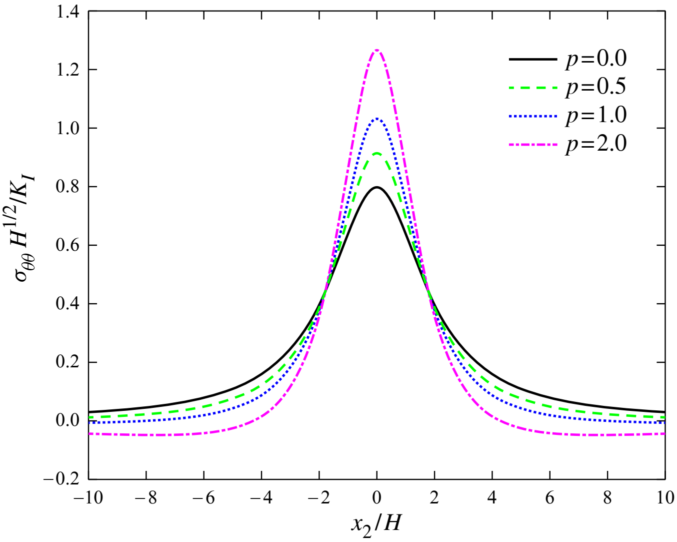

Figure 4. Effect of surface energy on the hoop stress along the parabola under mode I generalized stress intensity factor.

If the bulk material is under the generalized mode I stress intensity factor K_{I}, the hoop stress along the parabola is given by

$$\sigma_{\theta\theta} = \sqrt{\frac{2}{\pi}} K_{\mathrm{I}} h^{-1} \left\{ \mathrm{Re}\{h\xi^{-1}\} + p\,\mathrm{Re}\{h^{3/2}\xi^{-3/2}\} - \frac{p}{\sqrt{2}(2h^{-1}\xi - h^{-2}\xi^2)^{3/2}} \right\}, \quad \mathrm{Re}\{\xi\} = h, \tag{22}$$

where the dimensionless parameter p is defined by

$$p = \frac{\sqrt{\pi}\gamma}{2\sqrt{H} K_{\mathrm{I}}}. \tag{23}$$

We illustrate in Figure 4 the contribution of the surface energy to the hoop stress along the parabola. In the figure, $\sigma_{\theta\theta} = \sqrt{2/\pi}\, K_{\mathrm{I}} \,\mathrm{Re}\{z^{-1/2}\}$ for $p = 0$ along the parabolic boundary L is consistent with the observation in [Creager 1966; Creager and Paris 1967] that the hydrostatic stress (or mean stress) $\sigma_{11} + \sigma_{22}$ is independent of the finite curvature at the tip of a blunt crack in the absence of surface effects. It is seen from Figure 4 that when the geometric parameter H is of the order of $\gamma^2/K_{\mathrm{I}}^2$, the contribution from the surface is significant and cannot be disregarded. For example, when choosing $\gamma \approx 1\,\mathrm{J/m^2}$ [Yang 2004; Kim et al. 2011] and $K_{\mathrm{I}} = 10^4\,\mathrm{N}\sqrt{\mathrm{m}}/\mathrm{m}^2$, the surface effect becomes significant when $H \leq 10\,\mathrm{nm}$. Note that H is just half of the radius of curvature of the parabola at the vertex $(x_1, x_2) = (H, 0)$ (this fact can be seen clearly from (16) by setting $\xi = h = \sqrt{H}$). This implies that the surface effect cannot be ignored for a relatively sharp crack. This conclusion is in qualitative agreement with that in [Wang et al. 2008].

When the material containing a parabolic hole endowed with surface effects is subjected to far-field external loading characterized by the generalized mode I and mode II stress intensity factors, the two analytic functions $\varphi(\xi)$ and $\psi(\xi)$ can be obtained from (18) and the result given in [Wang and Schiavone

2019b] as

$$\varphi(\xi) = \frac{\bar{K}\xi}{\sqrt{2\pi}} + \gamma\sqrt{\frac{\xi}{2h}},$$
$$\psi(\xi) = \frac{(2K-\bar{K})\xi}{2\sqrt{2\pi}} - \frac{2h^2\bar{K}}{\sqrt{2\pi}\xi} + \gamma\sqrt{\frac{2h-\xi}{\xi}} - \gamma\sqrt{\frac{2h-\xi}{2h}} - \frac{\gamma(2h-\xi)^2}{4\xi\sqrt{2h\xi}}, \quad \mathrm{Re}\{\xi\} \geq h, \tag{24}$$

where $K = K_{\mathrm{I}} + \mathrm{i}K_{\mathrm{II}}$.

4. Conclusions

We have incorporated a simplified version of the Gurtin–Murdoch surface elasticity model into the plane strain deformations of an isotropic material with a parabolic hole. In the absence of external loading, a closed-form full-field solution is derived in (18) with the stress and displacement fields given by (19). In the presence of external loading, the closed-form full-field solution is given by (24).

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant 11272121) and through a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada (Grant RGPIN – 2017 - 03716115112).

References

[Creager 1966] M. Creager, *The elastic stress field near the tip of a blunt crack*, Master's thesis, Lehigh University, 1966.

[Creager and Paris 1967] M. Creager and P. C. Paris, "Elastic field equations for blunt cracks with reference to stress corrosion cracking", *Int. J. Fracture* **3** (1967), 247–251.

[Gurtin and Murdoch 1975] M. E. Gurtin and A. Murdoch, "A continuum theory of elastic material surfaces", *Arch. Rational Mech. Anal.* **57** (1975), 291–323.

[Gurtin et al. 1998] M. E. Gurtin, J. Weissmuller, and F. Larche, "A general theory of curved deformable interface in solids at equilibrium", *Philos. Mag. A* **78** (1998), 1093–1109.

[Kim et al. 2011] C. I. Kim, P. Schiavone, and C. Q. Ru, "The effect of surface elasticity on a mode-III interface crack", *Arch. Mech.* **63** (2011), 267–286.

[Miller and Shenoy 2000] R. E. Miller and V. B. Shenoy, "Size-dependent elastic properties of nanosized structural elements", *Nanotechnology* **11** (2000), 139–147.

[Muskhelishvili 1953] N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity: fundamental equations, plane theory of elasticity, torsion and bending*, P. Noordhoff, 1953.

[Ru 2010] C. Q. Ru, "Simple geometrical explanation of Gurtin-Murdoch model of surface elasticity with clarification of its related versions", *Sci. China* **53** (2010), 536–544.

[Sharma and Ganti 2004] P. Sharma and S. Ganti, "Size-dependent Eshelby's tensor for embedded nano-inclusions incorporating surface/interface energies", *ASME J. Appl. Mech.* **71** (2004), 663–671.

[Shenoy 2002] V. B. Shenoy, "Size-dependent rigidities of nanosized torsional elements", *Int. J. Solids Struct.* **39** (2002), 4039–4052.

[Ting 1996] T. C. T. Ting, *Anisotropic elasticity theory and applications*, Oxford Eng. Sci. Ser. **45**, Oxford University Press, 1996.

[Wang and Schiavone 2019a] X. Wang and P. Schiavone, "Elastic field for a blunt crack in a decagonal quasicrystalline material", *Eng. Fract. Mech.* **220** (2019a), art. id. 106657.

[Wang and Schiavone 2019b] X. Wang and P. Schiavone, "New interpretation of the elastic field for a blunt crack", 2019b. Unpublished.

[Wang and Wang 2006] G. F. Wang and T. J. Wang, "Deformation around a nanosized elliptical hole with surface effect", *Appl. Phys. Lett.* **89** (2006), art. id. 161901.

[Wang et al. 2008] G. F. Wang, X. Q. Feng, T. J. Wang, and W. Gao, "Surface effects on the near-tip stresses for mode-I and mode-III cracks", *ASME J. Appl. Mech.* **75** (2008), art. id. 011001.

[Yang 2004] F. Q. Yang, "Size-dependent effective modulus of elastic composite materials: spherical nanocavities at dilute concentrations", *J. Appl. Phys.* **95** (2004), 3516–3520.

Received 11 Oct 2019. Revised 12 Jan 2020. Accepted 29 Jan 2020.

XU WANG: xuwang@ecust.edu.cn
School of Mechanical and Power Engineering, East China University of Science and Technology, Shanghai, China

PETER SCHIAVONE: pschiavo@ualberta.ca
Department of Mechanical Engineering, University of Alberta, 10-203 Donadeo Innovation Center for Engineering, Edmonton, AB, Canada

JOURNAL OF MECHANICS OF MATERIALS AND STRUCTURES

msp.org/jomms

Founded by Charles R. Steele and Marie-Louise Steele

Cover photo: Mando Gomez, www.mandolux.com

See msp.org/jomms for submission guidelines.

JoMMS (ISSN 1559-3959) at Mathematical Sciences Publishers, 798 Evans Hall #6840, c/o University of California, Berkeley, CA 94720-3840, is published in 10 issues a year. The subscription price for 2020 is US $660/year for the electronic version, and $830/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues, and changes of address should be sent to MSP.

JoMMS peer-review and production is managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Journal of Mechanics of Materials and Structures

Volume 15, No. 2 March 2020